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BICOMPLEX UNIVALENT FUNCTIONS

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ABSTRACT. In this paper we introduce bicomplex univalent functions and also discuss the properties of a specific class of univalent functions.

Key words and phrases: $\mathbb{BC}$-univalent, $\mathbb{BC}$-conformal, unit disk, Koebe function, $\mathbb{BC}$-holomorphic function.

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1. Introduction and Preliminaries

Bicomplex numbers, just like quaternions, are a generalization of complex numbers. These two number systems are different from each other in two important ways, quaternions, which form a division algebra, are non commutative, whereas bicomplex numbers are commutative but do not form a division algebra.
For the sake of completion and to make the paper self contained, we first summarize some basic properties of bicomplex numbers and hyperbolic numbers which is used in this paper. Bicomplex numbers have two imaginary units $\mathbf{i}$ and $\mathbf{j}$ satisfying

$$\mathbf{i} \neq \mathbf{j};\ \mathbf{ij} = \mathbf{ji} = \mathbf{k};\ \mathbf{i^2} = \mathbf{j^2} = -1.$$

Now let $\mathbb{C}(\mathbf{i})$ be the set of complex numbers with imaginary units $\mathbf{i}$ and let $\mathbb{C}(\mathbf{j})$ be the set of complex numbers with imaginary units $\mathbf{j}$. We define set of bicomplex numbers denoted by $\mathbb{BC}$ as

$$\mathbb{BC} = \{z = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} : x_1, x_2, x_3, x_4 \in \mathbb{R}\} = \{z = z_1 + \mathbf{j}z_2 : z_1, z_2 \in \mathbb{C}(\mathbf{i})\}.$$

We refer to [1], [10] [14] and [16] for detailed introduction to the algebra, geometry and analysis of the bicomplex numbers. Due to the fact that the set $\mathbb{BC}$ has two imaginary units i.e., $\mathbf{i}$ and $\mathbf{j}$, $\mathbb{BC}$ has three conjugations. These conjugations are bar-conjugation, †-conjugation and $*$-conjugation defined as $\overline{z} = \overline{z}_1 + \mathbf{j}\overline{z}_2$, $z^\dagger = z_1 - \mathbf{j}z_2$ and $z^* = \overline{z}^\dagger = \overline{z}_1 - \mathbf{j}\overline{z}_2$, respectively. Where $\overline{z}_1, \overline{z}_2$ are the usual conjugations of complex numbers z_1, z_2 in $\mathbb{C}(\mathbf{i})$.
Accordingly three types of moduli arise. These are $z \cdot z^\dagger$, $z \cdot \overline{z}$ and $z \cdot z^*$. It is to be noted that these modulus are $\mathbb{C}(\mathbf{i}), \mathbb{C}(\mathbf{j})$ and $\mathbb{D}$-valued. For details of conjugations on set of bicomplex numbers see [1], [10] and [14]. However, the †-conjugation defined by $z^\dagger = z_1 - \mathbf{j}z_2$, where $z = z_1 + \mathbf{j}z_2; z_1, z_2 \in \mathbb{C}(\mathbf{i})$ with moduli

$$z \cdot z^\dagger = |z|^2_{\mathbf{i}} = z_1^2 + z_2^2 = (|\eta_1|^2 - |\eta_2|^2) + 2Re(\eta_1\eta_2^*)\mathbf{i}$$

is important as it is used to define the invertiblity of a bicomplex number. A bicomplex number z is said to be invertible if $z \cdot z^\dagger \neq 0$ and its inverse is given by

$$z^{-1} = \frac{z^\dagger}{z \cdot z^\dagger} = \frac{z^\dagger}{|z|^2_{\mathbf{i}}}.$$

Further, if $z \neq 0$, but $z \cdot z^\dagger = |z|^2_{\mathbf{i}} = 0$, then z is said to be a zero-divisor. We denote the set of all zero-divisors by

$$\mathbb{NC} = \left\{z = z_1 + \mathbf{j}z_2 : z \neq 0, z \cdot z^\dagger = z_1^2 + z_2^2 = 0\right\}$$

and is called the null cone of the set of bicomplex number $\mathbb{BC}$. Let $\mathbb{NC}_0 = \mathbb{NC} \cup \{0\}$ be the null cone along with zero.
Now there are two special zero divisors $\mathbf{e}_1 = \frac{1}{2}(1 + \mathbf{k})$ and $\mathbf{e}_2 = \frac{1}{2}(1 - \mathbf{k})$ and called them idempotent elements and having the following properties:-

$$\mathbf{e}_1 + \mathbf{e}_2 = 1;\ \mathbf{e}_1 - \mathbf{e}_2 = \mathbf{k}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = 0;\ \mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_1;\ \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_2.$$

The sets $\mathbb{BC}_{\mathbf{e}_1} = \mathbf{e}_1\mathbb{BC}$ and $\mathbb{BC}_{\mathbf{e}_2} = \mathbf{e}_2\mathbb{BC}$ are (principal) ideals in the ring $\mathbb{BC}$ and have the property that

$$\mathbb{BC}_{\mathbf{e}_1} \cap \mathbb{BC}_{\mathbf{e}_2} = \{0\}$$

and

$$\mathbb{BC} = \mathbb{BC}_{\mathbf{e}_1} + \mathbb{BC}_{\mathbf{e}_2}. \tag{1.1}$$

This equation is called the idempotent decomposition of the ring of bicomplex numbers $\mathbb{BC}$. Thus each $z \in \mathbb{BC}$ can uniquely be expressed as $z = z_1\mathbf{e}_1 + z_2\mathbf{e}_2$ and also it allows us with component wise addition, multiplication and taking inverse of elements in $\mathbb{BC}$. The Euclidean norm $|\cdot|$ of a bicomplex number z is defined as $|z| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \sqrt{|z_1|^2 + |z_2|^2}$ and for any z and w in $\mathbb{BC}$, we have

$$|z \cdot w| \leq \sqrt{2}|z||w|.$$

The $\mathbb{D}$-valued norm of the bicomplex number $z = z_1\mathbf{e}_1 + z_2\mathbf{e}_2$ denoted by $|z|_{\mathbf{k}}$ is defined as $|z|_{\mathbf{k}} = |z_1|\mathbf{e}_1 + |z_2|\mathbf{e}_2$, where $|z_1|$ and $|z_2|$ are the usual modulus of complex numbers z_1 and z_2. Further $|z \cdot w|_{\mathbf{k}} = |z|_{\mathbf{k}} \cdot |w|_{\mathbf{k}}$ i.e., the hyperbolic modulus of the product is equal to the product of the corresponding moduli which is not true for the norm in $\mathbb{R}^4$ and Euclidean norm and hyperbolic norm of a bicomplex number is related by $||z|_{\mathbf{k}}| = |z|$. For the above discussion we refer to [1] and [10].

The hyperbolic numbers denoted by $\mathbb{D}$ is a ring of all numbers of the form $z = a + b\mathbf{k}$, where a,b $\in \mathbb{R}$, with $\mathbf{k}$ satisfying $\mathbf{k^2} = 1$.

$$i.e.,\ \mathbb{D} = \left\{a + b\mathbf{k} : a, b \in \mathbb{R},\ \mathbf{k^2} = 1,\ \mathbf{k} \notin \mathbb{R}\right\}.$$

Also the set of hyperbolic numbers have idempotent decomposition as

$$\mathbb{D} = \mathbb{D}\mathbf{e}_1 + \mathbb{D}\mathbf{e}_2.$$

The $\overline{\mathbb{BC}}$ is not one point Alexendrov compactification but is the union of with three different types of infinitive elements:

$$\overline{\mathbb{BC}} = \mathbb{BC} \cup \{\infty\mathbf{e}_1 + \mathbb{C}(\mathbf{i})\mathbf{e}_2\} \cup \{\mathbb{C}(\mathbf{i})\mathbf{e}_1 + \infty\mathbf{e}_2\} \cup \{\infty\mathbf{e}_1 + \infty\mathbf{e}_2\}$$

i.e., $\overline{\mathbb{BC}}$ contains the elements of the form $\infty\mathbf{e}_1 + z_2\mathbf{e}_2$ and $z_1\mathbf{e}_1 + \infty\mathbf{e}_2$ with $z_1, z_2 \in \mathbb{C}(\mathbf{i})$ and unique element $\infty\mathbf{e}_1 + \infty\mathbf{e}_2$. Thus infinity in $\mathbb{BC}$ have three different type of elements. For more details we refer to [12].

If $Z = Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2 \neq 0 \notin \mathbb{NC}_0$, then it is invertible. Writing

$$Z = |Z|_{\mathbf{k}}|Z|_{\mathbf{k}}^{-1}\left(Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2\right) = |Z|_{\mathbf{k}}\left(|Z_1|^{-1}\mathbf{e}_1 + |Z_2|^{-1}\mathbf{e}_2\right)\left(Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2\right),$$

implies

$$Z = |Z|_{\mathbf{k}}\left(\frac{Z_1}{|Z_1|}\mathbf{e}_1 + \frac{Z_2}{|Z_2|}\mathbf{e}_2\right).$$

As $\mathbf{e}_1$ and $\mathbf{e}_2$ are the coefficient of complex numbers of modulus one, we takes for real numbers μ_1 and μ_2 : $\frac{Z_1}{|Z_1|} = e^{\mathbf{i}\mu_1}$ and $\frac{Z_2}{|Z_2|} = e^{\mathbf{i}\mu_2}$. Let $\Psi_Z = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2$ be the hyperbolic number. Then Ψ_Z is called the hyperbolic argument associated with the bicomplex number Z, c.f. [13]. It has trigonometric representation in hyperbolic terms given as:

$$\begin{aligned} Z &= |Z|_{\mathbf{k}} \cdot (cos\Psi_Z + \mathbf{i}sin\Psi_Z) = |Z|_{\mathbf{k}} \cdot (e^{\mathbf{i}\mu_1}\mathbf{e}_1 + e^{\mathbf{i}\mu_2}\mathbf{e}_2) \\ &= |Z|_{\mathbf{k}} \cdot e^{\mathbf{i}(\mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2)} = |Z|_{\mathbf{k}} \cdot e^{\mathbf{i}\Psi_Z}. \end{aligned}$$

A set $\Omega \subset \mathbb{BC}$ is said to be product-type set if Ω can be written as $\Omega = \Omega_1\mathbf{e}_1 + \Omega_2\mathbf{e}_2$ where $\Omega_1 = \Pi_{1,\mathbf{i}}(\Omega)$ and $\Omega_2 = \Pi_{2,\mathbf{i}}(\Omega)$ are the projections of $\mathbb{BC}$ on $\mathbb{C}(\mathbf{i})$. A set $\Omega \subset \mathbb{BC}$ is said to be product-type domain in $\mathbb{BC}$ if Ω_1 and Ω_2 are domains in the complex plane. Also if γ_1, γ_2 are curves in $\mathbb{C}$ then hyperbolic curves in $\mathbb{BC}$ are product-type and are denoted as $\gamma = \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2$ and a hyperbolic curve is said to be $\mathbb{BC}$-rectifiable, $\mathbb{BC}$-Jordan and $\mathbb{BC}$-closed if and only if γ_1 and γ_2 are rectifiable, Jordan and closed respectively, see [2] and [10]. A function $F : \Omega \to \mathbb{BC}$ is said to be product-type if there exists $F_i : \Omega_i \to \mathbb{C}$ for $i = 1, 2$ such that $F(Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2) = F_1(Z_1)\mathbf{e}_1 + F_2(Z_2)\mathbf{e}_2$ for all $Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2 \in \Omega$. For more details and examples refer to [10].

Definition 1.1. The bicomplex open ball with center $Z_o = Z_{1,0}\mathbf{e}_1 + Z_{2,0}\mathbf{e}_2$ and positive hyperbolic radius $r = r_1\mathbf{e}_1 + r_2\mathbf{e}_2, (r_1 \neq 0 \text{ and } r_2 \neq 0)$ is

$$\mathbb{B}(Z_o, r) := \{Z : |Z - Z_0|_{\mathbf{k}} \prec r\} = \{Z = Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2 : |Z_1 - Z_{1,0}| < r_1 \text{ and } |Z_2 - Z_{2,0}| < r_2\}.$$

The bicomplex circumference of this bi-disk has the shape of torous, but it is a torus that lives in the four dimensional world, that is, it is not usual torus that can exist in $\mathbb{R}^3$.

$$\mathbb{B}_{\mathbb{BC}}(Z_o, r) = \mathcal{B}_{\mathbf{e}_1} \times \mathcal{B}_{\mathbf{e}_2} \subset \mathbb{BC},$$

where $\mathcal{B}_{\mathbf{e}_1} \subset \mathbb{BC}_{\mathbf{e}_1}$ is a disk with center in $Z_{1,0}$ and radius r_1 and similarly $\mathcal{B}_{\mathbf{e}_2} \subset \mathbb{BC}_{\mathbf{e}_2}$ is disk with center in $Z_{2,0}$ and radius r_2.
Now, we define a bicomplex ball with centre at the origin and hyperbolic radius 1 as

$$\begin{aligned} \mathbb{B}_1 &= \{Z \in \mathbb{BC} : |Z|_{\mathbf{k}} \prec 1\} \\ &= \{Z = Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2 : |Z_1| < 1,\ |Z_2| < 1\}. \end{aligned}$$

Also, let

$$\mathbb{B}_1 = \mathbb{B}_{1,1} \times \mathbb{B}_{1,2} \tag{1.2}$$

where $\mathbb{B}_{1,l} = \{Z_l : |Z_l| < 1\},\ l = 1, 2$ is a cartesian product of unit ball in $\mathbb{C}(\mathbf{i})$.

Definition 1.2. Let $\Omega \subset \mathbb{BC}$ be a product-type domain, then a function $R : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ is $\mathbb{BC}$-rational if 'R' is the quotient of two continuous $\mathbb{BC}$-functions i.e.,

$$R(Z) = \frac{G(Z)}{H(Z)} \; such\ that\ H(Z) \notin \mathbb{NC}_0.$$

Also bicomplex holomorphic rational functions are product-type, i.e., there exist holomorphic $R_i : \Omega_i \to \mathbb{C}$ for $i = 1, 2$ such that $R(Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2) = R_1(Z_1)\mathbf{e}_1 + R_2(Z_2)\mathbf{e}_2$ for all $Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2 \in \Omega$.

Definition 1.3. We say that function $F : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ is $\mathbb{BC}$-holomorphic if for every $Z \in \Omega$ there exist derivative $F'(Z)$ for which the following limit exist

$$\lim_{Y \to Z} \frac{F(Y) - F(Z)}{Y - Z},\ where\ Y \in \Omega\ and\ (Y - Z) \notin \mathbb{NC}_0.$$

A function $F(Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2) = F_1(Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2)\mathbf{e}_1 + F_2(Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2)\mathbf{e}_2$ is $\mathbb{BC}$-holomorphic if and only if $F_1(Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2)$ and $F_2(Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2)$ are holomorphic functions with respect to only Z_1 and Z_2 respectively. For the above discussions we refer to [1], [2], [10] and [14].

In this paper we extend the theory of univalent functions to bicomplex version and analyze its various properties whether they hold in bicomplex number framework, particularly in bicomplex unit disk. In section 2, we define bicomplex univalent function and also analyze the properties of specific class of bicomplex univalent functions which we denote by $\mathcal{F}$ in bicomplex. Here we investigate the bicomplex version of Koebe function which is an important example in class $\mathcal{F}$. Section 3, deals with a brief discussion of $\mathbb{BC}$-Mobius invariant properties of class $\mathcal{F}$. For a study of the univalent functions, we refer to [4],[5],[15] and reference therein.

2. $\mathbb{BC}$-UNIVALENT FUNCTIONS

In this section, we introduce the class $\mathcal{F}$ of $\mathbb{BC}$-univalent functions. The property of univalence is much stronger in complex case than in real which led to the development of theory of univalent function. This theory was born around the past century and is still active field of research. Now we define $\mathbb{BC}$-univalent function.

Definition 2.1. A $\mathbb{BC}$-holomorphic function $F : \Omega \subseteq \overline{\mathbb{BC}} \to \overline{\mathbb{BC}}$ is said to be a $\mathbb{BC}$-univalent function on Ω if $F(Z_1) \neq F(Z_2),\ \forall\ Z_1, Z_2 \in \Omega$ with $Z_1 \neq Z_2$.

Definition 2.2. The upper half plane in bicomplex is denoted by $\prod_{\mathbb{BC}}^{+}$ and is define by:

$$\prod_{\mathbb{BC}}^{+} = \left\{ Z = Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2 : (Z_1, Z_2) \in \prod_{1}^{+} \times \prod_{2}^{+} \right\}, \tag{2.1}$$

where $\prod_{i}^{+} = \{Z_i \in \mathbb{C} : Im(Z_i) > 0\},\ i = 1, 2.$

Example 2.1. *Let $Z = Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2 \in \mathbb{BC}$ and $\mathbf{S} = \mathbf{S}_1 \times \mathbf{S}_2 \subset \mathbb{B}_{1,1} \times \mathbb{B}_{1,2}$ be a Cartesian domain in $\mathbb{BC}$ such that*

$$\mathbf{S} = \left\{ Z \in \mathbb{BC} : 0 \prec |Z|_{\mathbf{k}} \prec 1, 0 < arg_{\mathbb{C}(\mathbf{i})} Z < \frac{\pi}{2} \right\}. \tag{2.2}$$

Then $\mathbf{S}$ is a part of bicomplex unit disk in the first quadrant and

$$\mathbb{B}_{\mathbb{BC}} \cap \prod_{\mathbb{BC}}^{+} = \left\{ Z = Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2 : (Z_1, Z_2) \in \left(\mathbb{B}_{1,1} \cap \prod_{1}^{+}\right) \times \left(\mathbb{B}_{1,2} \cap \prod_{2}^{+}\right) \right\}.$$

Then the function $F : \mathbf{S} \to \mathbb{B}_{\mathbb{BC}} \cap \prod_{\mathbb{BC}}^{+}$ such that $F(Z) = Z^2$ is $\mathbb{BC}$-conformal mapping. When we separate the idempotent parts of the above system, we get two different systems, one with the complex variable Z_1 in the plane $\mathbb{BC}_{\mathbf{e}_1}$ and other with the complex variable Z_2 in the plane $\mathbb{BC}_{\mathbf{e}_2}$. Taking only the first idempotent component from (1.2),(2.1) and (2.2) we get:

$$\begin{aligned}
\mathbf{S}_{\mathbf{e}_1} &= \mathbf{e}_1\mathbf{S}_1 = \left\{ Z_1\mathbf{e}_1 \in \mathbb{C}(\mathbf{i})\mathbf{e}_1 : 0 < |Z_1| < 1, 0 < arg_{\mathbb{C}(\mathbf{i})}(Z_1) < \frac{\pi}{2} \right\}, \\
\mathbb{B}_{\mathbf{e}_1} &= \mathbf{e}_1\mathbb{B}_{1,1} = \{Z_1\mathbf{e}_1 : |Z_1| < 1\} \textit{ and} \\
\prod_{\mathbf{e}_1}^{+} &= \mathbf{e}_1 \prod_{1}^{+} = \{Z_1\mathbf{e}_1 \in \mathbb{C}(\mathbf{i})\mathbf{e}_1 : Im(Z_1) \geq 0\}, \textit{then} \\
\mathbb{B}_{\mathbf{e}_1} \cap \prod_{\mathbf{e}_1}^{+} &= \{Z_1\mathbf{e}_1 \in \mathbb{C}(\mathbf{i})\mathbf{e}_1 : 0 < |Z_1| < 1, Im(Z_1) > 0\}.
\end{aligned}$$

From this, we find a real two-dimensional surface in $\mathbb{R}^4$ and Fig.1 shows its idempotent projection on $\mathbb{BC}\mathbf{e}_1$. Then, clearly the mapping $F_1 : \mathbf{S}_{\mathbf{e}_1} \to \mathbb{B}_{\mathbf{e}_1} \cap \prod_{\mathbf{e}_1}^{+}$ such that $F_1(Z_1) = Z_1^2$ is conformal mapping. Its projection on $\mathbb{BC}\mathbf{e}_2$ is quite similar.
The mapping of $F(Z) = Z^2$ in $\mathbb{BC} \cong \mathbb{D}^2$ is shown in Fig 2, where planes $\mathbb{D}$ and $\mathbf{i}\mathbb{D}$ are seen as lines, although they are real two dimensional planes, as shown in [13, Fig 3].

From bicomplex Riemann mapping theorem [10, Theorem 8.6.2 page-190], for any product-type simply connected domain Ω in $\mathbb{BC}$, there exists a bijective $\mathbb{D}$-conformal mapping $F : \Omega \to \mathbb{B}_1$. Furthermore, for any fixed $Z_o \in \Omega$, we can find an F such that $F(Z_o) = 0$ and $F^{'}(Z_o)$ is strictly positive hyperbolic numbers with such a specification F is unique.
As a result, a statement about $\mathbb{BC}$-univalent function on arbitrary product-type simply connected domain can be translated to statement about $\mathbb{BC}$-univalent function on the unit ball. We shall examine the following class of $\mathbb{BC}$-univalent functions.

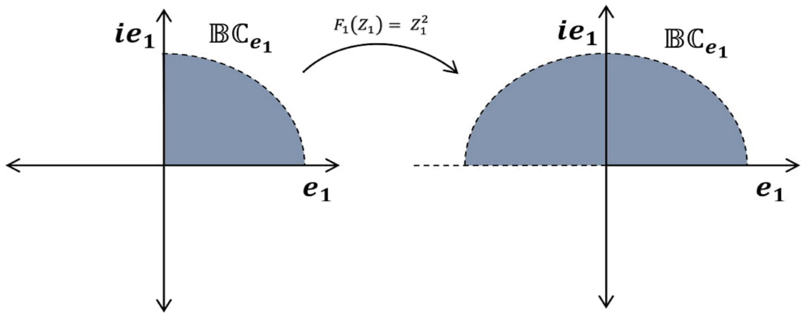

Figure 1: The projection of $F_1(Z_1) = Z_1^2$ *in* $\mathbb{BC}\mathbf{e}_1$

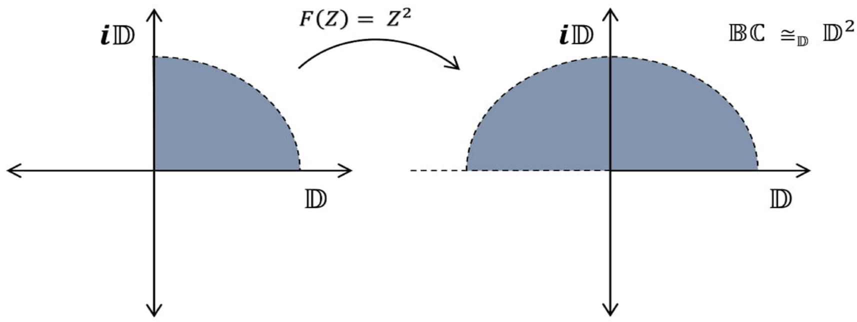

Figure 2: The mapping of $F(Z) = Z^2$ *in* $\mathbb{BC} \cong \mathbb{D}^2$

Definition 2.3. Let $\mathcal{F}$ denote the set of $\mathbb{BC}$-holomorphic, $\mathbb{BC}$-univalent functions on the unit disk $\mathbb{B}_{\mathbb{BC}}$ normalized by the condition $F(0) = 0$ and $F^{'}(0) = 1$. That is,

$\mathcal{F} = \{F : \mathbb{B}_{\mathbb{BC}} \to \mathbb{BC}$: F is $\mathbb{BC}$-holomorphic and $\mathbb{BC}$-univalent on $\mathbb{B}_{\mathbb{BC}}, F(0) = 0, F^{'}(0) = 1\}$.

Then it follows from [10, Theorem 10.5.2, page 208], that for every $F \in \mathcal{F}$ has a bicomplex Taylor series expansion of the form

$$F(Z) = Z + A_2 Z^2 + ..., \ |Z|_{\mathbf{k}} \prec 1,$$

where $A_n \in \mathbb{BC}, n \in \mathbb{N}$.

Now, we introduce the bicomplex Koebe function which is one of the most important member of $\mathcal{F}$. The Koebe function in complex plane is defined as:

$$\begin{aligned} K(z) &= \frac{z}{(1-z)^2} \\ &= \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4} \end{aligned}$$

where $z \in \mathbb{C}$.
Then the bicomplex Koebe function is given as

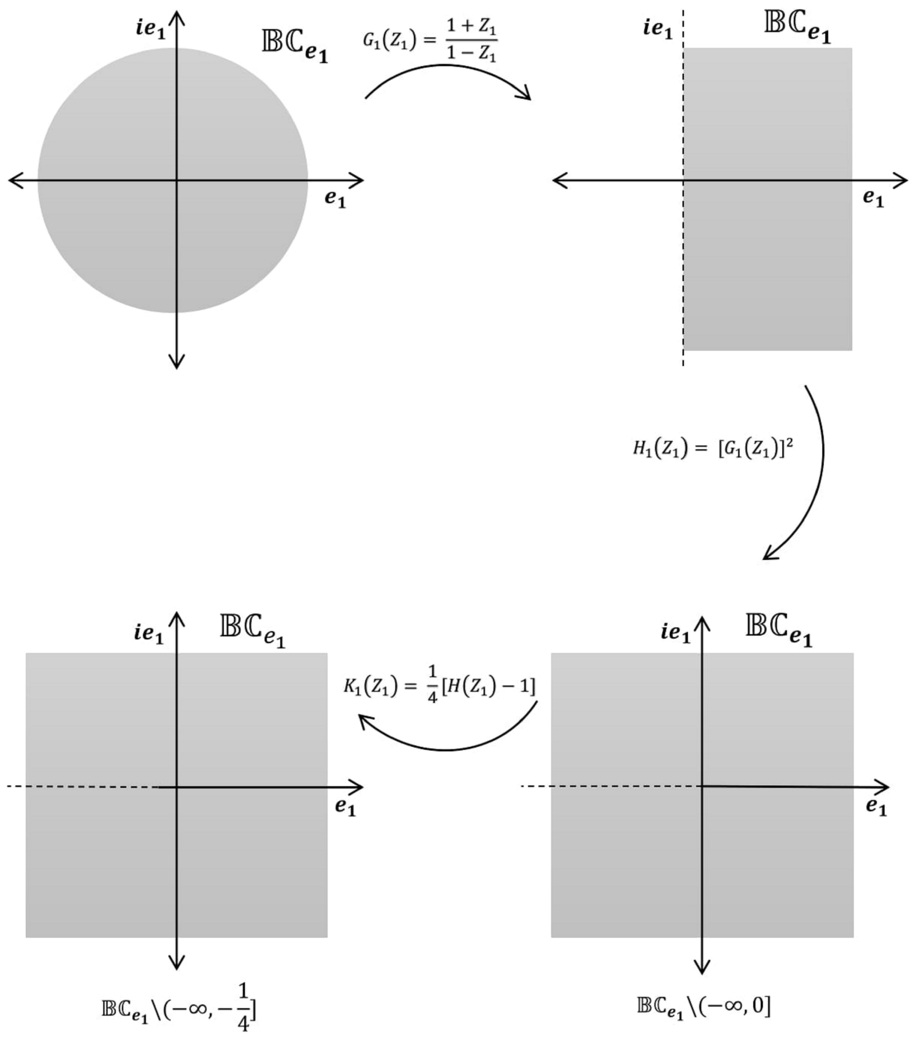

Figure 3: The projection of $K_1(Z_1)$ in $\mathbb{BC}\mathbf{e}_1$

$$\begin{aligned} K(Z) &= \frac{1}{4}\left(\frac{1+Z}{1-Z}\right)^2 - \frac{1}{4}, \\ &= \frac{Z}{(1-Z)^2} \end{aligned}$$

where $Z \in \mathbb{BC}$. Now,

$$\begin{aligned} K(Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2) &= \frac{1}{4}\left(\frac{1+(Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2)}{1-(Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2)}\right)^2 - \frac{1}{4} \\ &= \frac{1}{4}\left(\frac{1+Z_1}{1-Z_1}\right)^2 \mathbf{e}_1 - \frac{1}{4}\mathbf{e}_1 + \frac{1}{4}\left(\frac{1+Z_2}{1-Z_2}\right)^2 \mathbf{e}_2 - \frac{1}{4}\mathbf{e}_2 \\ &= \left(\frac{1}{4}\left(\frac{1+Z_1}{1-Z_1}\right)^2 - \frac{1}{4}\right)\mathbf{e}_1 + \left(\frac{1}{4}\left(\frac{1+Z_1}{1-Z_1}\right)^2 - \frac{1}{4}\right)\mathbf{e}_2 \qquad (2.3) \\ &= K_1(Z_1)\mathbf{e}_1 + K_2(Z_2)\mathbf{e}_2. \qquad (2.4) \end{aligned}$$

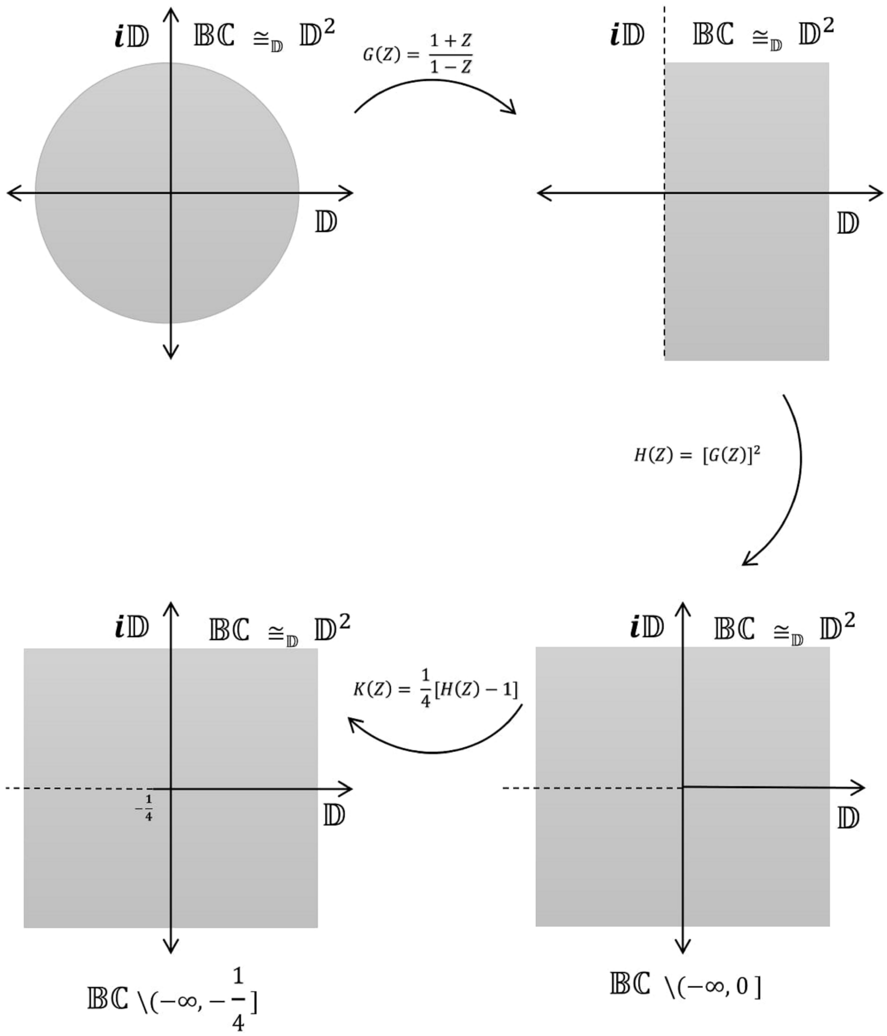

Figure 4: The mapping of $\mathbb{BC}$-Koebe function $K(Z)$ in $\mathbb{BC} \cong \mathbb{D}^2$

When we separate the idempotent parts of the above system, we get the two different systems, one with the complex variable Z_1 in the plane $\mathbb{C}(\mathbf{i})\mathbf{e}_1$ and other with the complex variable Z_2 in the plane $\mathbb{C}(\mathbf{i})\mathbf{e}_2$. Taking only the first idempotent components from (2.3) and (2.4), we obtain:

$$K_1(Z_1)\mathbf{e}_1 = \left(\frac{1}{4}\left(\frac{1+Z_1}{1-Z_1}\right)^2 - \frac{1}{4}\right)\mathbf{e}_1.$$

From this, we find a real two-dimensional surface in $\mathbb{R}^4$ and Fig.3 shows its idempotent projection on $\mathbb{BC}\mathbf{e}_1$. Its projection on $\mathbb{BC}\mathbf{e}_1$ is quite similar.

The mapping of $K(Z) = \frac{1}{4}\left(\frac{1+Z}{1-Z}\right)^2 - \frac{1}{4}$ in $\mathbb{BC} \cong \mathbb{D}^2$ is shown in Fig.4. As in [13, Fig 3], plane $\mathbb{D}$ and $\mathbf{i}\mathbb{D}$ are seen as lines, although they are real two dimensional planes.

3. $\mathbb{BC}$-MOBIUS INVARIANT CLASS $\mathcal{F}$

In this section, we will study $\mathbb{BC}$-Mobius invariant function of class $\mathcal{F}$.

If $F, G : \Omega = \Omega_1\mathbf{e}_1 + \Omega_2\mathbf{e}_2 \subseteq \mathbb{BC} \to \mathbb{BC}$ be $\mathbb{BC}$-holomorphic function. Then, for every $Z \in \Omega$, there exist $Z_1 \in \Omega_1$ and $Z_2 \in \Omega_2$ such that

$$\begin{aligned} F(Z) &= F_1(Z_1)\mathbf{e}_1 + F_2(Z_2)\mathbf{e}_2 \\ G(Z) &= G_1(Z_1)\mathbf{e}_1 + G_2(Z_2)\mathbf{e}_2 \end{aligned}$$

and so

$$(F \circ G)(Z) = (F_1 \circ G_1)(Z_1)\mathbf{e}_1 + (F_2 \circ G_2)(Z_2)\mathbf{e}_2,$$

where $F_1 \circ G_1 : \Omega_1 \to \mathbb{C}$ and $F_2 \circ G_2 : \Omega_2 \to \mathbb{C}$ are holomorphic functions.

Remark 3.1. $\mathcal{F}$ is not closed under addition. Here is the example:

Example 3.1. *Let $F(Z) = Z$ and $G(Z) = \frac{Z}{1-Z}$ so that $F, G \in \mathcal{F}$. However, $F^{'}(Z) = 1$ and $G^{'}(Z) = \frac{1}{(1-Z)^2}$. Then*

$$\begin{aligned} F^{'}(Z) + G^{'}(Z) &= 1 + \frac{1}{(1-Z)^2} \\ &= \frac{Z^2 - 2Z + 2}{(1-Z)^2}, \end{aligned}$$

from which we conclude that $F^{'}(Z) + G^{'}(Z) = 0$, if $Z = 1 + \mathbf{i},\ 1 - \mathbf{i},\ 1 + \mathbf{j},\ 1 - \mathbf{j}$. It follows that $F + G$ is not one-to-one in $\mathbb{B}_{\mathbb{BC}}$, hence $F + G \notin \mathcal{F}$.

Theorem 3.1. *The class $\mathcal{F}$ is preserved under the following $\mathbb{BC}$-transformation:*
(I) Rotation: If $F \in \mathcal{F}$, $\Theta \in \mathbb{BC}$ and $G(Z) = e^{-\mathbf{i}\Theta}F(e^{\mathbf{i}\Theta})$, then $G \in \mathcal{F}$.
(II) Dilation: If $F \in \mathcal{F}, 0 \prec r \prec 1$ and $G(Z) = \frac{1}{r}F(rZ)$, then $G \in \mathcal{F}$.

Proof. (I): Let $F = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 \in \mathcal{F}$ and also let $S(Z) = e^{\mathbf{i}\Theta_1}Z_1\mathbf{e}_1 + e^{\mathbf{i}\Theta_2}Z_2\mathbf{e}_2$ and $T(Z) = e^{-\mathbf{i}\Theta_1}Z_1\mathbf{e}_1 + e^{-\mathbf{i}\Theta_2}Z_2\mathbf{e}_2$. First, we have to show that $S : \mathbb{BC} \to \mathbb{BC}$ is one-to-one. For this, let $Z = Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2,\ Y = Y_1\mathbf{e}_1 + Y_2\mathbf{e}_2 \in \mathbb{BC}$ and suppose that

$$S(Z) = S(Y).$$

Then
$e^{\mathbf{i}\Theta_1}Z_1\mathbf{e}_1 + e^{\mathbf{i}\Theta_2}Z_2\mathbf{e}_2 = e^{\mathbf{i}\Theta_1}Y_1\mathbf{e}_1 + e^{\mathbf{i}\Theta_2}Y_2\mathbf{e}_2$
$\Longleftrightarrow (e^{\mathbf{i}\Theta_1}\mathbf{e}_1 + e^{\mathbf{i}\Theta_2}\mathbf{e}_2)(Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2) = (e^{\mathbf{i}\Theta_1}\mathbf{e}_1 + e^{\mathbf{i}\Theta_2}\mathbf{e}_2)(Y_1\mathbf{e}_1 + Y_2\mathbf{e}_2)$
$\Longleftrightarrow Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2 = Y_1\mathbf{e}_1 + Y_2\mathbf{e}_2$
$\Longleftrightarrow Z = Y.$
Therefore, S is one-to-one. Similarly, $T : \mathbb{BC} \to \mathbb{BC}$ is one-to-one. Now,

$$\begin{aligned} G(Z) &= e^{-\mathbf{i}\Theta}F(e^{\mathbf{i}\Theta}Z) \\ &= e^{-\mathbf{i}\Theta_1}F_1(e^{\mathbf{i}\Theta_1}Z_1)\mathbf{e}_1 + e^{-\mathbf{i}\Theta_2}F_2(e^{\mathbf{i}\Theta_2}Z_2)\mathbf{e}_2 \\ &= (T_1 \circ F_1 \circ S_1)(Z_1)\mathbf{e}_1 + (T_2 \circ F_2 \circ S_2)(Z_2)\mathbf{e}_2. \end{aligned}$$

Since, $(T_1 \circ F_1 \circ S_1)(Z_1)$ and $(T_2 \circ F_2 \circ S_2)(Z_2)$ are one-to-one mapping, see [7, Theorem 5, page-6]. So, $G(Z)$ is one-to-one mapping. Thus we conclude that G is $\mathbb{BC}$-univalent in $\mathbb{B}_{\mathbb{BC}}$. Now,

$$\begin{aligned} G^{'}(Z) &= e^{-\mathbf{i}\Theta_1} \cdot e^{\mathbf{i}\Theta_1} \cdot F_1^{'}(e^{\mathbf{i}\Theta_1}Z_1)\mathbf{e}_1 + e^{-\mathbf{i}\Theta_2} \cdot e^{\mathbf{i}\Theta_2} \cdot F_1^{'}(e^{\mathbf{i}\Theta_2}Z_2)\mathbf{e}_2 \\ &= F_1^{'}(e^{\mathbf{i}\Theta_1}Z_1)\mathbf{e}_1 + F_1^{'}(e^{\mathbf{i}\Theta_2}Z_2)\mathbf{e}_2. \end{aligned}$$

Since, $F_1^{'}(e^{\mathbf{i}\Theta_1}Z_1)$ and $F_2^{'}(e^{\mathbf{i}\Theta_2}Z_2)$ are holomorphic, see [7, Theorem 5, page-6]. We conclude that, G is also $\mathbb{BC}$-holomorphic in $\mathbb{B}_{\mathbb{BC}}$. Also, $G(0) = e^{-\mathbf{i}\Theta_1}F_1(0)\mathbf{e}_1 + e^{-\mathbf{i}\Theta_2}F_2(0)\mathbf{e}_2 = 0$ and $G^{'}(0) = F_1^{'}(0)\mathbf{e}_1 + F_2^{'}(0)\mathbf{e}_2 = \mathbf{e}_1 + \mathbf{e}_2 = 1$. Then $G \in \mathcal{F}$.

(II): Let $F \in \mathcal{F}$ and $0 < r_1, r_2 < 1$. Suppose $S(Z) = r_1Z_1\mathbf{e}_1 + r_2Z_2\mathbf{e}_2$ and $T(Z) = \frac{Z_1}{r_1}\mathbf{e}_1 + \frac{Z_2}{r_2}\mathbf{e}_2$. Then clearly $S, T : \mathbb{BC} \to \mathbb{BC}$ are one-to-one. Now,

$$\begin{aligned} G(Z) &= \frac{1}{r}F(rZ) \\ &= \frac{1}{r_1}F_1(r_1Z_1)\mathbf{e}_1 + \frac{1}{r_2}F_2(r_2Z_2)\mathbf{e}_2 \\ &= (T_1 \circ F_1 \circ S_1)Z_1\mathbf{e}_1 + (T_1 \circ F_2 \circ S_2)Z_2\mathbf{e}_2. \end{aligned}$$

Clearly, $G(Z)$ is a composition of one-to-one mappings, we conclude that G is $\mathbb{BC}$-univalent on $\mathbb{B}_{\mathbb{BC}}$.
Now,

$$\begin{aligned} G^{'}(Z) &= \frac{1}{r_1} \cdot r_1 \cdot F_1^{'}(r_1Z_1)\mathbf{e}_1 + \frac{1}{r_2} \cdot r_2 \cdot F_2^{'}(r_2Z_2)\mathbf{e}_2 \\ &= F_1^{'}(r_1Z_1)\mathbf{e}_1 + F_2^{'}(r_2Z_2)\mathbf{e}_2. \end{aligned}$$

Since, $F_1^{'}(r_1Z_1)\mathbf{e}_1$ and $F_2^{'}(r_2Z_2)\mathbf{e}_2$ are holomorphic. So, G is $\mathbb{BC}$-holomorphic on $\mathbb{B}_{\mathbb{BC}}$. Also, $G(0) = \frac{1}{r_1}F_1(0)\mathbf{e}_1 + \frac{1}{r_2}F_2(0)\mathbf{e}_2 = 0$ and $G^{'}(0) = F_1^{'}(0)\mathbf{e}_1 + F_2^{'}(0)\mathbf{e}_2 = \mathbf{e}_1 + \mathbf{e}_2 = 1$. Then $G \in \mathcal{F}$.

∎

We have three types of conjugations in bicomplex, bar-conjugation, †-conjugation and $*$-conjugation. By the combination of these three conjugations we get nine different conjugation. From those nine combination of conjugations, class $\mathcal{F}$ is preserved under three conjugations and is not preserved under six conjugations. This concept is explored in the following theorems:

Theorem 3.2. *Let $H(\mathcal{F}) = \{\overline{F(\overline{Z})}, (F(Z)^*)^*, (F(Z)^\dagger)^\dagger\}$ be a class of different combinations of conjugations. If $F \in \mathcal{F}$ and $G(Z) \in H(\mathcal{F})$, then $G \in \mathcal{F}$.*

Proof. If $F \in \mathcal{F}$, $G(Z) = \overline{F(\overline{Z})}$, and $W(Z) = \overline{Z} = \overline{Z}_1\mathbf{e}_2 + \overline{Z}_2\mathbf{e}_1$, then $W : \mathbb{BC} \to \mathbb{BC}$ is clearly one-to-one.
Now,

$$\begin{aligned} G(Z) &= \overline{F(\overline{Z})} \\ &= \overline{F_1(\overline{Z}_1)\mathbf{e}_2 + F_2(\overline{Z}_2)\mathbf{e}_1} \\ &= \overline{F_1(\overline{Z}_1)}\mathbf{e}_1 + \overline{F_2(\overline{Z}_2)}\mathbf{e}_2 \\ &= (W_1 \circ F_1 \circ W_1)(Z_1)\mathbf{e}_1 + (W_2 \circ F_2 \circ W_2)(Z_2)\mathbf{e}_2. \end{aligned}$$

Since $(W_1 \circ F_1 \circ W_1)(Z_1)$ and $(W_2 \circ F_2 \circ W_2)(Z_2)$ are one-to-one from using [7, Theorem 5, page-6], we conclude that $G(Z)$ is a composition of one-to-one mapping, so G is $\mathbb{BC}$-univalent on $\mathbb{B}_{\mathbb{BC}}$. Since $W(Z)$ is not $\mathbb{BC}$-holomorphic on $\mathbb{B}_{\mathbb{BC}}$, so we cannot simply use the assumption that a composition of $\mathbb{BC}$-holomorphic functions is $\mathbb{BC}$-holomorphic. Instead, we observe that the $\mathbb{BC}$-Taylor series of F, namely

$$Z + \sum_{n=2}^{\infty} A_n Z^n \tag{3.1}$$

has a radius of convergence 1, see [10, Theorem 10.5.2, page-208]. With the uniform convergence on every closed disk $|Z|_{\mathbf{k}} \preceq r \prec 1$, the $\mathbb{BC}$-Taylor series (3.1) converges to $F(Z),\ \forall\, |Z|_{\mathbf{k}} \prec 1$. It follows that the $\mathbb{BC}$-Taylor series

$$Z + \sum_{n=2}^{\infty} \overline{A}_n Z^n \tag{3.2}$$

has radius of convergence 1 and thus (3.2) defines an $\mathbb{BC}$-holomorphic function on $\mathbb{B}_{\mathbb{BC}}$. Hence, we conclude that

$$G(Z) = \overline{F(\overline{Z})} = \overline{\overline{Z} + A_2\overline{Z}^2 + A_3\overline{Z}^3 + ...} = Z + \overline{A}_2 Z^2 + \overline{A}_3 Z^3 + ...$$

is $\mathbb{BC}$-holomorphic on $\mathbb{B}_{\mathbb{BC}}$ with $G(0) = 0$ and $G^{'}(0) = 1$. Thus, $G \in \mathcal{F}$.
Similarly, we can show that for $G(Z) = (F(Z^*))^*,\ G(Z) = (F(Z^\dagger))^\dagger,\ G \in \mathcal{F}$. ∎

Theorem 3.3. *Suppose* $I(\mathcal{F}) = \{\overline{F(Z^*)}, \overline{(F(Z^\dagger))}, (F(\overline{Z}))^*, (F(Z^\dagger))^*, (F(\bar{Z}))^\dagger$ *and* $(F(Z^*))^\dagger\}$ *be a class of conjugations in* $\mathbb{BC}$*. If* $F \in \mathcal{F}$ *and* $G(Z) \in I(\mathcal{F})$*, then* $G \notin \mathcal{F}$*.*

Proof. Suppose $F \in \mathcal{F}$ and $G(Z) = \overline{F(Z^*)}$ and $W(Z) = Z^* = Z_1^*\mathbf{e}_1 + Z_2^*\mathbf{e}_2$ and $S(Z) = \overline{Z} = \overline{Z}_1\mathbf{e}_2 + \overline{Z}_2\mathbf{e}_1$. Then $W, S : \mathbb{BC} \to \mathbb{BC}$ are clearly one-to-one.
Now,

$$\begin{aligned} G(Z) &= \overline{F(Z^*)} \\ &= \overline{F_1(Z_1^*)\mathbf{e}_1 + F_2(Z_2^*)\mathbf{e}_2} \\ &= \overline{F_1(Z_1^*)}\mathbf{e}_2 + \overline{F_2(Z_2^*)}\mathbf{e}_1 \\ &= (S_1 \circ F_1 \circ W_1)(Z_1)\mathbf{e}_2 + (S_2 \circ F_2 \circ W_2)(Z_2)\mathbf{e}_1. \end{aligned}$$

Since $(S_1 \circ F_1 \circ W_1)(Z_1)$ and $(S_2 \circ F_2 \circ W_2)(Z_2)$ are one to one mappings. Thus, we conclude that G is $\mathbb{BC}$-univalent on $\mathbb{B}_{\mathbb{BC}}$.
Since $W(Z)$ is not $\mathbb{BC}$-holomorphic in $\mathbb{B}_{\mathbb{BC}}$, so we cannot simply use the assumption that a composition of $\mathbb{BC}$-holomorphic functions is $\mathbb{BC}$-holomorphic. Now,

$$G(Z) = \overline{F(Z^*)} = \overline{Z^* + A_2(Z^*)^2 + A_3(Z^*)^3 + ...} = Z^\dagger + \bar{A}_2(Z^\dagger)^2 + \bar{A}_3(Z^\dagger)^3 + ...$$

is not $\mathbb{BC}$-holomorphic in $\mathbb{B}_{\mathbb{BC}}$. So, $G \notin \mathcal{F}$.
Similarly, we can show that for $G(Z) = \overline{(F(Z^\dagger))},\ G(Z) = (F(\overline{Z}))^*,\ G(Z) = (F(Z^\dagger))^*, G(Z) = (F(\bar{Z}))^\dagger$ and $G(Z) = (F(Z^*))^\dagger,\ G \notin \mathcal{F}$. Therefore, these six conjugation is not preserved the class $\mathcal{F}$. ∎

Theorem 3.4. *The class* $\mathcal{F}$ *is preserved under the following* $\mathbb{BC}$*-transformation:*
(I) Disk automorphism: If $F \in \mathcal{F}$ *and* $G(Z) = \frac{F\left(\frac{Z+Z_o}{(1-Z_o^*Z)}\right)-F(Z_o)}{\left(1-|Z_o|_{\mathbf{k}}^2\right)F'(Z_o)}$ *for any* $|Z_o|_{\mathbf{k}} \prec 1,$ *then* $G \in \mathcal{F}$*.*
(II) Range transformation: If $F \in \mathcal{F},\ \Phi : F(\mathbb{B}_{\mathbb{BC}}) \to \mathbb{BC}$ *is* $\mathbb{BC}$*-holomorphic and* $\mathbb{BC}$*-univalent on* $F(\mathbb{B}_{\mathbb{BC}})$ *and*

$$G(Z) = \frac{(\Phi \circ f)(Z) - \Phi(0)}{\Phi'(0)},$$

then $G \in \mathcal{F}$*.*
(III) Omitted value transformation: If $F \in \mathcal{F}$ *with* $W - F(Z) \notin \mathbb{NC}_o$*,*

$$G(Z) = \frac{WF(Z)}{W - F(Z)},$$

then $G \in \mathcal{F}$*.*

Proof. (I): Let $F \in \mathcal{F}$ and $W(Z) = \frac{Z+Z_o}{1-Z_o^* Z}$ be the $\mathbb{BC}$-Mobius transformation which maps the unit disk $\mathbb{B}_{\mathbb{BC}}$ $\mathbb{BC}$-conformally onto itself with $W(0) = Z_0 = Z_{o,1}\mathbf{e}_1 + Z_{o,2}\mathbf{e}_2$. Since $Z_o \in \mathbb{BC}$, we conclude that

$$G(Z) = \frac{F_1(W_1(Z_1)) - F_1(Z_{o,1})}{(1-|Z_{o,1}|)F'(Z_{o,1})}\mathbf{e}_1 + \frac{F_2(W_2(Z_2)) - F_2(Z_{o,2})}{(1-|Z_{o,2}|)F'(Z_{o,2})}\mathbf{e}_2$$

is $\mathbb{BC}$-univalent on $\mathbb{B}_{1,1} \times \mathbb{B}_{1,2}$ with $G(0) = 0$.
Furthermore,

$$\begin{aligned} G^{'}(Z) &= \frac{W_1^{'}(Z_1)F_1^{'}(W_1(Z_1))}{(1-|Z_{o,1}|)F_1^{'}(Z_{o,1})}\mathbf{e}_1 + \frac{W_2^{'}(Z_2)F_2^{'}(W_2(Z_2))}{(1-|Z_{o,2}|)F_2^{'}(Z_{o,2})}\mathbf{e}_2 \\ &= \frac{F_1^{'}(W_1(Z_1))}{(1-\overline{Z}_{o,1}Z_1)^2F_1^{'}(Z_{o,1})}\mathbf{e}_1 + \frac{F_2^{'}(W_2(Z_2))}{(1-\overline{Z}_{o,2}Z_2)^2F_2^{'}(Z_{o,2})}\mathbf{e}_2, \end{aligned}$$

so that G is $\mathbb{BC}$-holomorphic on $\mathbb{B}_{\mathbb{BC}} = \mathbb{B}_{1,1} \times \mathbb{B}_{1,2}$ with $G^{'}(0) = 1$. Thus, $G \in \mathcal{F}$.

(II): Suppose $F \in \mathcal{F}$ and let $\Phi : F(\mathbb{B}_{\mathbb{BC}}) \to \mathbb{BC}$ be $\mathbb{BC}$-holomorphic and $\mathbb{BC}$-univalent on $F(\mathbb{B}_{\mathbb{BC}})$.
If

$$\begin{aligned} G(Z) &= \frac{(\Phi \circ F)(Z) - \Phi(0)}{\Phi'(0)} \\ &= \frac{(\Phi_1 \circ F_1)(Z_1) - \Phi_1(0)}{\Phi_1^{'}(0)}\mathbf{e}_1 + \frac{(\Phi_2 \circ F_2)(Z_2) - \Phi_2(0)}{\Phi_2^{'}(0)}\mathbf{e}_2, \end{aligned}$$

then G is clearly $\mathbb{BC}$-univalent on $\mathbb{B}_{\mathbb{BC}}$ with $G(0) = 0$.
Furthermore,

$$G^{'}(Z) = \frac{F_1^{'}(Z_1)\Phi_1^{'}(F_1(Z_1))}{\Phi_1^{'}(0)}\mathbf{e}_1 + \frac{F_2^{'}(Z_2)\Phi_2^{'}(F_2(Z_2))}{\Phi_2^{'}(0)}\mathbf{e}_2,$$

so that G is $\mathbb{BC}$-holomorphic on $\mathbb{B}_{1,1} \times \mathbb{B}_{1,2}$ with $G^{'}(0) = 1$. Thus, $G \in \mathcal{F}$.

(III): Suppose that $F \in \mathcal{F}$ with $W - F(Z) \notin \mathbb{NC}_o$ and let

$$G(Z) = \frac{W_1F_1(Z_1)}{W_1 - F_1(Z_1)}\mathbf{e}_1 + \frac{W_2F_2(Z_2)}{W_2 - F_2(Z_2)}\mathbf{e}_2.$$

Clearly, $T(\Gamma) = \frac{W_1\Gamma_1}{W_1-\Gamma_1}\mathbf{e}_1 + \frac{W_2\Gamma_2}{W_2-\Gamma_2}\mathbf{e}_2$ is one-to-one if $\Gamma_1 \neq W_1$ and $\Gamma_2 \neq W_2$. Then it follows that $G(Z) = (T_1 \circ F_1)(X_1)\mathbf{e}_1 + (T_2 \circ F_2)(X_2)\mathbf{e}_2$ is $\mathbb{BC}$-univalent on $\mathbb{B}_{1,1} \times \mathbb{B}_{1,2}$.
Furthermore,

$$G^{'}(Z) = \frac{W_1^2F_1^{'}(Z_1)}{(W_1 - F_1(Z_1))^2}\mathbf{e}_1 + \frac{W_2^2F_2^{'}(Z_2)}{(W_2 - F_2(Z_2))^2}\mathbf{e}_2,$$

and since $W - F(Z) \notin \mathbb{NC}_o$, it follows that G is $\mathbb{BC}$-holomorphic on $\mathbb{B}_{\mathbb{BC}}$ with $G^{'}(0) = 1$. Thus $G \in \mathcal{F}$.

∎

Lemma 3.5. *If F be $\mathbb{BC}$-holomorphic on $\mathbb{B}_{\mathbb{BC}}$ with $\mathbb{NC}_o \notin F(\mathbb{B}_{\mathbb{BC}})$, then there exist an $\mathbb{BC}$-holomorphic function H on $\mathbb{BC}$ with $H^2 = F$.*

Proof. Let $G(0)$ be any bicomplex number with $exp\{G(0)\} = F(0)$. For any other $W \in \mathbb{B}_{\mathbb{BC}}$, let $G(W) = G(0) + \int_\gamma \frac{F^{'}(Z)}{F(Z)}dZ$, where $\gamma : [0,1]_{\mathbb{D}} \to \mathbb{BC}$ is any curve from 0 to W. From the fundamental theorem of integral calculus in $\mathbb{BC}$ [14, Theorem 33.1, page 222], it follows that

$$G^{'}(W) = \frac{F^{'}(W)}{F(W)}. \tag{3.3}$$

Note that $F(Z) \notin \mathbb{NC}_o$ for $Z \in \mathbb{B}_{\mathbb{BC}}$ so that $G'(Z)$ is well defined for all $Z \in \mathbb{B}_{\mathbb{BC}}$ showing that G is $\mathbb{BC}$-holomorphic on $\mathbb{B}_{\mathbb{BC}}$, it follows from (3.3) that

$$\begin{aligned}
[Fe^{-G}]'(W) &= \left[F_1(W_1)e^{-G_1(W_1)}\right]' \mathbf{e}_1 + \left[F_2(W_2)e^{-G_2(W_2)}\right]' \mathbf{e}_2 \\
&= \left[F_1'(W_1)e^{-G_1(W_1)} - G_1'(W_1)e^{-G_1(W_1)}F_1(W_1)\right] \mathbf{e}_1 \\
&\quad + \left[F_2'(W_2)e^{-G_2(W_2)} - G_2'(W_2)e^{-G_2(W_2)}F_2(W_2)\right] \mathbf{e}_2 \\
&= e^{-G_1(W_1)} \left[F_1'(W_1) - G_1'(W_1)F_1(W_1)\right] \mathbf{e}_1 \\
&\quad + e^{-G_2(W_2)} \left[F_2'(W_2) - F_2'(W_2)F_2(W_2)\right] \mathbf{e}_2 \\
&= 0 + 0 = 0.
\end{aligned}$$

The equation $\left[Fe^{-G}\right]'(W) = 0$ implies that $F(W) = e^{-G(W)}$.

Hence, the proof is complete if we take $H(Z) = \exp\{\frac{G(Z)}{Z}\}$ so that H is $\mathbb{BC}$-holomorphic on $\mathbb{B}_{\mathbb{BC}}$ with $H^2(Z) = F(Z), \ \forall\, Z \in \mathbb{B}_{\mathbb{BC}}$. ∎

Lemma 3.6. *Suppose $F \in \mathcal{F}$. Then for every $Z \in \mathbb{B}_{\mathbb{BC}}$, there exist an odd function $H \in \mathcal{F}$ with $H^2(Z) = F(Z^2)$.*

Proof. If $F \in \mathcal{F}$, since F is $\mathbb{BC}$-holomorphic function, we can write

$$F(Z) = F_1(Z_1)\mathbf{e}_1 + F_2(Z_2)\mathbf{e}_2.$$

Then $\mathbb{BC}$-Taylor series of F can be written as

$$\begin{aligned}
F(Z) &= F_1(Z_1)\mathbf{e}_1 + F_2(Z_2)\mathbf{e}_2 \\
&= \left(Z_1 + \sum_{n=2}^{\infty} A_{1,n}Z_1^n\right)\mathbf{e}_1 + \left(Z_2 + \sum_{n=2}^{\infty} A_{2,n}Z_2^n\right)\mathbf{e}_2 \\
&= (Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2)\left(\left(1 + \sum_{n=2}^{\infty} A_{1,n}Z_1^n\right)\mathbf{e}_1 + \left(1 + \sum_{n=2}^{\infty} A_{2,n}Z_2^n\right)\mathbf{e}_2\right).
\end{aligned}$$

Therefore,

$$\frac{F(Z)}{Z} = \left(1 + \sum_{n=2}^{\infty} A_{1,n}Z_1^n\right)\mathbf{e}_1 + \left(1 + \sum_{n=2}^{\infty} A_{2,n}Z_2^n\right)\mathbf{e}_2$$

is non-zero, non-null-cone, $\mathbb{BC}$-holomorphic function on $\mathbb{B}_{\mathbb{BC}}$. Then by Lemma 3.5, there exist an $\mathbb{BC}$-holomorphic function G on $\mathbb{B}_{\mathbb{BC}}$ such that

$$G^2(Z) = \frac{F(Z)}{Z},$$

so their idempotent decomposition form is

$$Z_1G^2(Z_1)\mathbf{e}_1 + Z_2G^2(Z_2)\mathbf{e}_2 = F_1(Z_1)\mathbf{e}_1 + F_2(Z_2)\mathbf{e}_2.$$

If we define $H(Z) = Z_1G_1(Z_1^2)\mathbf{e}_1 + Z_2G_2(Z_2^2)\mathbf{e}_2$, then clearly H is odd function and

$$\begin{aligned}
H^2(Z) &= Z_1^2G_1^2(Z_1^2)\mathbf{e}_1 + Z_2^2G_2^2(Z_2^2)\mathbf{e}_2 \\
&= F_1(Z_1^2)\mathbf{e}_1 + F_2(Z_2^2)\mathbf{e}_2.
\end{aligned}$$

Also, $H(0) = 0$ and $H'(0) = G(0) = 1$.

Now, suppose that $Y, Z \in \mathbb{B}_{\mathbb{BC}}$ and let $H(Y) = H(Z)$. Then from the $\mathbb{BC}$-univalence of F implies that $Y^2 = Z^2$. So, there are two case arise that either $Y = Z$ or $Y = -Z$. If $Y = -Z$, then $H(Y) = -H(-Z)$. But we know that H is odd function, it contradicts the assumption that $H(Y) = H(Z)$. So we conclude that $Y = Z$. Hence $H \in \mathcal{F}$. ∎

Theorem 3.7. *The class $\mathcal{F}$ is preserved under the square root $\mathbb{BC}$-transformation that is, if $F \in \mathcal{F}$ and $G(Z) = \sqrt{F(Z^2)}$, then $G \in \mathcal{F}$.*

Proof. Suppose that $F \in \mathcal{F}$ and

$$\begin{aligned} G(Z) &= \sqrt{F(Z^2)} \\ &= \left(F_1(Z_1^2)\right)^{\frac{1}{2}} \mathbf{e}_1 + \left(F_2(Z_2^2)\right)^{\frac{1}{2}} \mathbf{e}_2. \end{aligned}$$

In order to define G, we must care some point. Since $F(Z) = 0$ if and only if $Z = 0$, so it is possible to choose a single-valued branch of the square root by writing

$$\begin{aligned} G(Z) &= \left(F_1(Z_1^2)\right)^{\frac{1}{2}} \mathbf{e}_1 + \left(F_2(Z_2^2)\right)^{\frac{1}{2}} \mathbf{e}_2 \\ &= \left(Z_1^2 + \sum_{n=2}^{\infty} A_{1,n} Z_1^{2n}\right)^{\frac{1}{2}} \mathbf{e}_1 + \left(Z_2^2 + \sum_{n=2}^{\infty} A_{2,n} Z_2^{2n}\right)^{\frac{1}{2}} \mathbf{e}_2 \\ &= \left(Z_1 + \sum_{n=2}^{\infty} B_{1,n} Z_1^{2n-1}\right) \mathbf{e}_1 + \left(Z_2 + \sum_{n=2}^{\infty} B_{2,n} Z_2^{2n-1}\right) \mathbf{e}_2 \end{aligned}$$

for $|Z_1|, |Z_2| < 1$ for some coefficients $B_{1,n}, B_{2,n} \in \mathbb{C}$. Then by Lemma 3.6, $G(Z)$ is $\mathbb{BC}$-univalent on $\mathbb{B}_{\mathbb{BC}} = \mathbb{B}_{1,1} \times \mathbb{B}_{1,2}$ and that $G(Z)$ is also $\mathbb{BC}$-holomorphic on $\mathbb{B}_{\mathbb{BC}}$ with $G(0) = 0$ and $G'(0) = 1$. That is, $G \in \mathcal{F}$ and the proof is complete. ∎

4. Conclusion

In $\mathbb{BC}$ unlike in $\mathbb{C}$, there are three types of conjugations. We see that in the bicomplex univalent function theory, there is a contrast in the closure of conjugations of $\mathbb{BC}$-univalent functions. In this paper, we study the behavior and geometric structure of $\mathbb{BC}$-univalent functions. We are also able to explore an aspect of $2D$ real surface, which is the cartesian product of lines that are playing role in the process of construction of Koebe function. We conclude that this theory will form a base for geometric function theory and bicomplex dynamics.

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