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APPROXIMATELY DUAL p-APPROXIMATE SCHAUDER FRAMES

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ABSTRACT. Approximately dual frame in Hilbert spaces was introduced by Christensen and Laugesen to overcome difficulties in constructing dual frames for a given Hilbert space frame. It becomes even more difficult in Banach spaces to construct duals. For this purpose, we introduce approximately dual frames for a class of approximate Schauder frames for Banach spaces and develop basic theory. Approximate dual for this subclass is completely characterized and its perturbation is also studied.

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1. INTRODUCTION

Works of Holub [13] and Li [15] classify frames and its duals for Hilbert spaces using the standard orthonormal basis for the standard separable Hilbert space. In the course of classifying approximate Schauder frames for Banach spaces using standard Schauder bases for classical sequence spaces, we [14] introduced the notion of p-approximate Schauder frames (p-ASFs) for Banach spaces (for approximate Schauder frames see [12, 18]) which behaves in much more similar way that of frames for Hilbert spaces whose definition reads as follows. Let $\mathcal{X}$ be a separable Banach space and $\mathcal{X}^*$ be its dual.

Definition 1.1. [14] Let $\{\tau_n\}_n$ be a collection in $\mathcal{X}$ and $\{f_n\}_n$ be a collection in $\mathcal{X}^*$. The pair $(\{f_n\}_n, \{\tau_n\}_n)$ for $\mathcal{X}$ is said to be a **p-approximate Schauder frame** (p-ASF), $p \in [1, \infty)$ if the following conditions hold.

1. The **frame operator** $S_{f,\tau} : \mathcal{X} \ni x \mapsto S_{f,\tau}x := \sum_{n=1}^\infty f_n(x)\tau_n \in \mathcal{X}$ is a well-defined bounded linear invertible operator.
2. The **analysis operator** $\theta_f : \mathcal{X} \ni x \mapsto \theta_f x := \{f_n(x)\}_n \in \ell^p(\mathbb{N})$ is a well-defined bounded linear operator.
3. The **synthesis operator** $\theta_\tau : \ell^p(\mathbb{N}) \ni \{a_n\}_n \mapsto \theta_\tau\{a_n\}_n := \sum_{n=1}^\infty a_n\tau_n \in \mathcal{X}$ is a well-defined bounded linear operator.

Constants $a, b > 0$ satisfying

$$a\|x\| \leq \left\|\sum_{n=1}^\infty f_n(x)\tau_n\right\| \leq b\|x\|, \quad \forall x \in \mathcal{X}$$

are called as lower ASF bound and upper ASF bound, respectively.

Since the frame operator $S_{f,\tau}$ is invertible, we have

$$x = \sum_{n=1}^\infty (f_n S_{f,\tau}^{-1})(x)\tau_n = \sum_{n=1}^\infty f_n(x) S_{f,\tau}^{-1}\tau_n, \quad \forall x \in \mathcal{X}. \tag{1.1}$$

It is also easy to see that both $(\{f_n S_{f,\tau}^{-1}\}_n, \{\tau_n\}_n)$ and $(\{f_n\}_n, \{S_{f,\tau}^{-1}\tau_n\}_n)$ are p-ASFs for $\mathcal{X}$. In general, there are other p-ASFs that satisfy Equation (1.1). This leads to the following notion.

Definition 1.2. [14] Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathcal{X}$. A p-ASF $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{X}$ is a **dual** for $(\{f_n\}_n, \{\tau_n\}_n)$ if

$$x = \sum_{n=1}^\infty g_n(x)\tau_n = \sum_{n=1}^\infty f_n(x)\omega_n, \quad \forall x \in \mathcal{X}.$$

Following theorem classifies duals of p-ASFs, where $\{e_n\}_n$ denotes the standard Schauder basis for $\ell^p(\mathbb{N})$, $p \in [1, \infty)$ and $\{\zeta_n\}_n$ denotes the coordinate functionals associated with $\{e_n\}_n$.

Theorem 1.1. [14] *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathcal{X}$. Then a p-ASF $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{X}$ is a dual for $(\{f_n\}_n, \{\tau_n\}_n)$ if and only if*

$$\begin{aligned} g_n &= f_n S_{f,\tau}^{-1} + \zeta_n U - f_n S_{f,\tau}^{-1}\theta_\tau U, \\ \omega_n &= S_{f,\tau}^{-1}\tau_n + V e_n - V\theta_f S_{f,\tau}^{-1}\tau_n, \quad \forall n \in \mathbb{N} \end{aligned}$$

such that the operator

$$S_{f,\tau}^{-1} + VU - V\theta_f S_{f,\tau}^{-1}\theta_\tau U$$

is bounded invertible, where $U : \mathcal{X} \to \ell^p(\mathbb{N})$ and $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ are bounded linear operators.

It is known even in the case of Hilbert spaces that (see [8]) it is difficult to construct or verify whether a given collection is a dual for a frame. This leads to the notion of approximately dual frames by Christensen and Laugensen [8] which is motivated from the work of Li and Yan [16]. Following the paper [8], there is a plenty of activities on the notion of approximately dual frames and its applications to wavelet frames, Gabor systems, shift-invariant systems, localized frames, cross Gram matrices [1, 2, 6, 7, 9, 10, 11, 17] etc. This notion is also useful in the study of famous Mexican hat problem [3, 4]. One can see Chapter 6 in the monograph [5] which gives a snapshot of approximately dual frames for Hilbert spaces.

The purpose of this paper is to study the notion of approximately dual frames for p-approximately Schauder frames for Banach spaces. One can naturally ask whether the notion of approximately dual frames can be defined for approximately Schauder frames. It seems that one can formulate the notion but it is difficult to study further due to the failure of factorization property of approximaly Schauder frames.

2. APPROXIMATELY DUAL P-APPROXIMATE SCHAUDER FRAMES

To define the notion of approximately dual frames for Hilbert spaces one need not consider frames, the notion of Bessel sequences suffices. To do the same thing for Banach spaces we first weaken Definition 1.1. Our motivation comes from a characterization of Bessel sequences for Hilbert spaces which reads as follows.

Theorem 2.1. [5] *For a collection $\{\tau_n\}_n$ in a separable Hilbert space $\mathcal{H}$, the following are equivalent.*

1. *$\{\tau_n\}_n$ is a Bessel sequence for $\mathcal{H}$, i.e., there exists $b > 0$ such that*
$$\sum_{n=1}^{\infty} |\langle h, \tau_n\rangle|^2 \leq b\|h\|^2, \forall h \in \mathcal{H}.$$
2. *The map $\mathcal{H} \ni h \mapsto \{\langle h, \tau_n\rangle\}_n \in \ell^2(\mathbb{N})$ is a well-defined bounded linear operator.*
3. *The map $\ell^2(\mathbb{N}) \ni \{a_n\}_n \mapsto \sum_{n=1}^{\infty} a_n\tau_n \in \mathcal{H}$ is a well-defined bounded linear operator.*

Theorem 2.1 leads to the following in Banach spaces.

Definition 2.1. A pair $(\{f_n\}_n, \{\tau_n\}_n)$ is said to be a **p-approximate Bessel sequence** for $\mathcal{X}$ (p-ABS), $p \in [1, \infty)$ if the following conditions hold.

1. The **analysis operator** $\theta_f : \mathcal{X} \ni x \mapsto \theta_f x := \{f_n(x)\}_n \in \ell^p(\mathbb{N})$ is a well-defined bounded linear operator.
2. The **synthesis operator** $\theta_\tau : \ell^p(\mathbb{N}) \ni \{a_n\}_n \mapsto \theta_\tau\{a_n\}_n := \sum_{n=1}^{\infty} a_n\tau_n \in \mathcal{X}$ is a well-defined bounded linear operator.

Constants $c, d > 0$ satisfying
$$\left(\sum_{n=1}^{\infty} |f_n(x)|^p\right)^{\frac{1}{p}} \leq c\|x\|, \quad \forall x \in \mathcal{X}, \quad \left\|\sum_{n=1}^{\infty} a_n\tau_n\right\| \leq d\left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}}, \quad \forall \{a_n\}_n \in \ell^p(\mathbb{N}),$$
are called as analysis and synthesis bounds, respectively.

Following result gives examples of p-ABS and it also characterizes them.

Theorem 2.2. *A pair $(\{f_n\}_n, \{\tau_n\}_n)$ is a p-ABS for $\mathcal{X}$ if and only if*
$$f_n = \zeta_n U, \quad \tau_n = V e_n, \quad \forall n \in \mathbb{N},$$
where $U : \mathcal{X} \to \ell^p(\mathbb{N})$, $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ are bounded linear operators.

Proof. ($\Leftarrow$) It is easy to see that θ_f and θ_τ are bounded linear operators.
($\Rightarrow$) Define $U := \theta_f$, $V := \theta_\tau$. Then $\zeta_n Ux = \zeta_n\theta_f x = \zeta_n(\{f_k(x)\}_k) = f_n(x)$, $\forall x \in \mathcal{X}$, $Ve_n = \theta_\tau e_n = \tau_n$, $\forall n \in \mathbb{N}$. ∎

The idea behind the notion of approximate dual frames is to weaken the duality condition in Definition 1.2.

Definition 2.2. Let $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ be two p-ABSs for $\mathcal{X}$. We say that they are **approximately dual p-ABSs** if

$$\|I_{\mathcal{X}} - \theta_\omega\theta_f\| < 1 \quad \text{and} \quad \|I_{\mathcal{X}} - \theta_\tau\theta_g\| < 1.$$

In addition, if $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ are p-ASFs for $\mathcal{X}$, then we say that they are **approximately dual p-ASFs**.

In the next example we see that both conditions in Definition 2.2 are independent, this is contrast to the situation of Hilbert spaces in which adjoint operation reveals that one condition is enough.

Example 2.1. *Let R be the right-shift and L be the left-shift operator on $\ell^p(\mathbb{N})$. Define $f_n := \zeta_n$, $\tau_n := Re_n$, $g_n := \zeta_n L$, $\omega_n := e_n$. Then $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ are p-ABSs for $\ell^p(\mathbb{N})$ but*

$$\|I_{\ell^p(\mathbb{N})} - \theta_\omega\theta_f\| = \|I_{\ell^p(\mathbb{N})} - I_{\ell^p(\mathbb{N})}\| < 1, \quad \|I_{\ell^p(\mathbb{N})} - \theta_\tau\theta_g\| = \|I_{\ell^p(\mathbb{N})} - RL\| = 1.$$

In the next example we show that there are p-ABSs which are not p-ASFs. This is in contrast with Hilbert space situation. Using Lemma 6.3.2 in [5] we can show that whenever two Bessel sequences for a Hilbert space are approximately duals, then they are necessarily frames.

Example 2.2. *Let R and L be as in Example 2.1. Define $f_n := \zeta_n$, $\tau_n := Le_n$, $g_n := \zeta_n R$, $\omega_n := e_n$. Then $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ are p-ABSs for $\ell^p(\mathbb{N})$ and*

$$\|I_{\ell^p(\mathbb{N})} - \theta_\omega\theta_f\| = \|I_{\ell^p(\mathbb{N})} - I_{\ell^p(\mathbb{N})}\| < 1, \quad \|I_{\ell^p(\mathbb{N})} - \theta_\tau\theta_g\| = \|I_{\ell^p(\mathbb{N})} - LR\| < 1.$$

Hence $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ are approximately dual p-ABSs for $\ell^p(\mathbb{N})$. Now note that $\theta_\tau\theta_f = L$ and $\theta_\omega\theta_g = R$. Since both left and right shift operators are not invertible, both $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ are not p-ASFs for $\ell^p(\mathbb{N})$.

Remark 2.1.
(1) Inequalities in Definition 2.2 say that both $S_{f,\omega}$ and $S_{g,\tau}$ are invertible. Thus if $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ are approximately dual p-ABSs for $\mathcal{X}$, then $(\{f_n\}_n, \{\omega_n\}_n)$ and $(\{g_n\}_n, \{\tau_n\}_n)$ are p-ASFs for $\mathcal{X}$.
(2) Proposition 2.13 in [14] says that dual p-ASFs are always approximately dual p-ASFs. Since dual p-ASF always exists for every p-ASF, it follows that every p-ASF has approximate dual p-ASF.

Definition 2.2 gives the following proposition.

Proposition 2.3. *If $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ are approximately dual p-ABSs for $\mathcal{X}$, then*

$$\left\|x - \sum_{n=1}^\infty f_n(x)\omega_n\right\| < \|x\| \quad \text{and} \quad \left\|x - \sum_{n=1}^\infty g_n(x)\tau_n\right\| < \|x\|, \quad \forall x \in \mathcal{X} \setminus \{0\}.$$

Proof. Given $x \in \mathcal{X} \setminus \{0\}$,

$$\left\|x - \sum_{n=1}^{\infty} f_n(x)\omega_n\right\| = \|I_{\mathcal{X}}x - \theta_\omega\theta_f x\| \leq \|I_{\mathcal{X}}x - \theta_\omega\theta_f\|\|x\| < \|x\|,$$
$$\left\|x - \sum_{n=1}^{\infty} g_n(x)\tau_n\right\| = \|I_{\mathcal{X}}x - \theta_\tau\theta_g x\| \leq \|I_{\mathcal{X}} - \theta_\tau\theta_g\|\|x\| < \|x\|. \blacksquare$$

Our next agenda gives a complete description of p-ABSs and p-AFSs. For this purpose, we weaken Definition 1.2.

Definition 2.3. Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ABS for $\mathcal{X}$. A p-ABS $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{X}$ is a **dual** for $(\{f_n\}_n, \{\tau_n\}_n)$ if

$$x = \sum_{n=1}^{\infty} g_n(x)\tau_n = \sum_{n=1}^{\infty} f_n(x)\omega_n, \quad \forall x \in \mathcal{X}.$$

In the case of Hilbert spaces, existence of dual Bessel sequence forces the Bessel sequence to a frame (Lemma 6.3.2 in [5]) However, Example 2.2 reveals that existence of dual Bessel sequences do not imply that they are frames. In the next proposition we show that approximate duals generate duals whose proof is a routine computation.

Proposition 2.4. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ be approximately dual p-ABSs for $\mathcal{X}$. Then $(\{g_nS_{g,\tau}^{-1}\}_n, \{S_{f,\omega}^{-1}\omega_n\}_n)$ is a dual p-ABS for $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{f_nS_{f,\omega}^{-1}\}_n, \{S_{g,\tau}^{-1}\tau_n\}_n)$ is a dual p-ABS for $(\{g_n\}_n, \{\omega_n\}_n)$.*

An operator-theoretic description of approximately duals can be given which is illustrated in the next result.

Theorem 2.5. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ABS for $\mathcal{X}$. A p-ABS $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{X}$ is an approximately dual p-ABS for $(\{f_n\}_n, \{\tau_n\}_n)$ if and only if there exist bounded linear operators $U, V : \mathcal{X} \to \mathcal{X}$ satisfying $\|I_{\mathcal{X}} - U\| < 1$ and $\|I_{\mathcal{X}} - V\| < 1$ such that $(\{h_n := g_nU^{-1}\}_n, \{\rho_n := V^{-1}\omega_n\}_n)$ is a dual for $(\{f_n\}_n, \{\tau_n\}_n)$. Statement holds even if approximately dual p-ABS is replaced by approximately dual p-ASF.*

Proof. $(\Rightarrow)$ Define $U := \theta_\tau\theta_g$ and $V := \theta_\omega\theta_f$. We then have $\|I_{\mathcal{X}} - U\| < 1$, $\|I_{\mathcal{X}} - V\| < 1$ and

$$\sum_{n=1}^{\infty} h_n(x)\tau_n = \sum_{n=1}^{\infty} g_n(U^{-1}x)\tau_n = \theta_\tau\theta_g U^{-1}x = UU^{-1}x = x,$$
$$\sum_{n=1}^{\infty} f_n(x)\rho_n = \sum_{n=1}^{\infty} f_n(x)V^{-1}\omega_n = V^{-1}\theta_\omega\theta_f = V^{-1}Vx = x, \quad \forall x \in \mathcal{X}.$$

Hence $(\{h_n\}_n, \{\rho_n\}_n)$ is a dual for $(\{f_n\}_n, \{\tau_n\}_n)$.

$(\Leftarrow)$ A direct computation says that $\theta_h = \theta_g U^{-1}$ and $\theta_\rho = V^{-1}\theta_\omega$. Since $(\{h_n\}_n, \{\rho_n\}_n)$ is a dual for $(\{f_n\}_n, \{\tau_n\}_n)$, we get that

$$\|I_{\mathcal{X}} - \theta_\omega\theta_f\| = \|I_{\mathcal{X}} - V\theta_\rho\theta_f\| = \|I_{\mathcal{X}} - V\| < 1,$$
$$\|I_{\mathcal{X}} - \theta_\tau\theta_g\| = \|I_{\mathcal{X}} - \theta_\tau\theta_h U\| = \|I_{\mathcal{X}} - U\| < 1.$$

Hence $(\{g_n\}_n, \{\omega_n\}_n)$ is an approximately dual p-ABS for $(\{f_n\}_n, \{\tau_n\}_n)$. $\blacksquare$

Theorem 2.5 combined with Theorem 1.1 gives the following result for approximate dual frames.

Theorem 2.6. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathcal{X}$. A p-ASF $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{X}$ is an approximately dual p-ASF for $(\{f_n\}_n, \{\tau_n\}_n)$ if and only if there exist bounded linear operators $U, V : \mathcal{X} \to \mathcal{X}$, $A : \mathcal{X} \to \ell^p(\mathbb{N})$, $B : \ell^p(\mathbb{N}) \to \mathcal{X}$ satisfying $\|I_{\mathcal{X}} - U\| < 1$, $\|I_{\mathcal{X}} - V\| < 1$,*

$$
\begin{aligned}
g_n &= f_n S_{f,\tau}^{-1} U + \zeta_n A U - f_n S_{f,\tau}^{-1} \theta_\tau A U, \\
\omega_n &= V S_{f,\tau}^{-1} \tau_n + V B e_n - V B \theta_f S_{f,\tau}^{-1} \tau_n, \quad \forall n \in \mathbb{N}
\end{aligned}
$$

such that the operator

$$S_{f,\tau}^{-1} + BA - V\theta_f S_{f,\tau}^{-1} \theta_\tau A$$

is bounded invertible.

In [8], Christensen and Laugensen constructed iterations which construct approximately duals at each stage and converge to the identity operator in operator norm. Here we have a similar result for p-ABSs.

Theorem 2.7. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ABS for $\mathcal{X}$ and $(\{g_n\}_n, \{\omega_n\}_n)$ be a p-ABS for $\mathcal{X}$ which is approximately dual for $(\{f_n\}_n, \{\tau_n\}_n)$.*

(1) *The dual p-ABS $(\{g_n S_{g,\tau}^{-1}\}_n, \{S_{f,\omega}^{-1}\omega_n\}_n)$ for $(\{f_n\}_n, \{\tau_n\}_n)$ can be written as*

$$
\begin{aligned}
g_n S_{g,\tau}^{-1} &= g_n + \sum_{m=1}^{\infty} g_n (I_{\mathcal{X}} - S_{g,\tau})^m, \\
S_{f,\omega}^{-1}\omega_n &= \omega_n + \sum_{m=1}^{\infty} (I_{\mathcal{X}} - S_{f,\omega})^m \omega_n, \quad \forall n \in \mathbb{N}.
\end{aligned}
$$

(2) *For $N \in \mathbb{N}$, define*

$$
\begin{aligned}
h_n^{(N)} &\coloneqq g_n + \sum_{m=1}^{N} g_n (I_{\mathcal{X}} - S_{g,\tau})^m, \\
\rho_n^{(N)} &\coloneqq \omega_n + \sum_{m=1}^{N} (I_{\mathcal{X}} - S_{f,\omega})^m \omega_n, \quad \forall n \in \mathbb{N}.
\end{aligned}
$$

Then $(\{h_n^{(N)}\}_n, \{\rho_n^{(N)}\}_n)$ is a p-ABS and is an approximately dual for $(\{f_n\}_n, \{\tau_n\}_n)$, for each $N \in \mathbb{N}$. Moreover,

$$
\begin{aligned}
\|I_{\mathcal{X}} - \theta_\rho^{(N)} \theta_f\| &\le \|I_{\mathcal{X}} - S_{f,\omega}\|^{N+1} \to 0 \quad \text{as } N \to \infty, \\
\|I_{\mathcal{X}} - \theta_\tau \theta_h^{(N)}\| &\le \|I_{\mathcal{X}} - S_{g,\tau}\|^{N+1} \to 0 \quad \text{as } N \to \infty.
\end{aligned}
$$

Proof. (1) This follows from the Neumann series

$$S_{g,\tau}^{-1} = \sum_{m=0}^{\infty} (I_{\mathcal{X}} - S_{g,\tau})^m, \quad S_{f,\omega}^{-1} = \sum_{m=0}^{\infty} (I_{\mathcal{X}} - S_{f,\omega})^m.$$

(2) Clearly $(\{h_n^{(N)}\}_n, \{\rho_n^{(N)}\}_n)$ is a p-ABS. Now consider

$$\begin{aligned}
\theta_\rho^{(N)}\theta_f x &= \sum_{n=1}^\infty f_n(x)\rho_n^{(N)} = \sum_{n=1}^\infty f_n(x)\sum_{m=0}^N (I_\mathcal{X} - S_{f,\omega})^m\omega_n \\
&= \sum_{m=0}^N (I_\mathcal{X} - S_{f,\omega})^m \sum_{n=1}^\infty f_n(x)\omega_n = \sum_{m=0}^N (I_\mathcal{X} - S_{f,\omega})^m S_{f,\omega}x \\
&= \sum_{m=0}^N (I_\mathcal{X} - S_{f,\omega})^m (I_\mathcal{X} - (I_\mathcal{X} - S_{f,\omega}))x = x - (I_\mathcal{X} - S_{f,\omega})^{N+1}x, \quad \forall x \in \mathcal{X} \\
&\implies \|I_\mathcal{X} - \theta_\rho^{(N)}\theta_f\| \le \|I_\mathcal{X} - S_{f,\omega}\|^{N+1}
\end{aligned}$$

and

$$\begin{aligned}
\theta_\tau\theta_h^{(N)}x &= \sum_{n=1}^\infty h_n^{(N)}(x)\tau_n = \sum_{n=1}^\infty \sum_{m=0}^N g_n((I_\mathcal{X} - S_{g,\tau})^m x)\tau_n \\
&= \sum_{m=0}^N \sum_{n=1}^\infty g_n((I_\mathcal{X} - S_{g,\tau})^m x)\tau_n = \sum_{m=0}^N S_{g,\tau}(I_\mathcal{X} - S_{g,\tau})^m x \\
&= \sum_{m=0}^N (I_\mathcal{X} - (I_\mathcal{X} - S_{g,\tau}))(I_\mathcal{X} - S_{g,\tau})^m x = x - (I_\mathcal{X} - S_{g,\tau})^{N+1}x, \quad \forall x \in \mathcal{X} \\
&\implies \|I_\mathcal{X} - \theta_\tau\theta_h^{(N)}\| \le \|I_\mathcal{X} - S_{g,\tau}\|^{N+1}.
\end{aligned}$$

Conclusion follows from $\|I_\mathcal{X} - S_{f,\omega}\| < 1$ and $\|I_\mathcal{X} - S_{g,\tau}\| < 1$. ∎

In [8], Christensen and Laugensen showed that by considering sequences close to a given frame one can generate approximate duals. Here is the similar result in the context of Banach spaces.

Theorem 2.8. *Let $\{\tau_n\}_n$ be a collection in $\mathcal{X}$ and $\{f_n\}_n$ be a collection in $\mathcal{X}^*$. Let $(\{h_n\}_n, \{\rho_n\}_n)$ be a p-ASF for $\mathcal{X}$ such that*

$$\left(\sum_{n=1}^\infty |f_n(x) - h_n(x)|^p\right)^{\frac{1}{p}} \le R\|x\|, \quad \forall x \in \mathcal{X} \tag{2.1}$$

and

$$\left\|\sum_{n=1}^\infty a_n(\rho_n - \tau_n)\right\| \le Q\left(\sum_{n=1}^\infty |a_n|^p\right)^{\frac{1}{p}}, \quad \forall \{a_n\}_n \in \ell^p(\mathbb{N}). \tag{2.2}$$

Let $(\{g_n\}_n, \{\omega_n\}_n)$ be a dual frame for $(\{h_n\}_n, \{\rho_n\}_n)$ with analysis bound c and synthesis bound d. If $dR, cQ < 1$, then $(\{g_n\}_n, \{\omega_n\}_n)$ is an approximately dual for $(\{f_n\}_n, \{\tau_n\}_n)$.

Proof. Inequalities (2.1) and (2.2) (using Minkowski's and triangle inequalities) say that $(\{f_n\}_n, \{\tau_n\}_n)$ is a p-ABS for $\mathcal{X}$. Note that Inequality (2.1) can be written as $\|\theta_h x - \theta_f x\| \le R\|x\|, \forall x \in \mathcal{X}$. Similarly, Inequality (2.2) can be written as $\|\theta_\tau\{a_n\}_n - \theta_\rho\{a_n\}_n\| \le Q\|\{a_n\}_n\|, \forall \{a_n\}_n \in \ell^p(\mathbb{N})$. Using these, we have

$$\|I_{\ell^p(\mathbb{N})} - \theta_\omega\theta_f\| = \|\theta_\omega\theta_h - \theta_\omega\theta_f\| \le \|\theta_\omega\|\|\theta_h - \theta_f\| \le dR < 1,$$

and

$$\|I_{\ell^p(\mathbb{N})} - \theta_\tau\theta_g\| = \|\theta_\rho\theta_g - \theta_\tau\theta_g\| \le= \|\theta_\rho - \theta_\tau\|\|\theta_g\| \le cQ < 1$$

which is required. ∎

REFERENCES

[1] P. BALAZS, M. SHAMSABADI, A. A. AREFIJAMAAL AND A. RAHIMI, U-cross Gram matrices and their invertibility, *J. Math. Anal. Appl.* ***476***, 2 (2019), pp. 367–390.

[2] A. BENAVENTE, O. CHRISTENSEN AND M. I. ZAKOWICZ, Generalized shift-invariant systems and approximately dual frames, *Ann. Funct. Anal.* ***8***, 2 (2017), pp. 177–189.

[3] H. Q. BUI AND R. S. LAUGESEN, Frequency-scale frames and the solution of the Mexican hat problem, *Constr. Approx.* ***33***, 2 (2011), pp. 163–189.

[4] H. Q. BUI AND R. S. LAUGESEN, Wavelets in Littlewood-Paley space, and Mexican hat completeness, *Appl. Comput. Harmon. Anal.* ***30***, 2 (2011), pp. 204–213.

[5] O. CHRISTENSEN, *An Introduction to Frames and Riesz Bases*, second ed, Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, [Cham], 2016.

[6] O. CHRISTENSEN, A. J. E. M. JANSSEN, H. O. KIM AND R. Y. KIM, Approximately dual Gabor frames and almost perfect reconstruction based on a class of window functions, *Adv. Comput. Math.* ***44***, 5 (2018), pp. 1519–1535.

[7] O. CHRISTENSEN, H. O. KIM AND R. Y. KIM, B-spline approximations of the Gaussian, their Gabor frame properties, and approximately dual frames, *J. Fourier Anal. Appl.* ***24***, 4 (2018), pp. 1119–1140.

[8] O. CHRISTENSEN AND R. S. LAUGESEN, Approximately dual frames in Hilbert spaces and applications to Gabor frames, *Sampl. Theory Signal Image Process* ***9***, 1-3 (2010), pp. 77–89.

[9] M. DÖRFLER AND E. MATUSIAK, Nonstationary Gabor frames—approximately dual frames and reconstruction errors, *Adv. Comput. Math.* ***41***, 2 (2015), pp. 293–316.

[10] H. G. FEICHTINGER, A. GRYBOS AND D. M. ONCHIS, Approximate dual Gabor atoms via the adjoint lattice method, *Adv. Comput. Math.* ***40***, 3 (2014), pp. 651–665.

[11] H. G. FEICHTINGER, D. M. ONCHIS AND C. WIESMEYR, Construction of approximate dual wavelet frames, *Adv. Comput. Math.* ***40***, 1 (2014), pp. 273–282.

[12] D. FREEMAN, E. ODELL, T. SCHLUMPRECHT AND A. ZSÁK, Unconditional structures of translates for $L_p(\mathbb{R}^d)$, *Israel J. Math.* ***203***, 1 (2014), pp. 189–209.

[13] J. R.HOLUB, Pre-frame operators, Besselian frames, and near-Riesz bases in Hilbert spaces, *Proc. Amer. Math. Soc.* ***122***, 3 (1994), pp. 779–785.

[14] K. M. KRISHNA AND P. S. JOHNSON, Towards characterizations of approximate Schauder frame and its duals for Banach spaces, *J. Pseudo-Differ. Oper. Appl.* ***12***, 1 (2021), Paper No. 9, 13.

[15] S. LI, On general frame decompositions, *Numer. Funct. Anal. Optim.* ***16***, 9-10 (1995), pp. 1181–1191.

[16] S. LI AND D. YAN, Frame fundamental sensor modeling and stability of one-sided frame perturbation, *Acta Appl. Math.* ***107***, 1-3 (2009), pp. 91–103.

[17] Q. LIAN AND M. YOU, Approximately dual frames of nonstationary Gabor frames for $l^2(\mathbb{Z})$ and reconstruction errors, *Math. Methods Appl. Sci.* ***43***, 11 (2020), pp. 6643–6662.

[18] S. M. THOMAS, Approximate Schauder frames for $\mathbb{R}^n$, Masters Thesis, St. Louis University, St. Louis, MO.