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JORDAN CANONICAL FORM OF INTERVAL MATRICES AND APPLICATIONS

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ABSTRACT. A square interval matrix over $\mathbb{IR}$ can be converted to diagonal form if certain prerequisites are satisfied. However not all square matrices can be diagonalized. As a consequence, we strive the next simplest form to which it can be reduced while retaining important properties such as eigenvalues, rank, nullity, and so on. It turns out that any real interval matrix has a Jordan Canonical Form (JCF) over E if it has n interval eigenvalues in $\mathbb{IR}$. We discuss in this paper a method for computing the Jordan canonical form of an interval matrix using a new pairing technique and a new type of interval arithmetic that will make classifying and analyzing interval matrices easier and more efficient. We conclude with a numerical example that supports the theory and application of predator-prey model.

Key words and phrases: Interval; Interval Eigenvalues; Interval Eigenvectors; Jordan canonical form; Differential equations.

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1. INTRODUCTION

An element of a set 'S' of quantities can be brought to standard (or) canonical form if each element of S satisfies certain conditions which make it amenable to be brought to the canonical form. For example, consider a set of $(n \times n)$ real matrices S. If each element of S has 'n' real eigenvalues (condition 1) and a set of 'n' linearly independent eigenvectors in $\mathbb{R}^n$ (condition 2), then every element of S can be diagonalized which is its canonical form. But most of the matrices we encounter in real life situations do not possess this stringent property. Hence we venture to relax the conditions imposed on the matrices by not requiring the 2-nd condition namely 'n' linearly independent eigenvectors.

This relaxation brings a lot more matrices into the ambit of canonicalization, although the canonical form in the second case, called the 'Jordan form' is slightly different from the canonical form in the first case which has two conditions leading to the diagonal form. The Jordan form however serves the purpose since it retains the important characteristics of the original matrix like rank, nullity, trace, determinant and eigenvalues. This is what we attempt to find for real interval matrices. We study the 'closest-to-diagonal' interval matrix that can be obtained through similarity transformations since not all interval matrices are similar to diagonal interval matrices. That is, a similarity transformation can convert any interval matrix into the interval Jordan canonical form when the interval eigenvalues are in $\mathbb{IR}$. The interval Jordan form is a significant concept in interval linear algebra. This can be applied for solving system of ordinary interval differential equations, state space models of continuous-time dynamic system etc. The Jordan Canonical Form of a matrix is incredibly susceptible to perturbations, even though numerical computation of it remains a great obstacle.

Hansen and Smith [12, 13] started the use of interval arithmetic in matrix computations. After this motivation and inspiration, several authors such as Alefeld and Herzberger [2], Luc Jaulin et al. [24], Neumaier [26], Assem Deif [3], Ganesan and Veeramani [9, 10], Nicolas Goze [27], Moore et al.[25], etc have studied interval matrices . Many authors such as Pastravanu et al. [28, 29], Berman et al. [4], Kaszkurewicz et al.[22] etc studied the diagonal stability of interval matrices and their applications.

Ganesan and Veeramani [9] introduced a new type of arithmetic operations on interval numbers. Ganesan [10] discussed some important properties of interval matrices. Assem Deif [3] studied interval eigenvalue problems. Abhirup Sit [1] studied eigenvalues of interval matrices using singularity property. Hema Surya et al. [14, 15] discussed the concept of interval linear algebra on a sound algebraic setting, by judiciously defining a field and a vector space over a field involving equivalence classes. They also discussed the diagonal canonical form of interval matrices and their applications. Pastravanu et al. [28, 29] discussed the diagonal stability of interval matrices and their applications in linear dynamical systems. Berman et al. [4] have also discussed the matrix diagonal stability and its implications. Gilbert Strang [11], Horn et al. [16] studied Matrix Analysis. Differential equations and stability of dynamic interval systems are discussed by Hsu et al. [17], John et al. [18], Kalman et al. [20], Perko [30] and Xu Daoyi [34]. Liao et al. [23], Juang et al. [19] studied stability of interval matrices and dynamic interval systems. Chen et al.[5] studied proof of controllability of Jordan form state equations with its applicable to both continuous and discrete state equations. Silverman and Meadows [33] discussed controllability and observability in time-variable linear systems and also provide the significant structural information including a necessary and sufficient condition for controllability and observability. Poljak and Schlegel [31] studied a generic sense, the Jordan canonical form of a matrix is uniquely determined by the structure of its zero and nonzero entries.

Shuai Liu et al.[32] analyzed algebraic requirements for the controllability of the multiagent network system with directed graph from two perspectives: leader-follower network attribute and coupling input disturbance, based on the Jordan form of the system matrix. Kameleh et al.[21] proposed a novel technique for figuring out the order of minimal realization based on the controllability and observability of Jordan canonical form systems. Yedavalli [35, 36, 37] discussed the stability robustness of linear systems in the time domain. The relationship between controllability, observability, transfer matrix, and minimal realizations is established by the canonical decompositions of state space equations. Four state space forms that are typically applied in modern control theory the phase variable form (controller form), the observer form, the modal form, and the Jordan form can be utilized to generate the state space model of a continuous-time dynamic system.

In this article, we will explain the quickest method for obtaining the Jordan interval matrix, which is closest to the diagonal. We discuss a real life application of Jordan canonical form on predator - prey models. Numerical examples are provided to explore the theory developed in this paper.

2. PRELIMINARIES

We recall some basic notions and notation on closed and bounded intervals in $\mathbb{R}$ [14].

Let $\mathbb{IR} = \{\tilde{a} = [a^L, a^U] : a^L \leq a^U \text{ and } a^L, a^U \in \mathbb{R}\}$ be the set of all closed and bounded intervals. If $a^L = a^U$, then $\tilde{a}$ is a degenerate interval.

The intervals are identified with an ordered pair $\langle m, w \rangle$ defined as follows: Let $\tilde{a} = [a^L, a^U] \subseteq \mathbb{R}$. Define $m(\tilde{a}) = \left(\dfrac{a^L + a^U}{2}\right)$ and $w(\tilde{a}) = \left(\dfrac{a^U - a^L}{2}\right)$ and hence $\tilde{a}$ can be uniquely expressed as $\langle m(\tilde{a}), w(\tilde{a})\rangle$. Conversly, when $\langle m(\tilde{a}), w(\tilde{a})\rangle$ is known, then $m(\tilde{a}) - w(\tilde{a}) = a^L$ and $m(\tilde{a}) + w(\tilde{a}) = a^U$ of $[a^L, a^U]$ and hence given $\langle m(\tilde{a}), w(\tilde{a})\rangle$, the interval $[a^L, a^U]$ is unique.

Note: If $m(\tilde{a}) = 0$, then $\tilde{a}$ is said to be a *zero interval*. In particular, if $m(\tilde{a}) = 0$ and $w(\tilde{a}) = 0$, then $\tilde{a} = [0, 0]$. If $m(\tilde{a}) = 0$ and $w(\tilde{a}) \neq 0$, then $\tilde{a} \approx \tilde{0}$. It is to be noted that if $\tilde{a} = [0, 0]$, then $\tilde{a} \approx \tilde{0}$, but the converse need not be true. If $\tilde{a} \not\approx \tilde{0}$, then $\tilde{a}$ is said to be a *non-zero interval*. If $m(\tilde{a}) > 0$ then $\tilde{a}$ is said to be a *positive interval* and is denoted by $\tilde{a} \succ \tilde{0}$.

3. INTERVAL MATRICES

We recall some basic notions and notation on interval matrices [14]. A classical matrix is a rectangular array of elements of a field $\mathbb{F}$. When matrix entries are inaccurate or unclear, we use Interval matrices, where each entry is a closed and bounded interval in $\mathbb{R}$. An interval matrix is expressed as $\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & ... & \tilde{a}_{1n} \\ ... & ... & ... \\ \tilde{a}_{m1} & ... & \tilde{a}_{mn} \end{pmatrix} = (\tilde{a}_{ij})$, where each $\tilde{a}_{ij} = [a_{ij}^L, a_{ij}^U]$. We use $E^{m \times n}$ to denote the set of all equivalence classes of $(m \times n)$ interval matrices. By $m(\tilde{A})$ we denote a matrix whose entries are the corresponding midpoints of the entries of $\tilde{A}$. i.e) $m(\tilde{A}) = \begin{pmatrix} m(\tilde{a}_{11}) & ... & m(\tilde{a}_{1n}) \\ ... & ... & ... \\ m(\tilde{a}_{m1}) & ... & m(\tilde{a}_{mn}) \end{pmatrix}$. The width of an interval matrix $\tilde{A}$ is the matrix of widths of its interval elements defined as $w(\tilde{A}) = \begin{pmatrix} w(\tilde{a}_{11}) & ... & w(\tilde{a}_{1n}) \\ ... & ... & ... \\ w(\tilde{a}_{m1}) & ... & w(\tilde{a}_{mn}) \end{pmatrix}$ which is always non-negative .

$\tilde{O}$ to denote the null interval matrix $\begin{pmatrix} \tilde{0} & ... & \tilde{0} \\ ... & ... & ... \\ \tilde{0} & ... & \tilde{0} \end{pmatrix}$.

Also we use $\tilde{I}$ to denote the identity interval matrix $\begin{pmatrix} \tilde{1} & ... & \tilde{0} \\ ... & \tilde{1} & ... \\ \tilde{0} & ... & \tilde{1} \end{pmatrix}$.

Remark 3.1. Let $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \cdots, \tilde{x}_n)^t$ is a interval column vector whose components are closed and bounded intervals. We use $\mathbb{IR}^n$ to denote the set of all n-dimensional interval vectors. By $m(\tilde{\mathbf{x}})$ we denote a column vector whose entries are the corresponding midpoints of the entries of $\tilde{\mathbf{x}}$. i.e) $m(\tilde{\mathbf{x}}) = (m(\tilde{x}_1), m(\tilde{x}_2), m(\tilde{x}_3), \cdots, m(\tilde{x}_n))^t$ and the width of interval vector is defined by $w(\tilde{\mathbf{x}}) = (w(\tilde{x}_1), w(\tilde{x}_2), w(\tilde{x}_3), \cdots, w(\tilde{x}_n))^t$. An interval vector is said to be non-negative if all of its components are non-neagative.

Definition 3.1. [15] Determinant of an interval matrix $\tilde{A}$ is an interval in $\mathbb{IR}$ and is defined by

$$det(\tilde{A}) = |\tilde{A}| = \left\langle |m(\tilde{A})|, \min_{w(\tilde{a}_{ij}) \neq 0} w(\tilde{A}) \right\rangle.$$

Remark 3.2. For an interval matrix $\tilde{A}$, $|m(\tilde{A})| = m(|\tilde{A}|)$.

Definition 3.2. [15] An Interval matrix $\tilde{A} \in M_n(\mathbb{IR})$ is said to be non-singular if $m(\tilde{A})$ is non-singular, (i.e) if $|m(\tilde{A})| \neq 0$.

Remark 3.3. For an interval matrix $\tilde{A}$, $|\tilde{A}| \not\approx \tilde{0}$ if $|m(\tilde{A})| \neq 0$.

Definition 3.3. [15] If $|\tilde{A}| \not\approx \tilde{0}$, then the inverse of $\tilde{A}$ exist and is defined by

$$\tilde{A}^{-1} = \left\langle (m(\tilde{A}))^{-1}, \min_{w(\tilde{a}_{ij}) \neq 0} w(\tilde{A}) \right\rangle.$$

Remark 3.4. For an interval matrix $\tilde{A}$, $(m(\tilde{A}))^{-1} = m(\tilde{A}^{-1})$.

Definition 3.4. A square interval matrix $\tilde{A}$ is called Nilpotent if $\tilde{A}^k \approx \tilde{0}$ for some $k \leq n$. A smallest positive integer k that gives $\tilde{A}^k \approx \tilde{0}$ is called the nilpotent index of $\tilde{A}$.

Definition 3.5. Suppose $\tilde{\lambda} = [\lambda^L, \lambda^U]$ is an interval eigenvalue of $\tilde{A}$. A $(j \times j)$ Jordan block corresponding to $\tilde{\lambda}$ is the $(j \times j)$ square matrix which is a square interval matrix in which all the leading diagonal entries are $\tilde{\lambda}$, all super diagonal entries are $\tilde{1}$ and all other entries are $\tilde{0}$.

For example,

(1×1) Jordan block is $[\ \tilde{\lambda}\ ]_{1\times 1}$

(2×2) Jordan block is $\begin{pmatrix} \tilde{\lambda} & \tilde{1} \\ \tilde{0} & \tilde{\lambda} \end{pmatrix}$

(3×3) Jordan block is $\begin{pmatrix} \tilde{\lambda} & \tilde{1} & \tilde{0} \\ \tilde{0} & \tilde{\lambda} & \tilde{1} \\ \tilde{0} & \tilde{0} & \tilde{\lambda} \end{pmatrix}$

$$\vdots$$

$$\vdots$$

$(j \times j)$ Jordan block is $\begin{pmatrix} \tilde{\lambda}_1 & \tilde{1} & \tilde{0} & \cdots & \cdots & \tilde{0} \\ \tilde{0} & \tilde{\lambda}_2 & \tilde{1} & \cdots & \cdots & \tilde{0} \\ \tilde{0} & \tilde{0} & \tilde{\lambda}_3 & \tilde{1} & \cdots & \tilde{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \tilde{1} \\ \tilde{0} & \tilde{0} & \tilde{0} & \cdots & \cdots & \tilde{\lambda}_j \end{pmatrix}$

4. PROCEDURE FOR FINDING THE JORDAN CANONICAL FORM (JCF) OF INTERVAL MATRICES

Let $\tilde{A}$ be a $(n \times n)$ real interval matrix. The Jordan canonical form of $\tilde{A}$ will exist iff $\tilde{A}$ has 'n' interval eigenvalues over $\mathbb{IR}$. So we assume that, $\tilde{A}$ has'n' interval eigenvalues say $\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_n$.

Suppose an interval eigenvalue $\tilde{\lambda}$ is repeated k times say with $A.M = k$ and $G.M = l(l < k)$. Then the interval eigenvalue $\tilde{\lambda}$ has 'l' Jordan blocks which are determined by aping the technique we have in the case of matrices over $\mathbb{R}$.

The k generalized eigenvectors are determined for the eigenvalue $\tilde{\lambda}$ which are made to form an $(n \times k)$ matrix and the smallest absolute value of the non-zero entries of this $(n \times k)$ matrix will give us the k generalized interval eigenvectors corresponding to $\tilde{\lambda}$.

This process is repeated for all interval eigenvalues to get an $(n \times n)$ interval matrix $\tilde{S}$ of generalized interval eigenvectors and this $\tilde{S}$ is invertible. Using the interval matrix arithmetic operation [14], it is easy to check that, $\tilde{S}^{-1}\tilde{A}\tilde{S}$ gives the Jordan form of $\tilde{A}$.

In numerical problems on interval matrices, we will be mainly interested in the JCF rather than finding $\tilde{S}$ and the JCF through $\tilde{S}$. So in such cases, we calculate $m_j = \eta(\tilde{A} - \tilde{\lambda}\tilde{I})^j$, $j = 1, 2, \cdots$ and $m_0 = 0$ always.

Using this m_j, we calculate $b_j = 2m_j - m_{j+1} - m_{j-1}$, $j = 1, 2, \cdots$, where b_j is the number of $(j \times j)$ Jordan blocks for the eigenvalue $\tilde{\lambda}$.

This process uniquely determines the JCF for a given(real) interval matrix $\tilde{A}$, provided $\tilde{A}$ has n interval eigenvalues.

We begin with an example to demonstrate how the Jordan form, or closest-to-diagonal form, of an interval matrix is constructed, when it exists. This approach is explained below:

Example 4.1. *Find the Jordan Canonical Form of an interval matrix*

$$\tilde{A} = \begin{pmatrix} [-1,7] & [1,1] & [0,2] \\ [0,4] & [-1,5] & [-1,3] \\ [-7,-5] & [-3,-3] & [-6,2] \end{pmatrix}.$$

Solution: Consider the midpoint matrix $m(\tilde{A}) = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 2 & 1 \\ -6 & -3 & -2 \end{pmatrix}$.

The eigenvalues of $m(\tilde{A})$ are 1, 1, 1

Therefore eigenvalue 1 is repeated three times and $(m(\tilde{A}) - I) = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ -6 & -3 & -3 \end{pmatrix}$.

Which implies, rank of $(m(\tilde{A}) - I)$ is 1 and nullity of $(m(\tilde{A}) - I)$ is 2.

Thus, the corresponding eigenvectors of $m(\tilde{\lambda}) = 1$ are $m(\tilde{\nu_1}) = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$, and $m(\tilde{\nu_3}) = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

To construct a Jordan basis, we need to know that $m(\tilde{A})$ has two linearly independent eigenvectors, so that the Jordan basis will have two Jordan blocks. Next to find the generalized eigenvector, we must solve $(m(\tilde{A}) - I)m(\tilde{\nu_2}) = m(\tilde{\nu_1})$. Then we get $m(\tilde{\nu_2}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Therefore, the Jordan basis of $m(\tilde{A})$ is $\{(1, 1, -3)^t, (0, 0, 1)^t, (-1, 1, 1)^t\}$

Also consider the width matrix $w(\tilde{A}) = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$.

The eigenvalues of $w(\tilde{A})$ are 3 , 3 , 5. All the eigenvalues are positive, so choose minimum positive eigenvalue 3. For $w(\tilde{\lambda}) = 3$, the corresponding eigenvectors are $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

We first need to form the (3×2) matrix whose columns are eigenvectors corresponding to $w(\tilde{\lambda}) = 3$. Let $w^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Choose an eigenvector pairing number[14]

$$\mu = \min\{|w^*_{ij}| \ / \ w^*_{ij} \neq 0, \ i = 1, 2, 3; j = 1, 2\} = 1$$

Now the interval eigenvalues of the given interval matrix $\tilde{A}$ are:

(1) $\langle 1, 3 \rangle = [-2, 4]$
(2) $\langle 1, 3 \rangle = [-2, 4]$
(3) $\langle 1, 3 \rangle = [-2, 4]$

This is justified because we are interested in minimising vagueness or error. The generalized interval eigenvectors of the given interval matrix $\tilde{A}$ are computed with the eigenvector pairing number 1 as follows:

Case 1: The interval eigenvector corresponding to the interval eigenvalue $\langle 1, 3 \rangle$ is

$$\left\langle \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}, 1 \right\rangle = \begin{pmatrix} [0, 2] \\ [0, 2] \\ [-4, -2] \end{pmatrix}.$$

Case 2: The interval eigenvector corresponding to the interval eigenvalue $\langle 1, 3 \rangle$ is

$$\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, 1 \right\rangle = \begin{pmatrix} [-1, 1] \\ [-1, 1] \\ [0, 2] \end{pmatrix}.$$

Case 3: The interval eigenvector corresponding to the interval eigenvalue $\langle 1, 3 \rangle$ is

$$\left\langle \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, 1 \right\rangle = \begin{pmatrix} [-2, 0] \\ [0, 2] \\ [0, 2] \end{pmatrix}.$$

Therefore the interval eigenvalues of $\tilde{A}$ are $\tilde{\lambda}_1 = [-2,4]$, $\tilde{\lambda}_2 = [-2,4]$ and $\tilde{\lambda}_3 = [-2,4]$ the corresponding generalized interval eigenvectors are $\tilde{\nu}_1 = \begin{pmatrix} [0,2] \\ [0,2] \\ [-4,-2] \end{pmatrix}$, $\tilde{\nu}_2 = \begin{pmatrix} [-1,1] \\ [-1,1] \\ [0,2] \end{pmatrix}$ and $\tilde{\nu}_3 = \begin{pmatrix} [-2,0] \\ [0,2] \\ [0,2] \end{pmatrix}$ respectively.

An invertible interval matrix $\tilde{S}$ is obtained by writing the generalized interval eigenvectors of $\tilde{A}$ as columns.

$$\tilde{S} = (\tilde{\nu}_1\ \tilde{\nu}_2\ \tilde{\nu}_3)_{3\times 3} = \begin{pmatrix} [0,2] & [-1,1] & [-2,0] \\ [0,2] & [-1,1] & [0,2] \\ [-4,-2] & [0,2] & [0,2] \end{pmatrix}$$

$$\text{Let } m(\tilde{S}^{-1}).m(\tilde{A}).m(\tilde{S}) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 4 & 2 & 2 \\ -1 & 1 & 0 \end{pmatrix}$$

$$.\begin{pmatrix} 3 & 1 & 1 \\ 2 & 2 & 1 \\ -6 & -3 & -2 \end{pmatrix}.\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ -3 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Also $w(\tilde{S}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = w(\tilde{S}^{-1})$ and hence

$$\max\left\{\min_{w(\tilde{s}_{ij})\neq 0} w(\tilde{S}^{-1}),\ \min_{w(\tilde{a}_{ij})\neq 0} w(\tilde{A}),\ \min_{w(\tilde{s}_{ij})\neq 0} w(\tilde{S})\right\} = \max\{1,1,1\} = 1$$

$$\text{Now } \tilde{S}^{-1}\tilde{A}\tilde{S} = \Big\langle m(\tilde{S}^{-1}).m(\tilde{A}).m(\tilde{S}),$$

$$\max\left\{\min_{w(\tilde{s}_{ij})\neq 0} w(\tilde{S}^{-1}),\ \min_{w(\tilde{a}_{ij})\neq 0} w(\tilde{A}),\ \min_{w(\tilde{s}_{ij})\neq 0} w(\tilde{S})\right\}\Big\rangle$$

$$= \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, 1 \right\rangle = \begin{pmatrix} [0,2] & [0,2] & [-1,1] \\ [-1,1] & [0,2] & [-1,1] \\ [-1,1] & [-1,1] & [0,2] \end{pmatrix}$$

$$\approx \begin{pmatrix} [0,2] & [0,2] & \tilde{0} \\ \tilde{0} & [0,2] & \tilde{0} \\ \tilde{0} & \tilde{0} & [0,2] \end{pmatrix} = \tilde{J}$$

Hence $\tilde{S}^{-1}\tilde{A}\tilde{S} \approx \tilde{J}$. By applying the pairing technique [14], we are able to diagonalize the given real interval matrix similar into Jordan diagonal form.

5. AN APPLICATION ON DYNAMICAL SYSTEM

System of first order interval linear differential equations can mathematically represent a wide range of real-world problems. Such a system can be resolved using the Jordan form.

Consider a system of first order interval linear differential equations in n differentiable functions $\tilde{y}_1(t), \tilde{y}_2(t) \cdots \tilde{y}_n(t)$

$$\begin{aligned}
\frac{d\tilde{y}_1(t)}{dt} =& \tilde{a}_{11}\tilde{y}_1(t) + \tilde{a}_{12}\tilde{y}_2(t) + \cdots + \tilde{a}_{1n}\tilde{y}_n(t)\\
\frac{d\tilde{y}_2(t)}{dt} =& \tilde{a}_{21}\tilde{y}_1(t) + \tilde{a}_{22}\tilde{y}_2(t) + \cdots + \tilde{a}_{2n}\tilde{y}_n(t)\\
\frac{d\tilde{y}_3(t)}{dt} =& \tilde{a}_{31}\tilde{y}_1(t) + \tilde{a}_{32}\tilde{y}_2(t) + \cdots + \tilde{a}_{3n}\tilde{y}_n(t)\\
&\cdots\cdots\\
&\cdots\cdots\\
&\cdots\cdots\\
\frac{d\tilde{y}_n(t)}{dt} =& \tilde{a}_{n1}\tilde{y}_1(t) + \tilde{a}_{n2}\tilde{y}_2(t) + \cdots + \tilde{a}_{nn}\tilde{y}_n(t)
\end{aligned} \tag{5.1}$$

with interval initial conditions $\tilde{y}_1(0), \tilde{y}_2(0) \cdots \tilde{y}_n(0)$ and the co-efficients $\tilde{a}_{ij} \in \mathbb{IR}$. We express the above system in matrix form as,

$\frac{d\tilde{\mathbf{y}}(t)}{dt} \approx \tilde{A}\tilde{\mathbf{y}}(t)$ subject to $\tilde{\mathbf{y}}(0) = \tilde{\mathbf{y}}_0$, where $\tilde{A} = (\tilde{a}_{ij})$ and $\tilde{\mathbf{y}}(t) = (\tilde{y}_1(t), \tilde{y}_2(t) \cdots \tilde{y}_n(t))^t$. Here $\tilde{\mathbf{y}}_0$ is the initial interval vector.

The solution of this system is given by, $\tilde{\mathbf{y}}(t) = e^{t\tilde{A}}\tilde{\mathbf{y}}_0$, where $e^{t\tilde{A}} \approx \tilde{I}_n + t\tilde{A} + \frac{t^2}{2!}\tilde{A}^2 + \frac{t^3}{3!}\tilde{A}^3 \cdots$ an infinite series of $(n \times n)$ interval matrices. So, to find the solution of system (5.1) we must to compute $e^{t\tilde{A}}$.

Theorem 5.1. *Let $\tilde{A}$ be an $(n \times n)$ interval matrix. Suppose $\tilde{A}$ has n interval eigenvalues over $\mathbb{IR}$, then there is a diagonalizable matrix $\tilde{D}$ on $\mathbb{IR}^{(n\times n)}$ and a nilpotent matrix $\tilde{N}$ on $\mathbb{IR}^{(n\times n)}$ such that*

$$\tilde{A} \approx \tilde{D} + \tilde{N} \tag{5.2}$$

$$\tilde{D}\tilde{N} \approx \tilde{N}\tilde{D} \tag{5.3}$$

The diagonalizable interval matrix $\tilde{D}$ and the nilpotent matrix $\tilde{N}$ are uniquely determined by (5.2) *and* (5.3) *and each of them is a polynomial in $\tilde{A}$.*

Proof. It is proved that $\tilde{A} \approx \tilde{D} + \tilde{N}$ where $\tilde{D}$ is diagonalizable and $\tilde{N}$ is nilpotent, where $\tilde{D}$ and $\tilde{N}$ not only commute but are polynomials in $\tilde{A}$. Now suppose that we also have $\tilde{A} \approx \tilde{D}' + \tilde{N}'$ where $\tilde{D}'$ is diagonalizable, $\tilde{N}'$ is nilpotent and $\tilde{D}'\tilde{N}' \approx \tilde{N}'\tilde{D}'$. We shall prove that $\tilde{D} \approx \tilde{D}'$ and $\tilde{N} \approx \tilde{N}'$ and hence $\tilde{D}$ and $\tilde{N}$ are unique. Since $\tilde{D}'$ and $\tilde{N}'$ commute with one another and $\tilde{A} \approx \tilde{D}' + \tilde{N}'$, we see that $\tilde{D}'$ and $\tilde{N}'$ commute with $\tilde{A}$. Thus $\tilde{D}'$ and $\tilde{N}'$ commute with any polynomial in $\tilde{A}$. Hence, they commute with $\tilde{D}$ and with $\tilde{N}$. Now we have $\tilde{D} + \tilde{N} \approx \tilde{D}' + \tilde{N}' \approx \tilde{A}$. Therefore $\tilde{D} - \tilde{D}' \approx \tilde{N}' - \tilde{N}$

All four of these interval matrices $\tilde{D}$,$\tilde{D}'$,$\tilde{N}$,$\tilde{N}'$ commute with one another. Since $\tilde{D}$ and $\tilde{D}'$ are both diagonalizable and they commute, they are simultaneously diagonalizable and hence $(\tilde{D} - \tilde{D}')$ is diagonalizable. Also, since $\tilde{N}$ and $\tilde{N}'$ are both nilpotent and they commute the matrix $(\tilde{N}' - \tilde{N})$ is nilpotent. Thus, the interval matrix $\tilde{D} - \tilde{D}'(\approx \tilde{N}' - \tilde{N})$ is both diagonalizable and nilpotent and hence must be equivalent to the zero interval matrix . That is,

$\tilde{D} - \tilde{D}' \approx \tilde{N}' - \tilde{N} \approx \tilde{0}$ and $\tilde{D} \approx \tilde{D}'$ and $\tilde{N}' \approx \tilde{N}$. Hence uniqueness. ∎

5.1. **Properties.**

(1) If $\tilde{A} \approx diag(\tilde{a}_{11}, \tilde{a}_{22}, \cdots \tilde{a}_{nn})$ is a diagonal interval matrix. Then $e^{t\tilde{A}} \approx diag(e^{t\tilde{a}_{11}}, e^{t\tilde{a}_{22}}, \cdots, e^{t\tilde{a}_{nn}})$
(2) $e^{\tilde{0}} \approx \tilde{I}_n$
(3) If $\tilde{A}$ is nilpotent, then $\tilde{A}^k \approx \tilde{0}$ for some positive integer k. So $e^{t\tilde{A}} \approx \tilde{I}_n + t\tilde{A} + \dfrac{t^2}{2!}\tilde{A}^2 + \dfrac{t^3}{3!}\tilde{A}^3 + \cdots + \dfrac{t^{k-1}}{(k-1)!}\tilde{A}^{(k-1)}$.
(4) $e^{s\tilde{A}}e^{t\tilde{A}} \approx e^{(s+t)\tilde{A}}$, where s and t are real numbers.
(5) $e^{\tilde{A}}$ is non-singular, and $(e^{\tilde{A}})^{-1} \approx e^{-\tilde{A}}$.
(6) If $\tilde{A}\tilde{B} \approx \tilde{B}\tilde{A}$, then $e^{\tilde{A}}e^{\tilde{B}} \approx e^{\tilde{A}+\tilde{B}}$.
(7) If $\tilde{B} \approx \tilde{P}^{-1}\tilde{A}\tilde{P}$, then $e^{\tilde{B}} \approx \tilde{P}^{-1}e^{\tilde{A}}\tilde{P}$.

Definition 5.1. Let $\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \tilde{a}_{n1} & \tilde{a}_{n1} & \cdots & \cdots & \tilde{a}_{nn} \end{pmatrix}$. Suppose $\tilde{A}$ has n real interval eigenvalues. Let $\tilde{\lambda}$ be one such real interval eigenvalues repeating p times. Then $\tilde{J} = \begin{pmatrix} \tilde{\lambda} & \tilde{1} & \tilde{0} & \cdots & \cdots & \tilde{0} \\ \tilde{0} & \tilde{\lambda} & \tilde{1} & \cdots & \cdots & \tilde{0} \\ \tilde{0} & \tilde{0} & \tilde{\lambda} & \tilde{1} & \cdots & \tilde{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \tilde{1} \\ \tilde{0} & \tilde{0} & \tilde{0} & \cdots & \cdots & \tilde{\lambda} \end{pmatrix}$ is called the $(p \times p)$ Jordan interval block. In this case we can prove that

$$e^{\tilde{J}t} = \begin{pmatrix} e^{\tilde{\lambda}t} & te^{\tilde{\lambda}t} & \dfrac{t^2}{2!}e^{\tilde{\lambda}t} & \cdots & \cdots & \dfrac{t^{p-1}}{(p-1)!}e^{\tilde{\lambda}t} \\ \tilde{0} & e^{\tilde{\lambda}t} & te^{\tilde{\lambda}t} & \cdots & \cdots & \dfrac{t^{p-2}}{(p-2)!}e^{\tilde{\lambda}t} \\ \tilde{0} & \tilde{0} & e^{\tilde{\lambda}t} & te^{\tilde{\lambda}t} & \cdots & \dfrac{t^{p-3}}{(p-3)!}e^{\tilde{\lambda}t} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & te^{\tilde{\lambda}t} \\ \tilde{0} & \tilde{0} & \tilde{0} & \cdots & \cdots & e^{\tilde{\lambda}t} \end{pmatrix}$$

Definition 5.2. Let $\tilde{J}$ be $(p \times p)$ Jordan interval block. We define the exponential of Jordan interval block as follows:

$$e^{\tilde{J}} = \left\langle e^{m(\tilde{J})}, \min_{w(\tilde{j}_{ij}) \neq 0} w(\tilde{J}) \right\rangle.$$

Example 5.1. *Solve the system of interval differential equations*

$$\begin{aligned}\frac{d\tilde{y}_1(t)}{dt} &= \langle\, 2,1\rangle\, \tilde{y}_1(t) + \langle -1,1\rangle\, \tilde{y}_2(t) + \langle 1,1\rangle \tilde{y}_3(t)\\ \frac{d\tilde{y}_2(t)}{dt} &= \langle\, 0,0\rangle\, \tilde{y}_1(t) + \langle 2,1\rangle\, \tilde{y}_2(t) + \langle 0,1\rangle \tilde{y}_3(t)\\ \frac{d\tilde{y}_3(t)}{dt} &= \langle\, 0,0\rangle\, \tilde{y}_1(t) + \langle 0,0\rangle\, \tilde{y}_2(t) + \langle 3,1\rangle \tilde{y}_3(t)\end{aligned} \tag{5.4}$$

with $\tilde{\mathbf{y}}(0) = \begin{pmatrix}\langle\, 1,1\rangle\\ \langle\, 1,0.5\rangle\\ \langle\, 1,0.75\rangle\end{pmatrix}$.

Solution: A matrix form of the above system is given by, $\tilde{A} = \begin{pmatrix}\langle 2,1\rangle & \langle -1,1\rangle & \langle 1,1\rangle\\ \langle 0,0\rangle & \langle 2,1\rangle & \langle 1,0\rangle\\ \langle 0,0\rangle & \langle 0,0\rangle & \langle 3,1\rangle\end{pmatrix}$.

Consider the midpoint matrix $m(\tilde{A}) = \begin{pmatrix}2 & -1 & 1\\ 0 & 2 & 1\\ 0 & 0 & 3\end{pmatrix}$. The eigenvalues of $m(\tilde{A})$ are $2, 2, 3$.

The generalized eigenvectors of $m(\tilde{A})$ are $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\1\end{pmatrix}$.

Also consider width matrix of $w(\tilde{A}) = \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$. The eigenvalues of $w(\tilde{A})$ are $1, 1, 1$. All the eigenvalues are positive, so choose minimum positive eigenvalue 1. For $w(\tilde{\lambda}) = 1$, the corresponding eigenvectors are $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\-1\end{pmatrix}$. Here we need to form the (3×2) matrix whose columns are eigenvectors corresponding to $w(\tilde{\lambda}) = 1$.

Let $w^* = \begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & -1\end{pmatrix}$. Now choose an eigenvector pairing number,

$$\mu = \min\{|w^*_{ij}| \ / \ w^*_{ij} \neq 0,\ i = 1, 2, \cdots, n; j = 1, 2, \cdots, k\} = 1.$$

Therefore, the interval eigenvalues of $\tilde{A}$ are $[1, 3]$, $[1, 3]$ and $[2, 4]$. The corresponding generalized interval eigenvectors are $\begin{pmatrix}[0,2]\\ [-1,1]\\ [-1,1]\end{pmatrix}$, $\begin{pmatrix}[0,2]\\ [-2,0]\\ [-1,1]\end{pmatrix}$ and $\begin{pmatrix}[-1,1]\\ [0,2]\\ [0,2]\end{pmatrix}$ respectively.

An invertible interval matrix $\tilde{S}$ is obtained by writing the generalized interval eigenvectors of $\tilde{A}$ as columns. $\tilde{S} = (\tilde{\nu}_1\ \tilde{\nu}_2\ \tilde{\nu}_3)_{3\times 3} = \begin{pmatrix}[0,2] & [0,2] & [-1,1]\\ [-1,1] & [-2,0] & [0,2]\\ [-1,1] & [-1,1] & [0,2]\end{pmatrix}$ and

$\tilde{S}^{-1} = \begin{pmatrix}[0,2] & [0,2] & [-2,0]\\ [-1,1] & [-2,0] & [0,2]\\ [-1,1] & [-1,1] & [0,2]\end{pmatrix}$. Let $m(\tilde{S}^{-1}).m(\tilde{A}).m(\tilde{S}) = \begin{pmatrix}2 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}$.

Also $w(\tilde{S}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = w(\tilde{S}^{-1})$ and hence

$$\max\left\{\min_{w(\tilde{s}_{ij})\neq 0} w(\tilde{S}^{-1}), \min_{w(\tilde{a}_{ij})\neq 0} w(\tilde{A}), \min_{w(\tilde{s}_{ij})\neq 0} w(\tilde{S})\right\} = \max\{1,1,1\} = 1$$

$$\begin{aligned}
\text{Now } \tilde{S}^{-1}\tilde{A}\tilde{S} &= \Big\langle m(\tilde{S}^{-1}).m(\tilde{A}).m(\tilde{S}), \\
&\qquad \max\left\{\min_{w(\tilde{s}_{ij})\neq 0} w(\tilde{S}^{-1}), \min_{w(\tilde{a}_{ij})\neq 0} w(\tilde{A}), \min_{w(\tilde{s}_{ij})\neq 0} w(\tilde{S})\right\}\Big\rangle \\
&= \left\langle \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, 1 \right\rangle = \begin{pmatrix} \langle 2,1\rangle & \langle 1,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 2,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 3,1\rangle \end{pmatrix} \approx \tilde{J}
\end{aligned}$$

$$\text{Let } \tilde{J} = \tilde{D} + \tilde{N} = \begin{pmatrix} \langle 2,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 2,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 3,1\rangle \end{pmatrix} + \begin{pmatrix} \langle 0,1\rangle & \langle 1,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix}$$

$$\begin{aligned}
\text{Hence } \quad e^{\tilde{J}t} &\approx e^{(\tilde{D}+\tilde{N})t} \approx e^{\tilde{D}t}.e^{\tilde{N}t} \\
&= \left\langle e^{m(\tilde{D})t}.e^{m(\tilde{N})t},\ \max\left\{\min_{w(\tilde{d}_{ij})\neq 0} w(\tilde{D}), \min_{w(\tilde{n}_{ij})\neq 0} w(\tilde{N})\right\}\right\rangle \\
&= \Bigg\langle \begin{pmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}, \\
&\qquad \max\left\{\min_{w(\tilde{d}_{ij})\neq 0} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \min_{w(\tilde{n}_{ij})\neq 0} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right\}\Bigg\rangle \\
&= \left\langle \begin{pmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}, \max\{1,1\} \right\rangle \\
&= \left\langle \begin{pmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}, 1 \right\rangle = \begin{pmatrix} e^{\langle 2,1\rangle t} & te^{\langle 2,1\rangle t} & \langle 0,1\rangle \\ \langle 0,1\rangle & e^{\langle 2,1\rangle t} & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & e^{\langle 3,1\rangle t} \end{pmatrix}.
\end{aligned}$$

The solution of the system is,

$$\begin{aligned}
\tilde{\mathbf{y}}(t) &= e^{t\tilde{A}}\tilde{\mathbf{y}}(0) \approx \tilde{P}e^{t\tilde{J}}\tilde{P}^{-1}\tilde{\mathbf{y}}(0) \\
&= \Big\langle m(\tilde{P}).m(e^{t\tilde{J}}).m(\tilde{P}^{-1}).m(\tilde{\mathbf{y}}(0)), \\
&\max\left\{\min_{w(\tilde{p}_{ij})\neq 0} w(\tilde{P}), \min_{w(\tilde{j}_{ij})\neq 0} w(\tilde{J}), \min_{w(\tilde{p}_{ij})\neq 0} w(\tilde{P}^{-1}), \min_{w(\tilde{y}_{ij})\neq 0} w(\tilde{\mathbf{y}}(0))\right\}\Big\rangle
\end{aligned} \tag{5.5}$$

.

Let $m(\tilde{P}).m(e^{t\tilde{J}}).m(\tilde{P}^{-1}).m(\tilde{\mathbf{y}}(0)) = \begin{pmatrix} e^{2t} \\ 0 \\ e^{3t} \end{pmatrix}$ and

$$\max\left\{ \min_{w(\tilde{p}_{ij})\neq 0} w(\tilde{P}), \min_{w(\tilde{j}_{ij})\neq 0} w(\tilde{J}), \min_{w(\tilde{p}_{ij})\neq 0} w(\tilde{P}^{-1}), \min_{w(\tilde{y}_{ij})\neq 0} w(\tilde{\mathbf{y}}(0)) \right\}$$
$$= \max\left\{ \min_{w(\tilde{p}_{ij})\neq 0} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \min_{w(\tilde{j}_{ij})\neq 0} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \min_{w(\tilde{p}_{ij})\neq 0} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \min_{w(\tilde{y}_{ij})\neq 0} \begin{pmatrix} 1 \\ 0.5 \\ 0.75 \end{pmatrix} \right\}$$
$$= \max\{1, 1, 1, 0.5\}$$
$$= 1$$

From equation (5.5), we get,

$$\tilde{\mathbf{y}}(t) = \left\langle \begin{pmatrix} e^{2t} \\ 0 \\ e^{3t} \end{pmatrix}, 1 \right\rangle \approx \begin{pmatrix} e^{\langle 2,1\rangle t} \\ \langle 0, 1\rangle\, t \\ e^{\langle 3,1\rangle t} \end{pmatrix}$$

Hence the solution is $\tilde{y}_1(t) = e^{\langle 2,1\rangle t}, \tilde{y}_2(t) = \langle 0, 1\rangle\, t$ and $\tilde{y}_3(t) = e^{\langle 3,1\rangle t}$.

6. An Application to Continuous evolution System

Consider a system of interval linear equations that models the predator and the prey as two different species. The prey is the predator's food. In this case, we assume that the predator solely eats its prey. So, if there had been no prey, the predator species would have disappeared. On the other hand, if there were no predator species, the prey species would have increased significantly (according to the Malthusian law). This system can be modelled by an almost interval linear system. A specific forest is the habitat of two species: foxes and chickens. The chicken serves as the food that foxes consume to survive. Assume that the chicken would have grown 20% more per time unit if the foxes were not present. The number of foxes would have reduced by 40% in each time unit if the chicken did not exist.

Let us look at how this can be used to analyze continuous models of system evolution using the Jordan form. We take into account population fluctuations at various points in time where a significant temporal difference occurs. It is frequently helpful to assume that the state s is defined for all time t by the function $s(t)$, and that information on state changes is provided in terms of the derivative $\dfrac{ds}{dt}$, especially if the time interval involved is extremely short or if the change in the population is extremely small relative to the time frame. Let $\tilde{F}(t)$ and $\tilde{C}(t)$ be the population of foxes and chickens at time t respectively. This situation can now be replaced by,

$$\begin{aligned} \frac{d\tilde{F}(t)}{dt} &= -\,[0.38, 0.42]\tilde{F}(t) + [0.46, 0.54]\tilde{C}(t) \\ \frac{d\tilde{C}(t)}{dt} &= -\,\tilde{p}(t) + [0.18, 0.22]\tilde{C}(t) \end{aligned} \tag{6.1}$$

Let $\tilde{p}$ is a predation rate that denotes the average number of chickens consumed each unit time by each fox. Here we assume that $\tilde{p} = [0.15, 0.17]$ and the initial population $\begin{pmatrix} \tilde{F}(0) \\ \tilde{C}(0) \end{pmatrix} = \begin{pmatrix} [95, 105] \\ [993, 1007] \end{pmatrix}$.

The above system can be written in matrix form as
$\frac{d\tilde{\mathbf{y}}(t)}{dt} = \tilde{A}\tilde{\mathbf{y}}(t)$, where $\tilde{A} = \begin{pmatrix} -[0.38, 0.42] & [0.46, 0.54] \\ -[0.15, 0.17] & [0.18, 0.22] \end{pmatrix}$, $\tilde{\mathbf{y}}(t) = \begin{pmatrix} \tilde{F}(t) \\ \tilde{C}(t) \end{pmatrix}$ and
$\tilde{\mathbf{y}}(0) = \begin{pmatrix} [95, 105] \\ [993, 1007] \end{pmatrix}$.

Solution: Consider the midpoint matrix $m(\tilde{A}) = \begin{pmatrix} \text{-0.40} & 0.50 \\ \text{-0.16} & 0.20 \end{pmatrix}$. The eigenvalues of $m(\tilde{A})$ are 0, -0.2000 and the corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0.80 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0.40 \end{pmatrix}$.

Also consider the width matrix $w(\tilde{A}) = \begin{pmatrix} 0.02 & 0.04 \\ 0.01 & 0.02 \end{pmatrix}$. The eigenvalues of $w(\tilde{A})$ are 0, 0.04. Choose minimum positive eigenvalue 0.040 and the corresponding eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

By applying the pairing techniques [14], the unique interval eigenvalues of the given interval matrix $\tilde{A}$ are $\tilde{\lambda}_1 = [-0.04, 0.04]$ and $\tilde{\lambda}_2 = [-0.24, -0.16]$ and the corresponding interval eigenvectors of the given interval matrix $\tilde{A}$ are $\tilde{\nu}_1 = \begin{pmatrix} [0, 2] \\ [-0.20, 1.80] \end{pmatrix}$ and $\tilde{\nu}_2 = \begin{pmatrix} [0, 2] \\ [-0.60, 1.40] \end{pmatrix}$ respectively .

An invertible matrix $\tilde{P} = \begin{pmatrix} [0, 2] & [0, 2] \\ [-0.20, 1.80] & [-0.60, 1.40] \end{pmatrix}$ is obtained by using columns as interval eigenvectors and $\tilde{P}^{-1} = \begin{pmatrix} [-2, 0] & [1.50, 3.50] \\ [1, 3] & [-3.50, -1.50] \end{pmatrix}$.

$$\begin{aligned}
\text{Now we compute}\quad \tilde{J} = \tilde{P}^{-1}\tilde{A}\tilde{P} &= \Big\langle m(\tilde{P}^{-1}).m(\tilde{A}).m(\tilde{P}), \max\Big\{ \min_{w(\tilde{p}_{ij})\neq 0} w(\tilde{P}^{-1}), \\
&\quad \min_{w(\tilde{a}_{ij})\neq 0} w(\tilde{A}), \min_{w(\tilde{p}_{ij})\neq 0} w(\tilde{P}) \Big\} \Big\rangle \\
&= \left\langle \begin{pmatrix} 0 & 0 \\ 0 & -0.20 \end{pmatrix}, \max\left\{ \min_{w(\tilde{p}_{ij})\neq 0} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \right.\right. \\
&\quad \left.\left. \min_{w(\tilde{a}_{ij})\neq 0} \begin{pmatrix} 0.02 & 0.04 \\ 0.01 & 0.02 \end{pmatrix}, \min_{w(\tilde{p}_{ij})\neq 0} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \right\rangle \\
&= \left\langle \begin{pmatrix} 0 & 0 \\ 0 & -0.20 \end{pmatrix}, \max\{1, 0.01, 1\} \right\rangle \\
&= \left\langle \begin{pmatrix} 0 & 0 \\ 0 & -0.20 \end{pmatrix}, 1 \right\rangle \approx \begin{pmatrix} [-1, 1] & [-1, 1] \\ [-1, 1] & [-1.20, 0.80] \end{pmatrix}
\end{aligned}$$

Then $e^{\tilde{J}t} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & e^{-0.20t} \end{pmatrix}, 1 \right\rangle = \begin{pmatrix} [0, 2] & [-1, 1] \\ [-1, 1] & [e^{-0.20t} - 1, e^{-0.20t} + 1] \end{pmatrix}$.
The general solution is

$$\begin{aligned}
\tilde{\mathbf{y}}(t) &= e^{t\tilde{A}}\tilde{\mathbf{y}}(0) \approx \tilde{P}e^{t\tilde{J}}\tilde{P}^{-1}\tilde{\mathbf{y}}(0) \\
&= \Big\langle m(\tilde{P}).m(e^{t\tilde{J}}).m(\tilde{P}^{-1}).m(\tilde{\mathbf{y}}(0)), \\
&\quad \max\Big\{ \min_{w(\tilde{p}_{ij})\neq 0} w(\tilde{P}), \min_{w(\tilde{j}_{ij})\neq 0} w(\tilde{J}), \min_{w(\tilde{p}_{ij})\neq 0} w(\tilde{P}^{-1}), \min_{w(\tilde{y}_{ij})\neq 0} w(\tilde{\mathbf{y}}(0) \Big\} \Big\rangle
\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{y}}(t) &= \left\langle \begin{pmatrix} 2400 - 2300e^{-0.20t} \\ 1920 - 920e^{-0.20t} \end{pmatrix}, \max\left\{ \min_{w(\tilde{p}_{ij})\neq 0} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \right.\right. \\ &\left.\left. \min_{w(\tilde{a}_{ij})\neq 0} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \min_{w(\tilde{p}_{ij})\neq 0} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \min_{w(\tilde{y}_{ij})\neq 0} \begin{pmatrix} 5 \\ 7 \end{pmatrix} \right\} \right\rangle \\ &= \left\langle \begin{pmatrix} 2400 - 2300e^{-0.20t} \\ 1920 - 920e^{-0.20t} \end{pmatrix}, \max\{1, 1, 1, 5\} \right\rangle \\ &= \begin{pmatrix} \langle 2400 - 2300e^{-0.20t}, 5\rangle \\ \langle 1920 - 920e^{-0.20t}, 5\rangle \end{pmatrix}.\end{aligned}$$

Thus, the population of foxes and chickens at time t are

$$\begin{aligned}\tilde{F}(t) =& \langle 2400 - 2300e^{-0.20t}, 5\rangle \\ =& [(2400 - 2300e^{-0.20t}) - 5, (2400 - 2300e^{-0.20t}) + 5] \\ =& [2395 - 2300e^{-0.20t}, 2405 - 2300e^{-0.20t}] \\ \tilde{C}(t) =& \langle 1920 - 920e^{-0.20t}, 5\rangle \\ =& [(1920 - 920e^{-0.20t}) - 5, (1920 - 920e^{-0.20t}) + 5] \\ =& [1915 - 920e^{-0.20t}, 1925 - 920e^{-0.20t}]\end{aligned}$$

7. Results and Discussion

Figure 1 and table 7.1 depicts the population of foxes and chickens over a period of time t.

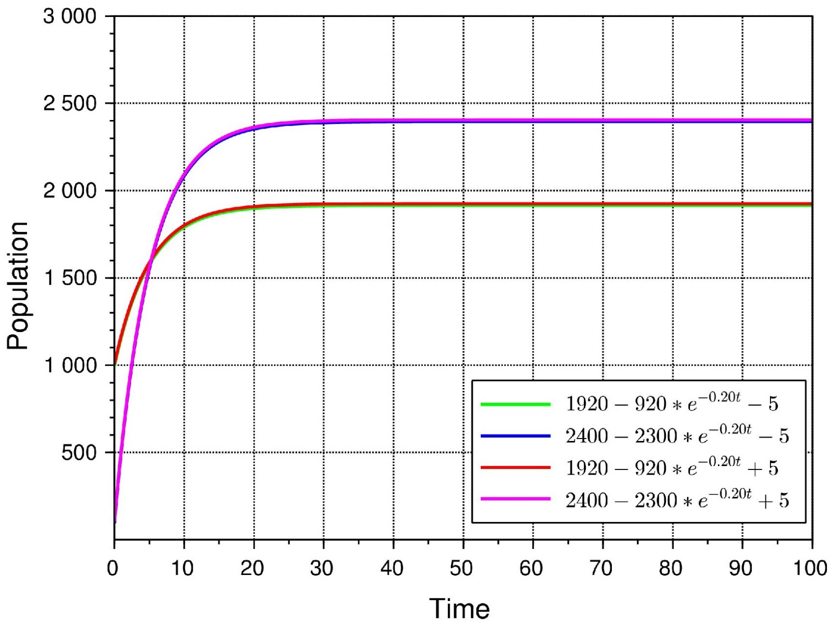

Figure 1: Population of Fox $\tilde{F}(t)$ and Chicken $\tilde{C}(t)$ upto 100 years

Table 7.1: Population of Foxes $\tilde{F}(t)$ and Chickens $\tilde{C}(t)$ upto $t = 65$ years

S.No	t	$\tilde{F}(t)$	$\tilde{C}(t)$	S.No	t	$\tilde{F}(t)$	$\tilde{C}(t)$
1	0	[95,105]	[995,1005]	8	35	[2393,2403]	[1914,1924]
2	5	[1549,1559]	[1577,1587]	9	40	[2394,2404]	[1915,1925]
3	10	[2084,2094]	[1790,1800]	10	45	[2395,2405]	[1915,1925]
4	15	[2280,2290]	[1869,1879]	11	50	[2395,2405]	[1915,1925]
5	20	[2353,2363]	[1898,1908]	12	55	[2395,2405]	[1915,1925]
6	25	[2380,2390]	[1909,1919]	13	60	[2395,2405]	[1915,1925]
7	30	[2389,2399]	[1913,1923]	14	65	[2395,2405]	[1915,1925]

We observed that, after a long period of time $(t \to \infty)$, we have $e^{-0.20t} \to 0$. The population of foxes and chickens becomes $\tilde{F}(t) = [2395, 2405]$ and $\tilde{C}(t) = [1915, 1925]$. In this case, the predation rate $\tilde{p} = [0.15, 0.17]$ is modest where $\tilde{F}(t)$ and $\tilde{C}(t)$ both fall within certain ranges. Therefore, this is an instance of "stable limiting populations" that coexist peacefully and in equilibrium with one another.

8. CONCLUSION

The hall mark of this article is that the JCF obtained through the interval arithmetic operations in the pairing technique is unique. The easiest technique to get the Jordan interval matrix, which is closest to the diagonal form is explained in this article. A significant theorem that we established is used to compute the exponential of $\tilde{A}$. Furthermore by applying the Jordan Canonical form of interval matrices, we discussed the solution of system of interval linear differential equations and continuous evolution system. We conclude with a numerical illustration that demonstrates how continuous evolution systems might be used and analyzed.

REFERENCES

[1] ABHIRUP SIT, Eigen Values of Interval Matrix, *American Journal of Applied Mathematics and Computing.*, **1** (2021), pp. 6–11. `https://doi.org/10.15864/ajamc.142.`

[2] G. ALEFELD and J. HERZBERGER, *Introduction to Interval Computations*, Academic Press, New York, first edition, 1983.

[3] ASSEM DEIF, The interval eigenvalue problem, *ZAMM Journal of applied mathematics and mechanics*, **71(1)** (1991), pp. 61–64. Doi:10.1002/zamm.19910710117.

[4] A. BERMAN and D. HERSHKOWITZ, Matrix diagonal stability and its implications, *SIAM J. Algebr, Discrete Methods*, **4** (1983), pp. 377–382.

[5] CHEN, CHI-TSONG and CHARLES A. DESOER, A Proof of Controllability of Jordan Form State Equations, *IEEE Transactions on Automatic Control*, **13(2)** (1968), pp. 195–196.

[6] J. B. DIAZ and F. T. METCALF, A complementary triangle inequality in Hilbert and Banach spaces, *Proc. Amer. Math. Soc.*, **17** (1966), pp. 88–97.

[7] S. S. DRAGOMIR, New Reverses of Schwarz, Triangle and Bessel Inequalities in Inner Product Spaces, *Aust. J. Math. Anal. Appl.*, **1** (2004), No. 1, Article 1, [`Online: https://ajmaa.org/cgi-bin/paper.pl?string=v1n1/nrstbiips.tex`].

[8] FREDERIC MAZENC and OLIVIER BERNARD, Interval observers for linear time-invariant systems with Disturbances, *Automatica*, **47(1)** (2010), pp. 140–147. Doi: 10.1016/j.automatica.2010.10.019. hal-00555464.

[9] K. GANESAN and P. VEERAMANI, On arithmetic operations of interval numbers, *International Journal of Uncertainty, Fuzziness and Knowledge Based Systems*, **13(6)** (2005), pp. 619–631. Doi:10.1142/S0218488505003710.

[10] K. GANESAN, On some properties of interval matrices, *International Journal of Computational and Mathematical Sciences*, **1(1)** (2007), pp. 35–42.

[11] GILBERT STRANG, Introduction to Linear Algebra, Wellesley-Cambridge Press, 3rd edition, 2003.

[12] E. HANSEN, Interval arithmetic in matrix computations, Part I, *SIAM Journal on Numerical Analysis*, **2(2)**, (1965), pp. 308–320.

[13] E. HANSEN and R. SMITH, Interval arithmetic in matrix computations, Part II, *SIAM Journal of Numerical Analysis*, **4(1)** (1967), pp. 1–9.

[14] S. HEMA SURYA, T. NIRMALA and K. GANESAN, Interval Linear Algebra - A New Perspective, *Journal of King Saud University-Science*, **35** (2023), 102502.

[15] S. HEMA SURYA, T. NIRMALA and K. GANESAN, Diagonal Canonical Form of Interval Matrices and Applications on Dynamical Systems, Phys. Scr., **98**, (2023), 075201.

[16] R.A. HORN, C.R. JOHNSON, *Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[17] S. B. HSU and K. C. CHEN, *Ordinary differential equations with applications*, Series on Applied Mathematics, Third edition, 2023.

[18] L. JOHN and CASTI, *Linear Dynamical Systems*, Academic Press, 1987.

[19] Y. T. JUANG and C. S. SHAO, Stability of dynamic interval systems, *Internat. J. Control*, **49(4)** (1989), pp. 1401–1408.

[20] KALMAN and E. RUDOLPH, Mathematical Description of Linear Dynamical Systems, *Journal of the Society of Industrial and Applied Mathematics-Control Series, Series A*, **1(2)** (1963), pp. 152–192.

[21] KAMELEH NASSIRI PIRBAZARI and MEHDI AZARI, The Order of Minimal Realization of Jordan Canonical Form Systems, *Bol. Soc. Paran. Mat*, **36(3)** (2018), pp. 81–88. Doi: 10.5269/bspm.v36i3.23030

[22] E. Kaszkurewicz, A. Bhaya, *Matrix Diagonal Stability in Systems and Computation*, Birkhauser, Boston, 2000.

[23] X. X. Liao and J. L. Qian, Some new results for stability of interval matrices, *Control Theory and Advances Technology*, **4(2)** (1988), pp. 265–270.

[24] LUC JAULIN, MICHEL KIEFFER, OLIVIER DIDRIT and ERIC WALTER, *Applied Interval Analysis*, Springer-Verlag, London, First edition, 2003.

[25] R. E. MOORE, R. B. KEARFOTT and M. J. CLOUD, *Introduction to Interval Analysis*, SIAM, Philadelphia, PA, 2009.

[26] A. NEUMAIER, *Interval Methods for Systems of Equations*, Cambridge University Press, Cambridge, first edition, 1990.

[27] NICOLAS GOZE, Linear algebra in the vector space of intervals, *arXiv math*, **1** (2010), pp. 1–11. Doi: 1006.5311.

[28] O. PASTRAVANU, M. VOICU, Generalized matrix diagonal stability and linear dynamical systems, *Linear Algebra Appl.*, **419**(1) (2006), pp. 299–310.

[29] O. PASTRAVANU, M.H. MATCOVSCH, Diagonal stability of interval matrices and applications, *Linear Algebra and its Applications*, **433** (2010), pp. 1646–1658.

[30] L. PERKO, *Differential Equations and Dynamical systems*, Text in Applied Mathematics, Springer-Verlag, 1991.

[31] S. POLJAK and M. SCHLEGEL, Computing the generic Jordan canonical form, *Linear Multilinear Algebra*, **28** (1991), pp. 241-249.

[32] SHUAI LIU, ZHIJIAN JI and HUIZI MA, Jordan Form-Based Algebraic Conditions for Controllability of Multiagent Systems under Directed Graphs, *Complexity*, Article ID 7685460, (2020), Doi: 10.1155/2020/7685460.

[33] L. M. SILVERMAN and H. E. MEADOWS, Controllability and Observability in Time-Variable Linear Systems, *SIAM Journal on Control and Optimization*, **5(1)** (1967), pp. 64–73.

[34] XU DAOYI, Simple criteria for stability of interval matrices, *Internat. J. Control*, **41(1)** (1985), pp. 289–295.

[35] R. K. YEDAVALLI, Improved measures of stability robustness for linear state space models, *IEEE Trans. Automat. Control*, **AC-30(6)** (1985), pp. 577–579.

[36] R. K. YEDAVALLI, Perturbation bounds for robust stability in linear state space models., *Internat. J. Control*, **42(6)** (1985), pp. 1507–1517.

[37] R. K. YEDAVALLI, Stability analysis of interval matrices: another sufficient condition, *Internat. J. Control*, **43(3)** (1986), pp. 767–772.