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# TOPOLOGICAL ASPECTS OF DISCRETE SWITCH DYNAMICAL SYSTEMS

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ABSTRACT. The notion of non-autonomous discrete dynamical systems is well studied in the literature. On the other hand, a similar idea exists in literature for continuous dynamical systems with the name switch dynamical systems. In this article, we interpret a non-autonomous dynamical system as a switch system and describe how the dynamics of a non-autonomous dynamical system can be better understood using the notion of switch.

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## 1. INTRODUCTION

A continuous dynamical system on a topological space $X$, is the action of a semigroup $\{f^t : t \in \mathbb{R}$ and $t \geq 0\}$ on $X$, where each $f^t$ is a continuous self map on $X$ (or the group $\{f^t : t \in \mathbb{R}\}$, if each $f^t$ is a homeomorphism) such that $f^0$ is the identity map and $f^{t+s} = f^t \circ f^s$, for every $t$ and $s$. Now, instead of considering this one-parameter family of maps, if we have more than one such family, say $\{f_i^t : t \in \mathbb{R}$ and $t \geq 0\}$, $i \in \{1, 2, ..., k\}$ and consider the action of different families at different instances of time, then we obtain a new notion of dynamics, called a continuous switch dynamical system. This idea of “action of different functions at different instances" is explained more precisely using a “switch function".

A continuous dynamical system arises naturally from the first order autonomous system of ordinary differential equations. Now, let $h_i : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$-map with bounded derivative for each $i \in \{1, 2, ..., k\}$. Consider the initial value problems

$$\begin{aligned} \frac{d}{dt}x^{(i)}(t) &= h_i(x^{(i)}), t \in \mathbb{R}, \\ x^{(i)}(0) &= x_0. \end{aligned} \tag{1.1}$$

As it is well known, for each $i \in \{1, 2, ..., k\}$, if $\phi_i(t, x_0)$ is the solution map of (1.1), then $f_i^t(x) = \phi_i(t, x)$ gives a continuous dynamical system.

Let $\sigma : \mathbb{R} \to \{1, 2, ..., k\}$ be a piecewise constant function i.e., $\sigma$ has finitely many discontinuities in any bounded interval and on the interval between any two consecutive discontinuities, $\sigma$ is constant. The function $\sigma$ is called a *switch function* and its discontinuities are called *switches*. For each $i \in \{1, 2, ..., k\}$, let $J_i = \sigma^{-1}(i)$, which is a union of intervals, the endpoints of each of which are the consecutive switches of $\sigma$. Note that $\mathbb{R}$ is the disjoint union of $J_i$’s.

Now, consider the following system.

$$\begin{aligned} \dot{x}(t) &= h_{\sigma(t)}(x),\ t \in \mathbb{R}, \\ x(0) &= x_0. \end{aligned} \tag{1.2}$$

By a solution $\phi(t, x_0)$ of this system (1.2), we mean $\phi(t, x_0) = \phi_i(t, x_0)$, where $i \in \{1, 2, ..., k\}$ is the unique index such that $t \in J_i$ and $\phi_i(t, x_0)$ is the solution of (1.1) for the respective value of $i$. These solutions of (1.2) corresponding to all values of $x_0$ give rise to a continuous switch dynamical system in the following way.

For each $i \in \{1, 2, ..., k\}$, consider the one-parameter family of functions, $\mathcal{F}_i = \{f_i^t : t \in \mathbb{R}\}$, where $f_i^t(x) = \phi_i(t, x)$. The triplet $(\mathbb{R}^n, \{\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_k\}, \sigma)$ is called a *continuous switch dynamical system*. The trajectory of a point $x \in \mathbb{R}^n$ in this system is given by $(x_t)_{t\in\mathbb{R}}$, where $x_t = f_{\sigma(t)}^t(x)$. Continuous switch dynamical systems are studied by many people in literature with main focus on stability (see, for instance [1], [3], [4] and [7]).

A *non-autonomous discrete dynamical system* (NDS), on the other hand, is defined as a topological space $X$ together with a sequence $(f_n)_{n\geq 1}$ of continuous self maps. In this system, the trajectory of a point $x \in X$ is defined as $(x_n)_{n\geq 0}$, where $x_n = f_n(x_{n-1})$ for every $n \geq 1$ and $x_0 = x$. The theory of NDS has an extensive literature (see [5], [6], [8] and [9]).

In this article, we interpret the dynamics of an NDS in the form of a switch system. We call it as a discrete switch dynamical system which will be defined in the next section followed by a discussion on other concepts of NDS using the idea of a switch.

The paper is organised as follows. Throughout this paper, in most of the instances, a discrete switch dynamical system will be referred to as a *switch system* whereas a discrete dynamical system will be called a *usual system*. In the next section, we introduce the terminology related to study of switch systems. We prove some results on periodicity, transitivity, recurrent points, conjugacy and minimality in Section 3. Then, in Section 4, we study switch systems of circle rotations. Here again, we discuss the periodicity of these switching rotations on the circle.

## 2. TERMINOLOGY

We now develop terminology for switch systems. Most of the concepts defined here for switch systems are generalizations of the corresponding notions in usual systems. However, there are instances, where a switch system $(X, \mathfrak{F}, \sigma)$ has a particular property with no usual individual system $(X, f)$ for $f \in \mathfrak{F}$ having it. For instance, Example 3.1 gives a switch system $(X, \{f_1, f_2\}, \sigma)$ with $X = [0, 2] \cup [4, 6]$, in which the point $x = 1$ is recurrent but it is recurrent neither in $(X, f_1)$ nor in $(X, f_2)$. We now begin with definitions of a switch system and the trajectory of a point in it.

**Definition 2.1.** Let $X$ be a topological space, $\mathfrak{F} = \{f_1, f_2, ..., f_k\}$ be a family of continuous self maps on $X$ and $\sigma : \mathbb{N} \to \{1, 2, ..., k\}$ be any map. The triplet $(X, \mathfrak{F}, \sigma)$ is called a *discrete switch dynamical system*. For each point $x$ in $X$, the trajectory $(x_n)_{n \geq 0}$ of $x$ is defined as $x_0 = x$ and $x_n = f_{\sigma(n)}(x_{n-1})$ for every $n \in \mathbb{N}$. The map $\sigma$ is called the switch function or simply as the switch of the system $(X, \mathfrak{F}, \sigma)$.

In a usual system $(X, f)$, the trajectory of a point $x \in X$ is defined as $(f^n(x))_{n \in \mathbb{N}_0}$ i.e, the $n^{th}$ term in the trajectory of $x$ is given by $f^n(x)$. In a switch system, the $n^{th}$ term of the trajectory of $x$ is given by $x_n = f_{(\sigma(n))} f_{(\sigma(n-1))} ... f_{(\sigma(1))}(x)$ i.e., the switch $\sigma$ specifies which function to apply at the $n^{th}$ time.

We assume that, in all the switch systems that are considered in this paper, the switch $\sigma$ is surjective and $\sigma^{-1}(i)$ is an infinite set for every $i \in \{1, 2, ..., k\}$, i.e., each $f_i$ in $\mathfrak{F}$ occurs infinitely many times in every trajectory. As a convention, we define $\sigma(0) = 0$ and $f_{\sigma(0)}(x) = x$, for every $x \in X$.

**Definition 2.2.** A switch function $\sigma$ is said to be a periodic function with a period $u$, if $\sigma(nu + l) = \sigma(l)$ for every $1 \leq l \leq u$ and for every $n \in \mathbb{N}$.

We will now define various dynamical notions for switch systems.

**Definition 2.3.** Let $(X, \mathfrak{F}, \sigma)$ be a switch system.

(1) A point $x \in X$ is called a periodic point, if there is an $m \in \mathbb{N}$ such that $x_{nm+l} = x_l$ for every $0 \leq l < m$ and for every $n \in \mathbb{N}$. Each such positive integer $m$ is called a period of $x$ and the least among them is called the least period of $x$.
(2) A point $x \in X$ is called a fixed point, if $x_n = x$ for every $n \in \mathbb{N}$.
(3) Let $x \in X$. An element $y \in X$ is called an $\omega$-limit point of $x$, if there is a sequence $(n_m)$ of positive integers such that $(n_m) \to \infty$ and $(x_{n_m}) \to y$. The set of all $\omega$-limit points of $x$ is denoted by $\omega(x)$. Further, if $x \in \omega(x)$, then $x$ is called a recurrent point.
(4) $(X, \mathfrak{F}, \sigma)$ is called topologically transitive, if there is an $x \in X$ such that $\overline{\{x_n : n \in \mathbb{N}_0\}} = X$.
(5) A subset $V \subset X$ is called invariant, if for every $x \in V$, $x_n \in V$ for every $n \in \mathbb{N}$.
(6) A closed non-empty invariant subset $V$ of $X$ is called a minimal set, if $V$ does not contain a proper closed, non-empty and invariant set. If $X$ itself is a minimal set, then $(X, \mathfrak{F}, \sigma)$ is a called a minimal switch system.

We now define the notion of topological conjugacy. In usual dynamical systems, $(X, f)$ is said to be topologically conjugate to $(Y, g)$ if there is a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. It follows that for every term $(f^n(x))$ in the trajectory of a point $x \in X$, we have $h(f^n(x)) = g^n(h(x))$. However, for switch systems, we need to ensure this as a part of the definition. After the following definition, we will simply use the words *conjugacy* and *conjugate* instead of *topological conjugacy* and *topologically conjugate* respectively.

**Definition 2.4.** Let $(X, \mathfrak{F}_1, \sigma_1)$ and $(Y, \mathfrak{F}_2, \sigma_2)$ be two switch dynamical systems. If there is a homeomorphism $h : X \to Y$ such that for any $x \in X$, $h(x_n) = (h(x))_n$ for every $n \in \mathbb{N}$, where $(x_n)$ and $((h(x))_n)$ are the trajectories of $x$ and $h(x)$ in $(X, \mathfrak{F}_1, \sigma_1)$ and $(Y, \mathfrak{F}_2, \sigma_2)$ respectively, then $h$ is called a topological conjugacy from $(X, \mathfrak{F}_1, \sigma_1)$ to $(Y, \mathfrak{F}_2, \sigma_2)$. In such a case, the two switch systems are said to be topologically conjugate.

## 3. MAIN RESULTS

In this section, we state and prove some results about the periodicity, transitivity, $\omega$-limit points, conjugacy and minimality. The following theorem gives a sufficient condition for existence of periodic points in a switch system. In a usual system $(X, f)$, for a point $x \in X$, if $f^m(x) = x$ for some $m \in \mathbb{N}$, then $x$ is a periodic point, but in a switch system $(X, \mathfrak{F}, \sigma)$, $x_m = x$ does not imply that $x$ is periodic. However, if $\sigma$ is a periodic function with the same integer $m$ as a period, then we prove in the following theorem that $x$ is a periodic point.

**Theorem 3.1.** *Let $(X, \mathfrak{F}, \sigma)$ be a switch dynamical system, where $\sigma$ is a periodic function with a period $m \in \mathbb{N}$. If $x \in X$ such that $x_m = x$, then $x$ is periodic in $(X, \mathfrak{F}, \sigma)$.*

*Proof.* It is enough to prove that $x_n = x_{n(mod\ m)}$ for every $n \in \mathbb{N}$. If $m = 1$, then $\sigma(n) = \sigma(1)$ for every $n \in \mathbb{N}$ i.e., $\sigma$ is a constant function. Then $\mathfrak{F}$ consists of only one function, say $f$ and we have a usual dynamical system $(X, f)$. Then $x_1 = x$ is same as saying that $f(x) = x$ and hence $x$ is periodic (in fact, a fixed point).

Now, consider the case where $m > 1$. It is obvious that $x_1 = x_{1(mod\ m)}$. Suppose that $x_n = x_{n(mod\ m)}$ for some $n \in \mathbb{N}$. We now claim that $x_{n+1} = x_{n+1(mod\ m)}$. We have $n = rm + l$ for some $0 \leq l < m$, so that $x_n = x_l$. Then, $n + 1 = rm + l + 1$ would imply that $\sigma(n + 1) = \sigma(l + 1)$ and thus, $x_{n+1} = f_{\sigma(n+1)}(x_n) = f_{\sigma(l+1)}(x_l) = x_{l+1}$. If $0 \leq l \leq m - 2$, then $l + 1 = n + 1(mod\, m)$ and thus we have $x_{n+1} = x_{n+1(mod\ m)}$. In case $l = m - 1$, we have $x_{n+1} = x_{l+1} = x_m$. Since it is given that $x_m = x$, it follows that $x_{n+1} = x = x_0 = x_{n+1(mod\ m)}$. Hence, by induction, we conclude that $x$ is periodic in $(X, \mathfrak{F}, \sigma)$. ∎

In the following proposition, we characterize the fixed points of a switch system $(X, \mathfrak{F}, \sigma)$ in terms of fixed points of the usual individual systems, $(X, f_i)$, where $\mathfrak{F} = \{f_1, f_2, ..., f_k\}$.

**Proposition 3.2.** *Let $(X, \mathfrak{F}, \sigma)$ be a switch system, where $\mathfrak{F} = \{f_1, f_2, ..., f_k\}$. An element $x \in X$ is a fixed point in $(X, \mathfrak{F}, \sigma)$ if and only if $x$ is a fixed point in $(X, f_i)$ for every $1 \leq i \leq k$.*

*Proof.* Suppose $x$ is a fixed point in $(X, f_i)$ for each $i$. Then $x_1 = f_{\sigma(1)}(x) = x$. Further, if $x_n = x$ for some $n \in \mathbb{N}$, then $x_{n+1} = f_{\sigma(n+1)}(x_n) = f_{\sigma(n+1)}(x) = x$. Thus, by induction, $x_n = x$ for every $n \in \mathbb{N}$ and hence $x$ is a fixed point in $(X, \mathfrak{F}, \sigma)$.

Now, assume that $x$ is a fixed point in $(X, \mathfrak{F}, \sigma)$. Fix an $f_i \in \mathfrak{F}$ for some $1 \leq i \leq k$. We know that $\sigma^{-1}(i)$ is an infinite subset of $\mathbb{N}$. Choose an $r \in \sigma^{-1}(i)$. Then, using the hypothesis that $x_n = x$ for every $n \in \mathbb{N}$, we get $f_i(x) = f_{\sigma(r)}(x) = f_{\sigma(r)}(x_{r-1}) = x_r = x$.

Thus, $x$ is a fixed point of $f_i$. ∎

In literature, a usual dynamical system is said to be topologically transitive, if it has a dense forward orbit. Notice that we have adopted the same definition for topological transitivity to switch systems. Under certain mild conditions on a usual dynamical system $(X, f)$, it can be proved that, if for any two non-empty open sets $U$ and $V$ in $X$, there exists $x \in U$ with $f^n(x) \in V$ for some $n \in \mathbb{N}$, then $(X, f)$ is topologically transitive (See Proposition 2.2.1, [2]).

Here we prove a similar result for switch systems assuming the same conditions. The proof given here uses the ideas, similar to those used in proving the above mentioned result for usual dynamical systems in [2].

**Theorem 3.3.** *Let $X$ be a second countable locally compact Hausdorff space. If for any two non-empty open sets $U$ and $V$ in $X$, there exists an $x \in U$ such that $x_n \in V$ for some $n \in \mathbb{N}$, then $(X, \mathfrak{F}, \sigma)$ is topologically transitive.*

*Proof.* Fix an open set $V$ in $X$. Define $V' = \bigcup\limits_{n=1}^{\infty} f_{\sigma(1)}^{-1} f_{\sigma(2)}^{-1} ... f_{\sigma(n)}^{-1}(V)$. If $U$ is any non-empty open set in $X$, then it is given that there is an $x \in U$ with $x_n \in V$ for some $n \in \mathbb{N}$. Now, $x_n \in V$ implies that $f_{\sigma(n)}(f_{\sigma(n-1)}(...(f_{\sigma(1)}(x))...)) \in V$ and thus $x \in V' \cap U$. Since this is true for any non-empty open set $U$, it follows that $V'$ is dense in $X$.

Choose a countable basis for $X$, say $\{V_i : i \in \mathbb{N}\}$. For each $V_i$, define $V_i' = \bigcup\limits_{n=1}^{\infty} f_{\sigma(1)}^{-1} f_{\sigma(2)}^{-1} .. ..f_{\sigma(n)}^{-1}(V_i)$. It follows from the above discussion that $V_i'$ is dense in $X$ for each $i$. Thus, the set $Y = \bigcap\limits_{i=1}^{\infty} V_i'$ is intersection of countably many open dense sets in $X$. Since $X$ is locally compact and Hausdorff, it is a Baire space. Therefore $Y \neq \emptyset$.

Now, choose $y \in Y$. Then $y \in V_i'$ for each $i \in \mathbb{N}$. This implies that $y \in f_{\sigma(1)}^{-1} f_{\sigma(2)}^{-1} ... f_{\sigma(n)}^{-1}(V_i)$ and thus $y_n \in V_i$ for some $n \in \mathbb{N}$. Thus, $\{y_n : n \in \mathbb{N}_0\} \cap V_i \neq \emptyset$ for each $i \in \mathbb{N}$. Since $\{V_i : i \in \mathbb{N}\}$ is a basis for $X$, we get $\overline{\{y_n : n \in \mathbb{N}_0\}} = X$. Thus, $(X, \mathfrak{F}, \sigma)$ is transitive. ∎

We now turn our attention towards the study of recurrent points and $\omega$-limit points. The following example shows that a point $x \in X$ can be a recurrent point in $(X, \mathfrak{F}, \sigma)$ without being a recurrent point in any individual usual system $(X, f_i)$ for $i \in \{1, 2, ..., k\}$. In other words, $\omega_{\mathfrak{F}}(x) \not\subset \bigcup\limits_{i=1}^{k} \omega_{f_i}(x)$, where $\omega_{\mathfrak{F}}(x)$ and $\omega_{f_i}(x)$ are the $\omega-$ limit sets of $x$ in $(X, \mathfrak{F}, \sigma)$ and $(X, f_i)$ respectively.

**Example 3.1.** *Let $X = [0, 2] \cup [4, 6]$. Define $f_1, f_2 : X \to X$ as*

$$f_1(x) = \begin{cases} x + 4, & \text{if } x \in [0, 2], \\ \frac{x}{2} + 2, & \text{if } x \in [4, 6], \end{cases} \quad \text{and } f_2(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, 2], \\ x - 4, & \text{if } x \in [4, 6]. \end{cases}$$

*Consider the switch system $(X, \mathfrak{F}, \sigma)$, where $\mathfrak{F} = \{f_1, f_2\}$ and $\sigma : \mathbb{N} \to \{1, 2\}$, defined as*

$$\sigma(n) = \begin{cases} 1, & \text{if } n \text{ is odd}, \\ 2, & \text{if } n \text{ is even}. \end{cases}$$

*Let $x = 1$. Then the trajectory of $x$ is given by*

$$x_n = \begin{cases} 1, & \text{if } n = 0 \text{ or } n \text{ is even}, \\ 5, & \text{if } n \text{ is odd}. \end{cases}.$$

*Thus, $(x_{n_{2m}}) \to 1$ and hence, $x = 1$ is a recurrent point in $(X, \mathfrak{F}, \sigma)$. However, $x = 1$ is a recurrent point neither in $(X, f_1)$ nor in $(X, f_2)$, because $\omega_{f_1}(1) = \{4\}$ and $\omega_{f_2}(1) = \{0\}$.*

However, we can ensure that, if $y \in \omega(x)$ in $(X, \mathfrak{F}, \sigma)$, then $y \in \overline{R(f_i)}$ for some $i \in \{1, 2, ..., k\}$, where $R(f_i)$ is the range of $f_i$. This is proved in the following theorem.

**Theorem 3.4.** *If $y \in \omega(x)$ in $(X, \mathfrak{F}, \sigma)$ for some $x \in X$, then $y \in \overline{R(f_i)}$ for some $i \in \{1, 2, ..., k\}$, where $R(f_i)$ is the range of $f_i$.*

*Proof.* By definition, for every $m \in \mathbb{N}$, there is an $n_m \in \mathbb{N}$ such that $x_{n_m} \in B(y, \frac{1}{m})$ and $(n_m) \to \infty$. In other words, $f_{\sigma(n_m)}(x_{n_m - 1}) \in B(y, \frac{1}{m})$ and thus $R(f_{\sigma(n_m)}) \cap B(y, \frac{1}{m}) \neq \emptyset$

for every $m \in \mathbb{N}$. Since $\sigma$ can take only finitely many values, there is an $i \in \{1, 2, ..., k\}$ such that $\sigma(n_m) = i$ for infinitely many $m$. We now claim that $y \in \overline{R(f_i)}$. For any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $B(y, \frac{1}{N}) \subset B(y, \epsilon)$. Since $\sigma(n_m) = i$ for infinitely many $m$, there is a $K \in \mathbb{N}$ such that $K > N$ and $\sigma(n_K) = i$. Then $B(y, \frac{1}{K}) \subset B(y, \epsilon)$ and thus $R(f_i) \cap B(y, \epsilon) = R(f_{\sigma(n_K)}) \cap B(y, \epsilon) \supset R(f_{\sigma(n_K)}) \cap B(y, \frac{1}{K}) \neq \emptyset$. Hence the claim. ∎

The following example is another instance to show the difference between usual and switch dynamical systems. Here, the switch systems $(X, \mathfrak{F}_1, \sigma_1)$ and $(Y, \mathfrak{F}_2, \sigma_2)$ are conjugate but the map $f_2 \in \mathfrak{F}_1$ is not conjugate to any of the maps in $\mathfrak{F}_2 = \{g_1, g_2\}$.

**Example 3.2.** *Let $X = Y = \mathbb{R}$. Define maps $f_1,\ f_2,\ g_1$ and $g_2$ on $\mathbb{R}$ as follows*

$$f_1(x) = \begin{cases} x+1, & \text{if } x \geq 0 \\ 1-x, & \text{if } x < 0 \end{cases}, \qquad f_2(x) = \begin{cases} x+4, & \text{if } x \geq 0 \\ 4, & \text{if } x < 0 \end{cases}$$
$$g_1(x) = \begin{cases} x+\frac{1}{2}, & \text{if } x \geq 0 \\ \frac{1}{2}-x, & \text{if } x < 0 \end{cases}, \qquad g_2(x) = \begin{cases} x+2, & \text{if } x \geq 0 \\ 2-x, & \text{if } x < 0 \end{cases}.$$

*Consider the switch systems $(X, \mathfrak{F}_1, \sigma_1)$ and $(Y, \mathfrak{F}_2, \sigma_2)$, where $\mathfrak{F}_1 = \{f_1,\ f_2\}$, $\mathfrak{F}_2 = \{g_1, g_2\}$ and $\sigma_1(n) = \sigma_2(n) = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 2, & \text{if } n \text{ is even} \end{cases}$. Let us denote $\sigma = \sigma_1 = \sigma_2$.*

*Now, define $h : X \to Y$ as $h(x) = \frac{x}{2}$. We now show that $h$ is a conjugacy from $(X, \mathfrak{F}_1, \sigma_1)$ to $(Y, \mathfrak{F}_2, \sigma_2)$.*
*For any $x \in X$, note that*

$$h(f_1(x)) = g_1(h(x)) = \begin{cases} \frac{x}{2}+\frac{1}{2}, & \text{if } x \geq 0 \\ \frac{1}{2}-\frac{x}{2}, & \text{if } x < 0 \end{cases} \tag{3.1}$$

*and*

$$h(f_2(f_1(x))) = g_2(g_1(h(x))) = \begin{cases} \frac{x}{2}+\frac{5}{2}, & \text{if } x \geq 0 \\ \frac{5}{2}-\frac{x}{2}, & \text{if } x < 0 \end{cases}. \tag{3.2}$$

*We now prove that $h(x_n) = (h(x))_n$ for every $n \in \mathbb{N}$ using induction. The above calculations show that the statement is true for $n = 1$ and $n = 2$. Assume now that $h(x_k) = (h(x))_k$ for every $k \leq n$.*
*If $n$ is even, then*

$$\begin{aligned} h(x_{n+1}) &= h(f_{\sigma(n+1)}(x_n)) \\ &= h(f_1(x_n)) \\ &= g_1(h(x_n)) \\ &= g_1((h(x))_n) \\ &= g_{\sigma(n+1)}((h(x))_n) \\ &= (h(x))_{n+1}. \end{aligned}$$

*If $n$ is odd, then*

$$\begin{aligned} h(x_{n+1}) &= h(f_{\sigma(n+1)}(x_n)) \\ &= h(f_{\sigma(n+1)} \circ f_{\sigma(n)}(x_{n-1})) \\ &= h(f_2 \circ f_1(x_{n-1})) \\ &= g_2 \circ g_1(h(x_{n-1})) \\ &= g_2 \circ g_1((h(x))_{n-1}) \\ &= g_{\sigma(n+1)} \circ g_{\sigma(n)}((h(x))_{n-1}) \\ &= (h(x))_{n+1}. \end{aligned}$$

*Hence, $h$ is a conjugacy from $(X, \mathfrak{F}_1, \sigma_1)$ to $(Y, \mathfrak{F}_2, \sigma_2)$.*

*However, $f_2$ is not conjugate to any of the maps $g_1$ and $g_2$. For, if $\alpha_i : (X, f_2) \to (Y, g_i)$, i=1,2 is a conjugacy, then $f_2(-1) = f_2(-2) = f_2(-3) = 4$ would imply that $g_i(\alpha_i(-1)) = g_i(\alpha_i(-2)) = g_i(\alpha_i(-3)) = \alpha_i(4)$, which is a contradiction because a point in $Y$ has at most two pre-images under any of the maps $g_1$ and $g_2$.*

However, it can be observed that if $h$ is a conjugacy from $(X, \mathfrak{F}_1, \sigma_1)$ to $(Y, \mathfrak{F}_2, \sigma_2)$, then for any $x \in X$, $h \circ f_{\sigma_1(1)}(x) = h(x_1) = (h(x))_1 = g_{\sigma_2(1)} \circ h(x)$. Thus, $(X, f_{\sigma_1(1)})$ and $(Y, g_{\sigma_2(1)})$ are conjugate. Hence, we have the following proposition.

**Proposition 3.5.** *If $(X, \mathfrak{F}_1, \sigma_1)$ and $(Y, \mathfrak{F}_2, \sigma_2)$ are two conjugate switch systems, then $(X, f_{\sigma_1(1)})$ and $(Y, g_{\sigma_2(1)})$ are conjugate (usual) dynamical systems.*

The following theorem can be easily proved using the definition of conjugacy. So, we state it without giving an explicit proof.

**Theorem 3.6.** *Let $h$ be a conjugacy from $(X, \mathfrak{F}_1, \sigma_1)$ to $(Y, \mathfrak{F}_2, \sigma_2)$ and let $x \in X$. Then,*
*(i) $x$ is periodic if and only if $h(x)$ is periodic.*
*(ii) $x$ is recurrent if and only if $h(x)$ is recurrent.*
*(iii) $(X, \mathfrak{F}_1, \sigma_1)$ is transitive if and only if $(Y, \mathfrak{F}_2, \sigma_2)$ is transitive.*
*(iii) $(X, \mathfrak{F}_1, \sigma_1)$ is minimal if and only if $(Y, \mathfrak{F}_2, \sigma_2)$ is minimal.*

Finally, we have the following theorem, which ensures the existence of a minimal set in a switch system on a compact space. The same is true for a usual system also (see Proposition 2.1.2, [2]). In fact, the proof given in [2] for usual systems, also holds for the following theorem. So, we simply state the theorem and omit the proof.

**Theorem 3.7.** *Let $(X, \mathfrak{F}, \sigma)$ be a switch system. If $X$ is compact, then $X$ contains a minimal set.*

## 4. SWITCHING ROTATIONS ON $S^1$

This section deals with a switch dynamical system $(X, \mathfrak{F}, \sigma)$, where $X = S^1$ and $\mathfrak{F}$ is a family of rotations on $S^1$. We consider the circle $S^1$ as $[0,1]/_\sim$, where only the end points 0 and 1 are identified under the equivalence relation $\sim$. In a usual dynamical system $(S^1, R_\alpha)$, where $R_\alpha$ denotes the rotation $x \mapsto x + \alpha(mod\ 1)$, the set of periodic points is either empty or the entire space $S^1$, depending upon whether $\alpha$ is irrational or rational respectively. We prove a similar result for switch systems of rotations also.

**Theorem 4.1.** *Let $k \in \mathbb{N}$ and for each $1 \le i \le k$, let $\alpha_i \in \mathbb{R}$ and define $f_i : S^1 \to S^1$ as $f_i(x) = x + \alpha_i(mod\ 1)$. Let $\mathfrak{F} = \{f_1, f_2, ..., f_k\}$ and $\sigma$ be any switch function. Then in the switch system $(S^1, \mathfrak{F}, \sigma)$, the set of periodic points, $P(\mathfrak{F})$ is either empty or $S^1$. If $P(\mathfrak{F}) = S^1$, then $\sigma$ is a periodic function and there exists $r_i \in \mathbb{Z}$ for each $1 \le i \le k$ such that $\sum_{i=1}^{k} r_i\alpha_i \in \mathbb{Z}$.*

*Proof.* Suppose $P(\mathfrak{F}) \neq \emptyset$ and $x \in S^1$ is periodic with period $m \in \mathbb{N}$. Then $x_{nm+l} = x_l$ for every $0 \le l < m$ and $n \in \mathbb{N}$. In particular, $x_{nm} = x_0$ for every $n \in \mathbb{N}$. This is the same as $x_0 + \sum_{i=1}^{nm} \alpha_{\sigma(i)}(mod\ 1) = x_0$ and then it follows that

$$\sum_{i=1}^{nm} \alpha_{\sigma(i)} \in \mathbb{Z} \tag{4.1}$$

for every $n \in \mathbb{N}$.

Proceeding along the same lines, for any $0 \le l < m$, we have $x_{nm+l} = x_l$, or
$(x_0 + \sum_{i=1}^{nm} \alpha_{\sigma(i)} + \sum_{i=nm+1}^{nm+l} \alpha_{\sigma(i)})(mod\ 1) = (x_0 + \sum_{i=1}^{l} \alpha_{\sigma(i)})(mod\ 1)$.
Hence, we have $\sum_{i=nm+1}^{nm+l} \alpha_{\sigma(i)} - \sum_{i=1}^{l} \alpha_{\sigma(i)} \in \mathbb{Z}$, or $\sum_{i=1}^{l} (\alpha_{\sigma(nm+i)} - \alpha_{\sigma(i)}) \in \mathbb{Z}$.
Since this is true for any $0 \le l < m$, it is easy to show by induction that $\alpha_{\sigma(nm+l)} - \alpha_{\sigma(l)} \in \mathbb{Z}$, for every $0 \le l < m$. This implies that $f_{\sigma(nm+l)} = f_{\sigma(l)}$ and thus $\alpha_{\sigma(nm+l)} = \alpha_{\sigma(l)}$ for every $0 \le l < m$. Therefore, $\sigma$ is a periodic function with a period $m$.

In view of (4.1), we obtain $\sum_{i=1}^{nm} \alpha_{\sigma(i)} \in \mathbb{Z}$ for every $n \in \mathbb{N}$. Moreover, it follows that, for any $y \in S^1$, $n \in \mathbb{N}$ and $0 \le l < m$,

$$\begin{aligned} y_{nm+l} &= (y_0 + \sum_{i=1}^{nm} \alpha_{\sigma(i)} + \sum_{i=nm+1}^{nm+l} \alpha_{\sigma(i)})(mod\ 1) \\ &= (y_0 + \sum_{i=nm+1}^{nm+l} \alpha_{\sigma(i)}(mod\ 1)) \\ &= (y_0 + \sum_{i=1}^{l} \alpha_{\sigma(i)})(mod\ 1) \\ &= y_l. \end{aligned}$$

The last equality follows because, $\sigma$ is periodic.

Thus, $y$ is periodic and hence $P(\mathfrak{F}) = S^1$.

Finally, the expression $\sum_{i=1}^{nm} \alpha_{\sigma(i)} \in \mathbb{Z}$ can be written as $\sum_{i=1}^{k} r_i \alpha_i \in \mathbb{Z}$, by making some re-arrangements and also taking some $r_i$'s to be 0, if necessary. ∎

**Theorem 4.2.** *Let $\mathfrak{F}$ be a family of rotations on $S^1$ as described in the above theorem. If $\sigma$ is a periodic function, then $(S^1, \mathfrak{F}, \sigma)$ is either minimal or every point in it is a periodic point.*

*Proof.* Let $\sigma(nm + l) = \sigma(l)$ for every $n \in \mathbb{N}$ and for every $1 \le l \le m$ and $x \in S^1$. Then, for any $n \in \mathbb{N}$, $x_{nm} = x_0 + n\beta$, where $\beta = \sum_{i=1}^{m} \alpha_{\sigma(i)}$.

If $\beta \in \mathbb{Q}$, then $q\beta \in \mathbb{Z}$ for some $q \in \mathbb{N}$ and thus $x_{qm} = x_0$. Since $qm$ is also a period for $\sigma$, it follows from Theorem 3.1, that $x$ is a periodic point. Thus, by Theorem 4.1, every point in $S^1$ is periodic.

If $\beta \in \mathbb{R} \setminus \mathbb{Q}$, then $\overline{\{x_0 + n\beta(mod 1) : n \in \mathbb{N}\}} = S^1$ and thus $\overline{\{x_n : n \in \mathbb{N}\}} = S^1$. Hence, $(S^1, \mathfrak{F}, \sigma)$ is a minimal system. ∎

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