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ON THE OLDEST PROBLEM IN THE CALCULUS OF VARIATIONS: A NEW MESSAGE FROM QUEEN DIDO

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ABSTRACT. We consider the problem of finding the optimal curve of given length linking two points in a plane such as it encloses a maximal area. We show that if the curve is not described by a single-valued function, its determination does not necessarily imply to work with a parametric representation of the curve. We show that a simpler approach is at hand – and, who knows? – this might well be the method Queen Dido used.

Key words and phrases: Calculus of variations; Isoperimetric constraints.

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1. INTRODUCTION

It would be difficult to find a text on the calculus of variations that does not introduce this beautiful area of applied mathematics with three classical problems. The first goes back to Johann Bernouilli who, in 1696, offered it as a challenge to his contemporaries: find, in a vertical plane, the curve joining two points A and B such that a bead slide frictionless from A to B in minimum time. Johann himself, his brother Jakob, Leibniz, L'Hospital, Newton, von Tschirnhaus solved the problem by subtle physical and geometrical considerations, discovering that the optimal curve, called a brachistochrone, was an arc of a cycloid. However, none of those leading mathematicians were able at that time to solve analytically the problem of extremizing a functional as simple as $\int_a^b f(x, y, y')dx$.

Nearly half a century went by before the genius of Euler came up, in 1744, with differential equations that constituted necessary conditions for the optimization not only of the integral just mentioned, but of much more complex functionals (notably involving constraints, higher order derivatives, or double integrals). One of Euler's illustrations constitutes the second classical problem always presented in texts: find the continuously differentiable curve that joins two points A and $B,$ such that rotating the curve around the abscissa generates a surface of minimum area. Euler showed that in a large part of (x, y) space, the solution was a catenary.

The third problem goes back much further in time; it is intimately linked to the legend of the foundation of Carthago by Queen Dido, nearly three thousand years ago, as Virgil relates for us in his Aeneid. Dido, offered to define between two points a curve of given length delimiting a maximal area for her future city, found that the optimal curve was an arc of a circle.

The reader going through the solutions given to these three problems will not fail to notice that the third one – the oldest one – receives by far the shortest shrift[1]. In particular, the task of identifying the constants resulting from the relevant differential equations is systematically left to the reader, sometimes with hardly encouraging warnings such as: "this identification involves some work", or: "the equations are quite messy"... To the best of our knowledge, the only author who took pains to go through the whole process is Mark Kot, in his excellent text [7]; see his neat, subtle exposition in pp. 186-189.

Such a rather frigid approach by most authors to a complete solution in their introduction may be explained by the fact that Dido's challenge presents two complications not shared by the other problems. The first is that it involves a new constraint, in addition to the fact that the solution has to go through two fixed points: the curve has a fixed length. The second difficulty stems from the nature of the curve: while the optimal curve resulting from the first two problems can always be expressed as a single-valued function $y = f(x)$, this seems not to be the case in Dido's problem where, depending on the initial conditions, the curve needs to be expressed in parametric form, or so it appears.

We will show that Dido's problem can be made simpler and that its solution can always be obtained without a parametric framework, thus leading in all cases to a straightforward identification of the relevant constants. To do this we will rely on and extend the very clear exposition given by Mark Kot in the case where the optimal curve is a single valued function $y = f(x)$ (see [7], pages 120-122).

Let us then go back some 3000 years in time and, with the Aeneid in hand, set our imagination free.

[1]For example, see Akhiezer [1], Bliss [2], Clegg [3], Elsgolc [4], Forsyth [4], Gelfand and Fomin [5], Weinstock [8].

2. QUEEN DIDO'S FOUNDATION OF CARTHAGO: THE TRUE STORY.

After Queen Dido, a Phoenician refugee, had been thrown by a frightful storm on the shores of Tunisia, she asked King Hiarbas to sell her a piece of his land for a fixed amount of gold. Mocking her, the king answered that he would oblige provided that the area of the plot would be delimited by a bull's skin. A wicked individual, Hiarbas told Dido that, in addition, the plot of land would have to follow the coastline from point A to B (see Figure 1). To the horror of her companions, Dido accepted the deal.

Cutting a bull's hide into extraordinarily slim slices, Dido realized not only that the thread she managed to form could follow the shoreline from A to B, but that she still had at her disposal a length l larger than the straight line distance between B and A.

What was then at stake for Dido was to figure out the shape she would give to the remaining part of her string. She addressed her companions as follows:

"First, use the straight line BA as an abscissa, oriented South-North; at its midpoint, consider the perpendicular as the ordinate, oriented East-West. The coordinates of A and B are $(a, 0)$ and $(-a, 0)$.

Let us maximize, in this system of axes, the area S equal to

$$S\,[y] = \int_{-a}^{a} y(x)dx \tag{2.1}$$

under the three constraints $y(a) = 0$, $y(-a) = 0$, and

$$\int_{-a}^{a} \sqrt{1 + y'^2}dx = l. \tag{2.2}$$

Denote the integrands of S and l as $F(x, y, y')$ and $G(x, y, y')$ respectively; the augmented integrand, $\varphi(x, y, y') \equiv F(x, y, y')+\lambda G(x, y, y')$, where λ designates a constant to be identified,

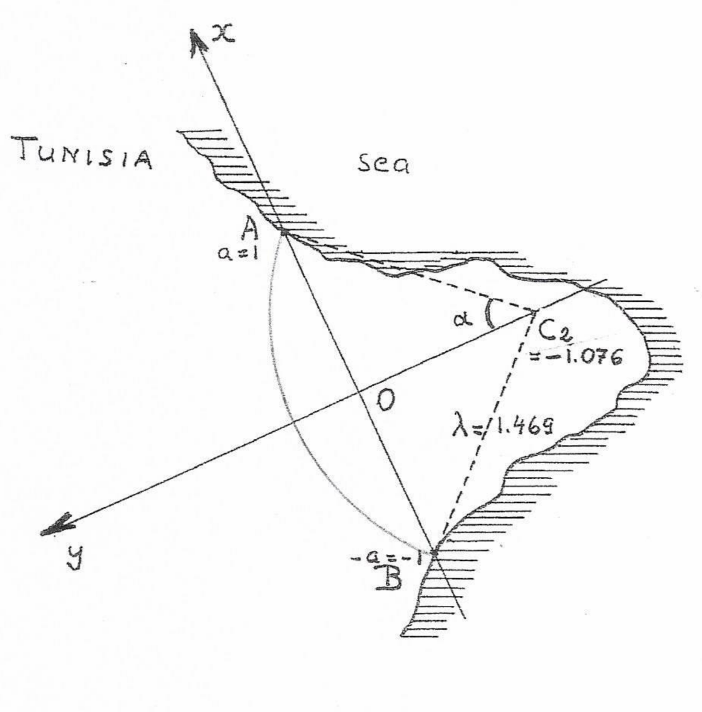

Fig. 1

is in this case

$$\varphi(y, y', \lambda) = y + \lambda\sqrt{1 + y'^2}, \tag{2.3}$$

leading to

$$\frac{\partial\varphi}{\partial y} - \frac{d}{dx}\frac{\partial\varphi}{\partial y'} = 1 - \lambda\frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2}} = 0, \tag{2.4}$$

which integrates into

$$x = \lambda\frac{y'}{\sqrt{1 + y'^2}} + C_1 \tag{2.5}$$

or, rearranging and squaring:

$$(x - C_1)^2 = \lambda^2\frac{y'^2}{1 + y'^2}, \tag{2.6}$$

equivalent to

$$y'^2\left[\lambda^2 - (x - C_1)^2\right] = (x - C_1)^2. \tag{2.7}$$

Separating the variables,

$$(dy)^2 = \frac{(x - C_1)^2}{\lambda^2 - (x - C_1)^2}(dx)^2, \tag{2.8}$$

we have

$$dy = \frac{\pm(x - C_1)}{\sqrt{\lambda^2 - (x - C_1)^2}}dx. \tag{2.9}$$

Let us set $u = \lambda^2 - (x - C_1)^2$; then $dx = du/(-2(x - C_1))$ and

$$dy = \pm\frac{1}{2}u^{-1/2}du. \tag{2.10}$$

Therefore

$$y = \pm u^{1/2} + C_2 = \pm\sqrt{\lambda^2 - (x - C_1)^2} + C_2 \tag{2.11}$$

and

$$(y - C_2)^2 = \lambda^2 - (x - C_1)^2, \tag{2.12}$$

or

$$(y - C_2)^2 + (x - C_1)^2 = \lambda^2, \tag{2.13}$$

the equation of a circle centered at C_1, C_2, with radius λ."

Dido now just needed to identify the constants C_1, C_2, and λ. She explained to her companions that since the circle had to go through $A(a, 0)$ and $B(-a, 0)$, its center must be located on the mediator of AB, i.e. at some point C_2 to be determined on the ordinate. This implied $C_1 = 0$.

Denoting α the acute angle between radius C_2A and the ordinate, Dido promised her audience that deriving the value of α from the constrained parameters a and l would be key to pin down the center of the optimal circle, as well as its radius. She defined the size of a as exactly one

Phoenician length unit; the length AB was then $2a = 2$, and the remaining length of the cord, which had turned out to be 10% larger than AB, was $l = 2.2$. She first wrote equations

$$\alpha\lambda = \frac{l}{2} \tag{2.14}$$

and

$$\frac{a}{\lambda} = \sin\alpha. \tag{2.15}$$

As soon as Dido, using $\lambda = l/\left(2\alpha\right)$, wrote down on her wax tablet

$$\frac{2a}{l}\alpha = \sin\alpha, \tag{2.16}$$

a feeling of embarrassment seized her small audience, everyone realizing that this equation, transcendental, had no algebraic solution; only numeric methods could apply. But all worries soon abated when they saw their Queen holding, miraculously saved from their wreckage, a beautifully accurate trigonometric table, from which she promptly computed the solution as $\alpha = 0.749$.

Dido had hardly scribbled on her diagram the resulting ordinate of the circle's center

$$C_2 = -a/\tan\alpha = -1.076 \tag{2.17}$$

and its radius

$$\lambda = l/\left(2\alpha\right) = 1.469 \tag{2.18}$$

when one of her companions raised yet more concern: "With all due respect, he said, how would Your Majesty determine those results if the length l of the string left at our disposal exceeded π? I can see that if the string describes exactly half of a circle between A and B, it implies that its length l must be equal to π. But if l exceeds π, the corresponding larger arc of circle is not a single-valued function $y(x)$ any more. How would we then proceed?"

"Your concern is perfectly justified" the Queen answered with the faintest of smiles; "I myself gave some thought to this issue. First, I was tempted to express our optimal curve parametrically, as I remembered my excellent teachers in Alexandria advising me to do in such circumstances; but I just realized that in this case it was simpler to proceed as follows. If $l > \pi$, simply interchange the axes. AB becomes the ordinate, now oriented North-South, and its mediator becomes the abscissa, oriented East-West (see Figure 2). The coordinates of points A and B are now $(0, -a)$ and $(0, a)$ respectively. Call $y(x)$ the new function we want to determine. What we now want to maximize is the integral

$$\Phi\left[y\right] = \int_0^d y(x)dx \tag{2.19}$$

where d is the abscissa of a point located on the abscissa; this point is still free, but can be determined together with the solution. The new constraints are

$$\int_0^d \sqrt{1+y'^2}dx = l/2, \tag{2.20}$$

together with $y(0) = a$. "The augmented integrand of the functional still remains

$$\varphi(x, y, y') = y + \lambda\sqrt{1+y'^2}. \tag{2.21}$$

Formally, the problem can thus be expressed with the same equations, and the solution of

$$\frac{\partial\varphi}{\partial y} - \frac{d}{dx}\frac{\partial\varphi}{\partial y'} = 1 - \lambda\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0, \tag{2.22}$$

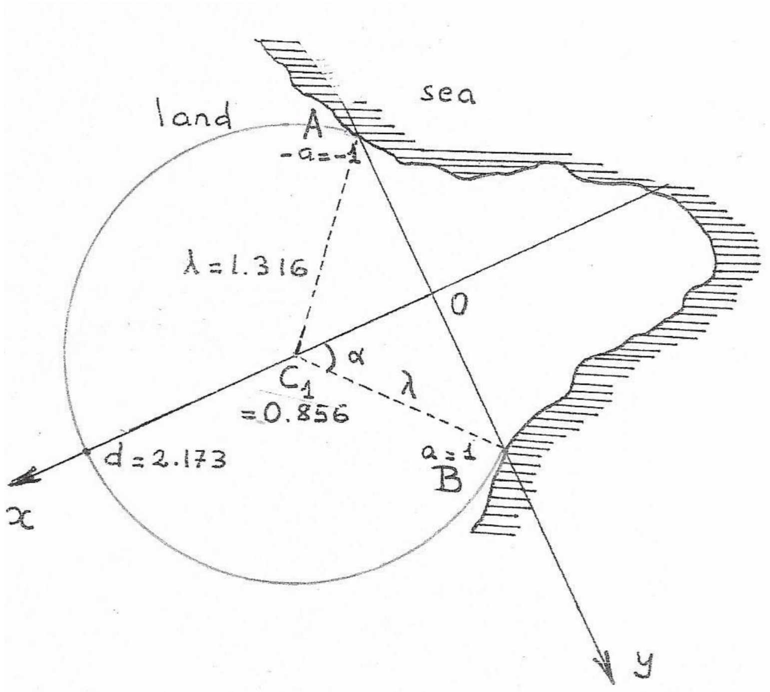

Fig. 2

is still

$$(y - C_2)^2 + (x - C_1)^2 = \lambda^2, \tag{2.23}$$

where C_1, C_2, and λ are new constants to be identified.

"From the symmetry of the problem we now have $C_2 = 0$. The center of the optimal circle will be located on the abscissa, at a point C_1 to be determined. Denote as α the acute angle between the ray C_1B and the abscissa. The constrained values l and a are now linked to α by

$$\lambda(\pi - \alpha) = \frac{l}{2} \tag{2.24}$$

and

$$\frac{a}{\lambda} = \sin\alpha. \tag{2.25}$$

From $\lambda = l/\left[2\left(\pi - \alpha\right)\right]$ and (2.25), we now have

$$\frac{2a}{l}\left(\pi - \alpha\right) = \sin\alpha. \tag{2.26}$$

"This time, said the Queen, instead of considering $l = 2.2$, let us take the much larger $l = 6$. Solving numerically (2.26) gives $\alpha = 0.8627$, from which the radius of the circle is

$$\lambda = l/\left[2\left(\pi - \alpha\right)\right] = 1.316, \tag{2.27}$$

and the abscissa of its center is

$$C_1 = \lambda\cos\alpha = 0.856, \tag{2.28}$$

so that the maximum area we could obtain, additional to the area between the coastline and AB, is equal to

$$(\pi - \alpha)\lambda^2 + aC_1 = 4.805 \text{ Phoenician length units squared,} \tag{2.29}$$

which can be verified, added the Queen, if we determine the upper bound of integration d as $C_1 + \lambda = 2.173$ and compute

$$2\Phi\left[y\right] = 2\int_0^d y(x)dx = 2\int_0^d \sqrt{\lambda^2 - (x - C_1)^2}dx \; = 4.805. \tag{2.30}$$"

Following the Queen's last words, you could almost hear a collective sigh of relief from her retinue.

This is how, nearly 3000 years ago, Carthago was founded. You might very well read or hear other stories: do not fall for them; they are just legends. Of course, in the account we give here, there are a few details that historians might still want to discuss. True, we do not yet have irrefutable proof that Dido, as a young student in Alexandria, did take a course on the calculus of variations or on optimal economic growth theory, fascinating as these topics may be. Nevertheless, if one day you visit Tunisia, and if your steps take you to Carthago, why wouldn't you want to locate points A and B on the Tunisian seashore?

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