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ON SUFFICIENT CONDITIONS FOR STRONG STARLIKENESS

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ABSTRACT. In the present investigation, we obtain some sufficient conditions for a normalized analytic function $f(z)$ defined on the unit disk to satisfy the condition

$$-\frac{\beta\pi}{2} < \arg\frac{zf'(z)}{f(z)} < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1; 0 < \beta \leq 1).$$

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1. INTRODUCTION

Let $\mathcal{A}$ be the class of *analytic functions* $f(z)$ defined on the unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$. Let $S^*(\alpha,\beta)$ be the class of all functions $f(z) \in \mathcal{A}$ satisfying

$$-\frac{\beta\pi}{2} < \arg\frac{zf'(z)}{f(z)} < \frac{\alpha\pi}{2}, \quad (z \in \Delta; 0 < \alpha \leq 1, 0 < \beta \leq 1).$$

This class was introduced by Takahashi and Nunokawa[1]. We note that $S^*(\alpha,\alpha) =: SS^*(\alpha)$ is the familiar class of *strongly starlike functions of order* α.

Tuneski[7] obtained a sufficient condition for function to be starlike in terms of the quantity $\frac{f(z)f''(z)}{f'(z)^2}$. Nunokawa and Owa[4] obtained the following sufficient condtion for function to be strongly starlike of order α $(0 < \alpha \leq 1)$:

Theorem 1.1. *If $f \in \mathcal{A}$ satisfies*

$$\left|\arg\left(1 - \frac{f(z)f''(z)}{f'(z)^2}\right)\right| \leq \frac{\pi\alpha}{2} + \tan^{-1}\alpha \quad (z \in \Delta; 0 < \alpha \leq 1),$$

then $f(z) \in SS^*(\alpha)$.

In our present investigation, we extend the Theorem 1.1 for the class $S^*(\alpha,\beta)$. Also we obtain some sufficient conditions for functions to be in the class $S^*(\alpha,\beta)$. Also we give a sufficient condition involving the argument of the quantity

$$\frac{z^2f'(z)}{f(z)^2} - \gamma z^2\left(\frac{f(z)}{z}\right)''$$

for functions to satisfy

$$-\frac{\beta\pi}{2} < \arg\frac{z^2f'(z)}{f(z)^2} < \frac{\alpha\pi}{2}.$$

We need the following result of Nunokawa, Owa, Saitoh, Cho and Takahashi [6] (see [1]) to prove our main result:

Lemma 1.2. *Let $p(z)$ be analytic in Δ with $p(0) = 1$ and $p(z) \neq 0$. If there exists two points $z_1, z_2 \in \Delta$ such that*

$$-\frac{\beta\pi}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\alpha\pi}{2} \tag{1.1}$$

for $\alpha > 0$, $\beta > 0$, and for $|z| < |z_1| = |z_2|$, then we have

$$\frac{z_1p'(z_1)}{p(z_1)} = -i\frac{\alpha+\beta}{2}m \tag{1.2}$$

and

$$\frac{z_2p'(z_2)}{p(z_2)} = i\frac{\alpha+\beta}{2}m \tag{1.3}$$

where

$$m \geq \frac{1-|a|}{1+|a|} \quad \textit{and} \quad a := i\tan\left(\frac{\pi}{4}\frac{\alpha-\beta}{\alpha+\beta}\right).$$

2. SUFFICIENT CONDITION

By making use of Lemma 1.2, we first prove the following:

Lemma 2.1. *Let $p(z)$ be analytic in Δ with $p(0) = 1$ and $p(z) \neq 0$. If*

$$(2.1)\qquad -\frac{\beta\pi}{2} - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha+\beta)\gamma}{2}\right) < \arg\left(p(z)+\gamma z p'(z)\right) < \frac{\alpha\pi}{2} + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha+\beta)\gamma}{2}\right)$$

for $\alpha, \beta, \gamma > 0$, then

$$-\frac{\beta\pi}{2} < \arg p(z) < \frac{\alpha\pi}{2}.$$

Proof. Assume that there exists points z_1 and z_2 such that the inequality (1.1) holds. Then by Lemma 1.2, (1.2) and (1.3) holds where m and a are as in Lemma 1.2. By making use of (1.2) and (1.3), we have

$$\begin{aligned}\arg(p(z_1)+\gamma z_1 p'(z_1)) &= \arg p(z_1) + \arg\left(1+\gamma\frac{z_1 p'(z_1)}{p(z_1)}\right)\\ &= -\frac{\beta\pi}{2} - \tan^{-1}\left(\frac{(\alpha+\beta)\gamma}{2}m\right)\\ &\le -\frac{\beta\pi}{2} - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha+\beta)\gamma}{2}\right)\end{aligned}$$

and

$$\arg(p(z_2)+\gamma z_2 p'(z_2)) \ge -\frac{\alpha\pi}{2} - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha+\beta)\gamma}{2}\right),$$

which contradicts (2.1). This completes the proof. ∎

Theorem 2.2. *If $f \in \mathcal{A}$ satisfies*

$$(2.2)\qquad -\frac{\alpha\pi}{2} - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha+\beta)}{2}\right) < \arg\left(1-\frac{f(z)f''(z)}{f'(z)^2}\right) < \frac{\beta\pi}{2} + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha+\beta)}{2}\right)$$

for $\alpha, \beta > 0$, then $f(z) \in S^*(\alpha, \beta)$.

Proof. Define the function $p(z)$ by

$$p(z) := \frac{f(z)}{zf'(z)}.$$

Then a computation shows that

$$p(z) + zp'(z) = 1 - \frac{f(z)f''(z)}{f'(z)^2}.$$

Our result now follows by an application of Lemma 2.1. ∎

Remark 2.1. When $\beta = \alpha$, Theorem 2.2 reduces to Theorem 1.1.

Theorem 2.3. *If $f \in \mathcal{A}$ satisfies*

$$-\frac{\beta\pi}{2} - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha+\beta)\gamma}{2}\right) < \arg\left(\frac{z^2 f'(z)}{f(z)^2} - \gamma z^2\left(\frac{f(z)}{z}\right)''\right) < \frac{\alpha\pi}{2} + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha+\beta)\gamma}{2}\right)$$

for $\alpha,\ \beta,\ \gamma > 0$, then

$$-\frac{\beta\pi}{2} < \arg\frac{z^2 f'(z)}{f(z)^2} < \frac{\alpha\pi}{2}.$$

Proof. Define the function $p(z)$ by

$$p(z) := \frac{z^2 f(z)}{f(z)^2}.$$

Then a computation shows that

$$p(z) + \gamma z p'(z) = \frac{z^2 f'(z)}{f(z)^2} - \gamma z^2\left(\frac{f(z)}{z}\right)''.$$

Our result now follows by an application of Lemma 2.1. ∎

3. Another Sufficient Condition

We need the following result to prove our main result of this section.

Lemma 3.1. *Let $\alpha, \beta, \mu > 0$,*

$$\gamma(\alpha,\beta,\mu) := \frac{2}{\pi}\tan^{-1}\left(\frac{(1-|a|)\sigma(\alpha,\beta)\mu\sin\left(\frac{\pi(1-\alpha)}{2}\right)}{1+|a|+(1-|a|)\sigma(\alpha,\beta)\cos\left(\frac{\pi(1-\alpha)}{2}\right)}\right),$$

$$\delta(\alpha,\beta,\mu) := \frac{2}{\pi}\tan^{-1}\left(\frac{(1-|a|)\sigma(\alpha,\beta)\mu\sin\left(\frac{\pi(1-\beta)}{2}\right)}{1+|a|+(1-|a|)\sigma(\alpha,\beta)\cos\left(\frac{\pi(1-\beta)}{2}\right)}\right),$$

where

$$\sigma(\alpha,\beta) := \frac{\alpha+\beta}{2-\alpha-\beta}\left(\frac{2-\alpha-\beta}{2+\alpha+\beta}\right)^{(2+\alpha+\beta)/4}.$$

If $p(z)$ is analytic in Δ and $p(0) = 1$ and

$$\delta(\alpha,\beta,\mu) \le \arg\left(1 + \gamma\frac{zp'(z)}{p(z)^2}\right) \le \gamma(\alpha,\beta,\mu),$$

then

$$-\frac{\beta\pi}{2} < \arg p(z) < \frac{\alpha\pi}{2}.$$

Proof. The proof of the Lemma 3.1 is essentially similar to the proof of [1, Theorem, p. 654] and therefore omitted. ∎

By taking $p(z) = zf'(z)/f(z)$ in Lemma 3.1, we obtain the following:

Theorem 3.2. *Let* $\alpha, \beta, \mu > 0$ *and* δ, γ *be as in Lemma 3.1. If* $f \in \mathcal{A}$ *satisfies*

$$\delta(\alpha, \beta, \mu) \leq \arg \left(\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \right) \leq \gamma(\alpha, \beta, \mu),$$

then $f(z) \in S^*(\alpha, \beta)$.

4. SUFFICIENT CONDITIONS INVOLVING α-CONVEX FUNCTIONS

Theorem 4.1. *Let* $0 < \alpha \leq 1,\ 0 < \beta \leq 1, \mu > 0, \lambda + \mu > 0$ *and* δ, γ *be as in Lemma 3.1. If* $f \in \mathcal{A}$ *satisfies*

$$\beta(\lambda + \mu) + \delta(\alpha, \beta, \mu) \leq \arg \left(\lambda \frac{zf'(z)}{f(z)} + \mu \left(1 + \frac{zf''(z)}{f'(z)} \right) \right) \leq \alpha(\lambda + \mu) + \gamma(\alpha, \beta, \mu),$$

then $f(z) \in S^*(\alpha, \beta)$.

Proof. Define $p(z)$ by

$$p(z) := \frac{zf'(z)}{f(z)}.$$

Then a computation shows that

$$\lambda \frac{zf'(z)}{f(z)} + \mu \left(1 + \frac{zf''(z)}{f'(z)} \right) = (\lambda + \mu) p(z) + \mu \frac{zp'(z)}{p(z)}.$$

The rest of the proof is similar to the proof of [1, Theorem, p. 654] and therefore omitted. ∎

Next if $\lambda = 1 - \mu$, we have the following:

Corollary 4.2. *Let* $0 \leq \alpha < 1,\ 0 \leq \beta < 1, \mu > 0$ *and* δ, γ *be as in Lemma 3.1. If* $f \in \mathcal{A}$ *satisfies*

$$\beta + \delta(\alpha, \beta, \mu) \leq \arg \left((1 - \mu) \frac{zf'(z)}{f(z)} + \mu \left(1 + \frac{zf''(z)}{f'(z)} \right) \right) \leq \alpha + \gamma(\alpha, \beta, \mu),$$

then $f(z) \in S^*(\alpha, \beta)$.

Remark 4.1. When $\mu = 1$, our result reduces to [1, Theorem, p. 654]. When $\beta = \alpha$, the result reduces to a recent result of Marjono and Thomas[2] and for $\mu = 1$, the result was proved earlier by Nunokawa [3, Main Theorem, p. 234]. See also Nunokawa and Thomas[5].

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