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POSITIVE SOLUTION FOR DISCRETE THREE-POINT BOUNDARY VALUE PROBLEMS

WING-SUM CHEUNG AND JINGLI REN

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF HONG KONG, POKFULAM, HONG KONG
wscheung@hku.hk

INSTITUTE OF SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING 100080, P.R. CHINA
renjl@mx.amss.ac.cn

ABSTRACT. This paper is concerned with the existence of positive solution to the discrete three-point boundary value problem

$$\nabla \Delta u(k) + \lambda f(k, u(k)) = 0,\ k \in \{1, \ldots, N\},$$
$$u(0) = 0,\ u(N+1) = \alpha u(l)$$

where $\lambda > 0$, $l \in \{1, \cdots, N\}$, and f is allowed to change sign. By constructing available operators, we shall apply the method of lower solution and the method of topology degree to obtain positive solution of the above problem for λ on a suitable interval. The associated Green's function is first given.

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1. INTRODUCTION

The multi-point boundary value problems (BVP) of differential equations or difference equations arise in a variety of areas in applied mathematics and physics. The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev [5], motivated by the work of Bitsadzeand Samarskii [1] on nonlocal linear elliptic boundary problems. Since then, nonlinear multi-point boundary value problems have been studied by several authors, for example, see [2], [3], [4], [6], [7], [8], [9] and the references cited therein. The main tools used are fixed-point theorems in cones. All the above works have been done under the assumption that the nonlinear term is nonnegative so as to make use of the concavity of solutions in the proofs.

In this paper, we consider the following discrete three-point boundary value problem

$$\nabla\Delta u(k) + \lambda f(k, u(k)) = 0,\ k \in \mathfrak{L}, \tag{1.1}$$

$$u(0) = 0,\ u(N+1) = \alpha u(l), \tag{1.2}$$

where $\lambda > 0$ and α are fixed constants, $N \geq 1$ is a fixed integer, $\mathfrak{L} = \{1, 2, \cdots, N\}$, $\mathfrak{L}^+ = \{0, 1, \cdots, N+1\}$, l is a fixed integer in $\mathfrak{L}$, and the nonlinear term f is continuous and is allowed to change sign. Here, as usual, Δ is the forward difference operator with stepsize 1, and ∇ is the backward difference operator with stepsize 1. We first establish the Green's function of the problem, then by constructing available operators, we combine the method of lower solution with the method of topology degree and show that BVP (1.1)-(1.2) has at least one positive solution with certain growth conditions imposed on f. In this way we removed the usual restriction $f \geq 0$.

2. MAIN RESULTS

Before the statement of our main results, we give some lemmas which are needed later.

Let $C(\mathfrak{L}^+)$ denote the class of real-valued maps ω on $\mathfrak{L}^+$ with norm $|\omega|_0 = \max\limits_{k\in\mathfrak{L}^+} |\omega(k)|$. Note that $C(\mathfrak{L}^+)$ is a Banach space.

Lemma 2.1. *Suppose that $N + 1 - \alpha l \neq 0$ and $y(k) \in C(\mathfrak{L}^+)$, then BVP*

$$\nabla\Delta u(k) + y(k) = 0,\ \ k \in \mathfrak{L}, \tag{2.1}$$

$$u(0) = 0,\ \ u(N+1) = \alpha u(l) \tag{2.2}$$

has a unique solution

$$\begin{aligned} u(k) \ &= \ -\sum_{j=1}^{k-1}(k-j)y(j) + \sum_{j=1}^{N}\frac{k}{N+1-\alpha l}(N+1-j)y(j) \\ &\quad -\sum_{j=1}^{l-1}\frac{\alpha k}{N+1-\alpha l}(l-j)y(j), \quad k \in \mathfrak{L}^+. \end{aligned} \tag{2.3}$$

Here we adopt the convention that $\sum\limits_{i=m_1}^{m_2} f(i) = 0$ *for* $m_2 < m_1$.

Proof. From (2.1), we have, for any $i \in \mathfrak{L}$,

$$\Delta u(i) - \Delta u(i-1) = -y(i),$$

thus for any $j \in \mathfrak{L}$,

$$\Delta u(j) - \Delta u(0) = \sum_{i=1}^{j} [\Delta u(i) - \Delta u(i-1)] = -\sum_{i=1}^{j} y(i).$$

Since $u(0) = 0$, we get

$$\Delta u(j) - u(1) = -\sum_{i=1}^{j} y(i), \quad j \in \mathfrak{L},$$

which implies that

$$u(k+1) - u(1) - ku(1) = \sum_{j=1}^{k} [\Delta u(j) - u(1)] = -\sum_{j=1}^{k} \sum_{i=1}^{j} y(i), \quad k \in \mathfrak{L}.$$

Equivalently, we have

$$u(k) = -\sum_{j=1}^{k-1} \sum_{i=1}^{j} y(i) + ku(1) = -\sum_{j=1}^{k-1} (k-j) y(j) + ku(1), \quad k \in \mathfrak{L}^+.$$

Using the condition $u(N+1) = \alpha u(l)$, we arrive at

$$u(1) = \frac{1}{N+1-\alpha l} \sum_{j=1}^{N} (N+1-j) y(j) - \frac{\alpha}{N+1-\alpha l} \sum_{j=1}^{l-1} (l-j) y(j),$$

and the lemma follows by putting this back into the last expression. □

Lemma 2.2. *Suppose $N+1-\alpha l \neq 0$, then the Green's function for the BVP*

$$-\nabla \Delta u(k) = 0, \quad k \in \mathfrak{L}, \tag{2.4}$$

$$u(0) = 0, \;\; u(N+1) = \alpha u(l), \tag{2.5}$$

is given by

$$G(k,j) = \begin{cases} \dfrac{j[N+1-k-\alpha(l-k)]}{N+1-\alpha l}, & j \le k, \; j \le l; \\[2ex] \dfrac{j(N+1-k) + \alpha l(k-j)}{N+1-\alpha l}, & l < j \le k; \\[2ex] \dfrac{k(N+1-j)}{N+1-\alpha l}, & j > k, \; j > l; \\[2ex] \dfrac{k[N+1-j-\alpha(l-j)]}{N+1-\alpha l}, & k < j \le l. \end{cases} \tag{2.6}$$

Proof. If $k \geq l$, the unique solution (2.3) can be rewritten as

$$\begin{aligned} u(k) &= -\sum_{j=1}^{l-1}(k-j)y(j) - \sum_{j=l}^{k-1}(k-j)y(j) \\ &\quad + \sum_{j=1}^{l-1}\frac{k}{N+1-\alpha l}(N+1-j)y(j) + \sum_{j=l}^{k-1}\frac{k}{N+1-\alpha l}(N+1-j)y(j) \\ &\quad + \sum_{j=k}^{N}\frac{k}{N+1-\alpha l}(N+1-j)y(j) - \sum_{j=1}^{l-1}\frac{\alpha k}{N+1-\alpha l}(l-j)y(j) \\ &= \sum_{j=1}^{l-1}\frac{j[N+1-k-\alpha(l-k)]}{N+1-\alpha l}y(j) + \sum_{j=l}^{k-1}\frac{j(N+1-k)+\alpha l(k-j)}{N+1-\alpha l}y(j) \\ &\quad + \sum_{j=k}^{N}\frac{k(N+1-j)}{N+1-\alpha l}y(j). \end{aligned}$$

Similarly, if $k < l$, the unique solution (2.3) becomes

$$\begin{aligned} u(k) &= \sum_{j=1}^{k-1}\frac{j[N+1-k-\alpha(l-k)]}{N+1-\alpha l}y(j) + \sum_{j=k}^{l-1}\frac{k[N+1-j-\alpha(l-j)]}{N+1-\alpha l}y(j) \\ &\quad + \sum_{j=l}^{N}\frac{k(N+1-j)}{N+1-\alpha l}y(j). \end{aligned}$$

Therefore, the unique solution of (2.1)-(2.2) is $u(k) = \sum\limits_{j=1}^{N} G(k,j)y(j)$. Lemma 2.2 now follows. $\square$

Lemma 2.3. (see [10]) *Let X be a real Banach space with norm* $\|\cdot\|$, Ω *is an open bounded subset in* X *with* $0 \in \Omega$. *Suppose* $A : \overline{\Omega} \to X$ *is a completely continuous operator. If*

$$\|Ax\| \leq \|x\|, \; Ax \neq x, \text{ for all } x \in \partial\Omega,$$

then

$$\deg\{I - A, \Omega, 0\} = 1.$$

Now let $X = C(\mathfrak{L}^+)$ and $K = \{u \in X : u \geq 0\}$. Throughout the rest of the paper we assume that the following hypotheses are satisfied:

(H1) $0 < \alpha < 1$;

(H2) $f : \mathfrak{L}^+ \times [0,\infty) \to R$ is continuous.

Observe that if (H1) holds, we have $G(k,j) \geq 0$. Moreover, if $u(k)$ is the solution of BVP (2.1)-(2.2), then $u(k) = \sum\limits_{j=1}^{N} G(k,j)y(j)$. In particular, for the special case where $y(k) \equiv 1$, define $w(k) = \sum\limits_{j=1}^{N} G(k,j)$, then from (2.3) we have

$$w(k) = \sum_{j=1}^{N} G(k,j) = -\frac{k(k-1)}{2} + \frac{k[(N+1)(N+2) - \alpha l(l-1)]}{2(N+1-\alpha l)}, \; k \in \mathfrak{L}^+. \tag{2.7}$$

Let $A = \max\limits_{k \in \mathfrak{L}^+} w(k)$. It is clear that $0 < A < \infty$.

Theorem 2.4. *Suppose there are real numbers $R > M > 0$ such that*

$$0 < \frac{M}{\min\limits_{k\in\mathfrak{L}^+} f(k, Mw(k))} = a < c = \frac{R}{A \max\limits_{\substack{k\in\mathfrak{L}^+ \\ Mw(k)\le u\le R}} f(k,u)}. \tag{2.8}$$

Then BVP (1.1)-(1.2) has at least one positive solution $y(k)$ satisfying

$$0 < Mw(k) \le y(k),\ k \in \mathfrak{L}^+ \text{ and } \|y\| < R$$

if $\lambda \in [a, c]$.

Proof. Let

$$f^*(k,u) = \begin{cases} f(k,u), & u \ge Mw(k), \\ f(k, Mw(k)), & u \le Mw(k), \end{cases} \tag{2.9}$$

and define $\Phi : K \to X$ by

$$(\Phi u)(k) = \lambda \sum_{j=1}^{N} G(k,j) f^*(j, u(j)),\ k \in \mathfrak{L}^+. \tag{2.10}$$

Then Φ is on K a completely continuous operator. Let $\theta : X \to K$ be defined by

$$(\theta u)(k) = \max\{u(k), 0\}, \tag{2.11}$$

it is clear that $\theta \circ \Phi : K \to K$ is also completely continuous.

Take $\Omega = \{u \in K : \|u\| < R\}$. Given $u \in \partial\Omega$, set $I = \{k \in \mathfrak{L}^+ : f^*(k, u(k)) \ge 0\}$. Then

$$\begin{aligned} (\theta\circ\Phi)u(k) &= \max\{\lambda \sum_{j=1}^{N} G(k,j) f^*(j,u(j)), 0\} \\ &\le \lambda \sum_{I} G(k,j) f^*(j,u(j)) \\ &\le c \max_{\substack{k\in\mathfrak{L}^+ \\ 0\le u\le r}} f^*(k,u) \sum_{I} G(k,j) \\ &\le Ac \max_{\substack{k\in\mathfrak{L}^+ \\ Mw(k)\le u\le r}} f(k,u) \\ &= R. \end{aligned}$$

If there is a $u \in \partial\Omega$ such that $(\theta\circ\Phi)u = u$, then $\theta\circ\Phi$ has a fixed point in $\overline{\Omega}$. On the other hand, if for any $u \in \partial\Omega$, $(\theta\circ\Phi)u \ne u$, it follows from Lemma 2.3 that

$$\deg\{I - \theta\circ\Phi, \Omega, 0\} = 1.$$

Then $\theta\circ\Phi$ has a fixed point in Ω. So in both cases $\theta\circ\Phi$ has a fixed point $y \in \overline{\Omega}$.

We claim that

$$(\Phi y)(k) \ge Mw(k),\ k \in \mathfrak{L}^+. \tag{2.12}$$

If not, there is $k_0 \in \mathfrak{L}^+$ such that

$$\gamma := Mw(k_0) - (\Phi y)(k_0) = \max_{k\in\mathfrak{L}^+}\{Mw(k) - (\Phi y)(k)\} > 0. \tag{2.13}$$

Now clearly $k_0 \ne 0$ by the first equation of (1.2). Besides, if $k_0 = N+1$, then from the second equation of (1.2),

$$\begin{aligned} Mw(N+1) - (\Phi y)(N+1) &= \alpha[Mw(l) - (\Phi y)(l)] \\ &< Mw(l) - (\Phi y)(l) \\ &\le \gamma, \end{aligned}$$

a contradiction. So $k_0 \in \{1, 2, \cdots, N\}$. It is obvious that

$$M\Delta w(k_0) - \Delta(\Phi y)(k_0) \leq 0$$

and

$$M\Delta w(k_0 - 1) - \Delta(\Phi y)(k_0 - 1) \geq 0.$$

Note that in this case we must have

$$Mw(k) \geq \Phi y(k),\ k \in \mathfrak{L}^+. \tag{2.14}$$

For if not, there exists $k_1 \in \{1, \cdots, k_0 - 1\} \cup \{k_0 + 1, k_0 + 2, \cdots, N + 1\}$ such that

$$Mw(k_1) - \Phi y(k_1) < 0 \tag{2.15}$$

and

(2.16) $Mw(k) - (\Phi y)(k) \geq 0,\ k \in \{k_1 + 1, k_1 + 2, \cdots, k_0\}$ or $k \in \{k_0, k_0 + 1, \cdots, k_1 - 1\}$.

If $k_1 \in \{1, \cdots, k_0 - 1\}$, then

$$\begin{aligned}
& M\Delta w(k_1) - \Delta(\Phi y)(k_1) \\
= \ & M\Delta w(k_0) - \Delta(\Phi y)(k_0) - \sum_{j=k_1+1}^{k_0} [M\nabla\Delta w(j) - \nabla\Delta(\Phi y)(j)] \\
\leq \ & \sum_{j=k_1+1}^{k_0} [M - \lambda f^*(j, y(j))] \\
\leq \ & \sum_{j=k_1+1}^{k_0} [M - a \min_{k\in\mathfrak{L}^+} f(k, Mw(k))] \\
= \ & 0,
\end{aligned}$$

which implies that

$$Mw(k_1) - (\Phi y)(k_1) \geq Mw(k_1 + 1) - (\Phi y)(k_1 + 1) \geq 0.$$

This contradicts (2.15). On the other hand, if $k_1 \in \{k_0 + 1, \cdots, N + 1\}$, then

$$\begin{aligned}
& M\Delta w(k_1) - \Delta(\Phi y)(k_1) \\
= \ & M\Delta w(k_0 - 1) - \Delta(\Phi y)(k_0 - 1) + \sum_{j=k_0}^{k_1} [M\nabla\Delta w(j) - \nabla\Delta(\Phi y)(j)] \\
\geq \ & \sum_{j=k_0}^{k_1} [\lambda f^*(j, y(j)) - M] \\
\geq \ & \sum_{j=k_0}^{k_1} [a \min_{k\in\mathfrak{L}^+} f(k, Mw(k)) - M] \\
= \ & 0,
\end{aligned}$$

which implies that

$$Mw(k_1) - (\Phi y)(k_1) \geq Mw(k_1 - 1) - (\Phi y)(k_1 - 1) \geq 0.$$

This also contradicts (2.15). So (2.14) holds.

However

$$\begin{aligned}
Mw(k_0) - \Phi y(k_0) &= \sum_{j=1}^{N} G(k_0, j)[M - \lambda f^*(j, y(j))] \\
&\leq [M - a \min_{k\in\mathfrak{L}^+} f(k, Mw(k))] \sum_{j=1}^{N} G(k_0, j) \\
&= 0.
\end{aligned}$$

This contradicts (2.13). Thus (2.12) holds. Then $(\theta \circ \Phi)y = \Phi y = y$ and $y(t)$ is a solution of BVP (1.1)-(1.2). □

Similarly, we have:

Theorem 2.5. *Suppose $f(k,0) \geq 0$, $f(k,0) \not\equiv 0$ for $k \in \mathfrak{L}^+$ and there is $R > 0$ such that*

$$c = \frac{R}{A \max\limits_{\substack{k\in\mathfrak{L}^+ \\ 0\leq u\leq R}} f(k,u)} > 0. \tag{2.17}$$

Then when $\lambda \leq c$, BVP (1.1)-(1.2) has at least one positive solution $y(t)$ satisfying

$$0 < \|y\| < R.$$

Proof. Let

$$f^*(k,u) = \begin{cases} f(k,u), & u \geq 0, \\ f(k,0) - u, & u < 0. \end{cases} \tag{2.18}$$

The theorem now follows from arguments similar to those used in the proof of Theorem 2.4. □

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