Aust. J. Math. Anal. Appl.
Vol. **21** (2024), No. 1, Art. 15, 11 pp.
AJMAA

USING AN EULER TYPE TRANSFORM FOR ACCELERATING CONVERGENCE OF SERIES

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Received 3 May, 2022; accepted 13 May, 2024; published 14 June, 2024.

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ABSTRACT. In the paper we give a property of an operator of generalised difference, defined earlier, linear on a set of sequences, and use it to establish Euler type transforms for alternating series. These transforms accelerate the convergence of series under the same conditions as the transforms of non-alternating series. We also give analysis of an algorithm for computing a partial sum of the transformed series by using a higher order operator of generalised difference, and prove a theorem stating its order of complexity.

Key words and phrases: Accelerating convergence; Euler transform; Algorithm analysis.

2010 *Mathematics Subject Classification.* Primary 65B10. Secondary 40A25, 40A05.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Let $\sum_{n=0}^{\infty}(-1)^n a_n x^n$ be a convergent real power series and $x>0$. The following identity is well known as Euler transform [2, pp. 384–386]

$$\sum_{n=0}^{\infty}(-1)^n a_n x^n = \frac{1}{1+x}\sum_{k=0}^{\infty}(-1)^k \Delta^k(a_0)\left(\frac{x}{1+x}\right)^k.$$

It is well known also that the Euler transform does not necessarily accelerate the convergence of a series, i.e. there are examples [4] where the transformed series converges faster as well as those where it converges slower than the original one.

In papers [3], [4] and [1] a linear operator on a set of number sequences $\{a_n\}_{n=0}^{\infty}$ was defined by

$$\begin{gathered}\Delta_{r_1}^1(a_n) = \Delta_{r_1}(a_n) = a_{n+1} - r_1 a_n,\\ \Delta_{r_1 r_2 \ldots r_{m+1}}^{m+1}(a_n) = \Delta_{r_{m+1}}^1(\Delta_{r_1 r_2 \ldots r_m}^m(a_n)) \quad (m=1,2,\ldots),\end{gathered}$$

where $\{r_m\}_{m=1}^{\infty}$ is a given sequence of real numbers. By means of this operator of generalised difference, modified Euler transforms stated by the following theorems were established.

Theorem 1 ([3]). *Let $\sum_{n=0}^{\infty} a_n$ be a real or complex convergent number series and $\{r_k\}_{k=1}^{\infty}$ a sequence of real numbers such that $r_k \neq 1$ $(k=1,2,\ldots)$. For every positive integer p the following equality holds*

$$\begin{aligned}\sum_{n=0}^{\infty} a_n = \frac{a_0}{1-r_1} + \sum_{k=1}^{p-1}\frac{\Delta_{r_1 r_2 \ldots r_k}^k(a_0)}{(1-r_1)\ldots(1-r_{k+1})}&\\ + \frac{1}{(1-r_1)\ldots(1-r_p)}\sum_{n=0}^{\infty}\Delta_{r_1 r_2 \ldots r_p}^p(a_n).&\end{aligned}$$

Theorem 2 ([3]). *Let $\sum_{n=0}^{\infty} a_n x^n$ be a real or complex convergent power series and $\{r_k\}_{k=1}^{\infty}$ a sequence of real or complex numbers such that $r_k x \neq 1$ $(k=1,2,\ldots)$. For every positive integer p the following equality holds*

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^n = \frac{a_0}{1-r_1 x} + \sum_{k=1}^{p-1}\frac{\Delta_{r_1 r_2 \ldots r_k}^k(a_0)x^k}{(1-r_1 x)\ldots(1-r_{k+1}x)}&\\ + \frac{x^p}{(1-r_1 x)\ldots(1-r_p x)}\sum_{n=0}^{\infty}\Delta_{r_1 r_2 \ldots r_p}^p(a_n)x^n.&\end{aligned}$$

Theorem 3 ([1]). *Let $\sum_{n=0}^{\infty} a_n \cos(\alpha n+\beta)x$ be a real or complex convergent cosine series and $\{r_k\}_{k=1}^{\infty}$ a sequence of real or complex numbers such that $r_k e^{\pm\alpha x i} \neq 1$ $(k=1,2,\ldots)$. For every* *positive integer p the following equality holds*

$$\sum_{n=0}^{\infty} a_n \cos(\alpha n+\beta)x = \frac{a_0 C_{r_1}^1(0)}{1-2r_1\cos\alpha x + r_1^2} + \sum_{k=1}^{p-1} \frac{\Delta_{r_1 r_2 \ldots r_k}^k(a_0) C_{r_1 r_2 \ldots r_{k+1}}^{k+1}(0)}{(1-2r_1\cos\alpha x+r_1^2)\ldots(1-2r_{k+1}\cos\alpha x + r_{k+1}^2)} + \frac{1}{(1-2r_1\cos\alpha x+r_1^2)\ldots(1-2r_p\cos\alpha x+r_p^2)} \times \sum_{n=0}^{\infty} \Delta_{r_1 r_2 \ldots r_p}^p(a_n) \Delta_{r_1 r_2 \ldots r_p}^p(\cos(\alpha n+\beta)x),$$

where

$$C_{r_1 r_2 \ldots r_m}^m(n) = \Delta_{r_1 r_2 \ldots r_m}^m(\cos(\alpha(n-1)+\beta)x) \quad (m=0,1,2,\ldots).$$

Theorem 4 ([1]). *Let $\sum_{n=0}^{\infty} a_n \sin(\alpha n+\beta)x$ be a real or complex convergent sine series and $\{r_k\}_{k=1}^{\infty}$ a sequence of real or complex numbers such that $r_k e^{\pm\alpha x i} \neq 1$ $(k=1,2,\ldots)$. For every positive integer p the following equality holds*

$$\sum_{n=0}^{\infty} a_n \sin(\alpha n+\beta)x = \frac{a_0 S_{r_1}^1(0)}{1-2r_1\cos\alpha x + r_1^2} + \sum_{k=1}^{p-1} \frac{\Delta_{r_1 r_2 \ldots r_k}^k(a_0) S_{r_1 r_2 \ldots r_{k+1}}^{k+1}(0)}{(1-2r_1\cos\alpha x+r_1^2)\cdots(1-2r_{k+1}\cos\alpha x + r_{k+1}^2)} + \frac{1}{(1-2r_1\cos\alpha x+r_1^2)\cdots(1-2r_p\cos\alpha x+r_p^2)} \times \sum_{n=0}^{\infty} \Delta_{r_1 r_2 \ldots r_p}^p(a_n) \Delta_{r_1 r_2 \ldots r_p}^p(\sin(\alpha n+\beta)x),$$

where

$$S_{r_1 r_2 \ldots r_m}^m(n) = \Delta_{r_1 r_2 \ldots r_m}^m(\sin(\alpha(n-1)+\beta)x) \quad (m=0,1,2,\ldots).$$

We say that a series $\sum_{n=0}^{\infty} a_n$ converges faster than a convergent series $\sum_{n=0}^{\infty} b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

The following remark gives the conditions under which these transforms accelerate the convergence of series. Notice that the conditions are given in terms of the operator $\Delta_{r_1 r_2 \ldots r_k}^k$ of generalised difference.

Remark 1. If there exist finite limits

$$\lim_{n\to\infty} \frac{\Delta_{r_1 r_2 \ldots r_k}^k(a_{n+1})}{\Delta_{r_1 r_2 \ldots r_k}^k(a_n)} \quad (k=0,1,2,\ldots,p-1),$$

then for

$$r_1 = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}, \quad r_{k+1} = \lim_{n\to\infty} \frac{\Delta_{r_1 r_2 \ldots r_k}^k(a_{n+1})}{\Delta_{r_1 r_2 \ldots r_k}^k(a_n)} \quad (k=1,2,\ldots,p-1)$$

the right–hand side series in Theorems 1, 2, 3 and 4 converge faster than the appropriate series on the left–hand side.

Notice that for $r_k = r$ $(k = 1, 2, \ldots, p)$ statements of Theorems 1, 2, 3 and 4, and Remark 1 are given in [8]. Furthermore, for $r_k = 1$ $(k = 1, 2, \ldots, p)$ the transform given by Theorem 2 is considered in [5].

In the present paper, we give a property of the operator $\Delta^k_{r_1 r_2 \ldots r_k}$ when applied on an alternating sequence $\{(-1)^n a_n\}_{n=0}^{\infty}$. Then, we use this property in order to establish modified Euler transforms for alternating number, power and trigonometric series.

Finally, we present algorithm analysis for all cases of computing the n-th patial sum of transformed series by using the operator of generalised difference of order p, and prove that its order of complexity is $\mathrm{O}(p^2 n)$.

2. STATEMENT OF RESULTS

Now we formulate our results.

Theorem 5. *Let $\{a_n\}_{n=0}^{\infty}$ and $\{r_m\}_{m=1}^{\infty}$ be arbitrary sequences of real or complex numbers. For every positive integers m and n the following equality holds*

$$\Delta^m_{r_1 r_2 \ldots r_m}((-1)^n a_n) = (-1)^{n+m} \Delta^m_{-r_1 -r_2 \ldots -r_m}(a_n).$$

If we put $a_n := (-1)^n a_n$ $(n = 0, 1, 2, \ldots)$ and $r_k := -r_k$ $(k = 1, 2, \ldots)$ in Theorem 1, and make use of Theorem 5, we obtain the following modified Euler transform for alternating number series.

Corollary 1. *Let $\sum_{n=0}^{\infty}(-1)^n a_n$ be a real or complex convergent number series and $\{r_k\}_{k=1}^{\infty}$ a sequence of real numbers such that $r_k \neq -1$ $(k = 1, 2, \ldots)$. For every positive integer p the following equality holds*

$$\sum_{n=0}^{\infty}(-1)^n a_n = \frac{a_0}{1+r_1} + \sum_{k=1}^{p-1}(-1)^k \frac{\Delta^k_{r_1 r_2 \ldots r_k}(a_0)}{(1+r_1)\cdots(1+r_{k+1})} + \frac{(-1)^p}{(1+r_1)\cdots(1+r_p)} \sum_{n=0}^{\infty}(-1)^n \Delta^p_{r_1 r_2 \ldots r_p}(a_n). \tag{2.1}$$

Specially, for $r_k = 1$ $(k = 1, 2, \ldots, p)$ Corollary 1 gives the classical Euler transform for number series [2, p. 386].

By putting $a_n := (-1)^n a_n$ $(n = 1, 2, \ldots)$ and $r_k := -r_k$ $(k = 1, 2, \ldots)$ in Theorem 2, and making use of Theorem 5 we obtain the modified Euler transform for alternating power series, given in [4].

If we put $a_n := (-1)^n a_n$ $(n = 1, 2, \ldots)$ and $r_k := -r_k$ $(k = 1, 2, \ldots)$ in Theorem 3, and make use of Theorem 5, we obtain the following modified Euler transform for alternating cosine series.

Corollary 2. *Let $\sum_{n=0}^{\infty}(-1)^n a_n \cos(\alpha n + \beta)x$ be a real or complex convergent cosine series and $\{r_k\}_{k=1}^{\infty}$ a sequence of real or complex numbers such that $r_k e^{\pm \alpha x i} \neq -1$ $(k = 1, 2, \ldots)$.*

For every positive integer p *the following equality holds*

$$\sum_{n=0}^{\infty}(-1)^n a_n \cos(\alpha n+\beta)x = \frac{a_0 C^1_{r_1}(0)}{1+2r_1\cos\alpha x + r_1^2} + \sum_{k=1}^{p-1}(-1)^k \frac{\Delta^k_{r_1 r_2\ldots r_k}(a_0) C^{k+1}_{r_1 r_2 \ldots r_{k+1}}(0)}{(1+2r_1\cos\alpha x+r_1^2)\cdots(1+2r_{k+1}\cos\alpha x + r_{k+1}^2)} + \frac{(-1)^p}{(1+2r_1\cos\alpha x+r_1^2)\cdots(1+2r_p\cos\alpha x+r_p^2)} \times \sum_{n=0}^{\infty}(-1)^n \Delta^p_{r_1 r_2\ldots r_p}(a_n)\Delta^p_{-r_1-r_2\ldots-r_p}(\cos(\alpha n+\beta)x),$$

where

$$C^m_{r_1 r_2 \ldots r_m}(n) = \Delta^m_{-r_1-r_2\ldots-r_m}(\cos(\alpha(n-1)+\beta)x) \quad (m=0,1,2,\ldots).$$

Finally, if we put $a_n := (-1)^n a_n$ $(n=1,2,\ldots)$ and $r_k := -r_k$ $(k=1,2,\ldots)$ in Theorem 4, and make use of Theorem 5, we obtain the following modified Euler transform for alternating sine series.

Corollary 3. *Let* $\sum_{n=0}^{\infty}(-1)^n a_n \sin(\alpha n+\beta)x$ *be a real or complex convergent sine series and* $\{r_k\}_{k=1}^{\infty}$ *a sequence of real or complex numbers such that* $r_k e^{\pm\alpha x i} \neq -1$ $(k=1,2,\ldots)$. *For every positive integer* p *the following equality holds*

$$\sum_{n=0}^{\infty}(-1)^n a_n \sin(\alpha n+\beta)x = \frac{a_0 S^1_{r_1}(0)}{1+2r_1\cos\alpha x + r_1^2} + \sum_{k=1}^{p-1}(-1)^k \frac{\Delta^k_{r_1 r_2\ldots r_k}(a_0) S^{k+1}_{r_1 r_2 \ldots r_{k+1}}(0)}{(1+2r_1\cos\alpha x+r_1^2)\cdots(1+2r_{k+1}\cos\alpha x + r_{k+1}^2)} + \frac{(-1)^p}{(1+2r_1\cos\alpha x+r_1^2)\cdots(1+2r_p\cos\alpha x+r_p^2)} \times \sum_{n=0}^{\infty}(-1)^n \Delta^p_{r_1 r_2\ldots r_p}(a_n)\Delta^p_{-r_1-r_2\ldots-r_p}(\sin(\alpha n+\beta)x),$$

where

$$S^m_{r_1 r_2 \ldots r_m}(n) = \Delta^m_{-r_1-r_2\ldots-r_m}(\sin(\alpha(n-1)+\beta)x) \quad (m=0,1,2,\ldots).$$

Notice that the conditions under which the transforms given in Corollaries 1, 2 and 3 accelerate the convergence of series are the same as those given in Remark 1.

3. ALGORITHM ANALYSIS FOR COMPUTING THE TRANSFORMED SERIES

By definition, a power of the generalised difference $\Delta^k_{r_1 r_2\ldots r_k}(a_n)$ could be computed recursively. However, such an approach would produce an algorithm of exponential complexity $O(2^p)$ (see also e.g., [7, Section 2.3]).

In a practical implementation augmented to our paper in a form of an application written in Java, whose source code is available, e.g. by writing to the authors, we apply a triangular scheme for computing the differences, as illustrated in Figure 1. We mention that a similar scheme is applied for computing ordinary differences in algorithms for interpolating functions by algebraical polynomials [6, pp. 118-120].

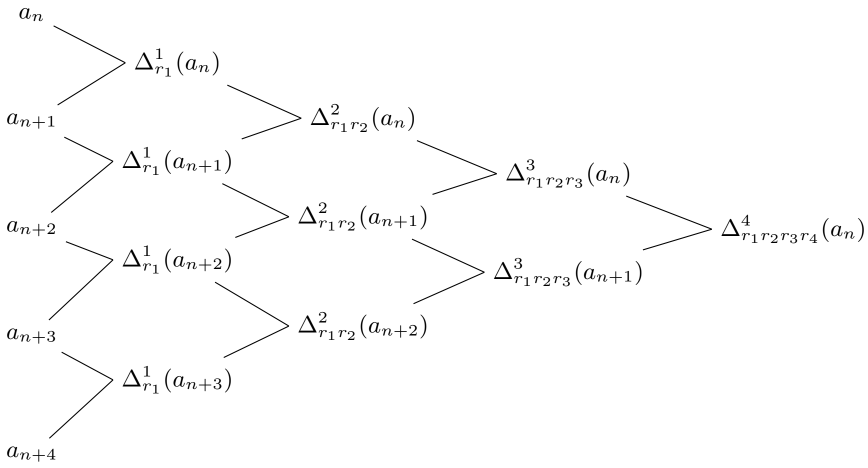

Figure 1: A triangular scheme for computing the differences

Notice that in order to implement the algorithm by using the triangular scheme in Figure 1 storage of different values of the operator (for different values of parameters p and n) is needed. For this reason, a binary tree data structure could be used.

Theorem 6. *Let $\{a_n\}_{n=0}^{\infty}$ be an arbitrary sequence of real or complex numbers and r a complex or real number. For any positive integers p and n, complexity of algorithm for computing a value $\Delta^p_{r_1 r_2 \ldots r_p}(a_n)$ of the generalised difference operator is $\mathrm{O}(p^2)$.*

Proof. By implementing the scheme given in Figure 1 in the algorithm, we start computing the value $\Delta^p_{r_1 r_2 \ldots r_p}(a_n)$ from $p+1$ values a_n, a_{n+1},…, a_{n+p}. Hence, the number of operations needed for implementing the algorithm is of order

$$\mathrm{O}(1+2+\cdots+p) = \mathrm{O}\left(\sum_{i=1}^{p} i\right) = \mathrm{O}\left(\frac{p(p+1)}{2}\right) = \mathrm{O}(p^2).$$

∎

Therefore, complexity of an algorithm implemented in this way for computing $\Delta^p_{r_1 r_2 \ldots r_p}(a_n)$ is quadratic with respect to the power p and does not depend on n.

Corollary 4. *For a fixed p, complexity of algorithm for computing a partial sum of order n of a transformed series by using a modified Euler transform (given by Theorems 1, 2, 3 and 4, and by Corollaries 1, 2 and 3) is of order $\mathrm{O}(n)$.*

Proof. Since in order to compute each term of transformed series computation of p-th power of difference operator for the sequence $\{a_n\}_{n=0}^{\infty}$ is needed (in the case of a trigonometric series, also for the appropriate sequences of sines and cosines), taking into consideration Theorem 6 the number of needed operations is of order

$$\mathrm{O}((n+1)\,\mathrm{O}(p^2)) = \mathrm{O}(p^2 n).$$

Hence, for fixed p, complexity of the algorithm is $\mathrm{O}(n)$. ∎

4. EXAMPLES

We illustrate in numerical examples the acceleration of convergence of series by the transforms. The examples also illustrate the scope of the class of sequences $\{a_n\}_{n=0}^{\infty}$ for which the conditions stated in Remark 1 are satisfied.

Example 1. In Corollary 1, we put

$$a_n := \frac{1}{Aa^n + Bb^n},$$

where A, B, a and b are real or complex numbers such that $B \neq 0$, $|a| < |b|$ and for all positive integers n holds $Aa^n + Bb^n \neq 0$.

If $Aa = 0$ and $|b| > 1$, we have $r_1 = \frac{1}{b}$, $\Delta^1_{r_1}(a_n) = 0$, and for $p = 1$ the transform gives the formula for summation of geometric series.

If $Aa \neq 0$, the following equalities can be proved by mathematical induction with respect to k applying them successively:

$$r_k = \lim_{n\to\infty} \frac{\Delta^{k-1}_{r_1 r_2 \dots r_{k-1}}(a_{n+1})}{\Delta^{k-1}_{r_1 r_2 \dots r_{k-1}}(a_n)} = \frac{a^{k-1}}{b^k} \quad (k = 1, 2, \dots) \tag{4.1}$$

and

$$\Delta^k_{r_1 r_2 \dots r_k}(a_n) = \frac{1}{b^{n+k}} \left(\frac{a}{b}\right)^{nk + \frac{1}{2}k(k-1)} \left(\frac{A}{B}\right)^k \times \frac{\prod_{j=1}^k \left(1 - \left(\frac{a}{b}\right)^j\right)}{\prod_{j=0}^k \left(1 + \frac{A}{B}\left(\frac{a}{b}\right)^{n+j}\right)} \quad (k = 1, 2, \dots).$$

Thus, for a given p the acceleration of convergence of the given series by the modified Euler transform, i.e. the speed by which the fraction $\frac{\Delta^p_{r_1 r_2 \dots r_p}(a_n)}{a_n}$ converges to 0 (as $n \to \infty$), is of order $\mathrm{O}\left(\left|\frac{a}{b}\right|^{pn}\right)$.

In particular, put $A := -1$, $B := 1$, $a = \frac{3}{2}$ and $b = 2$, we get the following alternating number series

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{4^n - 3^n}.$$

Making use of the preceding consideration we have $r_k = \frac{1}{2}\left(\frac{3}{4}\right)^{k-1}$ $(k = 1, 2, \dots)$.

Obviously, for every positive integer p the sequence $\{r_k\}_{k=1}^{\infty}$ satisfies the conditions given in Remark 1, which means that the acceleration of convergence of the given series provided by the transform from Corollary 1 is increased by increasing the value of p.

For practical implementation, we rewrite transform (2.1) as a sum of a finite part and an infinite remainder in the following way

$$\sum_{n=1}^{\infty} (-1)^n a_n = \frac{a_1}{1 + r_1} + \sum_{k=1}^{p-1} (-1)^k \frac{\Delta^k_{r_1 r_2 \dots r_k}(a_1)}{(1 + r_1) \cdots (1 + r_{k+1})} + \frac{(-1)^p}{(1 + r_1) \cdots (1 + r_p)} \sum_{n=1}^{q} (-1)^n \Delta^p_{r_1 r_2 \dots r_p}(a_n) + \tilde{R}_{q+1}, \tag{4.2}$$

where $\tilde{R}_{q+1}$ is the remainder, given by

$$\tilde{R}_{q+1} = \frac{(-1)^p}{(1 + r_1) \dots (1 + r_p)} \sum_{n=q+1}^{\infty} (-1)^n \Delta^p_{r_1 r_2 \dots r_p}(a_n).$$

Now, we choose a value of p, calculate the sum of the first two summands at the right–hand side of (4.2), and then we iterate with respect to q by calculating the third summand $\sum_{n=1}^{q} (-1)^n \Delta^p_{r_1 r_2 \dots r_p}(a_n)$ (and approximating the remainder $\tilde{R}_{q+1} \approx 0$).

Table 4.1: The number of iterations

ε	series				
	given	classical	modified		
			$p=1$	$p=2$	$p=3$
10^{-1}	3	3	1	1	1
10^{-2}	6	6	2	1	1
10^{-3}	9	9	4	2	1
10^{-4}	12	12	6	4	2
10^{-5}	16	15	9	5	3
10^{-6}	19	18	11	7	4
10^{-7}	22	21	13	9	6
10^{-8}	25	24	15	10	7
10^{-9}	29	27	18	12	9

Table 4.2: Relative errors

q	δ_q for series				
	given	classic	modified		
			$p=1$	$p=2$	$p=3$
1	0.265	0.142	$0.238 \cdot 10^{-1}$	$0.249 \cdot 10^{-2}$	$0.243 \cdot 10^{-3}$
2	$0.967 \cdot 10^{-1}$	$0.570 \cdot 10^{-1}$	$0.550 \cdot 10^{-2}$	$0.389 \cdot 10^{-3}$	$0.266 \cdot 10^{-4}$
3	$0.401 \cdot 10^{-1}$	$0.237 \cdot 10^{-1}$	$0.153 \cdot 10^{-2}$	$0.753 \cdot 10^{-4}$	$0.367 \cdot 10^{-5}$
4	$0.177 \cdot 10^{-1}$	$0.101 \cdot 10^{-1}$	$0.470 \cdot 10^{-3}$	$0.165 \cdot 10^{-4}$	$0.581 \cdot 10^{-6}$
5	$0.817 \cdot 10^{-2}$	$0.439 \cdot 10^{-2}$	$0.154 \cdot 10^{-3}$	$0.389 \cdot 10^{-5}$	$0.100 \cdot 10^{-6}$
6	$0.385 \cdot 10^{-2}$	$0.194 \cdot 10^{-2}$	$0.522 \cdot 10^{-4}$	$0.965 \cdot 10^{-6}$	$0.183 \cdot 10^{-7}$
7	$0.185 \cdot 10^{-2}$	$0.872 \cdot 10^{-3}$	$0.183 \cdot 10^{-4}$	$0.248 \cdot 10^{-6}$	$0.348 \cdot 10^{-8}$
8	$0.897 \cdot 10^{-3}$	$0.396 \cdot 10^{-3}$	$0.650 \cdot 10^{-5}$	$0.653 \cdot 10^{-7}$	$0.679 \cdot 10^{-9}$
9	$0.439 \cdot 10^{-3}$	$0.181 \cdot 10^{-3}$	$0.235 \cdot 10^{-5}$	$0.175 \cdot 10^{-7}$	$0.135 \cdot 10^{-9}$
10	$0.216 \cdot 10^{-3}$	$0.834 \cdot 10^{-4}$	$0.857 \cdot 10^{-6}$	$0.475 \cdot 10^{-8}$	$0.274 \cdot 10^{-10}$
11	$0.107 \cdot 10^{-3}$	$0.387 \cdot 10^{-4}$	$0.315 \cdot 10^{-6}$	$0.130 \cdot 10^{-8}$	$0.560 \cdot 10^{-11}$
12	$0.528 \cdot 10^{-4}$	$0.180 \cdot 10^{-4}$	$0.116 \cdot 10^{-6}$	$0.359 \cdot 10^{-9}$	$0.116 \cdot 10^{-11}$
13	$0.262 \cdot 10^{-4}$	$0.844 \cdot 10^{-5}$	$0.431 \cdot 10^{-7}$	$0.994 \cdot 10^{-10}$	$0.239 \cdot 10^{-12}$
14	$0.131 \cdot 10^{-4}$	$0.397 \cdot 10^{-5}$	$0.160 \cdot 10^{-7}$	$0.276 \cdot 10^{-10}$	$0.499 \cdot 10^{-13}$

Table 4.1 illustrates the dependence of the number of iterations needed for an approximate calculation of the sum of given series for the cases $p = 1$, $p = 2$ and $p = 3$.

It shows, for instance, that in order to calculate the approximate sum of the given series with an error not greater than 10^{-6} we must compute the sum of the first 19 terms. To obtain this accuracy for the classical Euler transform we need 18 summands. Applying the modified transform from Corollary 1, the same accuracy is obtained by computing the sum of the first 11 terms for $p = 1$, 7 terms for $p = 2$, and 4 terms for $p = 3$.

We mention that the number of operations needed for computing the first q individual summands for the classical Euler transform, as given in Section 1, is of order $\mathrm{O}(q^2)$.

Table 4.2 illustrates the relative errors of approximate sums of the series for a given number of iterations.

Example 2. We denote by P_k a polynomial of degree k. For non-negative integers l and m, and A, B, a and b satisfying conditions of Example 1 we put in Corollary 1

$$a_n := \frac{1}{Aa^n + Bb^n}\frac{P_l(n)}{P_m(n)}.$$

By the same reasoning as in Example 1, it can be proved that the acceleration of convergence of the given series by the modified Euler transform in this case is of order $\mathrm{O}\left(\frac{1}{n}\right)$.

For example, for the slowly convergent series $\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n}$ we apply the transform given in Corollary 2 with $p := 1$, $r_1 := 1$ to obtain

$$\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n} = -\frac{1}{2} + \frac{1}{2\cos\frac{x}{2}}\sum_{n=1}^{\infty}(-1)^n \frac{\cos(n+\frac{1}{2})x}{n(n+1)}.$$

It is obvious that the transformed series converges faster than the given one.

In particular, for $Aa = 0$, $|b| > 1$ and $m = 0$ we obtain $r_k = \frac{1}{b}$ $(k = 1, \dots, l+1)$, and $\Delta^k_{r_1 r_2 \dots r_k}(a_n) = \frac{1}{b^{n+k}}P_{l-k}(n)$ $(k = 1, \dots, l)$. Hence $\Delta^{l+1}_{r_1 r_2 \dots r_{l+1}}(a_n) = 0$, implying thus that for $p := l + 1$ the given series is transformed into a finite sum.

For example, by applying transform given in Corollary 3, for $|q| < 1$ we obtain

$$\sum_{n=0}^{\infty}(-1)^n q^n P_l(n)\sin(\alpha n + \beta)x$$
$$= \frac{P_l(0)S_q^1(0)}{1 + 2q\cos\alpha x + q^2} + \sum_{k=1}^{l}\frac{q^k \Delta^k(P_l(0))S_q^{k+1}(0)}{(1 + 2q\cos\alpha x + q^2)^{k+1}}.$$

A special case, for $l := 0$, $P_l(n) := P_0(n) := 1$ we obtain the well known formula

$$\sum_{n=1}^{\infty}(-1)^n q^n \sin nx = -\frac{q\sin x}{1 + 2q\cos\alpha x + q^2}.$$

5. PROOF OF THEOREM 5

Now we prove Theorem 5 given above.

Proof. First we prove that for every positive integers m and n the following equality holds

$$\Delta^m_{r_1 r_2 \dots r_m}(a_n) = \sum_{k=0}^{m}(-1)^{m-k}D_{m-k}a_{n+k}, \tag{5.1}$$

where are

$$D_0 = 1, \quad D_1 = \sum_{i=1}^{m} r_i, \quad D_2 = \sum_{1\le i<j\le m} r_i r_j,$$
$$D_3 = \sum_{1\le i<j<k\le m} r_i r_j r_k, \quad \dots, \quad D_m = r_1 r_2 \dots r_m$$

(i.e. the summation for D_p is performed over all combinations of distinct indices between 1 and m taken p at a time).

Indeed, for $m = 1$ the equality is implied by the definition of the operator $\Delta^m_{r_1 r_2 \dots r_m}$.

Under the assumption that equality (5.1) holds for m, we have

$$\begin{aligned}\Delta^{m+1}_{r_1r_2\dots r_{m+1}}(a_n) &= \Delta^1_{r_{m+1}}(\Delta^m_{r_1r_2\dots r_m}(a_n))\\ &= \Delta^1_{r_{m+1}}\left(\sum_{k=0}^{m}(-1)^{m-k}D_{m-k}a_{n+k}\right)\\ &= \sum_{k=0}^{m}(-1)^{m-k}D_{m-k}a_{n+1+k} - \sum_{k=0}^{m}(-1)^{m-k}r_{m+1}D_{m-k}a_{n+k}\\ &= a_{n+m+1} + \sum_{k=1}^{m}(-1)^{m+1-k}a_{n+k}(D_{m+1-k}+r_{m+1}D_{m-k})\\ &\qquad - (-1)^m r_{m+1}D_m a_n,\end{aligned}$$

wherefrom, taking into consideration that for a fixed k such that $1 \le k \le m$ holds

$$D_{m+1-k} + r_{m+1}D_{m-k} = \sum_{1\le i_1<i_2<\dots<i_{m+1-k}\le m+1} r_{i_1}r_{i_2}\dots r_{i_{m+1-k}},$$

we get

$$\begin{aligned}\Delta^{m+1}_{r_1r_2\dots r_{m+1}}(a_n) &= \sum_{k=0}^{m}(-1)^{m+1-k}a_{n+k}\\ &\qquad \times \sum_{1\le i_1<i_2<\dots<i_{m+1-k}\le m+1} r_{i_1}r_{i_2}\dots r_{i_{m+1-k}} + a_{n+m+1}.\end{aligned}$$

Since, relaying on the definition of numbers D_p, for $p \ge 1$ holds

$$D_p = \sum_{1\le i_1<i_2<\dots<i_p\le m} r_{i_1}r_{i_2}\dots r_{i_p},$$

equality (5.1) is true for every $m \in \mathbb{N}$.

Now, equality (5.1) yields

$$\Delta^m_{r_1r_2\dots r_m}((-1)^n a_n) = \sum_{k=0}^{m}(-1)^{m-k}D_{m-k}(-1)^{n+k}a_{n+k}.$$

Since for $m-k>0$ holds

$$\begin{aligned}D_{m-k} &= \sum_{1\le i_1<i_2<\dots<i_{m-k}\le m}\prod_{j=1}^{m-k} r_{i_j}\\ &= (-1)^{m-k}\sum_{1\le i_1<i_2<\dots<i_{m-k}\le m}\prod_{j=1}^{m-k} -r_{i_j} = (-1)^{m-k}D'_{m-k},\end{aligned}$$

we obtain

$$\begin{aligned}\Delta^m_{r_1r_2\dots r_m}((-1)^n a_n) &= (-1)^{n+m}\sum_{k=0}^{m}(-1)^{m-k}D'_{m-k}a_{n+k}\\ &= (-1)^{n+m}\Delta^m_{-r_1-r_2\dots-r_m}(a_n).\end{aligned}$$

This completes the proof of Theorem 5. ∎

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