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GROWTH AND APPROXIMATION OF ENTIRE SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATION IN TERMS OF BESSEL POLYNOMIAL APPROXIMATION ERRORS IN L^p-NORM, $1 \leq p \leq \infty$

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ABSTRACT. We deal with entire solutions of some special type linear homogeneous partial differential equations that are represented in convergent series of Bessel polynomials. We determine the growth orders and types of the solutions, in terms of Bessel polynomial approximation errors in both sup norm and L^p-norm, $1 \leq p \leq \infty$.

Key words and phrases: Bessel polynomial approximation error; Partial differential equation and entire functions; Order and type.

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1. INTRODUCTION

The Bessel polynomials

$$y_n(t) = \sum_{k=0}^{\infty} \frac{(n+k)!}{(n-k)!k!}(\frac{t}{2})^k$$

satisfy the differential equation

$$t^2\frac{d^2w}{dt^2} + (2t+2)\frac{dw}{dt} - n(n+1)w = 0.$$

Let us consider the following homogeneous partial differential equation of the second order:

$$t^2\frac{\partial^2 u}{\partial t^2} - z^2\frac{\partial^2 u}{\partial z^2} + (2t+2)\frac{\partial u}{\partial t} - 2z\frac{\partial u}{\partial z} = 0. \tag{1.1}$$

The existence and behavior of global meromorphic solutions of (1.1) was studied by Hu and Yang ([2], [3]). According to Hu and Yang ([3]) these solutions are related by Bessel functions and Bessel polynomials for $(t, z) \in \mathbb{C}^2$. Also, they proved that partial differential equation (1.1) has an entire solution $u = f(t, z)$ on $\mathbb{C}^2$ if and only if $u = f(t, z)$ has a series expansion

$$f(t, z) = \sum_{n=0}^{\infty} \frac{c_n}{n!} y_n(t) z^n \tag{1.2}$$

such that

$$\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = 0. \tag{1.3}$$

It was found ([3]) that the series (1.2) converges in the domain $D_R = \{(t, z) \in \mathbb{C}^2, 2|tz| < R\}$, where $\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}, 0 < R < \infty$.
The generating function of Bessel polynomials has the following expansion in the domain D_1 ([4]):

$$(1-2tz)^{-\frac{1}{2}} \exp(\frac{1}{t}\{1-(1-2tz)^{\frac{1}{2}}\}) = \sum_{n=0}^{\infty} \frac{y_n(t)}{n!} z^n.$$

Bernstein theorem identifies a real analytic function on the closed unit disk as the restriction of an analytic function defined on an open disk of radius $R > 1$ by computing R from the sequence of minimal errors generated from optimal polynomials approximates. The disk of maximum radius on which analytic function $f(t, z)$ exists is denoted by D_R. A function $f(t, z)$ is said to be regular in D_R if the series (1.2) converges uniformly on compact subsets of D_R. A class of functions $f(t, z)$ regular in D_R will be denoted by $A(D_R)$. If f is an entire function, then we write $f \in A(\mathbb{C}^2)$.

Let Π_n be a set of Bessel polynomials of degree no higher than n. Approximation of function $f(t, z) \in A(D_R)$ by Bessel polynomials $g_n(t, z) \in \Pi_n$ be determined as

$$E_n(f, R) = \inf_{g_n(t,z)\in\Pi_n} \{\max_{(t,z)\in\overline{D_R}} |f(t, z) - g_n(t, z)|\}, \tag{1.4}$$

where $\overline{D_R}$ be the closure of D_R.

In ([3]) Hu and Yang studied the growth order and type of entire solutions of the equation (1.1) in terms of series coefficients c_n given by (1.2). Here in this paper, we eloborate the growth order and type of entire function solutions of (1.1) in terms of Bessel polynomial approximation errors determined by (1.4).

We define the order of $f(t,z)$ by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r,f)}{\log r}.$$

If $0 < \rho(f) < \infty$, then the type of $f(t,z)$ defined by

$$T(f) = \limsup_{r\to\infty} \frac{\log^+ M(r,f)}{r^{\rho(f)}},$$

where $\log^+ x = \max\{\log x, 0\}$ and $M(r,f) = \max_{|z|\le r, |t|\le r} |f(t,z)|$.

2. AUXILIARY RESULTS

Lemma 2.1. *Let $f(t,z) \in A(D_R)$, then the following inequality holds:*

$$|c_n|R^n \le (-1)^{n+1}(2n)^n 2^{\frac{3}{2}} n!(n+\frac{1}{2})e^{-n-1}E_{n-1}(f,R),$$

where $E_{n-1}(f,R)$ is defined by (1.4).

Proof. In view of orthogonality property of Bessel polynomials ([1], [4]) with uniform convergence of series (1.2) on $\overline{D_\tau}, 0 < \tau < R$, it follows that the coefficients c_n given by integrating counter-clockwise around the unit circle

$$c_n\tau^n = (-1)^{n+1}\frac{n!}{2\pi i}(n+\frac{1}{2})\int_{|t|=1} f(t,z)y_n(t)e^{-\frac{2}{t}}dt. \tag{2.1}$$

Using the addition theorem of Bessel polynomials $y_n(t)$, we get

$$\int_{|t|=1} q(t,z)e^{-\frac{2}{t}}y_n(t)dt = 0, \tag{2.2}$$

where $q \in \Pi_{n-1}$. Bearing (2.2), we can rewrite (2.1) as

$$c_n\tau^n = (-1)^{n+1}\frac{n!}{2\pi i}(n+\frac{1}{2})\int_{|t|=1} (f(t,z)-q(t,z))y_n(t)e^{-\frac{2}{t}}dt. \tag{2.3}$$

In the consequence of Schwartz inequality and orthogonality of Bessel polynomials in (2.3), we obtain

$$c_n\tau^n \le \max_{t,z\in\overline{D_\tau}} |f(t,z)-q(t,z)|(-1)^{n+1}n^n 2^{n+\frac{1}{2}} n!(n+\frac{1}{2})e^{-n-1}. \tag{2.4}$$

Now it follows from the definition of $E_n(f,R)$ that there exists a Bessel polynomial $\hat{q} \in \Pi_{n-1}$, such that

$$\max_{t,z\in\overline{D_\tau}} |f(t,z)-\hat{q}(t,z)| \le 2E_{n-1}(f,\tau). \tag{2.5}$$

Putting $q = \hat{q}$ in (2.4) and using (2.5) with the arbitrariness of τ, we obtain the required result from (2.4). ∎

Lemma 2.2. *Let $f(t,z) \in A(\mathbb{C}^2)$, then*

$$E_n(f,R) \le KM(r,f)(\frac{R}{r})^{2n}t^n e^{\frac{1}{t}},$$

for $0 < r < R$ and all sufficiently large values of n. K is a constant independent of n and r.

Proof. First we consider the truncated polynomial

$$P_n^f(t,z) = \sum_{k=0}^{n} \frac{c_n}{n!} y_n(t) z^k,$$

where $r > 0$ and $P_n^f(t,z) \in \Pi_n$. From (1.4) for all $r, 0 < r < R$, we get

$$E_n(f,R) \leq \max_{t,z\in \overline{D_R}} |f(t,z) - P_n^f(t,z)| \leq \sum_{j=n+1}^{\infty} \frac{|c_j|}{j!} R^{2j} |y_j(t)|. \tag{2.6}$$

We have ([3]):

$$|c_k| \leq \frac{M(r,f)2^k(k!)^2}{r^{2k}(2k)!} \quad for \;\; every \;\; r < R. \tag{2.7}$$

From ([5]) for $t \neq 0$ and $n \to \infty$, we have

$$|y_j(t) \sim \sqrt{2}(\frac{2jt}{e})^j e^{\frac{1}{t}}. \tag{2.8}$$

Combining (2.6), (2.7) and (2.8), it gives

$$E_n(f,R) \leq M(r,f)e^{\frac{1}{t}}\sqrt{2} \sum_{j=n+1}^{\infty} \frac{j!2^j}{(2j)!}(\frac{2jt}{e})^j(\frac{R}{r})^{2j}.$$

Using Stirling formula for the factorials, we get

$$\frac{(2j)!}{j!2^j} \sim \sqrt{2}(\frac{2j}{e})^j \quad for \;\; j > 1.$$

Hence

$$\begin{aligned} E_n(f,R) \leq & 2M(r,f)e^{\frac{1}{t}} \sum_{j=n+1}^{\infty} t^j (\frac{R}{r})^{2j} \\ & \leq 2M(r,f)e^{\frac{1}{t}}(\frac{R}{r})^{2n} \sum_{j=n+1}^{\infty} t^j (\frac{R}{r})^{2j-2n}. \end{aligned}$$

For $r > eR$, substituting $\nu = j - n$, we have

$$\begin{aligned} E_n(f,R) \leq & 2M(r,f)e^{\frac{1}{t}}(\frac{R}{r})^{2n} \sum_{\nu=1}^{\infty} t^{\nu+n}(e)^{-2\nu} \\ & \leq 2M(r,f)e^{\frac{1}{t}}(\frac{R}{r})^{2n} t^n \sum_{\nu=1}^{\infty} t^{\nu}(e)^{-2\nu}. \end{aligned}$$

For $t < e^2$ the last series is convergent, therefore the required result is immediate. ∎

Lemma 2.3. *The partial differential equation (1.1) has an entire solution $u = f(t,z)$ on $\mathbb{C}^2$ if and only if $u = f(t,z)$ has a series expansion given by (1.2) such that*

$$\lim_{n\to\infty} (E_n(f,R))^{\frac{1}{n}} = 0. \tag{2.9}$$

Proof. Let $f(t,z) \in A(D_R)$ continues to an entire function $f(t,z)$. Then equality (2.9) follows from Lemma 2.2. Now let (2.9) holds and using the estimate (2.6) with Lemma 2.1, we have

(2.10)
$$
\begin{aligned}
|\sum_{n=0}^{\infty} \frac{c_n y_n(t)}{n!} z^n| \le & |c_o||y_o(t)| + e^{\frac{1}{t}} \sum_{n=1}^{\infty} \frac{|c_n|}{n!} r^n \sqrt{2} (\frac{2nt}{e})^n \\
& \le |c_o| + \sqrt{2} e^{\frac{1}{t}} 2^{\frac{3}{2}} \sum_{n=1}^{\infty} (-1)^{n+1} (2n)^n (n+\frac{1}{2}) e^{-n} (\frac{2nt}{e})^n E_{n-1}(f,R) (\frac{r}{R})^n \\
& = |c_o| + 2e^{\frac{1}{t}} \sum_{n=1}^{\infty} (-1)^{n+1} (2n)^n e^{-2n} t^n E_{n-1}(f,R) (\frac{r}{R})^n.
\end{aligned}
$$

Thus our assumption (2.9), a uniform convergence of the series in the right side of equality (1.2) on compact subsets of the complex plane follows. Hence the function $f(t,z) \in A(D_R)$ represented by a series (1.2) shall continue over the whole complex plane $\mathbb{C}$. ∎

3. MAIN RESULTS

Theorem 3.1. *If $f(t,z)$ is an entire solution of (1.1) defined by (1.2), then the order $\rho(f)$ is given by*

(3.1)
$$\rho(f) = \limsup_{n\to\infty} \frac{2\ln n}{\ln[R^{-1}E_n(f,R)]^{-\frac{1}{n}}},$$

where $E_{n-1}(f,R)$ is determined by (1.4).

Proof. First we consider the following functions:

$$f_1(t,z) = \sum_{k=0}^{\infty} t^{-k} e^{-\frac{1}{t}} E_k(f,R) (\frac{z}{R})^{2k};$$

and

$$f_2(t,z) = \sum_{k=1}^{\infty} (-1)^{k+1} (2k)^k k! (k+\frac{1}{2}) e^{-k-1} E_{k-1}(f,R) (\frac{z}{R})^{2k}.$$

It is clear from (2.9) that $f_1(t,z)$ and $f_2(t,z)$ are entire functions. By Lemma 2.2 and inequality (2.10), we get

(3.2)
$$\mu(r,f) \le M(r,f) \le c_o + 2e^{\frac{1}{t}} M(r,f_2),$$

where $\mu(r,f_1)$ is the maximum term of power series of $f_1(t,z)$ on the disk $D_R = \{(t,z) \in \mathbb{C}^2 : 2|tz| < R\}$, and $M(r,f_2) = \max_{|z|\le r, |t|\le r} |f_2(t,z)|$ is the maximum of the module of function $f_2(t,z)$. Hence by using [([3]), Thm. 1.5], we obtain

(3.3)
$$\rho(f_1) \le \rho(f) \le \rho(f_2).$$

Applying the formula that expresses the order of an entire solution of (1.1) defined by (1.2) and (1.3) in terms of coefficients c_n given by [([3]), Thm. 1.2], we get

(3.4)
$$\rho(f_1) = \rho(f_2) = \limsup_{n\to\infty} \frac{2\ln n}{\ln[R^{-1}E_n(f,R)]^{-\frac{1}{n}}}.$$

Combining (3.3) and (3.4) we complete the proof of theorem. ∎

Theorem 3.2. *If $f(t,z)$ is an entire solution of (1.1) defined by (1.2) and (1.3) such that $0 < \rho(f) < \infty$, then the type $T(f)$ of f satisfies*

$$e\rho(f)T(f) = 2^{\frac{\rho(f)}{2}} \limsup_{n\to\infty} 2n[R^{-1}E_n(f,R)]^{\frac{\rho(f)}{2n}}.$$

Proof. In view of inequality (3.2) with [([3]), Thm. 1.5] we obtain

$$T(f_1) \le T(f) \le T(f_2). \tag{3.5}$$

Now applying the formula of type of an entire solution of (1.1) in terms of coefficients c_n given by [([3]), Thm. 1.3], we have

$$T(f_1) = T(f_2) = 2^{\frac{\rho(f)}{2}} \limsup_{n\to\infty} 2n[R^{-2n}E_n(f,R)]^{\frac{\rho(f)}{2n}}. \tag{3.6}$$

The equality (3.6) and inequality (3.5) together complete the required proof. ∎

Theorem 3.2 fail to compare the Bessel polynomial approximation errors $E_n(f,R)$ of those entire solutions of (1.1) defined by (1.2) and (1.3), which have same order but their types are infinity. To include this important class the concept of proximate order has used.

Definition 3.1. A proximate order $\rho(r)$ (for the order $\rho \ge 0$) is a function $\rho(r) \ge 0$ defined for $r \in \mathbb{R}^+$ satisfying
1.$\lim_{r\to\infty} \rho(r) = \rho$,
2.$\lim_{r\to\infty} r\rho'(r)\ln r = 0$.

So from Theorem 3.2, it is possible to obtain:

Let $\rho(r)$ be the proximate order of entire solution $f(t,z)$ the formula for the type $T^*(f)$ with respect to the proximate order $\rho(r)$ is given by

$$\frac{R}{\sqrt{2}}(T^*(f)e\rho(f))^{\frac{1}{\rho(f)}} = \varphi(n)[E_n(f,R)]^{\frac{1}{2n}},$$

where $r = \varphi(\tau)$ is the function, inverse to $\tau = r^{\rho(r)}$.

4. L^p-APPROXIMATION ERRORS

In this section we will show that in the results of Section 3, $E_n(f,R)$ can be replaced by $E_n^p(f,R)$, the approximation error in L^p-norm, $1 \le p < \infty$.

The L^p-norm on $\overline{D_R}$ is defined as

$$\| f(t,z) \|_{p,R} = [\int\int_{\overline{D_R}} |f(t,z)|^p dt dz]^{\frac{1}{p}}, 1 \le p < \infty.$$

For $f(t,z) \in A(\overline{D_R})$, the L^p-approximation error is defined as

$$E_n^p(f,R) = \inf_{g_n(t,z)\in\Pi_n} \{\| f(t,z) - g_n(t,z) \|_{p,R}\}. \tag{4.1}$$

For each n there is an extremal Bessel polynomial $g_n^*(t,z) \in \Pi_n$ for which $\| f(t,z) - g_n^*(t,z) \|_{p,R} = E_n^p(f,R)$.

Lemma 4.1. *Let $f(t,z) \in A(D_R)$, then for all $n \in \mathbb{N}$ the following inequality holds:*

$$|c_n|R^n \le K(-1)^{n+1}(2n)^n 2^{\frac{3}{2}} n!(n+\frac{1}{2})(\pi R^2)^{\frac{1}{\eta}} E_{n-1}^p(f,R), \frac{1}{p} + \frac{1}{\eta} = 1, \tag{4.2}$$

where K is a constant depending on p and R only.

Proof. For $p > 1$ choose $\frac{1}{p} + \frac{1}{\eta} = 1$. For $f(t,z) \in A(\overline{D_R})$ there exists a Bessel polynomial $\hat{q}_{n-1} \in \Pi_{n-1}$, by using Holder's inequality we obtain

$$\int_{|t|=1} |f(t,z) - \hat{q}_{n-1}(t,z)|dt \le (\int_{|t|=1} |f(t,z) - \hat{q}_{n-1}(t,z)|^p dt)^{\frac{1}{p}} (\int_{|t|=1} dt)^{\frac{1}{\eta}}. \tag{4.3}$$

Also we know that

$$2E^p_{n-1}(f,R) \ge \| f(t,z) - \hat{q}_{n-1}(t,z) \|_{p,R} . \tag{4.4}$$

Since $y_n(t)e^{-\frac{1}{t}} \sim \sqrt{2}(\frac{2nt}{e})^n$ or $y_n(t)e^{-\frac{2}{t}} \sim 2^{n+\frac{1}{2}} n^n e^{-n-\frac{1}{t}} t^n$.
From (2.3), we have

$$\begin{aligned} |c_n|\tau^n \le& (-1)^{n+1} \frac{n!}{2\pi}(n+\frac{1}{2}) \int_{|\tau|=1} |f(\tau,z) - q_n(\tau,z)||y_n(\tau)e^{-\frac{2}{\tau}}|d\tau \\ &\sim (-1)^{n+1} \frac{n!}{2\pi}(n+\frac{1}{2}) 2^{n+\frac{1}{2}} n^n \int_{|\tau|=1} |f(\tau,z) - q_n(\tau,z)||e^{-n-\frac{1}{\tau}}\tau^n|d\tau \\ &\le K(-1)^{n+1} \frac{n!}{2\pi}(n+\frac{1}{2}) 2^{n+\frac{1}{2}} n^n \int_{\overline{D_R}} |f(\tau,z) - q_n(\tau,z)|d\tau. \end{aligned} \tag{4.5}$$

Now combining (4.3), (4.4) and (4.5) with $q_n = \hat{q}_n$ in (4.5), by using Holder's inequality we get

$$|c_n|\tau^n \le K(-1)^{n+1} \frac{n!}{2\pi}(n+\frac{1}{2}) 2^{n+\frac{1}{2}} n^n 2E^p_{n-1}(f,\tau)(\pi\tau^2)^{\frac{1}{\eta}}, \tag{4.6}$$

where K is a constant depending on p and τ only. For $p = 1$, (4.6) is obvious with $\eta = \infty$. Using the arbitrariness of τ, we obtain the required result from (4.6). ∎

Lemma 4.2. *Let $f(t,z) \in A(\mathbb{C}^2)$, then*

$$E^p_n(f,R) \le K_1 M(r,f)(\frac{R}{r})^{2n} t^n e^{\frac{1}{t}}, \tag{4.7}$$

for $0 < r < R$ and all sufficiently large values on n. Here K_1 is a constant depending on p and R only.

Proof. Using Lemma 2.2 and (4.1), the proof is immediate. ∎

Using (4.2) and (4.7) we see that $E_n(f,R)$ can be replaced by $E^p_n(f,R)$ in Section 3.

Conclusion 1. *Since the entire solutions of linear homogeneous partial differential equation represented by Bessel polynomials are used not only in mathematics, but also in physics, mechanics, and other applied sciences, it is important to estimate their growth by expansion coefficients in series or by other characteristics, such as approximation errors in both sup norm and L^p-norm, $1 \le p \le \infty$.*

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