Aust. J. Math. Anal. Appl.
Vol. **21** (2024), No. 1, Art. 10, 15 pp.
AJMAA

SOME NONLINEAR GRONWALL-BELLMAN TYPE RETARDED INTEGRAL INEQUALITIES WITH POWER AND THEIR APPLICATIONS

AMMAR BOUDELIOU

Received 26 November, 2023; accepted 29 March, 2024; published 26 April, 2024.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONSTANTINE 1 BROTHERS MENTOURI,
BP, 325, AIN EL BEY STREET, 25017, ALGERIA.
ammar_boudeliou@umc.edu.dz

ABSTRACT. In this paper, we investigate a certain class of nonlinear Gronwall-Bellman type integral inequalities with power in more general cases involving retarded term and more general nonlinearities. Our results generalize some known integral inequalities and other results obtained very recently. The inequalities given here can be used to estimate the bound on the solutions of retarded integral equation of Volterra type and integro-differential equations (IDE) with power. Two examples are given to show the validity of our established theorems.

Key words and phrases: Boundedness; Retarded Integral inequality; Power; Delay IDE..

2010 *Mathematics Subject Classification.* Primary 26D05; 26D07. Secondary 26D10; 26D15.

ISSN (electronic): 1449-5910

1. INTRODUCTION

The integral inequalities that provide explicit bounds on unknown functions have become major tools in the study of qualitative properties of solutions of differential and integral equations. The generalization of Gronwall-Bellman inequality has earned much consideration of many mathematicians and scientists (see [1],[3]-[5],[7],[10]-[12]). Bellman [2] proved and made use of the following variant of the Gronwall inequality given in [6] to study the asymptotic behavior of the solutions of linear differential-difference equations.

$$u(t) \leq a(t) + \int_a^t f(s)u(s)ds, \qquad t \in [a,b]\,, \tag{1.1}$$

where $a(t)$ is a continuous, positive, and nondecreasing function defined on $[a,b]$, then

$$u(t) \leq a(t) \exp\left(\int_a^t f(s)ds\right), \qquad t \in [a,b]\,.$$

Pachpatte in [10] gave a generalization of Gronwall-Bellman inequality as follows

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t g(s)\left(u(s) + \int\limits_0^s h(\tau)u(\tau)d\tau\right)ds, t \in R_+, \tag{1.2}$$

where u, f, g, and h are nonnegative continuous functions defined on R_+, and u_0 be a nonnegative constant, then

$$\begin{aligned} u(t) \;\leq\; & u_0\left[\exp\left(\int_0^t f(s)ds\right) + \right. \\ & \left.\int_0^t g(s)\exp\left(\int\limits_0^s \left[f(\tau)+g(\tau)+h(\tau)\right]d\tau\right)\exp\left(\int\limits_s^t f(\tau)d\tau\right)ds\right]. \end{aligned}$$

Very recently, some integral inequalities with power have been investigated.

Li and Wang [8] studied the power integral inequality

$$u(t) \leq a(t) + \int_{t_0}^{\alpha(t)} f(s)\left[u^m(s) + \int_0^s g(\tau)u^n(\tau)d\tau\right]^p ds, \tag{1.3}$$

where $u, a, f, g \in C(R_+, R_+)$, and $\alpha(t)$ be a continuous, differentiable, and increasing function on $[t_0, \infty]$ with $\alpha(t) \leq t, \alpha(t_0) = t_0$. $p, m, n \in (0,1]$ are positive constants, then

$$u(t) \leq a(t) + A(t)\exp\left(\int_{t_0}^{\alpha(t)} pmf(s)ds + \int_{t_0}^{\alpha(t)} pf(s)\left(\int_{t_0}^s ng(\tau)d\tau\right)ds\right), t \in R_+.$$

Tian and Fan [13] discussed the following integral inequalities

$$u(t) \leq a(t) + \int_{t_0}^{\alpha(t)} b(s)\left[u^m(s) + \int_0^s c(\tau)u^n(\tau)d\tau\right]^p ds, \tag{1.4}$$

where $0 < m, n \leq 1, p > 1$.

$$u^q(t) \leq a(t) + \int_{t_0}^{\alpha(t)} b(s)\left[u^m(s) + \int_0^s c(\tau)u^n(\tau)d\tau\right]^p ds, \tag{1.5}$$

where $q \geq m > 0, q \geq n > 0, p > 0$.

Motivated by the results (1.3),(1.4),(1.5), our main aim is to establish some interesting nonlinear Gronwall-Belleman-Pachpatte type integral inequalities with power. Many authors gave generalizations of integral inequalities with power in the case $p \in (0,1]$. In this paper, we establish some integral inequalities with power in more general cases where $p > 0$ which give more general nonlinearities. Furthermore, we show that some results of [2, 8, 9, 10, 13] can be deduced from our results in some special cases. Finally, we give two examples to study the boundedness of the solution of retarded integral equations of Volterra type and initial value problem of nonlinear IDE with delay.

2. MAIN RESULTS

Let $\mathbb{R}$ denotes the set of real number and $\mathbb{R}_+ = [0,\infty)$ is the given subset of $\mathbb{R}$, $C(A,B)$ and $C^1(A,B)$ denote the classes of continuous functions and continuously differentiable functions on set A with a range in the set B respectively. The following lemmas are very useful to prove the main result of our paper.

Lemma 2.1. *([8]) Let $a \geq 0, p \geq q \geq 0$ and $p \neq 0$. Then*

$$a^{\frac{q}{p}} \leq \frac{q}{p}a + \frac{p-q}{p}.$$

Lemma 2.2. *([13]) Assume that $u, v \geq 0$ and $p > 0$. Then*

$$(u+v)^p \leq K_p\left(u^p + v^p\right),$$

where $K_p = 1, 0 \leq p \leq 1$, and $K_p = 2^{p-1}, p > 1$.

Theorem 2.3. *Let $u, a, b, c, f, g \in C(\mathbb{R}_+, \mathbb{R}_+)$, and let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function with $\alpha(t) \leq t, \alpha(0) = 0$. $p > 0$ is a constant. If $u(t)$ satisfies*

$$u(t) \leq a(t) + \int_0^{\alpha(t)} f(s)u(s)ds + \int_0^{\alpha(t)} g(s)\left[b(s)u(s) + c(s)\right]^p ds, \tag{2.1}$$

then

$$u(t) \leq \begin{cases} a(t) + k(t) + \displaystyle\int_0^{\alpha(t)} h(s)k(s)\exp\left(\int_s^{\alpha(t)} h(\tau)d\tau\right)ds, \; 0 < p \leq 1 \\ a(t) + \Bigg\{ j^{1-p}(t)\exp\left((1-p)\displaystyle\int_0^{\alpha(t)} f(s)ds\right) + \\ (1-p)2^{p-1}\displaystyle\int_0^{\alpha(t)} g(s)b^p(s)\exp\left((1-p)\int_s^{\alpha(t)} f(\tau)d\tau\right)ds \Bigg\}^{\frac{1}{1-p}}, \; p > 1, \end{cases} \tag{2.2}$$

with

$$j^{1-p}(t)\exp\left((1-p)\int_0^{\alpha(t)} f(s)ds\right) > (p-1)2^{p-1}\int_0^{\alpha(t)} g(s)b^p(s)\exp\left((1-p)\int_s^{\alpha(t)} f(\tau)d\tau\right)ds, \tag{2.3}$$

where

$$k(t) = \int_0^{\alpha(t)} f(s)a(s)ds + \int_0^{\alpha(t)} g(s)\Big[p\big(a(s)b(s) + c(s)\big) + 1 - p\Big]ds, \tag{2.4}$$

$$h(s) = f(s) + p\,g(s)b(s), \tag{2.5}$$

$$j(t) = \int_0^{\alpha(t)} f(s)a(s)ds + 2^{p-1} \int_0^{\alpha(t)} g(s)\Big[a(s)b(s) + c(s)\Big]^p ds. \tag{2.6}$$

Proof. Denote

$$z(t) = \int_0^{\alpha(t)} f(s)u(s)ds + \int_0^{\alpha(t)} g(s)\Big[b(s)u(s) + c(s)\Big]^p ds,$$

then $z(0) = 0$, and $z(t)$ is nondecreasing function, and

$$u(t) \leq a(t) + z(t). \tag{2.7}$$

Therefore, using (2.7), we have

$$\begin{aligned} z(t) &\leq \int_0^{\alpha(t)} f(s)\big(a(s) + z(s)\big)ds + \int_0^{\alpha(t)} g(s)\Big[b(s)\left(a(s) + z(s)\right) + c(s)\Big]^p ds \\ &\leq \int_0^{\alpha(t)} f(s)a(s)ds + \int_0^{\alpha(t)} f(s)z(s)ds + \int_0^{\alpha(t)} g(s)\Big[b(s)z(s) + a(s)b(s) + c(s)\Big]^p ds. \end{aligned} \tag{2.8}$$

Case 1: $0 < p \leq 1$. Applying Lemma 2.1 to (2.8), we get

$$\begin{aligned} z(t) \leq & \int_0^{\alpha(t)} f(s)a(s)ds + \int_0^{\alpha(t)} f(s)z(s)ds + \\ & \int_0^{\alpha(t)} g(s)\Big[p\big(b(s)z(s) + a(s)b(s) + c(s)\big) + 1 - p\Big]ds \\ & \leq k(t) + \int_0^{\alpha(t)} h(s)z(s)ds, \end{aligned}$$

where $k(t), h(t)$ are defined by (2.4), (2.5) respectively. Define

$$v(t) = \int_0^{\alpha(t)} h(s)z(s)ds,$$

then $v(0) = 0$, $v(t)$ is nondecreasing and

$$z(t) \leq k(t) + v(t). \tag{2.9}$$

Differentiating $v(t)$, and using (2.9), we get

$$\begin{aligned} v'(t) &= \alpha'(t)h\left(\alpha(t)\right)z(\alpha(t)) \\ &\leq \alpha'(t)h\left(\alpha(t)\right)\left(k(\alpha(t)) + v(\alpha(t))\right) \\ &\leq \alpha'(t)h\left(\alpha(t)\right)\left(k(\alpha(t)) + v(t)\right), \end{aligned}$$

or

$$v'(t) - \alpha'(t)h\left(\alpha(t)\right)v(t) \leq \alpha'(t)h\left(\alpha(t)\right)k(\alpha(t)). \tag{2.10}$$

Multiplying both sides of (2.10) by $\exp\left(-\int\limits_0^{\alpha(t)} h(s)ds\right)$, then it will be restated as follows

$$\frac{d}{dt}\left(v(t)\exp\left(-\int\limits_0^{\alpha(t)} h(s)ds\right)\right) \le \alpha'(t)h\left(\alpha(t)\right)k(\alpha(t))\exp\left(-\int\limits_0^{\alpha(t)} h(s)ds\right).$$

Integrating the above inequality from 0 to t, we get

$$v(t) \le \int\limits_0^{\alpha(t)} h(s)k(s)\exp\left(\int\limits_s^{\alpha(t)} h(\tau)d\tau\right)ds. \tag{2.11}$$

From (2.11), (2.9) and (2.7) we obtain the estimate of $u(t)$ given in (2.2).
Case 2: $p > 1$. Applying Lemma 2.2 to (2.8), we have

$$\begin{aligned} z(t) &\le \int\limits_0^{\alpha(t)} f(s)a(s)ds + \int\limits_0^{\alpha(t)} f(s)z(s)ds \\ &\quad + \int\limits_0^{\alpha(t)} g(s)2^{p-1}\Big[\big(b(s)z(s)\big)^p + \big(a(s)b(s)+c(s)\big)^p\Big]ds \\ &\le j(t) + \int\limits_0^{\alpha(t)} f(s)z(s)ds + 2^{p-1}\int\limits_0^{\alpha(t)} g(s)b^p(s)z^p(s)ds, \end{aligned}$$

where $j(t)$ is defined by (2.6) which is nondecreasing function, then for fixed T, we have

$$z(t) \le j(T) + \int\limits_0^{\alpha(t)} f(s)z(s)ds + 2^{p-1}\int\limits_0^{\alpha(t)} g(s)b^p(s)z^p(s)ds, \forall t \in [0,T].$$

Define

$$\varphi(t) = j(T) + \int\limits_0^{\alpha(t)} f(s)z(s)ds + 2^{p-1}\int\limits_0^{\alpha(t)} g(s)b^p(s)z^p(s)ds,$$

then $\varphi(0) = j(T)$, and

$$z(t) \le \varphi(t), z\left(\alpha(t)\right) \le z(t) \le \varphi(t). \tag{2.12}$$

Differentiating $\varphi(t)$ and using (2.12), we get

$$\begin{aligned} \varphi'(t) &\le \alpha'(t)f\left(\alpha(t)\right)z\left(\alpha(t)\right) + 2^{p-1}\alpha'(t)g\left(\alpha(t)\right)b^p(\alpha(t))z^p(\alpha(t)) \\ &\le \alpha'(t)f\left(\alpha(t)\right)\varphi(t) + 2^{p-1}\alpha'(t)g\left(\alpha(t)\right)b^p(\alpha(t))\varphi^p(t). \end{aligned}$$

Dividing both sides of the above inequality by $\varphi^p(t)$, we get

$$\varphi^{-p}(t)\varphi'(t) \le \alpha'(t)f\left(\alpha(t)\right)\varphi^{1-p}(t) + 2^{p-1}\alpha'(t)g\left(\alpha(t)\right)b^p(\alpha(t)). \tag{2.13}$$

Let $\psi(t) = \varphi^{1-p}(t)$, then $\psi(0) = j^{1-p}(T)$, and $\varphi^{-p}(t)\varphi'(t) = \frac{1}{1-p}\psi'(t)$. (2.13) will be restated as follows

$$\psi'(t) - (1-p)\alpha'(t)f\left(\alpha(t)\right)\psi(t) \ge (1-p)2^{p-1}\alpha'(t)g\left(\alpha(t)\right)b^p(\alpha(t)). \tag{2.14}$$

Multiplying (2.14) by $\exp\left(-(1-p)\int\limits_0^{\alpha(t)} f(s)ds\right)$, then we have

$$\frac{d}{dt}\left(\psi(t)\exp\left(-(1-p)\int\limits_0^{\alpha(t)} f(s)ds\right)\right) \geq (1-p)2^{p-1}\alpha'(t)g\left(\alpha(t)\right)b^p(\alpha(t))\times \exp\left(-(1-p)\int\limits_0^{\alpha(t)} f(s)ds\right)$$

Integrating the above inequality from 0 to t, we have

$$\psi(t) \geq j^{1-p}(T)\exp\left((1-p)\int\limits_0^{\alpha(t)} f(s)ds\right) + (1-p)2^{p-1}\int\limits_0^{\alpha(t)} g\left(s\right)b^p(s)\exp\left((1-p)\int\limits_s^{\alpha(t)} f(\tau)d\tau\right)ds,$$

from the hypothesis (2.3) and $\psi(t) = \varphi^{1-p}(t)$, we get

$$\varphi(t) \leq \left\{j^{1-p}(T)\exp\left((1-p)\int\limits_0^{\alpha(t)} f(s)ds\right) + (1-p)2^{p-1}\int\limits_0^{\alpha(t)} g\left(s\right)b^p(s)\exp\left((1-p)\int\limits_s^{\alpha(t)} f(\tau)d\tau\right)ds\right\}^{\frac{1}{1-p}}.$$

Since T is chosen arbitrarily, then we have

$$\varphi(t) \leq \left\{j^{1-p}(t)\exp\left((1-p)\int\limits_0^{\alpha(t)} f(s)ds\right) + (1-p)2^{p-1}\int\limits_0^{\alpha(t)} g\left(s\right)b^p(s)\exp\left((1-p)\int\limits_s^{\alpha(t)} f(\tau)d\tau\right)ds\right\}^{\frac{1}{1-p}}. \tag{2.15}$$

From (2.15), (2.12) and (2.7), we obtain the estimate of $u(t)$ given in (2.2).
∎

Remark 2.1. We deduce the following inequalities by changing the given assumptions in Theorem 2.3:
1. If $\alpha(t) = t, g(s) = 0$, and $a(t)$ be a nondecreasing function, then (2.1) in Theorem 2.3 reduces to the well known Gronwall-Bellman inequality (1.1).
2. If $g(s) = 0$, Theorem 2.3 reduces to Lemma 2 in [8].
3. If $f(t) = 0$, Theorem 2.3 reduces to Lemma 2.3 in [13].

Theorem 2.4. *Let $u, a, f, g, h \in C(\mathbb{R}_+, \mathbb{R}_+)$, and let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function with $\alpha(t) \leq t$, $\alpha(0) = 0$. Suppose that $m, n \in (0, 1]$, $p > 0$ are constants. If $u(t)$*

satisfies

$$u(t) \le a(t) + \int_0^{\alpha(t)} f(s)u(s)ds + \int_0^{\alpha(t)} g(s)\left[u^m(s) + \int_0^s h(\tau)u^n(\tau)d\tau\right]^p ds, \tag{2.16}$$

then

$$u(t) \le \begin{cases} a(t) + a_1(t) + k_1(t) + \int_0^{\alpha(t)} h(s)k_1(s)\exp\left(\int_s^{\alpha(t)} h(\tau)d\tau\right)ds,\ 0 < p \le 1 \\ a(t) + a_1(t) + \left\{ j_1^{1-p}(t)\exp\left((1-p)\int_0^{\alpha(t)} f(s)ds\right) + \right. \\ \left. (1-p)2^{p-1}\int_0^{\alpha(t)} g(s)b_1^p(s)\exp\left((1-p)\int_s^{\alpha(t)} f(\tau)d\tau\right)ds \right\}^{\frac{1}{1-p}},\ p > 1, \end{cases} \tag{2.17}$$

with

$$j_1^{1-p}(t)\exp\left((1-p)\int_0^{\alpha(t)} f(s)ds\right) > (p-1)2^{p-1}\int_0^{\alpha(t)} g(s)b_1^p(s)\exp\left((1-p)\int_s^{\alpha(t)} f(\tau)d\tau\right)ds, \tag{2.18}$$

where

$$k_1(t) = \int_0^{\alpha(t)} f(s)a_1(s)ds + \int_0^{\alpha(t)} g(s)\left[p\left(a_1(s)b_1(s) + c_1(s)\right) + 1 - p\right]ds, \tag{2.19}$$

$$j_1(t) = \int_0^{\alpha(t)} f(s)a_1(s)ds + 2^{p-1}\int_0^{\alpha(t)} g(s)\left(a_1(s)b_1(s) + c_1(s)\right)^p ds, \tag{2.20}$$

$$a_1(t) = \int_0^{\alpha(t)} f(s)a(s)ds, \qquad b_1(s) = m + n\int_0^s h(\tau)d\tau, \tag{2.21}$$

$$c_1(s) = ma(s) + 1 - m + \int_0^s h(\tau)\left[na(\tau) + 1 - n\right]d\tau. \tag{2.22}$$

Proof. Define a nonnegative and nondecreasing function $z(t)$ by

$$z(t) = \int_0^{\alpha(t)} f(s)u(s)ds + \int_0^{\alpha(t)} g(s)\left[u^m(s) + \int_0^s h(\tau)u^n(\tau)d\tau\right]^p ds,$$

then, $z(0) = 0$ and

$$u(t) \le a(t) + z(t), \tag{2.23}$$

using (2.23), we have

$$z(t) \leq \int_0^{\alpha(t)} f(s)\left(a(s)+z(s)\right) ds + \int_0^{\alpha(t)} g(s)\left[\left(a(s)+z(s)\right)^m + \int_0^s h(\tau)\left(a(\tau)+z(\tau)\right)^n d\tau\right]^p ds.$$

By Lemma 2.1, we have

$$\begin{aligned} z(t) \leq \int_0^{\alpha(t)} f(s)\Big(a(s)+z(s)\Big) ds + \int_0^{\alpha(t)} g(s)\Big[m\Big(a(s)+z(s)\Big) \\ +1-m+\int_0^s h(\tau)\Big[n\Big(a(\tau)+z(\tau)\Big)+1-n\Big] d\tau\Big]^p ds, \end{aligned}$$

then above inequality can restated as follows

$$z(t) \leq a_1(t) + \int_0^{\alpha(t)} f(s)z(s)ds + \int_0^{\alpha(t)} g(s)\left[b_1(s)z(s)+c_1(s)\right]^p ds, \tag{2.24}$$

where $a_1(t), b_1(s)$ and $c_1(s)$ are defined in (2.21) and (2.22) respectively. Now applying Theorem 2.3 to (2.24), we get

$$z(t) \leq \begin{cases} a_1(t)+k_1(t)+\int_0^{\alpha(t)} h(s)k_1(s)\exp\left(\int_s^{\alpha(t)} h(\tau)d\tau\right) ds, \ 0<p\leq 1 \\ a_1(t)+\left\{ j_1^{1-p}(t)\exp\left((1-p)\int_0^{\alpha(t)} f(s)ds\right) + \right. \\ \left. (1-p)2^{p-1}\int_0^{\alpha(t)} g(s)b_1^p(s)\exp\left((1-p)\int_s^{\alpha(t)} f(\tau)d\tau\right) ds \right\}^{\frac{1}{1-p}}, p>1, \end{cases} \tag{2.25}$$

where $k_1(t)$ and $j_1(t)$ are defined by (2.19) and (2.20) respectively. Then, from (2.25) and (2.23) yields the estimate of $u(t)$ in (2.17). This completes the proof. ∎

Remark 2.2. Theorem 2.4 generalizes some famous results obtained in [9, 13] as follows:
1. If $f(t)=0$, and $p>1$, Theorem 2.4 reduces to Theorem 2.1 in [13].
2. If $f(t)=0$, and $0<p\leq 1$, Theorem 2.4 reduces to Theorem 2.1 in [9].

Theorem 2.5. *Let* $u,a,f,g,h \in C(\mathbb{R}_+,\mathbb{R}_+)$*, and let* $\alpha \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ *be a nondecreasing function with* $\alpha(t)\leq t$*,* $\alpha(0)=0$*. Suppose that* m,n,q,r,p *are constants satisfying* $r\geq q>0$*,* $r\geq m>0$*,* $r\geq n>0$*,* $p>0$*. If* $u(t)$ *satisfies*

$$u^r(t) \leq a(t) + \int_0^{\alpha(t)} f(s)u^q(s)ds + \int_0^{\alpha(t)} g(s)\left[u^m(s)+\int_0^s h(\tau)u^n(\tau)d\tau\right]^p ds, \tag{2.26}$$

and

(2.27)
$$j_2^{1-p}(t)\exp\left((1-p)\int_0^{\alpha(t)} f_2(s)ds\right) > (p-1)2^{p-1}\int_0^{\alpha(t)} g_2(s)b_2^p(s)\exp\left((1-p)\int_s^{\alpha(t)} f_2(\tau)d\tau\right)ds,$$

then

(2.28)
$$u(t) \leq \begin{cases} \left(a(t)+a_2(t)+k_2(t)+\int_0^{\alpha(t)} h_2(s)k_2(s)\exp\left(\int_s^{\alpha(t)} h_2(\tau)d\tau\right)ds\right)^{\frac{1}{r}}, 0<p\leq 1 \\ \left(a(t)+a_2(t)+\left\{j_2^{1-p}(t)\exp\left((1-p)\int_0^{\alpha(t)} f_2(s)ds\right)+\right.\right. \\ \left.\left.(1-p)2^{p-1}\int_0^{\alpha(t)} g(s)b_2^p(s)\exp\left((1-p)\int_s^{\alpha(t)} f_2(\tau)d\tau\right)ds\right\}^{\frac{1}{1-p}}\right)^{\frac{1}{r}}, p>1 \end{cases}$$

where

$$j_2(t) = \int_0^{\alpha(t)} f_2(s)a_2(s)ds + 2^{p-1}\int_0^{\alpha(t)} g(s)\left(a_2(s)b_2(s)+c_2(s)\right)^p ds, \tag{2.29}$$

$$\begin{cases} f_2(t) = \frac{q}{r}f(t), \qquad a_2(t) = \int_0^{\alpha(t)} f(s)\left(\frac{q}{r}a(s)+\frac{r-q}{r}\right)ds, \\ b_2(t) = \frac{m}{r}+\frac{n}{r}\int_0^t h(s)ds, \\ c_2(t) = \frac{m}{r}a(t)+\frac{r-m}{r}+\int_0^t h(\tau)\left(\frac{n}{r}a(\tau)+\frac{r-n}{r}\right)d\tau, \end{cases} \tag{2.30}$$

$$k_2(t) = \int_0^{\alpha(t)} f_2(s)a_2(s)ds + \int_0^{\alpha(t)} g(s)\left[p\left(a_2(s)b_2(s)+c_2(s)\right)+(1-p)\right]ds, \tag{2.31}$$

$$h_2(t) = f_2(t) + p.g(t)b_2(t). \tag{2.32}$$

Proof. Define

$$z(t) = \int_0^{\alpha(t)} f(s)u^q(s)ds + \int_0^{\alpha(t)} g(s)\left[u^m(s)+\int_0^s h(\tau)u^n(\tau)d\tau\right]^p ds,$$

then, $z(0)=0$, $z(t)$ is nondecreasing function and

$$u(t) \leq (a(t)+z(t))^{\frac{1}{r}}. \tag{2.33}$$

Using (2.33) and applying Lemma 2.1, we get

$$z(t) \leq \int_0^{\alpha(t)} f(s)\big(a(s)+z(s)\big)^{\frac{q}{r}} ds +$$
$$\int_0^{\alpha(t)} g(s) \left[\big(a(s)+z(s)\big)^{\frac{m}{r}} + \int_0^s h(\tau)\big(a(\tau)+z(\tau)\big)^{\frac{n}{r}} d\tau \right]^p ds$$
$$\leq \int_0^{\alpha(t)} f(s) \left[\frac{q}{r}\big(a(s)+z(s)\big) + \frac{r-q}{r} \right] ds +$$
$$\int_0^{\alpha(t)} g(s) \left[\frac{m}{r}\big(a(s)+z(s)\big) + \frac{r-m}{r} + \int_0^s h(\tau) \left[\frac{n}{r}\big(a(\tau)+z(\tau)\big) + \frac{r-n}{r} \right] d\tau \right]^p ds,$$

then

$$z(t) \leq a_2(t) + \int_0^{\alpha(t)} f_2(s) z(s) ds + \int_0^{\alpha(t)} g(s) \Big[b_2(s) z(s) + c_2(s) \Big]^p ds, \tag{2.34}$$

where $a_2(t), f_2(s), b_2(s), c_2(s)$ are defined by (2.30). Applying Theorem 2.3 to (2.34), we obtain

$$z(t) \leq \begin{cases} a_2(t) + k_2(t) + \displaystyle\int_0^{\alpha(t)} h_2(s) k_2(s) \exp\left(\int_s^{\alpha(t)} h_2(\tau) d\tau \right) ds, \quad 0 < p \leq 1 \\ a_2(t) + \Bigg\{ j_2^{1-p}(t) \exp\left((1-p) \displaystyle\int_0^{\alpha(t)} f_2(s) ds \right) + \\ \quad (1-p)2^{p-1} \displaystyle\int_0^{\alpha(t)} g(s) b_2^p(s) \exp\left((1-p) \int_s^{\alpha(t)} f_2(\tau) d\tau \right) ds \Bigg\}^{\frac{1}{1-p}}, \ p > 1, \end{cases} \tag{2.35}$$

where $k_2(t), h_2(t)$ are defined by (2.31) and (2.32) respectively. Then (2.35) with (2.33) gives (2.28). The proof is completed. ∎

Remark 2.3. If $f(t) = 0$, then Theorem 2.5 reduces to Theorem 2.2 in [13].

3. Applications

In this section, we apply our results to study the boundedness of the solution of retarded integral equation of Volterra type and initial value problem of nonlinear integro-differential equations with delay.

Example 3.1 Consider the following retarded integral equation of Volterra type

$$x^3(t) - H_1(t, x(\alpha(t))) - M\left(t, x(\alpha(t)), \int_0^t H_2(\tau, x(\tau)) d\tau \right) = a(t), \tag{3.1}$$

where $x, a \in C(\mathbb{R}_+, \mathbb{R})$, $H_1, H_2 \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $M \in C(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function with $\alpha(t) \leq t$, $\alpha(0) = 0$.

Corollary 3.1. *If H_1, H_2, M satisfy the following conditions*

$$|H_1(t, x(\alpha(t)))| \leq \int_0^t f(s) |x(\alpha(s))| \, ds, \tag{3.2}$$

$$|H_2(\tau, x)| \leq h(\tau) |x(\tau)|, \tag{3.3}$$

$$\left| M\left(t, x(\alpha(t)), \int_0^t H_2(\tau, x(\tau)) \, d\tau\right) \right| \leq \int_0^t g(s) \left[x^2(\alpha(s)) + \int_0^s |H_2(\tau, x(\tau))| \, d\tau \right]^2 ds, \tag{3.4}$$

where $f, g, h \in$ are as in Theorem 2.5. Then all solutions of (3.1) *are bounded on R_+ and*

$$|x(t)| \leq \left(|a(t)| + a_2(t) + \frac{j_2(t)}{\exp\left(-\int_0^{\alpha(t)} f_2(s) ds\right) - 2j_2(t) \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} b_2^p(s) \exp\left(-\int_s^{\alpha(t)} f_2(\tau) d\tau\right) ds} \right)^{\frac{1}{3}}, \tag{3.5}$$

with

$$\exp\left(-\int_0^{\alpha(t)} f_2(s) ds\right) > 2j_2(t) \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} b_2^p(s) \exp\left(-\int_s^{\alpha(t)} f_2(\tau) d\tau\right) ds,$$

where

$$j_2(t) = \int_0^{\alpha(t)} f_2(s) a_2(s) ds + 2 \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \Big(a_2(s) b_2(s) + c_2(s)\Big)^2 ds,$$

$$\begin{aligned} f_2(t) &= \frac{1}{3} \frac{f(\alpha^{-1}(t))}{\alpha'(\alpha^{-1}(t))}, \qquad a_2(t) = \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \left(\frac{1}{3}|a(s)| + \frac{2}{3}\right) ds, \\ b_2(t) &= \frac{2}{3} + \frac{1}{3} \int_0^t h(s) ds, \quad c_2(t) = \frac{2}{3}|a(t)| + \frac{1}{3} + \int_0^t h(\tau) \left(\frac{1}{3}|a(\tau)| + \frac{2}{3}\right) d\tau. \end{aligned}$$

Proof. Using the conditions (3.2)-(3.4) in (3.1), we obtain

(3.6)
$$\begin{aligned}
|x(t)|^3 &\leq |a(t)| + |H_1(t,x)| + \left| M\left(t, x\left(\alpha(t)\right), \int_0^t H_2\left(\tau, x(\tau)\right) d\tau \right)\right| \\
&\leq |a(t)| + \int_0^t f(s)x\left|x(\alpha(s))\right| ds + \int_0^t g(s)\left[x^2(\alpha(s)) + \int_0^s \left|H_2(\tau, x(\tau))\right| d\tau\right]^2 ds \\
&\leq |a(t)| + \int_0^t f(s)\left|x(\alpha(s))\right| ds + \int_0^t g(s)\left[x^2(\alpha(s)) + \int_0^s h(\tau)\left|x(\tau)\right| d\tau\right]^2 ds \\
&\leq |a(t)| + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'\left(\alpha^{-1}(s)\right)}\left|x(s)\right| ds + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'\left(\alpha^{-1}(s)\right)}\left[x^2(s) + \int_0^s h(\tau)\left|x(\tau)\right| d\tau\right]^2 ds.
\end{aligned}$$

An application of Theorem 2.5 to (3.6) with $u(t) = |x(t)|, r = 3, q = 1, m = 2, n = 1, p = 2$, yields the desired inequality (3.5). This completes the proof. ∎

Example 3.2 Consider the following initial value problem for the delay IDE.

$$\begin{cases} x'(t) = a'(t) + H_1(t, x\left(\alpha(t)\right)) + M\left(t, x\left(\alpha(t)\right), \int_0^t H_2\left(\tau, x(\tau)\right) d\tau\right) \\ x(0) = a(0), \end{cases} \tag{3.7}$$

where $a(0) \neq 0$ is a constant, $x, a \in C^1(\mathbb{R}_+, \mathbb{R})$, $H_1, H_2 \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $M \in C(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function with $\alpha(t) \leq t$, $\alpha(0) = 0$.

Corollary 3.2. *Consider the initial value problem* (3.7) *and suppose that H_1, H_2 and M satisfy the conditions*

$$\left|H_1(t, x\left(\alpha(t)\right))\right| \leq f(t)\left|x(\alpha(t))\right|, \tag{3.8}$$

$$\left|H_2\left(\tau, x(\tau)\right)\right| \leq h(\tau)\left|x(\tau)\right|^n, \tag{3.9}$$

$$M\left(t, x\left(\alpha(t)\right), \int_0^t H_2\left(\tau, x(\tau)\right) d\tau\right) \leq g(t)\left[\left|x(\alpha(t))\right|^m + \int_0^t \left|H_2\left(\tau, x(\tau)\right)\right| d\tau\right]^p, \tag{3.10}$$

where $m, n \in (0,1], p > 0$ *are constants, and* $f, g, h \in C(R_+, R_+)$. *Then all solutions of* (3.7) *exist and*

(3.11)
$$|x(t)| \le \begin{cases} |a(t)| + a_1(t) + k_1(t) + \int\limits_0^{\alpha(t)} h(s)k_1(s)\exp\left(\int\limits_s^{\alpha(t)} h(\tau)d\tau\right)ds, \quad 0 < p \le 1 \\ |a(t)| + a_1(t) + \left\{ j_1^{1-p}(t)\exp\left((1-p)\int\limits_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds\right) + (1-p)2^{p-1}\times \right. \\ \left. \int\limits_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} b_1^p(s)\exp\left((1-p)\int\limits_s^{\alpha(t)} \frac{f(\alpha^{-1}(\tau))}{\alpha'(\alpha^{-1}(\tau))}d\tau\right)ds \right\}^{\frac{1}{1-p}}, p > 1, \end{cases}$$

with

$$j_1^{1-p}(t)\exp\left((1-p)\int\limits_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds\right) > (p-1)2^{p-1}\int\limits_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} b_1^p(s) \times \exp\left((1-p)\int\limits_s^{\alpha(t)} \frac{f(\alpha^{-1}(\tau))}{\alpha'(\alpha^{-1}(\tau))}d\tau\right)ds,$$

where

$$k_1(t) = \int\limits_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} a_1(s)ds + \int\limits_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\left[p\left(a_1(s)b_1(s) + c_1(s)\right) + 1 - p\right]ds,$$
$$j_1(t) = \int\limits_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} a_1(s)ds + 2^{p-1}\int\limits_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\left(a_1(s)b_1(s) + c_1(s)\right)^p ds,$$
$$a_1(t) = \int\limits_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} a(s)ds, \qquad b_1(s) = m + n\int\limits_0^s h(\tau)d\tau,$$
$$c_1(s) = m|a(s)| + 1 - m + \int\limits_0^s h(\tau)\left[n|a(\tau)| + 1 - n\right]d\tau.$$

Proof. Integrating both sides of (3.7) from 0 to t, we obtain

$$x(t) = a(t) + \int\limits_0^t H_1(s, x(\alpha(s)))ds + \int\limits_0^t M\left(s, x(\alpha(s)), \int\limits_0^s H_2(\tau, x(\tau))d\tau\right)ds. \tag{3.12}$$

Using conditions (3.8), (3.7) and (3.10) in (3.12), we get

(3.13)

$$\begin{aligned}
|x(t)| &\leq |a(t)| + \int_0^t |H_1(s, x(\alpha(s)))|\, ds + \int_0^t \left| M\left(s, x(\alpha(s)), \int_0^s H_2(\tau, x(\tau))\, d\tau\right)\right| ds \\
&\leq |a(t)| + \int_0^t f(s)\, |x(\alpha(s))|\, ds + \int_0^t g(s) \left[|x(\alpha(s))|^m + \int_0^s |H_2(\tau, x(\tau))|\, d\tau \right]^p ds \\
&\leq |a(t)| + \int_0^t f(s)\, |x(\alpha(s))|\, ds + \int_0^t g(s) \left[|x(\alpha(s))|^m + \int_0^s h(\tau)\, |x(\tau)|^n\, d\tau \right]^p ds \\
&\leq |a(t)| + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\, |x(s)|\, ds + \\
&\int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \left[|x(s)|^m + \int_0^s h(\tau)\, |x(\tau)|^n\, d\tau \right]^p ds.
\end{aligned}$$

An application of Theorem 2.4 to (3.13) gives the estimate of $|x(t)|$ given in (3.11). The proof is completed. ∎

4. Conclusion

In this paper, we established new retarded nonlinear Gronwall-Bellman-Pachpatte type integral inequalities with power in more general cases (where $p > 0$) involving retarded term and more general nonlinearities. The inequalities of our main results can be used as a handy tool to study the qualitative properties to solutions of differential equations and integral equations. We show that some results of [2, 8, 9, 10, 13] can be deduced from our results in some special cases. In the last section, as an application, we give two examples to ullustrate how our inequalities can be used to give the boundedness of solution of retarded integral equations of Volterra type and initial value problem of nonlinear IDE with delay.

Acknowledgment The author is very grateful to the editor and the anonymous referees for their helpful comments and valuable suggestions.

References

[1] D. BAINOV and P. SIMEONOV, *Integral inequalities and applications*, Kluwer Academic Publishers, Dordrecht, 1992.

[2] R. BELLMAN, A symptotic series for the solutions of linear differential-difference equations, *Rendiconti de circolo Matematica Di Palermo*, **7** (1958), pp. 1-9.

[3] A. BOUDELIOU and H. KHELLAF, On some delay nonlinear integral inequalities in two independent variables, *J Inequal Appl*, **2015**, 313 (2015).

[4] A. BOUDELIOU, On certain new nonlinear retarded integral inequalities in two independent variables and applications, *Appl. Math. Comput.*, **335** (2018), pp. 103-111.

[5] A. BOUDELIOU, Some Generalized Nonlinear Volterra-Fredholm Type Integral Inequalities with Delay of Several Variables and Applications, *Nonlinear Dynamics and Systems Theory*, **23** (3) (2023), pp. 261–272

[6] T.H. GRONWALL, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, *Ann Math.*, **20** (1919), pp. 292-296.

[7] Z. A. KHAN, Analysis on some powered integral inequalities with retarded argument and application, *Journal of Taibah university for science*, **14**(2020), pp. 488-495.

[8] Z. LI and W.S. WANG, Some nonlinear Gronwall-Bellman type retarded integral inequalities with power and their applications, *Appl. Math. Comput.*, **347** (2019), pp. 839-852.

[9] Z. LI and W.-S. WANG, Some new nonlinear powered Gronwall-Bellman type retarded integral inequalities and their applications, *J. Math. Inequal.*, **13** (2019), pp. 553-564.

[10] B. G. PACHPATTE, *Inequalities for Differential and Integral Equations*, Academic Press, London, 1998.

[11] A. SHAKOOR, I. ALI, S.WALI and A. REHMAN, Some generalizations of retarded nonlinear integral inequalities and its applications, *J. Math. Inequal.*, **14** (2020), 1223-1235.

[12] H. SONG and F. MENG, Some genereralizations of delay integral inequalities of Gronwall-Bellman type with power and their applications, *Math. Found. Comput.*, **5** (2022), pp. 45-55.

[13] Y. TIAN and M. FAN, Nonlinear integral inequality with power and its application in delay integro-differential equations, *Adv Differ Equ* **2020**, 142 (2020).