*Aust. J. Math. Anal. Appl.*
Vol. **21** (2024), No. 1, Art. 14, 10 pp.
AJMAA

# A NEW LOOK AT THE EQUATIONS OF THE CALCULUS OF VARIATIONS

OLIVIER DE LA GRANDVILLE

*Received 12 November, 2023; accepted 25 April, 2024; published 14 June, 2024.*

FACULTY OF ECONOMICS, GOETHE UNIVERSITY FRANKFURT, THEODORE ADORNO PLATZ 4, 60323 FRANKFURT, GERMANY.
odelagrandville@gmail.com

ABSTRACT. We first offer an entirely new way to derive the celebrated Euler equation of the calculus of variations. The advantage of this approach is two-fold. On the one hand, it entirely eschews the two hurdles encountered by Lagrange, which become challenging in the case of elaborate functionals: getting rid of the arbitrary character of the perturbation given to the optimal function, and demonstrating the fundamental lemma of the calculus of variations. On the other hand, it leads in a direct way to the remarkable discovery made by Robert Dorfman ( 1969, [3]) when he introduced a modified Hamiltonian, which we called a Dorfmanian (2018, [8]) to honor his memory. In turn, extending the Dorfmanian enables to obtain readily the fundamental equations of the calculus of variations for the optimization of high-order functionals, or multiple integrals.

*Key words and phrases:* Calculus of variations; Euler equations; Optimal control theory; New Hamiltonians.

1991 *Mathematics Subject Classification.* Primary: 49.01, 49.02.

ISSN (electronic): 1449-5910



The author would like to thank Michael Binder, Ernst Hairer, Michalis Haliassos, Rainer Klump and Volker Wieland for their interest and their insightful remarks.

## 1. INTRODUCTION: THE FIRST HITCHES MET BY LAGRANGE, IN HIS CLASSIC APPROACH.

To the best of our knowledge, all texts on the calculus of variations derive a necessary condition for extremizing a functional such as

$$I\left[y\right] = \int_a^b F(x, y, y')\,dx$$

subject to $y(a) = y_b$ and $y(b) = y_b$ using the analytic method suggested by Lagrange in his famous letter to Euler in 1755. Eleven years before, Euler had obtained this necessary condition, the second-order differential equation

$$\frac{\partial F}{\partial y}(x, y, y') - \frac{d}{dx}\frac{\partial F}{\partial y'}(x, y, y') = 0, \tag{1.1}$$

as well as the equations corresponding to higher-order functionals, by relying on a geometrical argument. Euler enthusiastically embraced the analytic method suggested by the 19-year-old Italian youngster. This method consisted in giving to the optimal curve $y$ a variation in the form (in modern notation) of a variable $\alpha$ multiplying an arbitrary, fixed function $\eta\left(x\right)$ such that $\eta(a) = 0$ and $\eta(b) = 0$. By doing this, since $y$ (the solution) and $\eta\left(x\right)$ were both considered as fixed, Lagrange was able to transform the problem of optimizing a functional

$$I\left[y + \alpha\eta\right] = \int_a^b F(x, y + \alpha\eta, y' + \alpha\eta')\,dx$$

into the optimization of a function of a single variable $J\left(\alpha\right)$. That was a first, remarkable insight. He was then led to take to zero the expression $I'\left(0\right)$, amounting to

$$I'\left(0\right) = \int_a^b \left[\frac{\partial F}{\partial y}\eta(x) \ + \frac{\partial F}{\partial y'}\eta'(x)\right] dx \ = 0. \tag{1.2}$$

Lagrange then had a second, beautiful idea: in order to suppress the dependency of the solution of this equation on the perturbation $\eta(x)$, he integrated by parts the second term of the integrand in (1.2), which, thanks to $\eta(a) = 0$ and $\eta(b) = 0$, led him to write (1.2) as

$$I'\left(0\right) = \int_a^b \eta(x)\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right] dx \ \equiv \int_a^b \eta(x)g(x)dx = \ 0. \tag{1.3}$$

With the impetuosity of youth, Giuseppe concluded that since $\eta(x)$ was arbitrary, $g(x)$ must vanish, which amounted to the Euler equation.

At this point, Euler was not entirely convinced, and asked young Giuseppe to please prove that $g(x)$ should indeed vanish. It is only after much debate, in 1879, that a rigorous proof of this apparently evident property was obtained by Paul du Bois-Reymond [1, 2]. In addition, du Bois-Reymond had the ingenious idea of integrating by parts the *first* term of the integrand in (1.2), which led him to the integrated form of the Euler equation, showing that it was not necessary to assume that the extremal had to be twice differentiable; continuous differentiability was sufficient (see the excellent exposition by Mark Kot [6], 2014, pp. 38-44). The property that $g(x)$ should vanish then became to be known as the fundamental lemma of the calculus of variations.

Summing up, we can say that Lagrange had encountered two hurdles: the first, getting rid of the perturbation by factoring it out in the integrand of equation (1.2), he overcame brilliantly; the second one, demonstrating the fundamental lemma of the calculus of variations, both Euler and Lagrange had to leave to their followers.

Although these hurdles are relatively easy to overcome in the case considered here, it turns out that they become more and more cumbersome when optimizing more complex functionals (for instance, if the functional depends on derivatives of order $n$). And they are really challenging when the functional is a multiple integral: already in the case of a double integral, factoring out the perturbation requires a very subtle use of Green's theorem; and the case of $n$-uple integrals is for the brave only, because we need to generalize Green's theorem to *n*-space, definitely a serious challenge (see Gelfand and Fomin [4] pp. 153-154, and Troutman [10], pp. 179-181).

In this paper we will suggest an analytic derivation not only of the Euler equation, but also of all higher order Euler equations that suppresses the difficulties met by Lagrange. Our approach will lead in a natural way to the modified Hamiltonian introduced by Robert Dorfman (1969, [3], p. 822) in his remarkable economic interpretation of optimal control theory. To pay tribute to Professor Dorfman and to honor his memory, we see it fitting to call this new Hamiltonian a *Dorfmanian*.

We will then show that the Dorfmanian can in turn be extended into forms that enable to optimize easily more complex functionals. This will be especially important in the notoriously difficult case of $n$-uple integrals; we will thus be led to a general Euler equation (the Ostrogradski equation) with four lines of elementary calculations only. But before showing this, let us derive the basic Euler equation in a new, simple way.

## 2. OPTIMIZING THE FUNCTIONAL $I\left[y\right] = \int_a^b F(x, y, y')\, dx$: A NEW PATH.

The method we suggest has two levers: the first is to transform this functional into a *function of the single variable* $y'$ at *any* point $x$ in the interval $[a, b)$, and equate the derivative of this function to zero. Secondly we will use the concept of the *unit price of a function* $y(x)$ introduced by Robert Dorfman (1969, [3]), which we can define as follows.

**Definition 2.1.** We define, and denote $p_y\left(x\right)$, the *unit price of a function* $y(x)$ as the *derivative of the optimal value of the functional* $\int_x^b F(u, y, y')\, du$ *with respect to* $y$, *at any point* $x$.We thus have

$$p_y\left(x\right) = \frac{\partial}{\partial y} \int_x^b F(u, y, y')\, du. \tag{2.1}$$

This unit price of $y(x)$ can thus be seen as the contribution to the optimal value of the functional if one additional unit of $y$ is made available at point $x$.

Let us now transform the functional $I\left[y\right] = \int_a^b F(x, y, y')\, dx$ into a function of $y'$, to be denoted $I(y')$. Suppose that we know the optimal trajectory of $y$ on the whole integration interval $[a, b]$, except on a *very small, open, interval* $(x, x + \Delta x)$ within $[a, b]$. At point $x$, the optimal value $y(x)$ is supposed to be known. However, our knowledge of the solution $y$ from $x + \Delta x$ to $b$ is subject to the following proviso. The initial value $y\left(x + \Delta x\right)$ depends on the slope of the curve $y$ at a point of the abscissa located immediately before $x + \Delta x$, which is essentially $x$ since we can take $\Delta x$ as small as we want. Our problem is now to determine the optimal value of $y'\left(x\right)$. This will be key to obtain a necessary condition for $y$ to be optimal over the whole interval $[a, b]$.

Since we know, without any restriction, the optimal $y$ over the interval $[a, x]$, we are left with the task of maximizing the functional over $[x, b]$, i.e. $\int_x^b F(x, y, y')\, dx$, denoted $J\left[y\right]$, equal to

$$J\left[y\right] = \int_x^b F(u, y, y')\, du = F(x, y, y')\Delta x + o\left(\Delta x\right) + \int_{x+\Delta x}^b F(u, y, y')\, du. \tag{2.2}$$

The optimal integral on the right hand side, to be denoted $J^*$, depends solely on the variables $x+\Delta x$ and $y\left(x+\Delta x\right)$; we can write it as

$$\int_{x+\Delta x}^{b} F(u,y,y')\,du = J^*\left(x+\Delta x,\ y(x+\Delta x)\right),$$

$y(x+\Delta x)$ being a function of $y'$ thanks to the equality

$$y\left(x+\Delta x\right) = y\left(x\right) + y'(x)\Delta x + o\left(\Delta x\right). \tag{2.3}$$

Our functional $J\left[y\right]$ thus can be transformed into a function of the single variable $y' \equiv y'(x)$; denoted $J\left(y'\right)$, it is equal to

$$J\left(y'\right) = F(x,y,y')\Delta x + o\left(\Delta x\right) + J^*\left(x+\Delta x,\ y(x+\Delta x)\right) \tag{2.4}$$

Thus, $J\left(y'\right)$ depends upon $y'$ on two grounds: first directly, as expressed in the first term on the right-hand side of (2.4), $F(x,y,y')\Delta x$; secondly, indirectly through the variable $y(x+\Delta x)$ as given by (2.4).

Taking to zero the derivative of $J(y')$ with respect to $y'$, we get

$$\frac{dJ(y')}{dy'} = \frac{\partial F}{\partial y'}(x,y,y')\Delta x + \frac{\partial J^*\left(x+\Delta x,\ y(x+\Delta x)\right)}{\partial y\left(x+\Delta x\right)}\frac{\partial y\left(x+\Delta x\right)}{\partial y'} = 0. \tag{2.5}$$

The derivative $\frac{\partial J^*(x+\Delta x,\, y(x+\Delta x))}{\partial y(x+\Delta x)}$ will play a crucial role: referring to our definition of the unit price of $y$ at $x$ (given by (2.1) ), we recognize in this derivative the unit price of $y$ at $(x+\Delta x)$, and accordingly denote it as $p_y\left(x+\Delta x\right)$. We have

$$p_y\left(x+\Delta x\right) = \frac{\partial J^*\left(x+\Delta x,\ y(x+\Delta x)\right)}{\partial y\left(x+\Delta x\right)}. \tag{2.6}$$

On the other hand, in light of (2.3), $\frac{\partial y(x+\Delta x)}{\partial y'} = \Delta x$, and (2.5) becomes

$$\frac{dJ(y')}{dy'} = \frac{\partial F}{\partial y'}(x,y,y')\Delta x + p_y\left(x+\Delta x\right)\Delta x = 0. \tag{2.7}$$

Simplifying, and taking $\Delta x$ to zero, we have a first, important equation:

$$\frac{\partial F}{\partial y'}(x,y,y') + p_y\left(x\right) = 0. \tag{2.8}$$

Suppose that this equation has a solution $y'$; replace this value in (2.4). $J\left(y'\right)$ can now be written as $J^*\left(y'\right)$, and equation (2.4) then becomes the *identity*

$$J^*\left(y'\right) = F(x,y,y')\Delta x + o\left(\Delta x\right) + J^*\left(x+\Delta x,\ y(x+\Delta x)\right) \tag{2.9}$$

where, $y(x+\Delta x)$ is given by (2.3) and thus depends upon $y$.
Differentiating both sides of this identity with respect to $y$ gives

$$\frac{dJ^*(y')}{dy} = \frac{\partial F}{\partial y}(x,y,y')\Delta x + \frac{\partial J^*\left(x+\Delta x,\ y(x+\Delta x)\right)}{\partial y\left(x+\Delta x\right)}\frac{\partial y\left(x+\Delta x\right)}{\partial y} \tag{2.10}$$

or, equivalently, with our preceding notation (2.1),

$$p_y(x) = \frac{\partial F}{\partial y}(x,y,y')\Delta x + p_y(x+\Delta x)\frac{\partial y\left(x+\Delta x\right)}{\partial y}. \tag{2.11}$$

Using $p_y(x+\Delta x) = p_y(x) + p_y'(x)\Delta x + o(\Delta x)$ and equation (2.3), we have

$$p_y(x) = \frac{\partial F}{\partial y}(x,y,y')\Delta x + p_y(x) + p_y'(x)\Delta x + o\left(\Delta x\right). \tag{2.12}$$

Eliminating $p_y(x)$, simplifying and taking $\Delta x$ to zero leads to our second central equation

$$\frac{\partial F}{\partial y}(x,y,y') + p_y'(x) = 0. \tag{2.13}$$

Equations (2.8) and (2.13) are just the Euler equation in parametric form; differentiating (2.8) and replacing $p_y'(x)$ in (2.13) immediately yields

$$\frac{\partial F}{\partial y}(x,y,y') - \frac{d}{dx}\frac{\partial F}{\partial y'}(x,y,y') = 0 \tag{2.14}$$

the Euler equation (1.1).

We may wonder whether this method of deriving the Euler equation in this simple case can be extended to the optimization of more complex functionals. We will show that this indeed is the case.

## 3. OPTIMIZING $I[y] = \int_a^b F(x,y,y',y'')\,dx$.

Our aim is now to show that a necessary condition for $y(x)$ to be an extremal of the functional $\int_a^b F(x,y,y',y'')\,dx$ is that $y(x)$ solves the fourth order differential Euler-Poisson equation

$$\frac{\partial F}{\partial y}(x,y,y',y'') - \frac{d}{dx}\frac{\partial F}{\partial y'}(x,y,y',y'') + \frac{d^2}{dx^2}\frac{\partial F}{\partial y''}(x,y,y',y'') = 0.$$

This time we will transform this functional into a simple function of $y''$, as follows. Similarly to what we did before, we define, and denote $p_1(x)$ the unit price of $y'(x)$ as

$$p_1(x) = \frac{\partial}{\partial y'(x)}\int_x^b F(u,y,y',y'')\,du. \tag{3.1}$$

To clearly distinguish this price from the unit price of $y(x)$ introduced before as $p_y(x)$, we will henceforth designate the latter as $p_0(x)$. We thus have

$$p_y(x) \equiv p_0(x) = \frac{\partial}{\partial y(x)}\int_x^b F(u,y,y',y'')\,du. \tag{3.2}$$

We now suppose that *both* $y(x)$ and $y'(x)$ are known over interval $[a,b)$, except on a *very small, open, interval* $(x, x+\Delta x)$ within $[a,b]$. The only piece of information missing is the value of $y''(x)$. We need to optimize the functional $\int_x^b F(u,y,y',y'')du$, denoted $G[y]$, equal to

$$\begin{aligned} G[y] &= \int_x^b F(u,y,y',y'')\,du \\ &= F(x,y,y',y'')\Delta x + o(\Delta x) + \int_{x+\Delta x}^b F(u,y,y',y'')\,du. \end{aligned} \tag{3.3}$$

The optimal integral on the right hand side, depending on the three variables $x+\Delta x$, $y(x+\Delta x)$ and $y'(x+\Delta x)$ only, can be denoted $G^*(x+\Delta x, y(x+\Delta x), y'(x+\Delta x))$; $y'(x+\Delta x)$ depends upon $y''$ through

$$y'(x+\Delta x) = y'(x) + y''(x)\Delta x + o(\Delta x). \tag{3.4}$$

Our functional $G[y]$ thus can be transformed into a function of $y''$; denoted $G(y'')$ it is equal to

$$G(y'') = F(x,y,y',y'')\Delta x + o(\Delta x) + G^*(x+\Delta x,\ y(x+\Delta x), y'(x+\Delta x)) \tag{3.5}$$

and is a function of $y''$ for two reasons: first, directly, as expressed in $F(x,y,y',y'')$; secondly, as a composite function through the variable $y'(x+\Delta x)$ as given by (3.4).

Taking to zero the derivative of $G(y'')$ with respect to $y''$, we get

$$\frac{dG(y'')}{dy''} = \frac{\partial F}{\partial y''}(x,y,y',y'')\Delta x +$$

$$+\frac{\partial G^*\left(x+\Delta x,\ y(x+\Delta x), y'\left(x+\Delta x\right)\right)}{\partial y'\left(x+\Delta x\right)}\frac{\partial y'\left(x+\Delta x\right)}{\partial y''} = 0. \tag{3.6}$$

The derivative $\frac{\partial G^*(x+\Delta x,\, y(x+\Delta x), y'(x+\Delta x))}{\partial y'(x+\Delta x)}$, the rate of increase of the optimal value of $G^*$ per additional unit of $y'$ available at $x+\Delta x$ is the unit price of the variable $y'$ at $x+\Delta x$, to be denoted as $p_1\left(x+\Delta x\right)$. On the other hand, in light of (3.4), $\frac{\partial y'(x+\Delta x)}{\partial y''} = \Delta x$, and (3.6) becomes

$$\frac{dG(y'')}{dy''} = \frac{\partial F}{\partial y''}(x,y,y',y'')\Delta x + p_1\left(x+\Delta x\right)\Delta x = 0. \tag{3.7}$$

Simplifying, and taking $\Delta x$ to zero gives our first central equation

$$\frac{\partial F}{\partial y''}(x,y,y',y'') + p_1\left(x\right) = 0. \tag{3.8}$$

Suppose that this equation has a solution $y''$; replace this value into (3.5). $G(y'')$ can now be written as $G^*\left(y''\right)$, and equation (3.5) becomes the *identity*

$$G^*\left(y''\right) = F(x,y,y',y'')\Delta x + o\left(\Delta x\right) + G^*\left(x+\Delta x,\ y(x+\Delta x), y'\left(x+\Delta x\right)\right). \tag{3.9}$$

Let us first differentiate identity (3.9) with respect to $y'$; this gives

$$\begin{aligned}\frac{dG^*(y'')}{dy'} \quad &= \quad \frac{\partial F}{\partial y'}(x,y,y',y'')\Delta x + \frac{\partial G^*}{\partial y\left(x+\Delta x\right)}\frac{\partial y\left(x+\Delta x\right)}{\partial y'} + \\ &\quad + \frac{\partial G^*}{\partial y'\left(x+\Delta x\right)}\frac{\partial y'\left(x+\Delta x\right)}{\partial y'}\end{aligned} \tag{3.10}$$

or, equivalently, with our preceding notation,

$$p_1(x) = \frac{\partial F}{\partial y'}(x,y,y',y'')\Delta x + p_0\left(x+\Delta x\right)\frac{\partial y\left(x+\Delta x\right)}{\partial y'} +$$

$$+\, p_1\left(x+\Delta x\right)\frac{\partial y'\left(x+\Delta x\right)}{\partial y'}. \tag{3.11}$$

Using equations (2.3) and (3.4), and writing

$$p_1\left(x+\Delta x\right) = p_1(x) + p_1'(x)\Delta x + o\left(\Delta x\right)$$

we have

$$p_1(x) = \frac{\partial F}{\partial y'}(x,y,y',y'')\Delta x + p_0\left(x+\Delta x\right)\Delta x + p_1(x) + p_1'(x)\Delta x + o\left(\Delta x\right). \tag{3.12}$$

Eliminating $p_1(x)$, simplifying and taking $\Delta x$ to zero yields our second central equation

$$\frac{\partial F}{\partial y'}(x,y,y',y'') + p_0(x) + p_1'(x) = 0. \tag{3.13}$$

Differentiating identity (3.9) with respect to $y$ and operating in the same way leads to the third central equation

$$\frac{\partial F}{\partial y}(x, y, y', y'') + p_0'(x) = 0. \tag{3.14}$$

Consider the system of the three equations (3.8), (3.13) and (3.14). These are just the Euler-Poisson equation in parametric form. Indeed, differentiating – with respect to $x$ – twice (3.8) and once (3.13), we immediately get the fourth-order differential equation[1]

$$\frac{\partial F}{\partial y}(x, y, y', y'') - \frac{d}{dx}\frac{\partial F}{\partial y'}(x, y, y', y'') + \frac{d^2}{dx^2}\frac{\partial F}{\partial y''}(x, y, y', y'') = 0. \tag{3.15}$$

## 4. A WELCOME RESULT: NEW, HIGHLY EFFICIENT HAMILTONIANS LEADING DIRECTLY TO EULER EQUATIONS.

At this stage, it is only natural to wonder if we could construct algebraic expressions, akin to Hamiltonians, from which the central, parametric Euler equations could be *directly* derived, practically with no calculations. If this appeared to be possible, at least for the functionals $\int_a^b F(x, y, y')\, dx$ and $\int_a^b F(x, y, y', y'')\, dx$ we just took up, we would definitely wonder whether this would carry over to all possible configurations of the functional, including complex ones.

Consider first the functional $\int_a^b F(x, y, y')\, dx$. The Euler equation in parametric form was given by the system

$$\frac{\partial F}{\partial y'}(x, y, y') + p\,(x) = 0 \tag{4.1}$$

and

$$\frac{\partial F}{\partial y}(x, y, y') + p'(x) = 0. \tag{4.2}$$

It is almost immediate to determine an expression $D(x, y, y')$ whose gradient with respect to $y$ and $y'$, when taken to zero, yields (4.1) and (4.2) . This is

$$D(x, y, y') = F(x, y, y') + p(x)y' + p'(x)y = F(x, y, y') + \frac{d}{dx}\left[p(x)y(x)\right] \tag{4.3}$$

This expression turns out to have exactly the structure of the concept Robert Dorfman had introduced as a "modified Hamiltonian" (1969 [3], p. 822), that we called a *Dorfmanian* and denoted $D$ in [8] and [9]. Indeed, referring to the main lines of Dorfman's essay given in this appendix, we can see that (4.3) is given by (6.6) when (a) the state variable is the function $y(x)$; (b) the derivative $y'(x)$ plays the role of the control variable; and (c) $p(x)$ replaces $\lambda(t)$.

In the simple case of the functional $I\,[y] = \int_a^b F(x, y, y')\, dx$ considered in section 2, the Dorfmanian is the traditional Hamiltonian augmented by $p'(x)y$. It offers remarkable advantages. First, the Dorfmanian has an immediate interpretation. Multiplying (4.3) by $dx,$ we have

$$\begin{aligned} D(x, y, y')dx &= F(x, y, y')dx + p(x)y'dx + p'(x)ydx \\ &= F(x, y, y')dx + d\left[p(x)y(x)\right]. \end{aligned} \tag{4.4}$$

This expression represents the sum of *all* effects generated on the functional by the optimal function $y$ and its derivative $y'$ at any point $x$, in an interval $dx$. The first effect is measured by the integrand of the functional, $F(x, y, y')dx$. In addition, we recall that $p(x)y(x)$ is the value

[1] An alternate way of obtaining the Euler - Poisson equation can be found in [7].

of the extremal $y$ at $x$; therefore the second part of the Dorfmanian, $d\,[p(x)y(x)] = p(x)y'dx + p'(x)ydx$, is the *increase in the value of* $y$ generated both by an increase in the quantity of $y$ (equal to $p(x)y'dx$) and by a change in its price (given by $p'(x)ydx$). A necessary condition for $y$ and $y'$ to optimize the functional is of course that the derivatives of $D$ with respect to $y$ and $y'$ vanish. Taking to zero the gradient of these total effects generates parametrically the Euler equation in just two lines. Indeed, we get

$$\frac{\partial D(x,y,y')}{\partial y'} = \frac{\partial F}{\partial y'}(x,y,y') + p\,(x) = 0$$

and

$$\frac{\partial D(x,y,y')}{\partial y} = \frac{\partial F}{\partial y}(x,y,y') + p'\,(x) = 0,$$

corresponding to our equations 2.8 and 2.13.

In the case of the functional $\int_a^b F(x,y,y',y'')\,dx$, consider the three equations we derived before:

$$\frac{\partial F}{\partial y''}(x,y,y',y'') + p_1\,(x) = 0, \tag{4.5}$$

$$\frac{\partial F}{\partial y'}(x,y,y',y'') + p_0(x) + p_1'(x) = 0, \tag{4.6}$$

and

$$\frac{\partial F}{\partial y}(x,y,y',y'') + p_0'(x) = 0, \tag{4.7}$$

leading to the Euler-Poisson equation. Let us ask whether we can find a modified Hamiltonian (an extended Dorfmanian) whose gradient, when equated to zero, would yield these equations.

When we took up this case, we supposed that *both* $y$ *and* $y'$ were known except on a small interval $(x, x+\Delta x)$; therefore we just needed to optimize $y''$ at $x$. We thus can infer that both functions, $y(x)$ and $y'(x)$, now hold the role solely played earlier by $y(x)$. Therefore it seems logical to extend the Dorfmanian (4.4) to the expression

$$D(x,y,y',y'') = F(x,y,y',y'') + \frac{d}{dx}\,[p_0(x)y(x)] + \frac{d}{dx}\,[p_1(x)y'(x)] =$$

$$= F(x,y,y',y'') + p_0(x)y' + p_0'(x)y + p_1(x)y'' + p_1'(x)y' \tag{4.8}$$

Equating to zero the gradient of (4.8) with respect to $y''$, $y'$, and $y$ yields equations (3.9), (3.13), and (3.14), and hence the Euler-Poisson equation in just three lines.

We are now on sure footing to infer that the Dorfmanian can be extended to handle in a most efficient way the optimization of more complex functionals, thus yielding generalized Euler equations without any concern to overcome the difficulties met by Lagrange. This is what we did in [8] and [9].

## 5. CONCLUSION.

Several paths can be considered if our goal is to optimize functionals defined by integrals. First, we can follow the analytical approach offered by Lagrange, enthusiastically embraced by Euler as a definite improvement upon his geometric reasoning. Innovative as it may be, the Lagrange method has its own drawbacks: it implies overcoming two hurdles, one of them particularly challenging in the case of multiple integrals, since it involves the difficult extension of Green's theorem to $n-$space.

Today, shortcuts avoiding these hurdles are available if we have recourse to optimal control theory, at least for simple functionals. But in turn, the Pontryagin principle is not devoid of its own problems. The first difficulty is of an interpretative nature. Even in the simplest cases, it is not easy to understand the significance of the implied equations; in particular, it is hard to figure out why the derivative of the Hamiltonian with respect to the state variable should be equal to minus the derivative of the costate variable with respect to time. Secondly, applying the Pontryagin principle to multiple integrals is admittedly challenging. Finally, to the best of our knowledge, the principle has never been extended to cover high-order functionals.

It is our view that among all possible approaches, the one resting on the Dorfmanian is by far the most useful – we could even say the most beautiful. Introducing new concepts: (1) the value of a variable and (2) a new Hamiltonian with an immediate, meaningful appeal, Dorfman's reasoning leads to the Euler equation in just two lines, without any hurdles to overcome – remember that the equation eluded even Newton, Leibniz, as well as the Bernouilli brothers. In its extensions, the Dorfmanian proves to be highly efficient: the Euler-Poisson equation is reached in three lines; and if we set out to address the difficult problem of optimizing $n-$uple integrals, we will obtain the Ostrogradski equation in just four lines of elementary calculations. It is only to be hoped that the path opened up by Robert Dorfman will be continued.

## 6. APPENDIX: ROBERT DORFMAN'S MODIFIED HAMILTONIAN: THE BIRTH OF A BEAUTIFUL, LONG NEGLECTED IDEA.

We believe that it is important to recall how a highly fruitful idea in dynamic optimization was born (1969, [3]). Taking up the basic problem of optimal control theory, Dorfman defined a state variable as $k(t)$ and a control variable as $x(t)$, and considered the problem of determining a necessary condition for the maximization of a functional

$$\int_0^T u(k(t), x(t), t)\, dt \tag{6.1}$$

under the constraints

$$\dot{k} = f(k, x, t) \tag{6.2}$$

and $k(0) = k_0$; $k(T) \leq k_T$. (We have kept Dorfman's original notation; for clarity, we emphasize the fact that in Dorfman's notation, $x(t)$ designates a *function* of time and $t$ is the integration variable; in our paper $x$ is an integration variable. The state variable $k(t)$ becomes our function $y(x)$, and $y'(x)$ plays the role of the control variable.)

The traditional Pontryagin approach requires to introduce a time dependent function $\lambda(t)$ and a Hamiltonian defined by

$$H = u(k, x, t) + \lambda(t) f(k, x, t) = u(k, x, t) + \lambda(t)\dot{k}. \tag{6.3}$$

The fundamental equations of the Pontryagin maximum principle then are

$$\frac{\partial H}{\partial x}=\frac{\partial u}{\partial x}+\lambda(t)\frac{\partial f}{\partial x}=0 \tag{6.4}$$

and

$$\frac{\partial H}{\partial k}=\frac{\partial u}{\partial k}+\lambda(t)\frac{\partial f}{\partial k}=-\dot{\lambda}(t), \tag{6.5}$$

together with the constraint (6.2).

Dorfman defined $\lambda(t)$ as the *unit price* of $k(t)$, equal to the derivative of the optimal value of the functional with respect to $k$ at time $t$. From economic reasoning, Dorfman first gave an impressive derivation of equations (6.4) and (6.5) (see [3], pp. 817-822).

He then made a crucial observation (p. 822): those equations could equally be derived by taking to zero the gradient with respect to $k$ and $x$ of a "modified Hamiltonian" (as he called it), denoted $H^*$ and defined as

$$\begin{aligned} H^* &= u(k,x,t)+\frac{d}{dt}[\lambda(t)k(t)] \\ &= u(k,x,t)+\lambda(t)\dot{k}(t)+\dot{\lambda}(t)k(t)=H+\dot{\lambda}(t)k(t). \end{aligned} \tag{6.6}$$

This modified Hamiltonian – this Dorfmanian as we called it – is thus the traditional Hamiltonian $H$ augmented by $\dot{\lambda}(t)k(t)$. As we have shown, it is a highly valuable concept: applied to the calculus of variations, not only does it yield the Euler equation in just two lines, but its extensions readily lead to all equations governing the optimisation of high-order functionals.

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