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A NEW ITERATIVE APPROXIMATION OF A SPLIT FIXED POINT CONSTRAINT EQUILIBRIUM PROBLEM

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ABSTRACT. The purpose of this paper is to introduce an iterative algorithm for approximating an element in the solution set of the common split feasibility problem for fixed points of demimetric mappings and equilibrium problem for monotone mapping in real Hilbert spaces. Motivated by self-adaptive step size method, we incorporate the inertial technique to accelerate the convergence of the proposed method and establish a strong convergence of the sequence generated by the proposed algorithm. Finally, we present a numerical example to illustrate the significant performance of our method. Our results extend and improve some existing results in the literature.

Key words and phrases: Equilibrium problem; Demimetric mappings; Fixed point problems; Iterative method.

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1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a nonlinear mapping. The Fixed Point Problem (FPP) is to a find point $x \in C$ such that

$$Sx = x, \forall\, x \in C. \tag{1.1}$$

The fixed point set of the mapping S is denoted by $Fix(S)$. The fixed point theory finds its application in the prove of existence of solution of many nonlinear problems arising in many real life situations. From the existence of solution of differential equation to integral equations and evolutionary equations. The fixed point of many linear and nonlinear operators have been considered in the literature (see [16, 17, 18, 25]).

Let H_1 and H_2 be two real Hilbert spaces, $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be nonlinear mappings. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*, then, the following Split Common Fixed Point Problem (SCFPP) is to find:

$$\text{Finding } x \in Fix(S) \text{ such that } Ax \in Fix(T). \tag{1.2}$$

The SCFPP (1.2) initially introduced and studied by Censor and Segal [13], is a generalization of the Split Feasibility Problem (SFP) arising from signal processing and image restoration (see [12, 35]). Note that solving (1.2) can be translated to solving the following fixed point equation (see [1, 2, 3, 4, 5, 19]).

$$x^* = S(x^* - \tau A^*(I - T)Ax^*),\ \tau \geq 0.$$

Recently, Censor and Segal [13] proposed the following algorithm to solve SCFP (1.2):

Algorithm 1.1. *: Initialization: Let $x^* \in H_1 := \mathbb{R}_n$ be arbitrary . Iterative step: let*

$$x_{n+1} = S(x_n - \tau A^*(I - T)Ax_n),\ n \geq 0$$

where $S : \mathbb{R}_n \to \mathbb{R}_n$ and $T : \mathbb{R}_m \to \mathbb{R}_m$ are two directed mappings and $\tau \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A. There has been growing interest in the (SCFPP) due to its various applications, (see for example, [11, 36]*).*

In 2019, Chen *et al.* [14] introduced the following self-adaptive algorithm for solving SCFPP for demimetric mappings in real Hilbert spaces as follows

Algorithm 1.2. *Initialization: Let $x_0 \in H_1$ be arbitrary. For $n \geq 0$, assume the current iterate x_n has been constructed. If*

$$\|x_n - Sx_n + A^*(I - T)Ax_n\| = 0,$$

then stop (in this case x_n solves problem (1.2)*). Otherwise, calculate the next iterate x_{n+1} by the following formula*

$$\begin{cases} y_n = x_n - Sx_n + A^*(I - T)Ax_n, \\ x_{n+1} = x_n - \alpha\tau_n y_n,\ \forall\, n \geq 0, \end{cases} \tag{1.3}$$

where $\alpha \in (0, \min\{1 - \beta, 1 - \mu\})$ is a positive constant and τ_n is chosen self adaptively as

$$\tau_n = \frac{\|x_n - Sx_n\|^2 + \|(I - T)Ax_n\|^2}{\|y_n\|^2}.$$

They assumed the sequence $\{x_n\}$ generated by (1.3) is infinite and prove a weak convergence theorem for approximating the solution of the SCFPP.

In 1994, Blum and Oettli [10] introduced the notion of Equilibrium Problem (EP) as a generalization of certain optimization and variational inequality problems. It has received much attention from many researchers since its establishment due to its application to many problems arising from finance, physics, economics and so on. For this reason, several authors have introduced various generalizations of EP and numerous iterative algorithms have been developed, to solve these problems. The EP consists of finding a point $x \in C$ such

$$F(x, y) \geq 0, \ \forall\, y \in C, \tag{1.4}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction.

The Mixed Equilibrium Problem (MEP) which is a generalization of the EP (1.3) is known to include fixed point problem, optimization problem, variational inequality problem,and Nash equilibrium problem as special cases; (see [10, 20]). Some methods have been proposed to solve the MEP, see, for example, [20, 22]. The Mixed Equilibrium Problem (MEP) is to find $x^* \in C$ such that

$$F(x^*, x) + \phi(x) - \phi(x^*) \geq 0 \text{ for all } x \in C. \tag{1.5}$$

If in (1.5) $\phi = 0$, then the MEP (1.5) reduces to the EP (1.4).

In 2017, Wang [34] introduced the following new iterative algorithm for the SCFPP of directed mappings

Algorithm 1.3. *choose an arbitrary initial guess x_0. Assume x_n has been constructed. If*

$$\|x_n - Sx_n + A^*(I - T)Ax_n\| = 0,$$

then stop; otherwise, continue and construct x_{n+1} via the formula;

$$x_{n+1} = x_n - \tau_n\{\|x_n - Sx_n + A^*(I - T)Ax_n\|\}, \ \forall\, n \geq 0,$$

where τ_n is choose self-adaptively as

$$\tau_n = \frac{\|x_n - Sx_n\|^2 + \|(I - T)Ax_n\|^2}{\|x_n - Sx_n + A^*(I - T)Ax_n\|^2}$$

Algorithm 1.4. *Let $u \in H$ and start an initial guess $x_0 \in H$, assume x_n has been constructed. If*

$$\|x_n - Sx_n + A^*(I - T)Ax_n\| = 0,$$

then stop; otherwise, continue and construct x_{n+1} via the formula;

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)\{x_n - Sx_n + A^*(I - T)Ax_n\}, \ \forall\, n \geq 0,$$

where stepsize sequence τ_n is choose self-adaptively as

$$\tau_n = \frac{\|x_n - Sx_n\|^2 + \|(I - T)Ax_n\|^2}{\|x_n - Sx_n + A^*(I - T)Ax_n\|^2}$$

Wang obtained a weak and a strong convergence of Algorithms 1.3 and 1.4, respectively. Wang's results in [34] from the directed mappings to the demicontractive mappings. Further, they construct the following two self-adaptive algorithms for solving the split common fixed point problem (1.2).

Algorithm 1.5. . *Initialization: Let $x_0 \in H_1$ be arbitrary . For $n \geq 0$, assume the current iterate x_n has been constructed. If*

$$\|x_n - Sx_n + A^*(I-T)Ax_n\| = 0,$$

then stop; otherwise , calculate the next iterate x_{n+1} by the following formula

$$\begin{cases} y_n = x_n - Sx_n + A^*(I-T)Ax_n, \\ z_n = x_n - \alpha_n \tau_n y_n, \ \forall\, n \geq 0, \end{cases}$$

where $\alpha \in (0, \min\{1-\beta, 1-\mu\})$ is a positive constant and τ_n is chosen self-adaptively as

$$\tau_n = \frac{\|x_n - Sx_n\|^2 + \|(I-T)Ax_n\|^2}{\|y_n\|^2}$$

Algorithm 1.6. *Initialization: let $u \in H_1$ be a fixed point and let $x_0 \in H$ be arbitrary. Iterative step: for $n \geq 0$, assume the current iterate x_n has been constructed . If*

$$\|x_n - Sx_n + A^*(I-T)Ax_n\| = 0,$$

then stop; otherwise , calculate the next iterate x_{n+1} by the following formula

$$\begin{cases} y_n = x_n - Sx_n + A^*(I-T)Ax_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)(x_n - \alpha \tau_n y_n), \ \forall\, n \geq 0, \end{cases}$$

where $\alpha \in (0, \min\{1-\beta, 1-\mu\})$ is a positive constant and τ_n is chosen self-adaptively as

$$\tau_n = \frac{\|x_n - Sx_n\|^2 + \|(I-T)Ax_n\|^2}{\|y_n\|^2}$$

They also obtained a weak and a strong convergence result of Algorithms 1.5 and 1.6, respectively. They present two self-adaptive algorithms for solving the split common fixed point problem (1.2).

Recently, Shehu introduced a hybrid method for finding a common fixed point of infinite family of k-strictly pseudocontractive mappings, the set of common solutions to a system of generalized mixed equilibrium problem, and the set of solutions to variational inequality problem in Hilbert space. Starting with an arbitrary $x_0 \in C, C_{1,i} = C, C_1 = \cap_{i=1}^{\infty} C_{1,i}$, and $x_1 = P_{C_1} x_0$ define sequence $\{x_n\}, \{w_n\}, \{u_n\}, \{z_n\}$, and $\{y_{n,i}\}$ as follows:

$$\begin{cases} z_n & = T_{r_n}^{F_1,\phi_1}(x_n - r_n A x_n), \\ y_n & = T_{r_n}^{F_2,\phi_2}(z_n - \lambda_n B x_n), \\ w_n & = P_C(u_n - s_n D u_n), \\ y_{n,i} & = \alpha_{n_i} w_n + (1-\alpha_{n_i}) T_i w_n, \ n \geq 1, \\ C_{n+1,i} & = \{z \in C_{n,i} : \|y_{n,i} - z\| \leq \|x_n - z\|\}, n \geq 1, \\ C_{n+1} & = \cap_{i=1}^{\infty} C_{n+1}, \\ x_{n+1} & = P_{C_{n+1}}, x_0, \ n \geq 1, \end{cases} \tag{1.6}$$

where T_i is a k_i-strictly pseudocontrative mapping and for some $0 \leq k_i < 1,\ A, B$ is α, β-inverse- strongly monotone mapping of C into H. He proved that if the sequence $\{\alpha_{n_i}\}, \{r_n\}, \{s_n\}$ and $\{\lambda_n\}$ of parameters satisfies appropriate conditions, then $\{x_n\}$ generated by (1.6)

Moudafi [24] recently studied the convergence properties of a relaxed algorithm for SCFP for a class of quasi-nonexpansive operators T such that $I - T$ is demiclosed at zero. He also proved a weak convergence theorem as shown below.

Theorem 1.1. *Given a bounded linear operator $A : H_1 \to H_2$, let $U : H_1 \to H_1$and $T : H_2 \to H_2$ be two quasi-nonexpansive operators with nonempty sets Fix(U) = C and Fix(T) = Q. Assume that $I - U$ and $I - T$ are demiclosed at zero. Suppose $\Gamma := \{x \in C : Ax \in Q\} \neq \emptyset$ and define an iterative sequence x_n by*

$$\begin{cases} x_0 \in H_1, \\ u_n = x_n + \alpha\beta A^*(T - I)Ax_n, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n U(u_n). \end{cases} \tag{1.7}$$

Polyak [26] first proposed an inertial extrapolation as an acceleration process to solve the smooth convex minimization problem. The inertial algorithm is a two-step iteration where the next iterate is defined by making use of the previous two iterates. Recently, several researchers have constructed some fast iterative algorithms by using inertial extrapolation (see, e.g., [6, 7]

Motivated by the above results and the current research interest in this direction,in this article, we propose a new iterative scheme for approximating an element in the solution set of the common split feasibility problem for fixed points of demimetric mappings and equilibrium problem for monotone mapping in a real Hilbert space. We incorporate self-adaptive step size method and inertial technique to accelerate the convergence of the proposed method, we establish the strong convergence of the sequence generated by the proposed algorithm. We finally, establish some applications and numerical examples to illustrate the significant performance of our method.

Subsequent sections of this work are organized as follows: In Section 2, we recall some basic definitions and Lemmas that are relevant in establishing our main results. In Section 3, we state some Lemmas that are useful in establishing the strong convergence of our proposed algorithm and also prove the strong convergence theorem for the algorithm.

2. PRELIMINARIES

In this paper, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the norm given by $\|\cdot\|$ respectively. We denote the weak and strong convergence of a sequence x_n to a point $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$ respectively.

Definition 2.1. A mapping $A : H \to H$ is said to be:

(i) monotone if:
$$\langle Ax - Ay, x - y \rangle \geq 0, \ \forall\, x, y \in H;$$
(ii) λ-inverse strongly monotone (co-coercive) if there exists $\lambda > 0$ such that:
$$(Ax - Ay, x - y) \geq \lambda \|Ax - Ay\|^2, \ \forall\, x, y \in H;$$
(iii) nonexpansive if:
$$\|Ax - Ay\| \leq \|x - y\|, \ \forall\, x, y \in H;$$
(iv) firmly nonexpansive if:
$$\langle Ax - Ay, x - y \rangle \geq \|Ax - Ay\|^2, \ \forall\, x, y \in H.$$

For $x \in H,$ there exists the unique nearest point $P_C x$ in K such that
$$\|x - y\| \leq \|x - P_C x\|, \ \forall y \in C.$$
P_C is called metric projection of H onto C. It is known that P_C is nonexpansive.

Lemma 2.1. [15] *Let C be a closed and convex subset in a real Hilbert space H, for any $x \in H$ and $z \in C$, we have*

$$z = P_C x \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \ \forall \ y \ \in C.$$

Lemma 2.2. [15] *Let C be a closed and convex subset in a real Hilbert space H and let $x \in H$, then we have the following*

(i) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle \ \forall \ y \in H.$
(ii) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2 \ \forall \ y \in C.$

For more properties of the metric projection, refer [15, Section 3].

We need the following assumptions to solve a mixed equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$ and a mapping $\phi : C \to \mathbb{R}$.

(A1) $F(x, x) = 0, \ \forall \ x \in C,$
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0 \ \forall \ x, y \in C,$
(A3) $\lim_{\alpha \to \infty} F(\alpha z + (1 - \alpha)x, y) \leq F(x, y) \ \forall \ x, y, z \in C,$
(A4) $\forall \ x \in C, y \mapsto F_1(x, y)$ is convex and lower semicontinuous,
(A5) for each $x \in C, \alpha \in (0, 1]$, and $r > 0$, there exist a bounded subset $D \subseteq C$ and $y \in C$ such that for any $z \in C/D$,

$$F(z, y) + \phi(y) - \phi(z) + \frac{1}{r}\langle y - z, z - x \rangle < 0.$$

(A6) C is a bounded set.

Lemma 2.3. [28] *Let C be a nonempty closed convex subset of a Hilbert space H_1 and $\phi : C \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous and convex mapping such that $C \cap$ dom $\phi = \emptyset$. Suppose that bifunction $F : C \times C \to \mathbb{R}$ and a mapping ϕ satisfy Conditions $(A1)(A6)$. For $r > 0$ and $x \in H$, let $T_r^{F,\phi} : H \to C$ be a mapping defined by*

$$T_r^{F,\phi}(x) = \{z \in C : F(z, y) + \phi(y) - \phi(z) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall \ y \in C\}. \tag{2.1}$$

Assume that either (A5) or (A6) holds. Then:

(i) *for each $x \in H, T_r^{F,\phi} x \neq \emptyset$,*
(ii) *$T_r^{F,\phi}$ is single valued,*
(iii) *$T_r^{F,\phi}$ is firmly nonexpansive,*
(iv) *$Fix(T_r^{F,\phi}) = MEP(F, \phi)$ and it is closed and convex.*

Lemma 2.4. [21] *Let H be a real Hilbert space. Then, we have*

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle,$$
$$\|x + y\|^2 \leq \|x\| + 2\langle y, x + y \rangle,$$

and

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$$

for all $x, y \in H$ and $\alpha \in [0, 1]$. Also, if $\{x_n\}$ is a sequence in H weakly converging to $z \in H$, then

$$\limsup_{n \to \infty} \|x_n - y\|^2 = \limsup_{n \to \infty} \|x_n - z\|^2 + \|z - y\|^2, \ \forall \ y \in H.$$

Lemma 2.5. [29] *Let $\{a_n\} \subset \mathbb{R}_+, \{b_n\} \subset \mathbb{R}$ and $\{\xi_n\} \subset (0, 1)$ be such that $\sum_{n=1}^{\infty} \xi_n = \infty$ and*

$$a_{n+1} \leq (1 - \xi_n)a_n + \xi_n b_n, \ \forall \ n \in \mathbb{N}.$$

If $\limsup_{i\to\infty} b_{n_i} \leq 0$ *for every subsequence* a_{n_i} *of* a_n *satisfying* $\liminf_{i\to\infty}(a_{n_i+1} - a_{n_i}) \geq 0$, *then* $\lim_{n\to\infty} a_n = 0$.

3. MAIN RESULT

We study the problem of finding a common solution of the split common fixed points problem and mixed equilibrium problem(SCFPPMEP). We denote by Γ the solution set of the SCFPP-MEP. That is $\Gamma := \{x \in C : x \in Fix(S) \cap MEP(F, \phi)$ such that $Ax \in Fix(T)$. For solving the SCFPPMEP, we make the following assumptions:

(B1) Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively;
(B2) $S : C \to C$ and $T : Q \to Q$ are two demimetric mappings with constants $\beta \in (-\infty, 1)$ and $\mu \in (-\infty, 1)$, respectively;
(B3) $A : H_1 \to H_2$ is bounded linear operator with its adjoint operator A^*;
(B4) The bifunction $F : C \times C \to \mathbb{R}$ and $\phi : C \to \mathbb{R}$ satisfy condition (A1) - (A6);
(B5) Γ is nonempty.

Next we proof the following self adaptive algorithm for solving the SCFPPMEP:

Algorithm 1. *Inertial Algorithm for SCFPPMEP*

Initialization: *Choose* $x_0, x_1 \in C$, $\theta \in (0, 1)$, $\beta, \mu \in (0, 1)$, $\{r_n\}$ *a sequence of nonnegative real numbers and* $\alpha_n,\ \beta_n \subset (0, 1)$ *satisfying*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(C3) $0 < \liminf_{n\to\infty} r_n$;
(C4) $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, *where* ϵ_n *is a sequence of nonnegative real numbers.*

Step 1: *Compute the inertial step*

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise}. \end{cases} \tag{3.1}$$

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 2: *Compute*

$$\begin{cases} y_n = \beta_n w_n + (1 - \beta_n) T_{r_n}^{F,\phi} w_n, \\ z_n = y_n - Sy_n + A^*(Ay_n - TAy_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)(y_n - \alpha\tau_n z_n), \end{cases} \tag{3.2}$$

where $\alpha \in (0, \min\{1 - \beta, 1 - \mu\})$ is a positive constant and τ_n is chosen self-adaptively as

$$\tau_n = \frac{(\|y_n - Sy_n\|^2 + \|Ay_n - TAy_n\|^2)}{\|z_n\|^2}.$$

Remark 1. *We assume that the sequence* $\{x_n\}$ *generated by Algorithm 1 is infinite. In other words, Algorithm 1 does not terminate in a finite number of iterations.*

Lemma 3.1. *If* $\|y_n - Sy_n + A^*(I - T)Ay_n\| = 0$, *then we arrive at the solution of the SCFPP.*

Proof: If y_n solves split common fixed points problem, then $y_n = Sy_n$ and $(I-T)Ay_n = 0$, Therefore, we have $\|y_n - Sy_n + A^*(I-T)Ay_n\| = 0$. To see the converse , suppose that $\|y_n - Sy_n + A^*(I-T)Ay_n\| = 0$. Then we have $y \in \Omega$ such that

$$\begin{aligned} 0 &= \|y_n - Sy_n + A^*(I-T)Ay_n\|\|y_n - y\| \\ &\geq \langle y_n - Sy_n + A^*(I-T)Ay_n, y_n - y\rangle \\ &\geq \langle y_n - Sy_n, y_n - y\rangle + \langle A^*(I-T)Ay_n, y_n - y\rangle \\ &\geq \langle y_n - Sy_n, y_n - y\rangle + \langle (I-T)Ay_n, Ay_n - Ay\rangle. \end{aligned} \tag{3.3}$$

Since S and T are demimetric, we have that

$$\langle y_n - Sy_n, y_n - y\rangle \geq \frac{1-\beta}{2}\|y_n - Sy_n\|^2 \tag{3.4}$$

and

$$\langle (I-T)Ay_n, Ay_n - Ay\rangle \geq \frac{1-\mu}{2}\|Ay_n - TAy_n\|^2 \tag{3.5}$$

we obtain the following by combining (3.3), (3.4) and (3.5),

$$0 \geq \frac{1-\beta}{2}\|y_n - Sy_n\|^2 + \frac{1-\mu}{2}\|Ay_n - TAy_n\|^2 \tag{3.6}$$

Since $\beta, \mu \in (-\infty, 1)$, we infer that $y_n \in Fix(S)$ and $Ay_n \in Fix(T)$ by (3.6). Therefore, y_n solves problem of common fixed point problem. This completes the proof.

Theorem 3.2. *Suppose assumption (B1)-(B5) hold. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to an element $p = P_\Gamma f(p) \in \Gamma$.*

Proof: Fix $p \in \Omega$, then from (1), we have

$$\begin{aligned} \langle z_n, y_n - p\rangle &= \langle y_n - Sy_n + A^*(Ay_n - TAy_n), y_n - p\rangle \\ &= \langle y_n - Sy_n, y_n - p\rangle + \langle A^*(Ay_n - TAy_n), y_n - p\rangle \\ &\geq \frac{1-\beta}{2}\|y_n - Sy_n\|^2 + \frac{1-\mu}{2}\|Ay_n - TAy_n\|^2 \\ &\geq \frac{1}{2}\min\{1-\beta, 1-\mu\}(\|y_n - Sy_n\|^2 + \|Ay_n - TAy_n\|^2). \end{aligned}$$

Now let $v_n = y_n - \alpha\tau_n z_n - p$, we get

$$\begin{aligned} \|v_n - p\|^2 &= \|y_n - \alpha\tau_n z_n - p\|^2 \\ &= \|y_n - p\|^2 - 2\alpha\tau_n\langle z_n, y_n - p\rangle + \alpha^2\tau_n^2\|y_n\|^2 \\ &= \|y_n - p\|^2 + \alpha^2\frac{(\|y_n - Sy_n\|^2 + \|Ay_n - TAy_n\|^2)}{\|z_n\|^2} \\ &\quad - \alpha\min\{1-\beta, 1-\mu\}\frac{(\|y_n - Sy_n\|^2 + \|Ay_n - TAy_n\|^2)}{\|z_n\|^2} \\ &\leq \|y_n - p\|^2 - \alpha\min\{1-\beta, 1-\mu\}\frac{(\|y_n - Sy_n\|^2 + \|Ay_n - TAy_n\|^2)}{\|z_n\|^2} \qquad (3.7) \\ &\leq \|y_n - p\|^2. \end{aligned}$$

Thus, we obtain that

$$\|v_n - p\| \leq \|y_n - p\|.$$

Observe that

$$\begin{aligned}\|y_n - p\| &= \|\beta_n w_n + (1-\beta_n)T_{r_n}^{F,\phi} w_n - p\| \\ &\le \beta_n\|w_n - p\| + (1-\beta_n)\|T_{r_n}^{F,\phi} w_n - p\| \\ &\le \beta_n\|w_n - p\| + (1-\beta_n)\|w_n - p\| \\ &= \|w_n - p\|.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)v_n - p\| \\ &\le \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|v_n - p\| \\ &\le \alpha_n c\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|v_n - p\| \\ &\le \{1-\alpha_n(1-c)\}\|x_n - p\| + \alpha_n\|f(p) - p\| + \theta_n(1-\alpha_n)\|x_n - x_{n-1}\| \\ &\le \{1-\alpha_n(1-c)\}\|x_n - p\| + \alpha_n\left(\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|f(p) - p\|\right) \\ &\le \max\left\{\|x_n - p\|, \frac{\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|f(p) - p\|}{1-c}\right\}.\end{aligned}$$

Therefore, we obtain by (3.1) and (C4), that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$ and there exist $M > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \le M$ for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned}\|x_{n+1} - p\| &\le \max\left\{\|x_n - p\|, \frac{M + f(p) - p\|}{1-c}\right\} \\ &\le \max\left\{\|x_n - p\|, \frac{M + f(p) - p\|}{1-c}\right\} \ \forall\, n \in \mathbb{N}.\end{aligned}$$

Hence, $\{x_n\}$ is bounded. It is easy to see that operator $P_\Gamma f$ is a contraction. Thus by the Banach contraction principle, there exists a unique point $p = P_\Gamma f(p)$. It follows from the characterization of P_Γ that

(3.8) $$\langle f(p) - p, q - p\rangle \ \le 0,\ \forall\, q \in \Gamma.$$

Using Lemma 2.4 and (3.7)

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \|(1-\alpha_n)(v_n-p)+\alpha_n(f(x_n)-f(p)\|^2+2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\leq (1-\alpha_n)\|v_n-p\|^2+\alpha_n\|f(x_n)-f(p)\|^2+2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\leq (1-\alpha_n)\|y_n-p\|^2-\alpha(\min\{1-\beta,1-\mu\})\frac{(\|y_n-Sy_n\|^2+\|Ay_n-TAy_n\|^2)}{\|z_n\|^2} \\
&+\alpha_n\|f(x_n)-f(p)\|^2+2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\leq (1-\alpha_n)\|w_n-p\|^2-\alpha(\min\{1-\beta,1-\mu\})\frac{(\|y_n-Sy_n\|^2+\|Ay_n-TAy_n\|^2)}{\|z_n\|^2} \\
&+\alpha_n\|f(x_n)-f(p)\|^2+2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\leq (1-\alpha_n)\|x_n-p\|^2+2\theta_n\langle x_n-p, x_n-x_{n-1}\rangle+\theta_n\|x_n-x_{n-1}\|^2 \\
&-\alpha(\min\{1-\beta,1-\mu\})\frac{(\|y_n-Sy_n\|^2+\|Ay_n-TAy_n\|^2)}{\|z_n\|^2} \\
&+\alpha_n\|f(x_n)-f(p)\|^2+2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\leq (1\alpha_n(1c)\|x_n-p\|^2+\alpha_n(1-c)b_n \\
&-\alpha(\min\{1-\beta,1-\mu\})\frac{(\|y_n-Sy_n\|^2+\|Ay_n-TAy_n\|^2)}{\|z_n\|^2},
\end{aligned} \tag{3.9}
$$

where

$$b_n := \frac{1}{1-c}(2\langle f(p)-p, x_{n+1}-p\rangle+\frac{\theta_n^2}{\alpha_n}\|x_n-x_{n-1}\|^2+2\frac{\theta_n}{\alpha_n}\|x_n-p\|\|x_n-x_{n-1}\|).$$

It follows that

(3.10)

$$\alpha(\min\{1-\beta,1-\mu\})\frac{(\|y_n-Sy_n\|^2+\|Ay_n-TAy_n\|^2)}{\|z_n\|^2} \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n(1-c)M',$$

where $M' = \sup\{b_n : n \in \mathbb{N}\}$.

Now, set $a_n = \|x_n-p\|^2$ and $\eta_n := \alpha_n(1-c)$. From (3.9) we have the following inequality:

$$a_{n+1} \leq (1-\eta_n)a_n+\eta_n b_n.$$

To apply Lemma 2.5, we have to show that $\limsup_{i\to\infty} b_{n_i} \leq 0$ for every subsequence $\{a_{n_i}\}$ of $\{a_n\}$ satisfying

$$\liminf_{i\to\infty}(a_{n_i+1}-a_{n_i}) \geq 0. \tag{3.11}$$

To do this, suppose that $\{a_{n_i}\} \subseteq \{a_n\}$ is a subsequence satisfying (3.11). Therefore, by (3.10) and (Cii), we have

$$
\begin{aligned}
&\limsup_{i\to\infty}\alpha(\min\{1-\beta,1-\mu\})\frac{(\|y_{n_i}-Sy_{n_i}\|^2+\|Ay_{n_i}-TAy_{n_i}\|^2)}{\|z_{n_i}\|} \\
&\leq \limsup_{i\to\infty}(a_{n_i+1}-a_{n_i})+(1-c)M'\lim_{i\to\infty}\alpha_{n_i} \\
&= -\liminf_{i\to\infty}(a_{n_i+1}-a_{n_i}) \\
&\leq 0,
\end{aligned}
$$

which implies

$$\frac{(\|y_{n_i} - Sy_{n_i}\|^2 + \|Ay_{n_i} - TAy_{n_i}\|^2)}{\|z_{n_i}\|^2} = 0. \tag{3.12}$$

Taking into consideration that

$$\frac{(\|y_{n_i} - Sy_{n_i}\|^2 + \|Ay_{n_i} - TAy_{n_i}\|^2)}{2\max\{1, \|A\|^2\}} \le \frac{(\|y_{n_i} - Sy_{n_i}\|^2 + \|Ay_{n_i} - TAy_{n_i}\|^2)}{\|z_{n_i}\|^2}. \tag{3.13}$$

We deduce from (3.12), that

$$\lim_{i\to\infty} \|y_{n_i} - Sy_{n_i}\|^2 = \lim_{i\to\infty} \|Ay_{n_i} - TAy_{n_i}\|^2 = 0. \tag{3.14}$$

Observe from (3.2) and the nonexpansive property of $T_{r_n}^{F,\phi}$, that

$$\begin{aligned}
\|y_n - p\|^2 &= \|\beta_n w_n + (1-\beta_n)T_{r_n}^{F,\phi} w_n - p\|^2 \\
&= \beta_n\|w_n - p\|^2 + (1-\beta_n)\|T_{r_n}^{F,\phi} w_n - p\|^2 - \beta_n(1-\beta_n)\|w_n - T_{r_n}^{F,\phi} w_n\|^2 \\
&\le \beta_n\|w_n - p\|^2 + (1-\beta_n)\|w_n - p\|^2 - \beta_n(1-\beta_n)\|w_n - T_{r_n}^{F,\phi} w_n\|^2 \\
&= \|w_n - p\|^2 - \beta_n(1-\beta_n)\|w_n - T_{r_n}^{F,\phi} w_n\|^2. \qquad (3.15)
\end{aligned}$$

Again, by using Lemma 2.4, (3.7) and (3.15), we have that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|(1-\alpha_n)(v_n - p) + \alpha_n(f(x_n) - f(p))\|^2 + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\le (1-\alpha_n)\|v_n - p\|^2 + \alpha_n\|f(x_n) - f(p)\|^2 + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\le (1-\alpha_n)\|y_n - p\|^2 + \alpha_n\|f(x_n) - f(p)\|^2 + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\le (1-\alpha_n)(\|w_n - p\|^2 - \beta_n(1-\beta_n)\|w_n - T_{r_n}^{F,\phi} w_n\|^2) + \alpha_n\|f(x_n) - f(p)\|^2 \\
&\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\le (1-\alpha_n)\|x_n - p\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle + \theta_n\|x_n - x_{n-1}\|^2 \\
&\quad - \beta_n(1-\beta_n)\|w_n - T_{r_n}^{F,\phi} w_n\|^2 + \alpha_n\|f(x_n) - f(p)\|^2 + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\le [1 - \alpha_n(1-c)]\|x_n - p\|^2 + \alpha_n(1-c)b_n - \beta_n(1-\beta_n)\|w_n - T_{r_n}^{F,\phi} w_n\|^2, \qquad (3.16)
\end{aligned}$$

where

$$b_n := \frac{1}{1-c}(2\langle f(p) - p, x_{n+1} - p\rangle + \frac{\theta_n^2}{\alpha_n}\|x_n - x_{n-1}\|^2 + 2\frac{\theta_n}{\alpha_n}\|x_n - p\|\|x_n - x_{n-1}\|).$$

Thus, we obtain

$$\beta_n(1-\beta_n)\|w_n - T_{r_n}^{F,\phi} w_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n(1-c)M', \tag{3.17}$$

where $M' = \sup\{b_n : n \in \mathbb{N}\}$.

Again as before, let $a_n = \|x_n - p\|^2$ and $\eta_n := \alpha_n(1-c)$. From (3.16), we have that

$$a_{n+1} \le (1-\eta_n)a_n + \eta_n b_n.$$

Therefore, by (C1) and (C2), we have

$$\begin{aligned}
&\limsup_{i\to\infty} \beta_{n_i}(1-\beta_{n_i})\|w_{n_i} - T_{r_{n_i}}^{F,\phi} w_{n_i}\|^2 \\
&\le \limsup_{i\to\infty}(a_{n_i+1} - a_{n_i}) + (1-c)M' \lim_{i\to\infty} \alpha_{n_i} \\
&= -\liminf_{i\to\infty}(a_{n_i+1} - a_{n_i}) \\
&\le 0,
\end{aligned}$$

which implies

$$\lim_{i\to\infty} \|w_{n_i} - T^{F,\phi}_{r_{n_i}} w_{n_i}\| = 0. \tag{3.18}$$

Now,

$$\begin{aligned}\|w_n - x_n\| &= \|x_n + \theta_n(x_n - x_{n-1}) - x_n\| \\ &\leq \theta_n \|x_n - x_{n-1}\| \\ &= \theta_n . \frac{\alpha_n}{\theta_n}\|x_n - x_{n-1}\| \to 0 \text{ as } n \to \infty,\end{aligned}$$

hence,

$$\lim_{i\to\infty} \|w_{n_i} - x_{n_i}\| = 0. \tag{3.19}$$

From (3.2) and (3.18), we have

$$\begin{aligned}\|y_n - w_n\| &= \|\beta_n w_n + (1-\beta_n) T^{F,\phi}_{r_n} w_n - w_n\| \\ &\leq (1-\beta_n)\|T^{F,\phi}_{r_n} w_n - w_n\| \to 0 \text{ as } n \to \infty,\end{aligned}$$

that is

$$\lim_{i\to\infty} \|y_{n_i} - w_{n_i}\| = 0. \tag{3.20}$$

It is easy to see from (3.19) and (3.20), that

$$\|y_{n_i} - x_{n_i}\| \leq \|y_{n_i} - w_{n_i}\| + \|w_{n_i} - x_{n_i}\| \to 0 \text{ as } i \to \infty. \tag{3.21}$$

Again from (3.2), we have

$$\begin{aligned}\|v_{n_i} - y_{n_i}\| &\leq \alpha \tau_n \|z_n\| \\ &= \alpha \frac{(\|y_n - Sy_n\|^2 + \|Ay_n - TAy_n\|^2)}{\|z_n\|},\end{aligned}$$

thus by (3.12), we get

$$\|v_{n_i} - y_{n_i}\| \to 0 \text{ as } i \to \infty.$$

It follows from this and (3.19), that

$$\lim_{i\to\infty} \|v_{n_i} - w_{n_i}\| = 0. \tag{3.22}$$

By using (C1), (3.2) and (3.12), we derive

$$\begin{aligned}\|x_{n_i+1} - x_{n_i}\| &\leq \alpha_{n_i}\|x_{n_i} - f(x_{n_i})\| + (1-\alpha_{n_i})\alpha\tau_{n_i}\|z_{n_i}\| \\ &\leq \alpha_{n_i}\|x_{n_i} - f(x_{n_i})\| + (1-\alpha_{n_i})\alpha \frac{(\|y_{n_i} - Sy_{n_i}\|^2 + \|Ay_{n_i} - TAy_{n_i}\|^2)}{\|z_{n_i}\|^2},\end{aligned}$$

which shows

$$\lim_{i\to\infty} \|x_{n_i+1} - x_{n_i}\| = 0. \tag{3.23}$$

We now show that $\limsup_{i\to\infty} b_{n_i} \leq 0$. Indeed, it suffices to show that

$$\limsup_{i\to\infty} \langle f(p) - p, x_{n_i+1} - p\rangle \leq 0.$$

Let $\{x_{n_{i_j}}\}$ be a sequence of $\{x_{n_i}\}$ such that

$$\lim_{j\to\infty} \langle f(p) - p, x_{n_{i_j}} - p\rangle = \limsup_{i\to\infty} \langle f(p) - p, x_{n_i} - p\rangle$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \rightharpoonup q \in C$. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup q$, we obtain by (3.19), that $w_{n_i} \rightharpoonup q$. We also have by (3.21) and (3.22) that y_{n_i} and v_{n_i} both converge weakly to q. Hence, by (3.14) and demiclosedness principle we have that $q \in Fix(S)$. Also, since A is a bounded linear operator we have that $Ay_{n_i} \rightharpoonup Aq$, thus by (3.14) again we obtain that $Aq \in Fix(T)$. Finally, we show that $q \in MEP(F, \phi)$. Let $u_n = T_{r_n}^F w_n$, we have by Lemma 2.3, that

$$F(u_n, y) + \phi(y) - \phi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \geq 0, \forall\, y \in H_1.$$

Now, since F is a monotone mapping, we obtain $\phi(y)-\phi(u_n)+\frac{1}{r_n}\langle y-u_n, u_n-w_n\rangle \geq F(y, u_n)$ and hence $\phi(y) - \phi(u_{n_i}) + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - w_{n_i}\rangle \geq F(y, u_{n_i})$ for all $y \in H_1$. It follows from (3.18), that $u_{n_i} \rightharpoonup q$. We obtain by (C3), (3.18) and the proper lower semicontinuity of ϕ that

$$F(y, q) + \phi(q) - \phi(y) \leq 0,\ \forall\, y \in H_1. \tag{3.24}$$

Let $y_t = ty + (1 - t)q$, for all $0 \leq t \leq 1$ and $y \in H_1$. It is easy to see that $y_t \in H_1$, thus (3.24) hold for $y = y_t$. that is

$$F(y_t, q) + \phi(q) - \phi(y_t) \leq 0. \tag{3.25}$$

From assumption (A1-A6) and (3.25), we have

$$\begin{aligned}
0 &= F(y_t, y) + \phi(y_t) - \phi(t)\\
&\leq tF(y_t, y) + (1 - t)F(y_t, q) + t\phi(y) + (1 - t)\phi(q) - t\phi(y_t) - (1 - t)\phi(y_t)\\
&= t[F(y_t, y) + \phi(y) - \phi(y_t)] + (1 - t)[F(y_t, q) + \phi(q) - \phi(y_t)]\\
&\leq t[F(y_t, y) + \phi(y) - \phi(y_t)].
\end{aligned}$$

Therefore, we obtain

$$t[f(y_t, y) + \phi(y) - \phi(y_t)] \geq 0,\ \forall\, y \in H_1. \tag{3.26}$$

Letting $t \to 0$ in (3.26), obtain $f(q, y) + \phi(y) - \phi(q) \geq 0,\ \forall\, y \in H_1$, thus we have $q \in MEP(F, \phi)$. Hence $q \in \Gamma$.
From (3.8) and (3.23), we have

$$\begin{aligned}
\limsup_{i\to\infty}\langle f(p) - p, x_{n_i+1} - p\rangle &\leq \limsup_{i\to\infty}\langle f(p) - p, x_{n_i+1} - x_{n_i}\rangle\\
&\quad + \limsup_{i\to\infty}\langle f(p) - p, x_{n_i} - p\rangle\\
&= \limsup_{j\to\infty}\langle f(p) - p, x_{n_j} - p\rangle\\
&= \langle f(p) - p, q - p\rangle\\
&\leq 0.
\end{aligned}$$

We conclude by Lemma 2.5 that $\{x_n\}$ converges strongly to a point $p \in \Gamma$, where $p = P_\Gamma f(p)$, The proof is complete.

4. NUMERICAL EXAMPLE

In this section, we provide some numerical examples to illustrate the efficiency of Algorithm 1 and we compare the accelerated and non accelerated method for the SCFPPMEP.

Example 4.1. *Let $C = H_1 = \mathbb{R} = H_2$ and define the bounded linear operator $A : H_1 \to H_2$ by $Ax = 2x$ for all $x \in \mathbb{R}$. Define the bifunction $F : C \times C \to \mathbb{R}$ by $F(x, y) = 3x^2 + xy + 2y^2$*

and $\phi : C \to \mathbb{R}$ *by* $\phi(x) = 0$. *Now, we compute* $u = T_r^{F,\phi}(x)$. *That is, we find* $u \in C$ *such that for all* $z \in C$

$$\begin{aligned} 0 &\geq F_1(u,z) + \phi(u) + \frac{1}{r_n}\langle z-u, u-x\rangle \\ &= -3u^2 + uz + 2z^2 + \frac{1}{r_n}\langle z-u, u-x\rangle \end{aligned}$$

that is

$$\begin{aligned} 0 &\geq -3r_nu^2 + r_nuz + 2r_nz^2 + \langle z-u, u-x\rangle \\ &= -3r_nu^2 + r_nuz + 2r_nz^2 + uz - xz - u^2 + ux \\ &= 2r_nz^2 + (r_nu + u + z)z + (-3r_nu - u^2 + ux). \end{aligned}$$

Let $h(z) = 2r_nz^2 + (r_nu + u - x)z + (-3r_nu^2 - u^2 + ux)$. *Then* $h(z)$ *is a quadratic function of z with coefficients* $a = 2r_n, b = r_nu + u - x,$ *and* $c = -3r_nu^2 - u^2 + ux$. *We determine the discriminant* Δ *of* $h(z)$ *as follows:*

$$\begin{aligned} \Delta &= (r_nu + u + u)^2 - 4(2r_n)(-3r_nu^2 - u^2 + ux), \\ &= 25r_n^2u^2 + 10r_nu^2 + u^2 - 10r_nux - 2ux + x^2, \\ &= ((5r_n + 1)u - x)^2. \end{aligned} \tag{4.1}$$

By Lemma 2.3, $T_{r_n}^{F,\phi}$ *is single-valued. Hence, it follows that* $h(z)$ *has at most one solution in* $\mathbb{R}$. *Therefore, from* (4.1) *we have that* $u = \frac{x}{5r_n+1}$. *This implies* $T_{r_n}^{F,\phi}(x) = \frac{x}{5r_n+1}$ *for all* $x \in H_1$.

Define the mappings $S : \mathbb{R} \to \mathbb{R}$ *and* $T : \mathbb{R} \to \mathbb{R}$ *by* $S(x) = -2x$ *and* $T(x) = -3x$, *respectively. We set* $f(x) = \frac{x}{4}, \beta_n = \frac{1}{2n+1}, \alpha_n = \frac{1}{n+1}, \epsilon_n = \frac{1}{3}, \theta = \frac{1}{3}, r_n = \frac{n+1}{2n}$ *in Algorithm 1 for each* $n \in \mathbb{N}$. *It can easily be verified that all the condition of Theorem* (3.2) *are satisfied. We choose different initial values as follows:*

Case 1 $x_0 = 1.78,\ x_1 = 1.5;$
Case 2 $x_0 = 0.5,\ x_1 = 0.15;$
Case 3 $x_0 = 0.05,\ x_1 = 0.95.$

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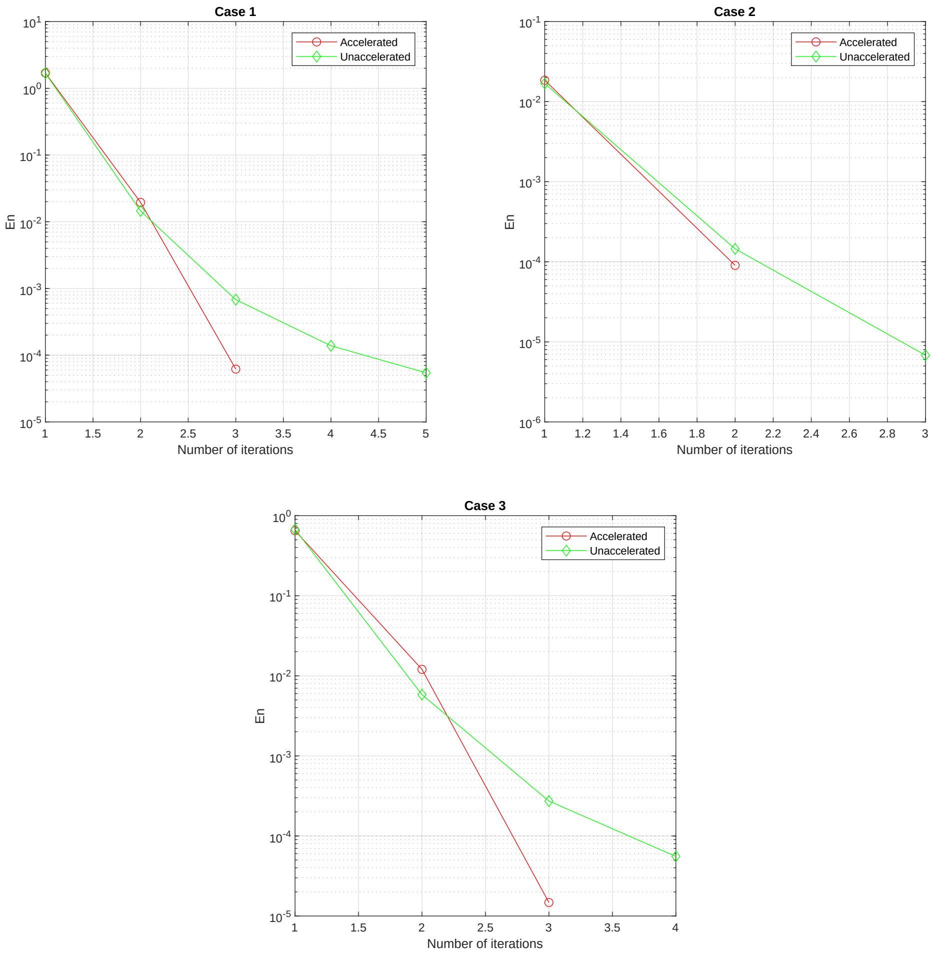

Figure 1: Example 4.1. Top left: **Case 1**, *Top right* **Case 2**, *Bottom:* **Case 3**.

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