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(p, q)-LUCAS POLYNOMIAL AND THEIR APPLICATIONS TO A CERTAIN FAMILY OF BI-UNIVALENT FUNCTIONS DEFINED BY WANAS OPERATOR

M MUSTHAFA IBRAHIM[1], SALEEM AHMED[2]

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COLLEGE OF ENGINEERING, UNIVERSITY OF BURAIMI[1,2], AL BURAIMI, P.O.BOX 512, OMAN,
musthafa.i@uob.edu.om, saleem.a@uob.edu.om

ABSTRACT. In this article, by making use of (p, q)-Lucas polynomials, we introduce and investigate a certain family of analytic and biunivalent functions associated with Wanas operator which defined in the open unit disk $\mathcal{U}$. Also, the upper bounds for the initial Taylor-Maclaurin coefficients and the Fekete-Szegö inequality of functions belonging to this family are obtained.

Key words and phrases: (p, q)-Lucas polynomials, Bi-univalent functions , Subordination.

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1. INTRODUCTION

In mathematics, Lucas polynomials are a polynomial sequence which can be considered as a generalization of the lucas number. Ths ploynomials are of wide spectra in a variety of branches such as Physics, Engineering, Architecture, Nature, Art, Number Theory, Combinatorics and Numerical analysis. The well-known (p,q)-Lucas polynomials are defined by the following definition:

Definition 1.1. ([9]) Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The (p,q)-Lucas polynomials $L_{p,q,n}(x)$ are established by the recurrence relation

$$L_{p,q,n}(x) = p(x)L_{p,q,n-1}(x) + q(x)L_{p,q,n-2}(x) \quad (n \geq 2)$$

from which the first few Lucas polynomials can be found as

$$L_{p,q,0}(x) = 2, \;\; L_{p,q,1}(x) = p(x), \;\; L_{p,q,2}(x) = p^2(x) + 2q(x), \\ L_{p,q,3}(x) = p^3(x) + 3p(x)q(x), \; \cdots . \tag{1.1}$$

Remark 1.1. By selecting the particular values of (p,q)-Lucas polynomials reduces to several polynomials. Some of these special cases are recorded below.

(1) Taking $p(x) = x$ and $q(x) = 1$, we obtain the Lucas polynomials $L_n(x)$.

(2) Taking $p(x) = 2x$ and $q(x) = 1$, we obtain the Pell-Lucas polynomials $D_n(x)$.

(3) Taking $p(x) = 1$ and $q(x) = 2x$, we obtain the Jacobsthal-Lucas polynomials $j_n(x)$.

(4) Taking $p(x) = 3x$ and $q(x) = -2$, we obtain the Fermat-Lucas polynomials $f_n(x)$.

(5) Taking $p(x) = 2x$ and $q(x) = -1$, we obtain the Chebyshev polynomials first kind $T_n(x)$.

Theorem 1.1. *(see [9]) Let $\mathcal{S}_{\{L_{p,q,n}(x)\}}(z)$ the generating function of the (p,q)-Lucas polynomial sequence $L_{p,q,n}(x)$.Then*

$$\mathcal{S}_{L_{p,q,n}(x)}(z) = \sum_{n=0}^{\infty} L_{p,q,n}(x)z^n = \frac{2 - p(x)z}{1 - p(x)z - q(x)z^2}. \tag{1.2}$$

Let $\mathcal{A}$ denote the class of functions of the form

$$f\,(z) = z + \sum_{n=2}^{\infty} a_n\, z^n, \tag{1.3}$$

consisting of functionas which are holomorphic and univalent in the unit disk $\mathcal{U}$. Let f^{-1} be inverse of the function $f\,(z)$, then we have

$$f^{-1}[f(z)] = z\,;\,(z \in \mathcal{U})$$

and

$$f[f^{-1}(w)] = w\,; \qquad (\,|w| < r_0(f)\,;\, r_0(f) \geq \frac{1}{4}\,).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots , \tag{1.4}$$

analytic in the open unit disk $\mathcal{U}$. Also we let Σ denote the class of all function in $\mathcal{A}$ which are univalent in $\mathcal{U}$. The well known example in this class is the Koebe function $k(z)$, defined by

$$k(z) = \frac{z}{(z-1)^2} = z + \sum_{n=2}^{\infty} n\, z^n.$$

The Bieberbach conjecture about the coefficient of the univalent functions in the unit disk was formulated by Bieberbach [3] in the year 1916. The conjecture states that for every function $f \in S$ given by (1.1), we have $|a_n| \leq n$, for every n. Strictly inequality holds for all n unless f is the Koebe function or one of its rotation. For many years, this conjecture remained as a challenge to mathematicians. After the proof of $|a_3| \leq 3$ by Lowner in 1923, Fekete-Szegö[6] surprised the mathematicians with the complicated inequality

$$\left|a_3 - \mu\, a_2^2\right| \leq 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right),$$

which holds good for all values $0 \leq \mu \leq 1$. Note that this inequality region was thoroughly investigated by Schaefer and Spencer [13].

For a class functions in $\mathcal{A}$ and a real (or more generally complex) number μ, the Fekete-Szegö problem is all about finding the best possible constant $C(\mu)$ so that $|a_3 - \mu\, a_2^2| \leq C(\mu)$ for every function in $\mathcal{A}$. For a brief history and interesting examples in the class Σ, (see [15]) (see also [7], [4], [5],[8]).

Recently, Wanas (see [17]) introduced the following operator (so-called Wanas operator) $\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} : \mathcal{A} \to \mathcal{A}$ de
fined by

$$\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z) = z + \sum_{n=2}^{\infty} \left[\chi_n(\kappa,\alpha,\beta)\right]^{\gamma} a_n z^n, \tag{1.5}$$

Where

$$\chi_n(\kappa,\alpha,\beta) = \sum_{m=1}^{k} \frac{k!}{m!(k-m)!}(-1)^{m+1}\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m}$$

$$\left(\alpha \in \mathbb{R}; \beta \geq 0 \;\; with \;\; \alpha+\beta > 0; m,\gamma \in \mathbb{N}_0 = \mathbb{N} \cup 0\right).$$

In the present paper, by using the $L_{p,q,n}(x)$ functions, our methodology intertwine to yield the Theory of Geometric Functions and that of Special Functions, which are usually considered as very different fields. Thus, we aim at introducing a new class of bi-univalent functions defined through the $(p\,,q)$ -Lucas polynomials. Furthermore, we derive coefficient inequalities and obtain Fekete-Szegö problem for this new function class.

Definition 1.2. For $0 \leq \lambda \leq 1$, A function $f \in \Sigma$ is said to be in the class $T_\Sigma(\lambda,\alpha,\beta,\kappa,\gamma;x)$ if it fullfills the subordinations:

$$1 + \frac{z(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z))'}{\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z)} + \frac{z(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z))''}{(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z))'} - \frac{\lambda z^2(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z))'' + z(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z))'}{\lambda z(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z))' + (1-\lambda)\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z)} \prec \mathcal{S}_{\{L_{p,q,n}(x)\}}(z) - 1, \tag{1.6}$$

and

$$1 + \frac{w(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w))'}{\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w)} + \frac{w(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w))''}{(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w))'} - \frac{\lambda w^2(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w))'' + z(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w))'}{\lambda w(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w))' + (1-\lambda)\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w)} \prec \mathcal{S}_{\{L_{p,q,n}(x)\}}(w) - 1, \tag{1.7}$$

where $g = f^{-1}$ given by (1.3).
It is interesting to note that the special values of λ, γ lead the class $T_{\Sigma}(\lambda, \alpha, \beta, \kappa, \gamma; x)$ to various subclasses, we illustrate the following subclasses:

(1) For $\lambda = \gamma = 0$, a function $f(z) \in \mathcal{A}$ is in the family $T_{\Sigma}(0, \alpha, \beta, \kappa, 0; x) =: T_{\Sigma}(x)$ which was considered recently by Magesh et al. in [1], if the following conditions are satisfied:
$$1 + \frac{zf''(x)}{f'(x)} \prec \Pi(x, y) + 1 - a,$$
and
$$1 + \frac{wg''(w)}{g'(w)} \prec \Pi(x, y) + 1 - a,$$
Where $z, w \in \mathcal{U}$ and the function g is described in (1.3).

(2) For $\lambda = 1 and \gamma = 0$, a function $f(z) \in \mathcal{A}$ is in the family $T_{\Sigma}(1, \alpha, \beta, \kappa, 0; x) =: \mathcal{W}_{\Sigma}(x)$ which was considered recently by Srivastava et al. in [14], if the following conditions are satisfied:
$$\frac{zf'(x)}{f(x)} \prec \Pi(x, y) + 1 - a,$$
and
$$\frac{wg'(w)}{g(w)} \prec \Pi(x, y) + 1 - a,$$
Where $z, w \in \mathcal{U}$ and the function g is described in (1.3).

2. COEFFICIENT BOUNDS

In this section, we shall make use of the (p, q)-Lucas polynomials to get the estimates on the coefficients $| a_2 |$ and $| a_3 |$ for functions in the class $T_{\Sigma}(\lambda, \alpha, \beta, \kappa, \gamma; x)$ proposed by Definition (1.2).

Theorem 2.1. *Let the function f given by (1.3) be in the class$T_{\Sigma}(\lambda, \alpha, \beta, \kappa, \gamma; x)$. Then*
(2.1)
$$| a_2 | = \frac{| p(x) | \sqrt{| p(x) |}}{\sqrt{| \left[\vartheta(\lambda, \gamma, \kappa, \alpha, \beta,) - (2-\lambda)^2 \chi_2^{2\gamma}(\kappa, \alpha, \beta)\right] p(x)^2 - 2(2-\lambda)^2 \chi_2^{2\gamma}(\kappa, \alpha, \beta) q(x) |}},$$
and
$$| a_3 | \leq \frac{| p(x)^2 |}{(2-\lambda)^2 \chi_2^{2\gamma}(\kappa, \alpha, \beta)} + \frac{| p(x) |}{(3-2\lambda) \chi_3^{\gamma}(\kappa, \alpha, \beta)} \tag{2.2}$$
Where
$$\vartheta(\lambda, \gamma, \kappa, \alpha, \beta,) = 2(3-2\lambda)\chi_3^{\gamma}(\kappa, \alpha, \beta) - (5 - (\lambda+1)^2)\chi_2^{2\gamma}(\kappa, \alpha, \beta).$$

Proof. Let $f \in T_{\Sigma}(\lambda, \alpha, \beta, \kappa, \gamma; x)$ From Definition (1.2), for some analytic function Φ and Ψ such that $\Phi(0) = \Psi(0) = 0$ and $| \Phi(x) | < 1$ and $| \Psi(x) | < 1$ for all $z, w \in \mathcal{U}$, we can write
$$1 + \frac{z(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z))'}{\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z)} + \frac{z(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z))''}{(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z))'} - \frac{\lambda z^2(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z))'' + z(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z))'}{\lambda z(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z))' + (1-\lambda)\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} f(z)} = \mathcal{S}_{\{L_{p,q,n}(x)\}}(\Phi(z)) - 1, \tag{2.3}$$
and
$$1 + \frac{w(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w))'}{\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w)} + \frac{w(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w))''}{(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w))'} - \frac{\lambda w^2(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w))'' + z(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w))'}{\lambda w(\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w))' + (1-\lambda)\mathcal{W}_{\alpha,\beta}^{\kappa,\gamma} g(w)} = \mathcal{S}_{\{L_{p,q,n}(x)\}}(\Psi(w)) - 1, \tag{2.4}$$

or equivalently

$$1 + \frac{z(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z))'}{\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z)} + \frac{z(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z))''}{(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z))'} - \frac{\lambda z^2(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z))'' + z(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z))'}{\lambda z(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z))' + (1-\lambda)\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z)} = -1 + L_{p,q,0}(x) + L_{p,q,1}(x)\Phi(z) + L_{p,q,2}(x)\Phi^2(z) + \cdots, \tag{2.5}$$

and

$$1 + \frac{w(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w))'}{\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w)} + \frac{w(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w))''}{(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w))'} - \frac{\lambda w^2(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w))'' + z(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w))'}{\lambda w(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w))' + (1-\lambda)\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w)} = -1 + L_{p,q,0}(x) + L_{p,q,1}(x)\Psi(z) + L_{p,q,2}(x)\Psi^2(z) + \cdots, \tag{2.6}$$

From the equalities (2.5) and (2.6), we obtain that

$$1 + \frac{z(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z))'}{\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z)} + \frac{z(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z))''}{(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z))'} - \frac{\lambda z^2(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z))'' + z(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z))'}{\lambda z(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z))' + (1-\lambda)\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} f(z)} = 1 + L_{p,q,1}(x)c_1 z + \{L_{p,q,2}(x)c_2 + L_{p,q,1}(x)c_1^2\}z^2 + \cdots, \tag{2.7}$$

and

$$1 + \frac{w(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w))'}{\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w)} + \frac{w(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w))''}{(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w))'} - \frac{\lambda w^2(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w))'' + z(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w))'}{\lambda w(\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w))' + (1-\lambda)\mathcal{W}^{\kappa,\gamma}_{\alpha,\beta} g(w)} = 1 + L_{p,q,1}(x)d_1 w + \{L_{p,q,2}(x)d_2 + L_{p,q,1}(x)d_1^2\}w^2 + \cdots. \tag{2.8}$$

It is fairly well known that if

$$|\phi(z)| = \left|c_1 z + c_2 z^2 + c_3 z^3 + \cdots\right| < 1, \ (z \in \mathcal{U})$$
$$|\varphi(z)| = \left|d_1 w + d_2 w^2 + d_3 w^3 + \cdots\right| < 1, \ (w \in \mathcal{U})$$

and it is well known that

$$|c_n| \le 1, |d_n| \le 1, \ n \in \mathcal{N}. \tag{2.9}$$

Thus, upon comparing the corresponding coefficients in (2.7) and (2.8), we have

$$(2-\lambda)\chi_2^{\gamma}(\kappa,\alpha,\beta)a_2 = L_{p,q,1}(x)c_1, \tag{2.10}$$

$$2(3-2\lambda)\chi_3^{\gamma}(\kappa,\alpha,\beta)a_3 - (5-(\lambda+1)^2)\chi_2^{2\gamma}(\kappa,\alpha,\beta)a_2^2 = L_{p,q,1}(x)c_2 + L_{p,q,2}(x)c_1^2, \tag{2.11}$$

$$-(2-\lambda)\chi_2^{\gamma}(\kappa,\alpha,\beta)a_2 = L_{p,q,1}(x)d_1, \tag{2.12}$$

$$2(3-2\lambda)\chi_3^{\gamma}(\kappa,\alpha,\beta)(2a_2^2 - a_3) - (5-(\lambda+1)^2)\chi_2^{2\gamma}(\kappa,\alpha,\beta)a_2^2 = L_{p,q,1}(x)d_2 + L_{p,q,2}(x)d_1^2. \tag{2.13}$$

From the equations (2.10) and (2.12) we can easily see that

$$c_1 = -d_1, \tag{2.14}$$

and From the equations (2.10) and (2.12) we can easily see that

$$2(2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)a_2^2 = L_{p,q,1}^2(x)(c_1^2 + d_1^2). \tag{2.15}$$

If we add (2.11) and (2.13), we get

$$2\left[2(3-2\lambda)\chi_3^{\gamma}(\kappa,\alpha,\beta)-(5-(\lambda+1)^2)\chi_2^{2\gamma}(\kappa,\alpha,\beta)\right]a_2^2 = L_{p,q,1}(x)(c_2+d_2)+ L_{p,q,2}(x)(c_1^2+d_1^2). \tag{2.16}$$

Clearly, by using (2.15) in the equality (2.16) , we have

$$a_2^2 = \frac{L_{p,q,1}^3(x)(c_2+d_2)}{2\left[L_{p,q,1}^2(x)\vartheta(\lambda,\gamma,\kappa,\alpha,\beta)-L_{p,q,2}(2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)\right]}, \tag{2.17}$$

Where

$$\vartheta(\lambda,\gamma,\kappa,\alpha,\beta,)=2(3-2\lambda)\chi_3^{\gamma}(\kappa,\alpha,\beta)-(5-(\lambda+1)^2)\chi_2^{2\gamma}(\kappa,\alpha,\beta).$$

Which gives

$$|\, a_2 \,| = \frac{|\, p(x)\,|\sqrt{|\, p(x)\,|}}{\sqrt{\left|\,\left[\vartheta(\lambda,\gamma,\kappa,\alpha,\beta,)-(2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)\right]p(x)^2-2(2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)q(x)\,\right|}}.$$

Moreover, if we subtract (2.13) from (2.11), we obtain

$$4(3-2\lambda)\chi_3^{\gamma}(\kappa,\alpha,\beta)(a_3-a_2^2)=L_{p,q,2}(x)(c_1^2-d_1^2)+L_{p,q,1}(x)(c_2-d_2). \tag{2.18}$$

Then, in view of (2.14) and (2.15) , (2.18) becomes

$$|\, a_3 \,| = \frac{L_{p,q,1}^2(x)(c_1^2+d_1^2)}{2(2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)}+\frac{L_{p,q,1}(x)(c_2-d_2)}{4(3-2\lambda)\chi_3^{\gamma}(\kappa,\alpha,\beta)}.$$

Thus applying (1.1) we obtain

$$|\, a_3 \,| \le \frac{|\, p(x)^2\,|}{(2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)}+\frac{|\, p(x)\,|}{(3-2\lambda)\chi_3^{\gamma}(\kappa,\alpha,\beta)}.$$

This completes the proof of Theorem 1. ∎

Corollary 2.2. *(see* [1]*) By taking $\lambda=\gamma=0$ in theorem 1, we state*

$$|\, a_2 \,| = \frac{|\, p(x)\,|\sqrt{|\, p(x)\,|}}{\sqrt{|\, -4p(x)^2-8q(x)\,|}}, \tag{2.19}$$

and

$$|\, a_3 \,| \le \frac{|\, p(x)^2\,|}{4}+\frac{|\, p(x)\,|}{6}. \tag{2.20}$$

Corollary 2.3. *(see* [16]*) By taking $\lambda=1$ and $\gamma=0$ in theorem 1, we state*

$$|\, a_2 \,| = \frac{|\, p(x)\,|\sqrt{|\, p(x)\,|}}{\sqrt{|\, -2q(x)\,|}}, \tag{2.21}$$

and

$$|\, a_3 \,| \le |\, p(x)^2\,| + \frac{|\, p(x)\,|}{2}. \tag{2.22}$$

3. FEKETE-SZEGÖ

In the next theorem, the Fekete-Szegö inequality for the family $T_\Sigma(\lambda, \alpha, \beta, \kappa, \gamma; x)$ is obtain.

Theorem 3.1. *For* $0 \leq \lambda \leq 1$ *and* $x, \mu \in \mathbb{R}$, *let* $f \in \mathcal{A}$ *be in the family* $T_\Sigma(\lambda, \alpha, \beta, \kappa, \gamma; x)$. *Then*

(3.1)
$$| a_3 - \mu a_2^2 | \leq \left\{ \begin{array}{l} \frac{|p(x)|}{2(3-2\lambda)\chi_3^\gamma(\kappa,\alpha,\beta)}, \\ |(1-\mu)| \leq | \vartheta(\lambda,\gamma,\kappa,\alpha,\beta) + (2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)\left(1+\frac{2q(x)}{p(x)}\right) | \\ \frac{|p(x)||(1-\mu)|}{|\left(\vartheta(\lambda,\gamma,\kappa,\alpha,\beta)-(2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)\right)p^2(x)-2(2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)q(x)|}, \\ |(1-\mu)| \geq | \vartheta(\lambda,\gamma,\kappa,\alpha,\beta) + (2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)\left(1+\frac{2q(x)}{p(x)}\right) | \end{array} \right\}.$$

Proof.

(3.2)
$$a_3 - \mu a_2^2 = \frac{L_{p,q,1}(x)(c_2 - d_2)}{4(3-2\lambda)\chi_3^\gamma(\kappa,\alpha,\beta)} + (1-\mu)\times \left(\frac{L_{p,q,1}^3(x)(c_2+d_2)}{2\left[L_{p,q,1}^2(x)\vartheta(\lambda,\gamma,\kappa,\alpha,\beta) - L_{p,q,2}(x)(2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)\right]}\right)$$

(3.3)
$$= \frac{L_{p,q,1}(x)}{2}\left[\left(H(\mu,x) + \frac{1}{2(3-2\lambda)\chi_3^\gamma(\kappa,\alpha,\beta)}\right)c_2 + \left(H(\mu,x) - \frac{1}{2(3-2\lambda)\chi_3^\gamma(\kappa,\alpha,\beta)}\right)d_2\right],$$

Where
$$H(\mu,x) = \frac{L_{p,q,1}^2(x)(1-\mu)}{L_{p,q,1}^2(x)\vartheta(\lambda,\gamma,\kappa,\alpha,\beta) - L_{p,q,2}(x)(2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)}.$$
Along the way, in view of (1.1), we conclude that

(3.4)
$$| a_3 - \mu a_2^2 | \leq \left\{ \begin{array}{l} \frac{|p(x)|}{2(3-2\lambda)\chi_3^\gamma(\kappa,\alpha,\beta)}, 0 \leq | H(\mu,x) | \leq \frac{1}{(3-2\lambda)\chi_3^\gamma(\kappa,\alpha,\beta)} \\ 2 \mid p(x) \mid\mid H(\mu,x) \mid, \mid H(\mu,x) \mid \geq \frac{1}{(3-2\lambda)\chi_3^\gamma(\kappa,\alpha,\beta)} \end{array} \right\},$$

After some computations, we obtain

(3.5)
$$| a_3 - \mu a_2^2 | \leq \left\{ \begin{array}{l} \frac{|p(x)|}{2(3-2\lambda)\chi_3^\gamma(\kappa,\alpha,\beta)}, \\ |(1-\mu)| \leq | \vartheta(\lambda,\gamma,\kappa,\alpha,\beta) + (2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)\left(1+\frac{2q(x)}{p(x)}\right) | \\ \frac{|p(x)||(1-\mu)|}{|\left(\vartheta(\lambda,\gamma,\kappa,\alpha,\beta)-(2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)\right)p^2(x)-2(2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)q(x)|}, \\ |(1-\mu)| \geq | \vartheta(\lambda,\gamma,\kappa,\alpha,\beta) + (2-\lambda)^2\chi_2^{2\gamma}(\kappa,\alpha,\beta)\left(1+\frac{2q(x)}{p(x)}\right) | \end{array} \right\}.$$

∎

Corollary 3.2. *(see* [16]*) By taking* $\lambda = 0$ *and* $\gamma = 0$ *in theorem 1, we state*

(3.6)
$$| a_3 - \mu a_2^2 | \leq \left\{ \begin{array}{l} \frac{|p(x)|}{6}, \\ |(1-\mu)| \leq | 6\left(1+\frac{2q(x)}{p(x)}\right) | \\ \frac{|p(x)||(1-\mu)|}{|-2p^2(x)-8q(x)|}, \\ |(1-\mu)| \geq | 6\left(1+\frac{2q(x)}{p(x)}\right) | \end{array} \right\}.$$

4. CONCLUSION

In this paper making use of Wanas operator, We introduced and investigated the bi-univalent function class $T_{\Sigma}(\lambda, \alpha, \beta, \kappa, \gamma; x)$ related to the (p, q) -Lucas polynomials. Thus, we obtained second and third TaylorâĂŞMaclaurin coefficients of functions for this class. These results were an improvement on the estimates obtained in the recent studies.

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