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A SELF ADAPTIVE METHOD FOR SOLVING SPLIT BILEVEL VARIATIONAL INEQUALITIES PROBLEM IN HILBERT SPACES

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ABSTRACT. In this work, we study the split bilevel variational inequality problem in two real Hilbert spaces. We propose a new modified inertial projection and contraction method for solving the aforementioned problem when one of the operators is pseudomonotone and Lipschitz continuous while the other operator is α-strongly monotone. The use of the weakly sequential continuity condition on the Pseudomonotone operator is removed in this work. A Strong convergence theorem of the proposed method is proved under some mild conditions. In addition, some numerical experiments are presented to show the efficiency and implementation of our method in comparison with other methods in the literature in the framework of infinite dimensional Hilbert spaces. The results obtained in this paper extend, generalize and improve several.

Key words and phrases: Bilevel variational inequality; Split variational inequality problem; Split feasibility problem; inertial iterative scheme; Fixed point problem.

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1. INTRODUCTION

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, C be a nonempty closed convex subset of H and $F_1 : H \to H$ be an operator. The classical Variational Inequality Problem (VIP) is formulated as: Find $x \in C$ such that

$$\langle F_1 x, y - x \rangle \geq 0 \; \forall \, y \in C. \tag{1.1}$$

The notion of VIP was introduced independently by Stampacchia [30] and Fichera [12, 13] for modeling problems arising from mechanics and for solving Signorini problems. It is well-known that many problems in economics, mathematical sciences, and mathematical physics can be formulated as VIP. Censor et al. in [10] extended the concept of VIP (1.1) to the following Split Variational Inequality Problem (SVIP): Find

$$x^* \in C \text{ that solves } \langle F_1 x^*, x - x^* \rangle \geq 0 \; \forall \, x \in C \tag{1.2}$$

such that $y^* = Ax^* \in Q$ solves

$$\langle F_2 y^*, y - y^* \rangle \geq 0 \; \forall \, y \in Q, \tag{1.3}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, $F_1 : H_1 \to H_1, \; F_2 : H_2 \to H_2$ are two operators and $A : H_1 \to H_2$ is a bounded linear operator.

Remark 1.1. When $F_1 = F_2 = 0$, the SVIP reduces to the Split Feasibility Problem (SFP).

The concept of SFP was introduced by Censor and Elfving [8] in the framework of finite-dimensional Hilbert spaces. The SFP has found applications in many real-life problems such as image recovery, signal processing, control theory, data compression, computer tomography and so on (see [11, 9] and the references therein). The fixed point problem finds application in proving the existence of solutions of many nonlinear problems arising in many real life problems. From the existence of solutions of differential, partial differential, integral, random differential and random integral equations, and evolutionary equations. For details about fixed point problems (see [14, 15, 17]). Furthermore, a common solution of a VIP and a fixed point problem find applications in real life problems like network resource allocation, image recovery, signal processing, for further details, (see [2, 6, 32, 31, 33] and the references therein).
Mainge in [21] proposed and studied a new type of optimization problem. Find

$$x^* \in VI(F_1, C) \cap F(T) \text{ such that } \; \langle F_2 x^*, x - x^* \rangle \geq 0, \; \forall \, x \in VI(F_1, C) \cap F(T), \tag{1.4}$$

where $F_1 : H \to H$ is monotone and L-Lipschitz continuous, $F_2 : H \to H$ is η-strongly monotone and k-Lispchitz continuous and $T : H \to H$ is a γ-demicontractive mapping and demiclosed at zero. He proposed the following iterative algorithm

$$\begin{cases} x_0 = H_1 \\ y_n = P_C(x_n - \lambda_n F_1 x_n) \\ z_n = P_C(x_n - \lambda_n F_1 y_n) \\ t_n = z_n - \alpha_n F_2(z_n) \\ x_{n+1} = (1 - \omega) t_n + \omega T(t_n), \end{cases} \tag{1.5}$$

where $\lambda_n \subset [a, b] \subset (0, \frac{1}{L})$, $\{\alpha_n\} \subset (0, 1)$, $\lim\limits_{n \to \infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$ and $\omega \in (0, \frac{1-\gamma}{2})$. He established that the sequence generated by algorithm (1.5) converges strongly to the solution set.

Remark 1.2. However, in Algorithm (1.5), the projection P_C onto feasible set C is evaluated two times in each iteration and this have adverse effect on the performance of the algorithm. In addition, the Lipschitz constant is required which is very difficult or impossible to estimate. Thus, the above iterative scheme is not easily applicable.

$$\begin{cases} x_0, x_1 = H_1 \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ T_n = \{x \in H : \langle w_n - \lambda_n A w_n - y_n, x - y_n\rangle \leq 0\} \\ z_n = P_{T_n} P(w_n - \lambda_n A y_n), \\ x_{n+1} = z_n - \alpha_n \gamma F(z_n), \end{cases} \tag{1.6}$$

and

$$\begin{cases} x_0, x_1 = H_1 \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ z_n = y_n - \lambda_n(A y_n - A w_n), \\ x_{n+1} = z_n - \alpha_n \gamma F(z_n), \end{cases} \tag{1.7}$$

Minh, Van and Anh in [26], also studied the following Split Bilevel Variational Inequality Problem (SBVIP): Find

$$x^* \in \Gamma \text{ such that } \langle F_2 x^*, x - x^*\rangle \geq 0, \tag{1.8}$$

for any $x \in \Gamma$, where

$$\Gamma = \{x^* \in VI(F_1, C) : Ax^* \in F(S)\}.$$

Using the following iterative method, they established a strong convergence theorem.

Algorithm 1.1. Initialization: *Let* $x_0 \in H_1$. *Set* $n := 0$.

Step 1. *Compute* $u_n = A(x_n)$ *and*

$$y_n = x_n + \omega_n A^*(S(u_n) - u_n). \tag{1.9}$$

Step 2. *Compute*

$$z_n = P_C(y_n - \lambda_n F_1 y_n), \tag{1.10}$$

$$t_n = P_{T_n}(y_n - \lambda_n F_1 z_n), \tag{1.11}$$

where $T_n = \{\omega \in H : \langle y_n - \lambda_n F_1 y_n - z_n, \omega - z_n\rangle \leq 0\}$.

Step 3. *Compute*

$$x_{n+1} = t_n - \alpha_n F_2 t_n, \tag{1.12}$$

where $A : H_1 \to H_2, F_2 : H_1 \to H_1$ is η-strongly monotone and L-Lipschitz continuous on H_1, with $L > 0$, $F_1 : H_1 \to H_1$ is pseudomonotone on C and L-Lipschitz continuous on H_1 with $\limsup_{n\to\infty}\langle F_1 x_n, y - y_n\rangle \leq \langle F_1 \overline{x}, y - \overline{y}\rangle$. They prove that the sequence $\{x_n\}$ generated by Algorithm 1.1 converges weakly to a unique solution of (1.8).

Remark 1.3. It is well-known that stepsizes play essential roles in the convergence properties of iterative methods, since the efficiency of the methods depends heavily on it. When the step size depends on the knowledge of either the operator norm or the coefficient of an operator, it usually slows down the convergence rate of the method. Moreover, in many practical cases, the operator norm or the coefficient of a given operator may not be known or may be difficult to estimate, thus, making the applicability of such method to be questionable. Therefore, iterative methods that do not depend on any of these, are more applicable in practice. From Algorithm 1.1, we have that

$$\{\omega_n\} \subset [\underline{\omega}, \overline{\omega}] \subset \left(0, \frac{1-\gamma}{\|A\|+1}\right),$$

which require computing the norm of $\|A\|$ and makes the algorithm difficult to compute and apply to real-life problems.

The inertial extrapolation method has proven to be an effective way for accelerating the rate of convergence of many iterative algorithms. The technique is based on a discrete version of a second order dissipative dynamical system [4, 3]. The inertial type algorithms use its two previous iterates to obtain its next iterate [1, 20]. For details on inertia extrapolation, see [5, 27, 28] and the references therein.
Based on Remark 1.2, Remark 1.3, the research works described above and the recent research interests in this direction, we propose a new self adaptive iterative method for solving SBVIP (1.8) that is free of the setbacks highlighted in Remark 1.2 and Remark 1.3. Furthermore, we prove that the proposed method converges strongly to a minimum norm solution of the BSVIP (1.8) in real Hilbert spaces. More so, some examples and numerical experiments were given to show the efficiency and implementation of our method in comparison with other methods in the literature in the framework of infinite dimensional Hilbert spaces. We emphasize that one of the novelties of this work is the introduction of a modified inertial technique and the removal of the weakly sequential continuity condition used by some authors to obtain strong convergence. The rest of this paper is organized as follows: In Section 2, we shall recall some useful definitions and Lemmas. In Section 3, we present our proposed method and highlight some of its features. Strong convergence analysis of our method is investigated in Section 4. Moreover, some numerical experiments to show the efficiency and implementation of our method (in comparison with other methods in the literature) are also discussed in the framework of infinite dimensional Hilbert spaces in Section 5. Lastly, in Section 6 we give a conclusion of the paper.

2. PRELIMINARIES

In this section, we begin by recalling some known and useful results which are needed in the sequel.
Let H be a real Hilbert space. The set of fixed points of a nonlinear mapping $T : H \to H$ will be denoted by $F(T)$, that is $F(T) = \{x \in H : Tx = x\}$. We denotes strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively. For any $x, y \in H$ and $\alpha \in [0, 1]$, it is well-known that

$$\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2. \tag{2.1}$$

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2. \tag{2.2}$$

$$\|x-y\|^2 \leq \|x\|^2 + 2\langle y, x-y\rangle. \tag{2.3}$$

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{2.4}$$

Definition 2.1. Let $T : H \to H$ be an operator. Then the operator T is called

(a) L-Lipschitz continuous if there exists $L > 0$ such that
$$\|Tx - Ty\| \leq L\|x - y\|,$$
for all $x, y \in H$;

(b) monotone if
$$\langle Tx - Ty, x - y\rangle \geq 0,\ \forall x, y \in H;$$

(c) α- strongly monotone on H if there exists $\alpha > 0$, such that
$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2,\ \forall\, x, y \in H.$$

(d) pseudomonotone if
$$\langle Ty, x - y\rangle \geq 0 \Rightarrow \langle Tx, x - y\rangle \geq 0,\ \forall\, x, y \in H.$$

Definition 2.2. A mapping $T : H \to H$ is said to be

(a) δ-demicontrative if $F(T) \neq 0$ and there exists $\delta \in (0, 1)$ such that
$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \delta\|x - Tx\|^2\ \forall\, x \in H,\ x^* \in F(T);$$

(b) directed if
$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 - \|x - Tx\|^2,$$
equivalently
$$\langle x^* - Tx, x - Tx\rangle \leq 0\ \forall\, x \in H,\ x^* \in F(T);$$

(c) nonexpansive it
$$\|Tx - Ty\| \leq \|x - y\|\ \forall\, x, y \in H;$$

(d) sequentially weakly continuous if for each sequence $\{x_n\}$ we have $\{x_n\}$ converges weakly to x implies that $\{Tx_n\}$ converges to Tx;

(e) demiclosed at zero if for every sequence $\{x_n\}$ contained in H, the following implications holds
$$x_n \rightharpoonup x \text{ and } (I - T)x_n \to 0$$
implies that $x \in F(T)$.

Lemma 2.1. [16]. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Given $x \in H$ and $z \in K$. Then*
$$z = P_C x \iff \langle x - z, z - y\rangle \geq 0,\ \forall\, y \in C.$$

Lemma 2.2. [16, 18]. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Given $x \in H$, then*

(a) $\|P_C x - P_C y\| \leq \langle P_C x - P_C y, x - y\rangle,\ \forall\, y \in C$;
(b) $\|x - y\| - \|x - P_C x\| \geq \|P_C x - y\|$;
(c) $\|(I - P_C)x - (I - P_C)y\|^2 \leq \langle (I - P_C)x - (I - P_C)y, x - y\rangle,\ \forall\, y \in C.$

Lemma 2.3. [18]. *Consider $VI(F_1, C)$* (1.1) *with C being a nonempty, closed and convex subset of a real Hilbert space H and $F_1 : K \to H$ being a pseudomonotone and continuous operator. Then $x^* \in VI(F_1, C)$ if and only if*
$$\langle F_1 x, x - x^*\rangle \geq 0,\ \forall\, x \in C.$$

Lemma 2.4. [2] *Let $T : H \to H$ be an operator. Then the following statements are equivalent:*

(1) *T is directed;*

(2) *there holds the relation*

$$\|x - Tx\|^2 \leq \langle x - p, x - Tx\rangle \ \forall p \in F(T),\ x \in H; \tag{2.5}$$

(3) *there holds the relation*

$$\|Tx - p\|^2 \leq \|x - p\|^2 - \|x - Tx\|^2 \ \forall p \in F(T),\ x \in H. \tag{2.6}$$

Lemma 2.5. [29] *Let* $\{a_n\}$ *be a sequence of positive real numbers,* $\{\alpha_n\}$ *be a sequence of real numbers in* $(0,1)$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\{d_n\}$ *be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n, n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ *for all subsequences* $\{a_{n_k}\}$ *of* $\{a_n\}$ *satisfying the condition*

$$\liminf_{k\to\infty}\{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then, $\lim\limits_{n\to\infty} a_n = 0$.

3. PROPOSED ALGORITHM

In this section, we present our proposed method for solving a class of bi-level split variational inequality and composed fixed point problem and highlight some of its important features.

Assumption 3.1. Condition A. *Suppose*

(1) H_1 *and* H_2 *are two real Hilbert spaces.*
(2) *The feasible set* C *is a nonempty closed and convex subset of* H_1.
(3) $\{S_n\}$ *is a sequence of nonexpansive mapping.*
(4) $A : H_1 \to H_2$ *is a bounded linear operator with the adjoint operator* A^* *and* $T : H_2 \to H_2$ *be a directed mapping, such that* T *is demiclosed at zero.*
(5) $F_1 : H_1 \to H_1$ *is pseudomonotone,* L_1*- Lipschitz continuous operator (Lipschitz constant need not to be known) and* $F_2 : H_1 \to H_1$ *is* α*-strongly monotone and* L_2*-Lipschitz continuous operator, where* $L_1, L_2 > 0$ *and* $\alpha > 0$.
(6) *The mapping* F_1 *satisfies the following; whenever*

$$\{x_n\} \subset C, x_n \rightharpoonup x^* \text{ we get } \|F_1 x^*\| \leq \liminf_{k\to\infty} \|F_1 x_n\|. \tag{3.1}$$

(7) *The solution set of problem* (1.8) *is denoted by* Ω *and* Ω *is not empty.*

Condition B. *Suppose that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences such that*

(1) $\beta_n \subset (0,1)$, $\lim\limits_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$.
(2) ϵ_n *is a positive integer such that* $\circ(\beta_n) = \epsilon_n$, $\mu \in (0,1)$, $\{\alpha_n\} \subset (a, 1-a)$ *for some* $a > 0, \alpha \geq 3$.

We present the following iterative algorithm.

Algorithm 3.2. Iterative steps: *Choose* $x_0, x_1 \in H_1$, *given the iterates* x_{n-1} *and* x_n *for all* $n \in \mathbb{N}$, *choose* θ_n *such that* $0 \leq \theta_n \leq \bar{\theta}_n$, *where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise}. \end{cases} \tag{3.2}$$

Step 1. *Set*

$$w_n = x_n + \theta_n(S_n x_n - S_n x_{n-1}).$$

Then, compute

$$y_n = w_n + \gamma_n A^*(T(Aw_n) - Aw_n), \tag{3.3}$$

where γ_n *be chosen in such a way that for some* $\varepsilon > 0$*,*

$$\gamma_n = \left(\epsilon, \frac{\|T(Aw_n) - Aw_n\|}{\|A^*(T(Aw_n) - Aw_n)\|} - \epsilon\right) \tag{3.4}$$

for $T(Aw_n) \neq Aw_n$*, otherwise* $\gamma_n = \gamma$.
Step 2. *Compute*

$$z_n = P_C(y_n - \lambda_n F_1 y_n). \tag{3.5}$$

$$v_n := y_n - \gamma_n b_n, \tag{3.6}$$

where $b_n = y_n - z_n - \lambda_n(F_1 y_n - F_1 z_n)$*,*

$$\gamma_n = \frac{\langle y_n - z_n, b_n\rangle}{\|b_n\|^2} \text{ if } b_n \neq 0, \text{ else } \gamma_n = 0. \tag{3.7}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|y_n - z_n\|}{\|F_1 y_n - F_1 z_n\|},\ \lambda_n\right\}, & \text{if } F_1 y_n \neq F_1 z_n, \\ \lambda_n & \text{otherwise}. \end{cases} \tag{3.8}$$

Step 3. *Compute*

$$x_{n+1} = \alpha_n y_n + (1 - \alpha_n) v_n - \beta_n F v_n. \tag{3.9}$$

Remark 3.1. (1) The sequentially weakly continuous assumption usually used in the literature is replaced with a weaker assumption.

(2) The extra projection onto the convex set and the projection into the half space technieques used in literature are dispensed with our new approach. In addition, comparing our algorithm with [26, 21], the implementation of our method does not require the knowledge of norm of the bounded linear operator $\|A\|$. We emphasize that this attribute is very important, because iterative algorithms that depends on the operator norm require the computation of the norm of the bounded linear operator, which in general is impossible or very difficult to compute.

(3) In Algorithm 3.2, it is easy to compute Step 1 since the value of $\|x_n - x_{n-1}\|$ is known before choosing θ_n. It is also easy to see from (3.2) that $\lim_{n\to\infty} \frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| = 0$. Since, $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\beta_n)$, which means that $\lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0$. Also $\theta_n\|x_n - x_{n-1}\| \leq \epsilon_n\ \forall\, n \in \mathbb{N}$, and with $\lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0$, yields

$$\lim_{n\to\infty} \frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0.$$

In addition, our numerical experiment (that is Section 5), we shall consider the sensitivity of θ_n in order to find numerically the optimum choice for θ_n with respect to the convergence speed of our proposed iterative method.

(4) Step 4 of our algorithm guarantee the strong convergence to the minimum norm solution of the problem.

4. CONVERGENCE ANALYSIS

In this section, we establish strong convergence result of our proposed method.

Lemma 4.1. [33] *The sequence $\{\lambda_n\}$ generated by Algorithm 3.2 is nonincreasing and*

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \{\frac{\mu}{L_1}, \lambda_1\}. \tag{4.1}$$

Lemma 4.2. *The setepsize sequence γ_n defined by* (3.4) *is well defined.*

Proof. Let $p \in \Omega$, then $Ap \in F(T)$, since T is a directed mapping and $F(T) \neq \emptyset$, using (2.5), we obtain

$$\begin{aligned} \|A^*(T-I)Aw_n\|\|w_n - p\| &\geq \langle A^*(T-I)Aw_n, w_n - p\rangle \\ &= \langle (T-I)Aw_n, Aw_n - Ap\rangle \\ &\geq \|(T-I)Aw_n\|^2. \end{aligned} \tag{4.2}$$

Since $T(Aw_n) \neq Aw_n$, then $\|(T-I)Aw_n\| > 0$, then $\|w_n - p\|\|A^*(T-I)Aw_n\| > 0$, hence, $\|A^*(T-I)Aw_n\| \neq 0$. Therefore, γ_n is well defined. ∎

Lemma 4.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2. Then, under Assumption 3.1, we have that $\{x_n\}$ is bounded.*

Proof. Let $p \in \Omega$ and since $\lim_{n\to\infty} \frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| = 0$, there exists $N_1 > 0$ such that $\frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| \leq N_1$, for all $n \in \mathbb{N}$. Then from **Step** 2, we have

$$\begin{aligned} \|w_n - p\| &= \|x_n + \theta_n(S_n x_n - S_n x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \theta_n\|S_n x_n - S_n x_{n-1}\| \\ &\leq \|x_n - p\| + \beta_n\frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| \\ &\leq \|x_n - p\| + \beta_n N_1. \end{aligned} \tag{4.3}$$

Also, using the fact that T is directed mapping, (2.2), $F(T) \neq \emptyset$, (2.5) and the stepsize of γ_n in (3.4), we have

$$\begin{aligned} \|y_n - p\|^2 &= \|w_n + \gamma_n A^*(T-I)Aw_n p\|^2 \\ &= \|w_n - p\|^2 + \gamma_n^2\|T^*(J_\lambda^{B_2} - I)Tw_n\|^2 + 2\gamma_n\langle w_n - p, A^*(T-I)Aw_n\rangle \\ &= \|w_n - p\|^2 + \gamma_n^2\|A^*(T-I)Aw_n\|^2 + 2\gamma_n\langle A(w_n - p), (T-I)Aw_n\rangle \\ &\leq \|w_n - p\|^2 + \gamma_n^2\|A^*(T-I)Aw_n\|^2 - \gamma_n\|(T-I)Aw_n\|^2 \\ &= \|w_n - p\|^2 - \gamma_n(\|T-I)Aw_n\|^2 - \gamma_n\|A^*(T-I)Aw_n\|^2) \\ &\leq \|w_n - p\|^2 - \gamma_n\varepsilon\|A^*(T-I)Aw_n\|^2 \\ &\leq \|w_n - p\|^2. \end{aligned} \tag{4.4}$$

Since $z_n = P_C(y_n - \lambda F_1 y_n)$, then by the characteristics of the P_C, we obtain

$$\langle y_n - z_n - \lambda F_1 y_n, z_n - p\rangle \geq 0. \tag{4.5}$$

Also, since $p \in VI(F_1, C)$ and $z_n \in C$, we obtain

$$\langle F_1 p, z_n - p\rangle \geq 0,$$

thus using the pseudomonotonicity of F_1, we obtain

$$\langle F_1 z_n, z_n - p\rangle \geq 0. \tag{4.6}$$

From (4.5) and (4.6), we obtain

$$\langle z_n - p, y_n - z_n - \lambda_n(F_1 y_n - F_1 z_n)\rangle \geq 0. \tag{4.7}$$

Thus, we have

$$\begin{aligned}\langle y_n - p, b_n\rangle &= \langle y_n - z_n, b_n\rangle + \langle z_n - p, b_n\rangle \\ &= \langle y_n - z_n, b_n\rangle + \langle z_n - p, y_n - z_n - \lambda_n(F_1 y_n - F_1 z_n)\rangle \\ &\geq \langle y_n - z_n, b_n\rangle \end{aligned}\tag{4.8}$$

From Step 3 of Algorithm 3.2 and (4.8), we have

$$\begin{aligned}\|v_n - p\|^2 &= \|y_n - \gamma_n b_n - p\|^2 \\ &= \|y_n - p\|^2 + \gamma_n^2\|b_n\|^2 - 2\gamma_n\langle y_n - p, b_n\rangle \\ &\leq \|y_n - p\|^2 + \gamma_n^2\|b_n\|^2 - 2\gamma_n\langle y_n - z_n, b_n\rangle \\ &= \|y_n - p\|^2 + \gamma_n^2\|b_n\|^2 - 2\gamma_n^2\|b_n\|^2 \\ &= \|y_n - p\|^2 - \|\gamma_n b_n\|^2 \\ &= \|y_n - p\|^2 - \|v_n - y_n\|^2 \\ &\leq \|y_n - p\|^2, \end{aligned}\tag{4.9}$$

which implies

$$\|v_n - p\| \leq \|y_n - p\|. \tag{4.10}$$

Now, observe

$$\begin{aligned}&\|[(1-\alpha_n)v_n - \beta_n F_2 v_n] - [(1-\alpha_n)p - \beta_n F_2 p]\| \\ &\leq (1-\alpha_n-\beta_n)\|v_n - p\| + \beta_n\|(v_n - p) - (F_2 v_n - F_2 p)\|, \end{aligned}\tag{4.11}$$

using the fact that F_2 is L_2-Lipschitz continuous and α-strongly monotone on H_1, we have that

$$\begin{aligned}\|(v_n - p) - (F_2 v_n - F_2 p)\|^2 &= \|v_n - p\|^2 - 2\langle v_n - p, F_2 v_n - F_2 p\rangle + \|F_2 v_n - F_2 p\|^2 \\ &\leq \|v_n - p\|^2 - 2\alpha\|v_n - p\|^2 + L_2^2\|v_n - p\|^2 \\ &= (1 - 2\alpha + L_2^2)\|v_n - p\|^2, \end{aligned}\tag{4.12}$$

which implies that

$$\|(v_n - p) - (F_2 v_n - F_2 p)\| \leq \sqrt{(1 - 2\alpha_n + L_2^2)}\|v_n - p\|.$$

Thus, we have (4.11) become

$$\begin{aligned}&\|[(1-\alpha_n)v_n - \beta_n F_2 v_n] - [(1-\alpha_n)p - \beta_n F_2 p]\| \\ &\leq (1-\alpha_n-\beta_n)\|v_n - p\| + \beta_n\sqrt{(1 - 2\alpha + L_2^2)}\|v_n - p\| \\ &= (1-\alpha_n-\beta_n + \beta_n\sqrt{1 - (2\alpha - L_2^2)})\|v_n - p\| \\ &= (1-\alpha_n-\beta_n(1 - \sqrt{1 - (2\alpha - L_2^2)})\|v_n - p\| \\ &= (1-\alpha_n-\beta_n\tau)\|v_n - p\|, \end{aligned}\tag{4.13}$$

where $\tau = 1 - \sqrt{1 - (2\alpha - L_2^2)} \in (0, 1)$.

Now, using Step 4 of Algorithm 3.2 and (4.13), we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n y_n + (1-\alpha_n)v_n - \beta_n F v_n - p\| \\ &= \|\alpha_n(y_n-p) + (1-\alpha_n)v_n - \beta_n F v_n - (1-\alpha_n)p + \beta_n Fp - \beta_n Fp\| \\ &\leq \|[(1-\alpha_n)v_n - \beta_n F v_n] - [(1-\alpha_n)p - \beta_n Fp]\| + \alpha_n\|y_n-p\| + \beta_n\|Fp\| \\ &\leq (1-\alpha_n-\beta_n\tau)\|v_n-p\| + \alpha_n\|y_n-p\| + \beta_n\|Fp\| \\ &\leq (1-\alpha_n-\beta_n\tau)\|y_n-p\| + \alpha_n\|y_n-p\| + \beta_n\|Fp\| \\ &\leq (1-\beta_n\tau)\|y_n-p\| + \beta_n\|Fp\| \\ &\leq (1-\beta_n\tau)\|w_n-p\| + \beta_n\|Fp\| \\ &\leq (1-\beta_n\tau)\|x_n-p\| + \beta_n N_1 + \beta_n\|Fp\| \\ &\leq (1-\beta_n\tau)\|x_n-p\| + \tau\beta_n\frac{(N_1+\|Fp\|)}{\tau} \qquad (4.14)\end{aligned}$$

It follows by induction

$$\|x_n-p\| \leq \max\{\|x_0-p\|, \frac{N_1+\nu\|Fp\|}{\tau}\}. \qquad (4.15)$$

Thus, we have $\{x_n\}$ is bounded. ∎

Lemma 4.4. *Let Assumption 3.1 hold and let $\{x_n\}$ be a sequence generated by Algorithm 3.2. Assume that the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to a point x^*, and $\lim_{k\to\infty}\|y_{n_k} - w_{n_k}\| = \lim_{k\to\infty}\|y_{n_k}-z_{n_k}\| = 0$, then, $x^* \in \Gamma$.*

Proof. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ which converges weakly to $x^* \in H$. It is easy to see that

$$\|w_{n_k}-x_{n_k}\| = \alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_n-1}\| \to 0 \text{ as } k\to\infty. \qquad (4.16)$$

It follows that

$$\|y_{n_k}-x_{n_k}\| \leq \|y_{n_k}-w_{n_k}\| + \|w_{n_k}-x_{n_k}\| \to 0 \text{ as } k\to\infty. \qquad (4.17)$$

Since A is a bounded linear operator, it follows from (4.16) that $\{Aw_{n_k}\}$ converges weakly to $Ax^* \in H_2$. Also, by (4.17), we obtain that y_{n_k} converges weakly to x^*. In addition, we have

$$\|z_{n_k}-x_{n_k}\| \leq \|z_{n_k}-y_{n_k}\| + \|y_{n_k}-x_{n_k}\| \to 0 \text{ as } k\to\infty. \qquad (4.18)$$

From (4.4), we have that

$$\begin{aligned}\|y_n-p\|^2 &\leq \|w_n-p\|^2 - \gamma_n\epsilon\|A^*(T(Aw_n)-Aw_n)\|^2 \\ &\leq \|w_n-p\|^2 - \epsilon^2\|A^*(T(Aw_n)-Aw_n)\|^2 \qquad (4.19)\end{aligned}$$

which implies that

$$\begin{aligned}\epsilon^2\|A^*(T(Aw_{n_k})-Aw_{n_k})\|^2 &\leq \|w_{n_k}-p\|^2 - \|y_{n_k}-p\|^2 \\ &\leq \|w_{n_k}-y_{n_k}\|^2 + 2\|y_{n_k}-p\|\|w_{n_k}-y_{n_k}\|, \qquad (4.20)\end{aligned}$$

thus, we have that

$$\lim_{k\to\infty}\|A^*(T(Aw_{n_k})-Aw_{n_k})\| = 0. \qquad (4.21)$$

Also from (4.4), we have

$$\begin{aligned}\|y_n-p\|^2 &\leq \|w_n-p\|^2 + \gamma_n^2\|A^*(T(Aw_n)-Aw_n)\|^2 - \gamma_n\|T(Aw_n)-Aw_n\|^2 \\ &\leq \|w_n-p\|^2 + \epsilon^2\|A^*(T(Aw_n)-Aw_n)\|^2 - \epsilon\|T(Aw_n)-Aw_n\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}
&\epsilon\|T(Aw_{n_k}) - Aw_{n_k}\|^2 \\
&\leq \|w_{n_k} - p\|^2 - \|y_{n_k} - p\|^2 + \epsilon^2\|A^*(T(Aw_{n_k}) - Aw_{n_k})\|^2 \\
&\leq \|w_{n_k} - y_{n_k}\|^2 + 2\|y_{n_k} - p\|\|w_{n_k} - y_{n_k}\| + \epsilon^2\|A^*(T(Aw_{n_k}) - Aw_{n_k})\|^2,
\end{aligned}$$

thus, using (4.21), we have that

$$\lim_{k\to\infty} \|T(Aw_{n_k}) - Aw_{n_k}\| = 0. \tag{4.22}$$

Thus, using demicloseness property and (4.22), we have

$$Ax^* \in F(T). \tag{4.23}$$

In addition, by the definition of $\{z_n\}$ and Lemma 2.1, that

$$\langle y_{n_j} - \lambda_{n_j} F_1 y_{n_j} - z_{n_j}, v - z_{n_j}\rangle \leq 0, \ \forall\, v \in C,$$

which implies

$$\begin{aligned}
\langle y_{n_j} - z_{n_j}, v - z_{n_j}\rangle &\leq \lambda_{n_j}\langle F_1 y_{n_j}, v - z_{n_j}\rangle \\
&= \lambda_{n_j}\langle F_1 y_{n_j}, y_{n_k} - z_{n_j}\rangle + \lambda_{n_j}\langle F_1 y_{n_j}, v - z_{n_j}\rangle.
\end{aligned} \tag{4.24}$$

As such, we have

$$\frac{1}{\lambda_{n_j}}\langle y_{n_j} - z_{n_j}, v - z_{n_k}\rangle + \langle F_1 y_{n_j}, z_{n_j} - y_{n_j}\rangle \leq \langle F_1 y_{n_j}, v - y_{n_j}\rangle, \ \forall\, v \in C. \tag{4.25}$$

Since $\{y_{n_j}\}$ is converges weakly to a point $x^* \in H_1$, thus, it is bounded. Then, since F_1 is Lipschitz continuous, $\{F_1 y_{n_j}\}$ is bounded. In addition, we have that $\{z_{n_j}\}$ is bounded since $\|y_{n_j} - z_{n_j}\| \to 0$ as $j \to \infty$ and $\lambda_{n_j} \in \min\left\{\lambda_1, \frac{\mu}{L}\right\}$. Taking limit as $j \to \infty$ in (4.25) we obtain

$$\liminf_{j\to\infty}\langle F_1 y_{n_j}, v - y_{n_j}\rangle \geq 0. \tag{4.26}$$

Now, note that

$$\begin{aligned}
\langle F_1 z_{n_j}, v - z_{n_j}\rangle &= \langle F_1 z_{n_j}, v - y_{n_j}\rangle + \langle F_1 z_{n_j}, y_{n_j} - z_{n_j}\rangle \\
&= \langle F_1 z_{n_j} - F_1 y_{n_j}, v - y_{n_j}\rangle + \langle F_1 y_{n_j}, v - y_{n_j}\rangle + \langle F_1 z_{n_j}, y_{n_k} - z_{n_j}\rangle.
\end{aligned} \tag{4.27}$$

Using the fact that $\lim\limits_{j\to\infty} \|y_{n_j} - z_{n_j}\| = 0$ and the continuity of F_1, we have

$$\lim_{j\to\infty} \|F_1 y_{n_j} - F_1 z_{n_j}\| = 0. \tag{4.28}$$

Thus, using (4.26),(4.27) and (4.28), we have

$$\liminf_{j\to\infty}\langle F_1 y_{k_j}, v - y_{k_j}\rangle \geq 0. \tag{4.29}$$

We choose a subsequence $\{\epsilon_j\}$ of positive number decreasing in $(0,1)$, such that $\epsilon_j \to 0$ as $j \to \infty$. For each j, let N_j be the smallest nonnegative integer such that

$$\langle F_1 z_{n_i}, v - z_{n_i}\rangle + \epsilon_j \geq 0, \ \forall\, i \geq N_j. \tag{4.30}$$

Since $\{\epsilon_j\}$ is decreasing, it is obvious that N_j is increasing. Further, for each $j \in \mathbb{N}$, the subsequence $\{z_{N_j}\} \subset C$ we obtain $F_1 z_{N_j} \neq 0$ so that z_{N_j} is not a solution of the $VIP(C, F_1)$. Now, we set

$$\nu_{N_j} = \frac{F_1 z_{N_j}}{\|F_1 z_{N_j}\|^2},$$

such that $\langle F_1 z_{N_j}, \nu_{N_j}\rangle = 1$ for each j. It follows from this and (4.30), that $\langle F_1 z_{N_j}, v + \epsilon_j \nu_{N_j} - z_{N_j}\rangle \geq 0$. Since F_1 is pseudomonotone, we have $\langle F_1(v+\epsilon_j\nu_{N_j}), v+\epsilon_j\nu_{N_j} - z_{N_j}\rangle \geq 0$ and thus

$$\langle F_1 v, v - z_{N_j}\rangle \geq \langle F_1 v - F_1(v+\epsilon_j\nu_{N_j}), v+\epsilon_j\nu_{N_j} - z_{N_j}\rangle - \epsilon_j\langle F_1 v, \nu_{N_j}\rangle. \tag{4.31}$$

Next, we show that $\epsilon_j\nu_{N_j} \to 0$ as $j \to \infty$. To see this, using our hypothesis, that is $\lim_{k\to\infty}\|z_{n_k} - y_{n_k}\| = 0$, we have $\{z_{n_k}\}$ converges weakly to x^*. Since $\{z_{n_k}\} \subset C$ and C is closed, then $x^* \in C$. We suppose that $F_1 x^* \neq 0$, if not, we obtain that x^* is a solution. Now, using condition (3.1), we obtain

$$0 < \|F_1 x^*\| \leq \liminf_{j\to\infty} \|F_1 z_{N_j}\|. \tag{4.32}$$

More so, using the fact that $\{z_{N_j}\} \subset \{z_{n_j}\}$ and $\epsilon_j \to 0$ as $j \to \infty$, we get

$$0 \leq \limsup_{j\to\infty} \|\epsilon_j \nu_{N_j}\| = \limsup_{j\to\infty}\left(\frac{\epsilon_j}{\|F_1 z_{n_j}\|}\right) \leq \frac{0}{\|F_1 x^*\|} = 0,$$

that is

$$\lim_{j\to\infty} \|\epsilon_j\nu_{N_j}\| = 0.$$

Thus from (4.31), we have

$$\liminf_{j\to\infty}\langle F_1 v, v - z_{N_j}\rangle \geq 0.$$

Therefore, for all $v \in C$, we have

$$\langle F_1 v, v - x^*\rangle = \lim_{j\to\infty}\langle F_1 v, v - z_{N_j}\rangle = \liminf_{j\to\infty}\langle F_1 v, v - z_{N_j}\rangle \geq 0.$$

Hence, by Lemma 2.3 we have $x^* \in VI(F_1, C)$. The proof is thus complete. ∎

Theorem 4.5. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.2. Then, under the Assumption 3.1, if $\lim\limits_{n\to\infty}\beta_n = 0, \sum_{n=1}^{\infty}\beta_n = \infty$. Then, $\{x_n\}$ converges strongly to $p \in \Omega$, where $\|p\| = \min\{\|x^*\| : x^* \in \Omega\}$.*

Proof. Let $p \in \Omega$, observe that

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \theta_n(S_n x_n - S_n x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + 2\theta_n\langle x_n - p, S_n x_n - S_n x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - p\|^2 + 2\theta_n\|S_n x_n - S_n x_{n-1}\|\|x_n - p\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \theta_n\|x_n - x_{n-1}\|] \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \beta_n\frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\|] \\
&\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \alpha_n N_1] \\
&\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2, \qquad (4.33)
\end{aligned}$$

for some $N_2 > 0$. Furthermore, we have that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n y_n + (1-\alpha_n)v_n - \beta_n F_2 v_n - p\|^2 \\
&= \|\alpha_n(y_n - p) + (1-\alpha_n)v_n - \beta_n F v_n - (1-\alpha_n)p + \beta_n Fp - \beta_n Fp\|^2 \\
&= \|[(1-\alpha_n)v_n - \beta_n F v_n] - [(1-\alpha_n)p - \beta_n Fp] + \alpha_n(y_n - p) + \beta_n Fp\|^2 \\
&= \Big(\|[(1-\alpha_n)v_n - \beta_n F v_n] - [(1-\alpha_n)p - \beta_n Fp]\| + \alpha_n\|y_n - p\|\Big)^2 \\
&\quad + 2\beta_n\langle Fp, p - x_{x_{n+1}}\rangle \\
&\le \Big([1-\alpha_n - \beta_n\tau]\|v_n - p\| + \alpha_n\|y_n - p\|\Big)^2 + 2\beta_n\langle Fp, p - x_{x_{n+1}}\rangle \\
&\le [1-\alpha_n - \beta_n\tau]\|v_n - p\|^2 + \alpha_n\|y_n - p\|^2 + 2\beta_n\langle Fp, p - x_{x_{n+1}}\rangle \\
&\le [1-\alpha_n - \beta_n\tau]\|y_n - p\|^2 + \alpha_n\|y_n - p\|^2 + 2\beta_n\langle Fp, p - x_{x_{n+1}}\rangle \\
&\le [1-\beta_n\tau]\|y_n - p\|^2 + 2\beta_n\langle Fp, p - x_{x_{n+1}}\rangle \\
&\le [1-\beta_n\tau]\|w_n - p\|^2 + 2\beta_n\langle Fp, p - x_{x_{n+1}}\rangle \\
&\le [1-\beta_n\tau]\|x_n - p\|^2 + [1-\beta_n\tau]\theta_n\|x_n - x_{n-1}\|N_2 + 2\beta_n\langle Fp, p - x_{x_{n+1}}\rangle \\
&\le [1-\beta_n\tau]\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2 + 2\beta_n\langle Fp, p - x_{x_{n+1}}\rangle \\
&= [1-\beta_n\tau]\|x_n - p\|^2 + \beta_n\tau\left(\frac{\theta_n}{\beta_n\tau}\|x_n - x_{n-1}\|N_1 + \frac{2}{\tau}\langle F_2 p, p - x_{x_{n+1}}\rangle\right) \\
&= [1-\beta_n\tau]\|x_n - p\|^2 + \beta_n\tau\Psi_n, \qquad (4.34)
\end{aligned}$$

where $\Psi_n = \frac{\theta_n}{\beta_n\tau}\|x_n - x_{n-1}\|N_1 + \frac{2}{\tau}\langle Fp, p - x_{x_{n+1}}\rangle$. According to Lemma 2.5, to conclude our proof, it is sufficient to establish that $\limsup_{k\to\infty}\Psi_n \le 0$ for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying the condition:

$$\liminf_{k\to\infty}\{\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\} \ge 0. \qquad (4.35)$$

From (4.34) and (4.9), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2 - [1-\alpha_n-\beta_n\tau]\|v_n - y_n\|^2 \\
+ 2\beta_n\langle Fp, p - x_{x_{n+1}}\rangle,
\end{aligned}$$

(4.36)

this implies that

$$\begin{aligned}
[1-\alpha_n - \beta_n\tau]\|v_n - y_n\|^2 &\le \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2 \\
&\quad - \|x_{n+1} - p\|^2 + 2\beta_n\langle Fp, p - x_{x_{n+1}}\rangle. \qquad (4.37)
\end{aligned}$$

Thus, we have

(4.38)

$$\begin{aligned}
\limsup_{k\to\infty}\Big([1-\alpha_{n_k} - \beta_{n_k}\tau]\|v_{n_k} - y_{n_k}\|^2\Big) &\le \limsup_{k\to\infty}\Big[\|x_{n_k} - p\|^2 + \beta_{n_k}\frac{\theta_{n_k}}{\beta_{n_k}}\|x_{n_k} - x_{n_k-1}\|N_2 \\
&\quad + 2\beta_{n_k}\langle F_2 p, p - x_{n_k+1}\rangle - \|x_{n_k+1} - p\|^2\Big] \qquad (4.39) \\
&\le -\liminf_{k\to\infty}[\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2] \le 0, \qquad (4.40)
\end{aligned}$$

as such, we have that

$$\lim_{k\to\infty} \|v_{n_k} - y_{n_k}\| = 0. \tag{4.41}$$

From Step 3 of Algorithm 3.2, we obtain that

$$\begin{aligned}
\|b_{n_k}\| &= \|y_{n_k} - z_{n_k} - \lambda_n(F_1 y_{n_k} - F_1 z_{n_k})\| \\
&\le \|y_{n_k} - z_{n_k}\| + \lambda_n \|F_1 y_{n_k} - F_1 z_{n_k}\| \\
&\le (1 + \frac{\lambda_n \mu}{\lambda_{n+1}})\|y_{n_k} - z_{n_k}\|,
\end{aligned} \tag{4.42}$$

also, we have

$$\begin{aligned}
\langle y_{n_k} - z_{n_k}, b_{n_k}\rangle &= \langle y_{n_k} - z_{n_k}, y_{n_k} - z_{n_k} - \lambda(Ay_{n_k} - Az_{n_k})\rangle \\
&= \|y_{n_k} - z_{n_k}\|^2 - \lambda\langle y_{n_k} - z_{n_k}, Ay_{n_k} - Az_{n_k}\rangle \\
&\ge \|y_{n_k} - z_{n_k}\|^2 - \lambda_n \|y_{n_k} - z_{n_k}\|\|F_1 y_{n_k} - F_1 z_{n_k}\| \\
&\ge (1 - \frac{\lambda_n \mu}{\lambda_{n+1}})\|y_{n_k} - z_{n_k}\|^2.
\end{aligned} \tag{4.43}$$

In addition, from the definition of v_n, we have that

$$\begin{aligned}
\|y_{n_k} - v_{n_k}\| = \gamma_{n_k}\|b_{n_k}\| = \frac{\langle y_{n_k} - z_{n_k}, b_{n_k}\rangle}{\|b\|} &\ge \frac{\lambda_{n_k+1} - \mu\lambda_{n_k}}{\lambda_{n_k+1} + \lambda_{n_k}\mu}\|y_{n_k} - z_{n_k}\|, \\
\Rightarrow \|y_{n_k} - z_{n_k}\| &\le \frac{\lambda_{n_k+1} + \mu\lambda_{n_k}}{\lambda_{n_k+1} - \mu\lambda_{n_k}}\|v_{n_k} - y_{n_k}\|.
\end{aligned} \tag{4.44}$$

From Lemma 4.1, we obtain

$$\liminf_{k\to\infty} \frac{\lambda_{n_k+1} + \mu\lambda_{n_k}}{\lambda_{n_k+1} - \mu\lambda_{n_k}} = \frac{1+\mu}{1-\mu}, \tag{4.45}$$

thus, $\{\frac{\lambda_{n_k+1} + \mu\lambda_{n_k}}{\lambda_{n_k+1} - \mu\lambda_{n_k}}\}$ is bounded and (4.41), we have

$$\lim_{k\to\infty} \|y_{n_k} - z_{n_k}\| = 0. \tag{4.46}$$

It is easy to see from (4.34) that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le \|y_n - p\|^2 + 2\beta_n\langle Fp, p - x_{n+1}\rangle \\
&\le \|w_n - p\|^2 - \epsilon^2\|A^*(T-I)Aw_n\|^2 + 2\beta_n\langle Fp, p - x_{n+1}\rangle \\
&\le \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2 - \epsilon^2\|A^*(T-I)Aw_n\|^2 + 2\beta_n\langle Fp, p - x_{n+1}\rangle,
\end{aligned} \tag{4.47}$$

which implies

$$\begin{aligned}
&\limsup_{k\to\infty} \epsilon^2\|A^*(T-I)Aw_n\|^2 \\
&\le \limsup_{k\to\infty}\Bigg[\|x_{n_k} - p\|^2 + \beta_{n_k}\frac{\theta_{n_k}}{\beta_{n_k}}\|x_{n_k} - x_{n_k-1}\|N_2 \\
&+ 2\beta_{n_k}\langle Fp, p - x_{n_k+1}\rangle - \|x_{n_k+1} - p\|^2\Bigg] \\
&\le -\liminf_{k\to\infty}[\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2] \le 0.
\end{aligned} \tag{4.48}$$

We obtain

$$\lim_{k\to\infty}\|A^*(T-I)Aw_{n_k}\|=0. \tag{4.49}$$

Thus, using (4.2), (4.49) and and the boundedness of $\{w_n\}$, we obtain

$$\lim_{k\to\infty}\|(T-I)Aw_{n_k}\|=0. \tag{4.50}$$

Using (4.49), we have that

$$\|y_{n_k}-w_{n_k}\|=\gamma_n\|A^*(T(Aw_{n_k})-Aw_{n_k})\|\to 0 \text{ as } k\to\infty. \tag{4.51}$$

It is easy to see that, as $k\to\infty,$ we have

$$\|w_{n_k}-x_{n_k}\|=\theta_{n_k}||x_{n_k}-x_{n_k-1}||=\alpha_{n_k}\cdot\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k}-x_{n_k-1}||\to 0. \tag{4.52}$$

It is easy to see that, as $k\to\infty,$ we have

$$\|w_{n_k}-x_{n_k}\|=\theta_{n_k}||x_{n_k}-x_{n_k-1}||=\beta_{n_k}\cdot\frac{\theta_{n_k}}{\beta_{n_k}}||x_{n_k}-x_{n_k-1}||\to 0. \tag{4.53}$$

In addition, we have that

$$\|y_{n_k}-x_{n_k}\|\le\|y_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k}\|\to 0 \text{ as } k\to\infty.$$

$$\|z_{n_k}-x_{n_k}\|\le\|z_{n_k}-y_{n_k}\|+\|y_{n_k}-x_{n_k}\|\to 0 \text{ as } k\to\infty.$$

$$\|v_{n_k}-x_{n_k}\|\le\|v_{n_k}-y_{n_k}\|+\|y_{n_k}-x_{n_k}\|\to 0 \text{ as } k\to\infty.$$

$$\|v_{n_k}-y_{n_k}\|\le\|v_{n_k}-x_{n_k}\|+\|x_{n_k}-y_{n_k}\|\to 0 \text{ as } k\to\infty.$$

$$\|x_{n_k+1}-y_{n_k}\|\le(1-\alpha_{n_k})\|v_{n_k}-y_{n_k}\|+\beta_{n_k}\|F_2v_{n_k}\|\to 0 \text{ as } k\to\infty.$$

$$\|x_{n_k+1}-x_{n_k}\|\le\|x_{n_k+1}-y_{n_k}\|+\|y_{n_k}-x_{n_k}\|\to 0 \text{ as } k\to\infty.$$

Since $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ converges weakly to x^*. In addition, from (4.53), we obtain that $\{Tw_{n_k}\}$ converges weakly to Tx^* and with (4.50) and the demiclosedness principle, we have

$$Ax^*\in F(T).$$

More so, we have

$$\limsup_{k\to\infty}\langle F_2p,p-x_{n_k}\rangle=\lim_{j\to\infty}\langle F_2p,p-x_{n_{k_j}}\rangle=\langle F_2p,p-x^*\rangle.$$

Also, we obtain from (4.46), (4.51) and Lemma 4.4 that $x^*\in\Omega$. Since p is a unique solution of $\Omega,$ we have obtain from (4) that

$$\limsup_{k\to\infty}\langle F_2p,p-x_{n_k}\rangle=\langle F_2p,p-x^*\rangle\le 0,$$

which implies that

$$\limsup_{k\to\infty}\langle F_2p,p-x_{n_k+1}\rangle\le 0,$$

Using using our assumption, (4.41) and the above inequality, we have that $\limsup_{k\to\infty}\Psi_{n_k}=\limsup_{k\to\infty}\left(\frac{\theta_{n_k}}{\beta_{n_k}\tau}\|x_{n_k}-x_{n_k-1}\|N_1+\frac{2}{\tau}\langle Fp,p-x_{x_{n_k+1}}\rangle\right)\le 0$. Thus, the last part of Lemma

2.5 is achieved. Hence, we have that $\lim_{n\to\infty} \|x_n - p\| = 0$. Thus, $\{x_n\}$ converges strongly to $p \in \Omega$. ∎

5. NUMERICAL EXAMPLES

In this section, we give some numerical examples in both finite and infinite dimensional Hilbert spaces.

Example 5.1. *Let $H_1 = \mathbb{R}^4$ be the four-dimensional Euclidean space of the real number with a norm defined by $\|x\| = \sqrt{\sum_{i=1}^{4} \|x_i\|^2}$ where $x = \{x_i\}_{i=1}^4 \in \mathbb{R}^4$ and $H_2 = \mathbb{R}^2$ be the two-dimensional Euclidean space of the real number with a norm defined by $\|x\| = \sqrt{\sum_{i=1}^{2} \|x_i\|^2}$ where $x = \{x_i\}_{i=1}^2 \in \mathbb{R}^2$. Define the feasible set C by $C := \{x \in \mathbb{R}^4 : \|x\| \leq 1\}$ where $x = \{x_i\}_{i=1}^\infty$. Consider the mapping $F_1 : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $F_1(x) = (\sin\|x\| + 2)b$, where $b = (12, -4, 4, -4)^T$ for all $x \in \mathbb{R}^4$. It is easy to verify that F_1 is pseudomonotone with $8\sqrt{3}$ Lipschitz constant (see* [26]*). Also, let the mapping $F_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F_2(x) = (x_1, x_2, x_3, x_4)^T$ for all $x = \{x_i\}_{i=1}^2$. It is obvious that F_2 is 1-strongly monotone with a Lipschitz constant 1. Now, define the operator $A : \mathbb{R}^4 \to \mathbb{R}^2$ by $A(x) = (2x_1 + x_2 + x_3 + 3x_4, x_1 + x_2 + x_3 + 2x_4)$, then A is a bounded linear operator with $\|A\| = \frac{1955}{419}$. Let the mapping $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $S(x) = \frac{2x}{3}$ for all $x = (x_1, x_2)^T \in \mathbb{R}^2$. For this example, we choose the following parameters, $\lambda_0 = 0.5$, $\alpha_n = \frac{n+1}{4n+17}$, $\beta_n = \frac{1}{13n+5}$, $\mu = 0.5$. Also, $\alpha = 5$, $\epsilon = \frac{1}{n^{1.2}}$. We make a comparison of our method with Algorithm 1* [26]*, with the following extra conditions $w_n \in [\bar{w}, \underline{w}] = [\frac{209}{2380}, \frac{209}{2390}]$ and $\lambda_n = \frac{n+1}{16n+18}$. Let $\|x_{n+1} - x_n\|^2 = 10^{-5}$, we consider the following cases of initial values of x_0 and x_1;*

Case 1 $x_0 = (2, 3, 9, 8)^T$ *and* $x_1 = (10, 3, 4, 5)^T$;
Case 2 $x_0 = (0.10, 0.20, 0.30, 0.40)^T$ *and* $x_1 = (0.15, 0.60, 0.50, 0.70)^T$;
Case 3 $x_0 = (10, 20, 35, 20)^T$ *and* $x_1 = (15, 60, -35, 40)^T$.

The results of this experiment are reported in Figure 1.

Example 5.2. *Let $H_1 = H_2 = \ell_2$ be the linear space whose elements consist of all 2-summable sequences $(x_1, x_2, \cdots, x_i, \cdots)$ scalars, that is*

$$\ell_2 = \left\{ x : x = (x_1, x_2, \cdots, x_i, \cdots) \text{ and } \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\},$$

with an inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$ where $x = \{x_i\}_{i=1}^\infty$ and $y = \{y_i\}_{i=1}^\infty$ and a norm $\|\cdot\| : \ell_2 \to \mathbb{R}$ defined by $\|x\|_2 = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$ where $x = \{x_i\}_{i=1}^\infty$. Define the feasible set C by $C := \{x \in \ell_2 : \|x\| \leq 1\}$ where $x = \{x_i\}_{i=1}^\infty$. Consider the mapping $F_1 : \ell_2 \to \ell_2$ defined by $F_1(x) = (\sin\|x\| + 2)b$, where $b = (1, 0, \cdots, 0, \cdots)^T$ for all $x \in \ell_2$. It is easy to verify that F_1 is pseudomonotone (see [26]*). Also, let the mapping $F_2 : \ell_2 \to \ell_2$ be defined by $F_2(x) = x$ for all $x = \{x_i\}_{i=1}^\infty$. It is obvious that F_2 is 1-strongly monotone with a Lipschitz constant 1. Now, define the operator $A : \ell_2 \to \ell_2$ by $A(x) = (x_1, x_2, \cdots, x_i, \cdots)$, then A is a bounded linear operator with $\|A\| = 3$. Let the mapping $S : \ell_2 \to \ell_2$ be defined by $S(x) = \frac{2x}{3}$ for all $x = \{x_i\}_{i=1}^\infty \in \ell_2$. For this example, we choose parameters as in Example 5.1 with $\lambda_0 = 2$.*

I $x_0 = (1, 0, \cdots, 3, \cdots)^T$ *and* $x_1 = (2, 0, \cdots, 3, \cdots)^T$;

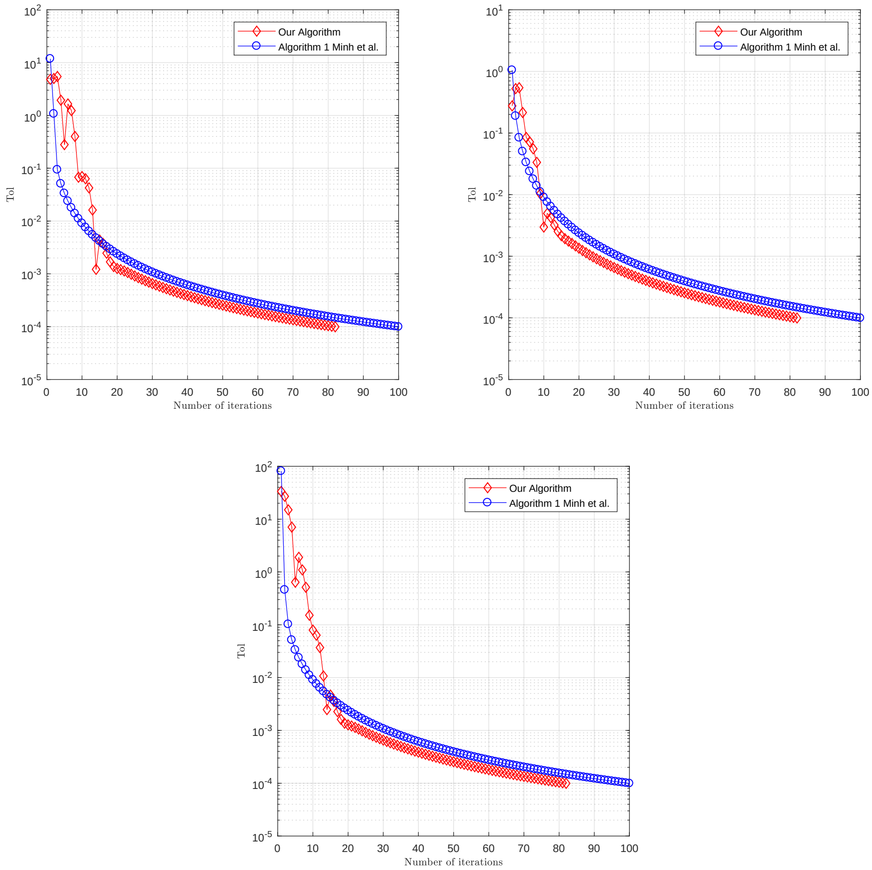

Figure 1: Example 5.1. Top left: Case 1, Top right: Case 2, Bottom: Case 3.

II $x_0 = (0.876, 0.567, 0, \cdots, 0, \cdots)^T$ *and* $x_1 = (0.576, 0.333, 0, \cdots, 0, \cdots)^T$;

III $x_0 = (10, 20, \cdots, 0, \cdots)^T$ *and* $x_1 = (15, 10, \cdots, 0, \cdots)^T$.

The results of this experiment are reported in Figure 2.

6. CONCLUSION

A modified inertial extrapolation projection and contraction iterative method is introduced and studied for solving the SBVIP (1.8) in two real Hilbert spaces in which one of the cost operators is pseudomonotone and Lipschitz continuous. As seen from our convergence analysis, we prove that the proposed algorithm converges strongly to the unique solution of the SBVIP (1.8). Lastly, we considered some numerical examples of our proposed method in comparison with the iterative method proposed by Minh et al. The numerical experiments validate that our iterative method converges faster and its more applicable to real life situation.

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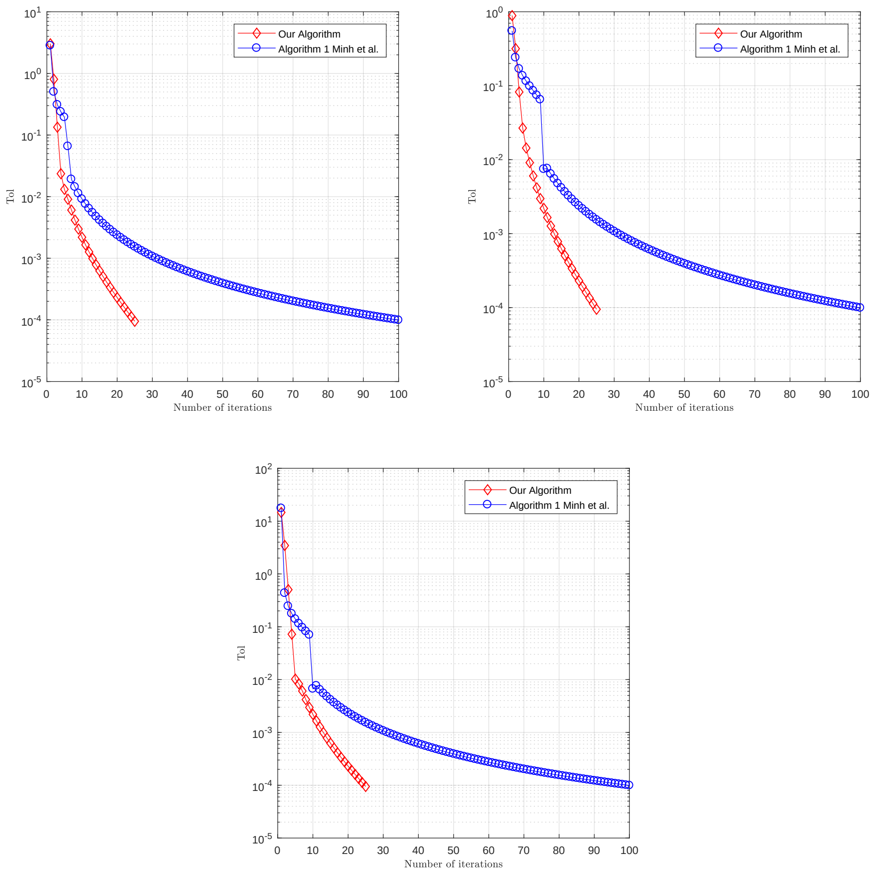

Figure 2: Example 5.2. Top left: I, Top right: II, Bottom: III.

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