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ANALYTICAL AND NUMERICAL SOLUTIONS OF THE INHOMOGENOUS WAVE EQUATION

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ABSTRACT. In this paper, by a new concept and method we give approximate solutions of the inhomogenous wave equation on multidimensional spaces. Numerical experiments are conducted as well.

Key words and phrases: Wave equation, Approximation of functions, Reproducing kernel, Tikhonov regularization, Sobolev space, Generalized inverse, Approximate inverse, Error estimate, Noise, Weighted convolution inequality.

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1. INTRODUCTION AND MAIN RESULTS

We give simple approximate solutions for the inhomogenous wave equation,

$$\Box_c u(x,t) = \partial_t^2 u(x,t) - c^2 \Delta_x u(x,t) = g \quad \text{on} \quad \mathbf{R}^n \quad (c>0) \tag{1.1}$$

for any $L_2(\mathbf{R}^n)$ function g in the class of the functions of the s order Sobolev Hilbert space H^s on the whole real space $\mathbf{R}^n (n \geq 2, s \geq 2, s > n/2)$. In this paper, we use the notation

$$x' = (x_1, x_2, ..., x_{n-1}), x_n = t$$

and similarly

$$p = (p', p_n) \in \mathbf{R}^n.$$

This equation is, of course, fundamental and has many applications to mathematical sciences.

Recently, in [1], [5], [6], we were able to obtain surprisingly simple and practical approximates of real inversion formulas for the Gaussian convolution equation by using the theory of reproducing kernels and with the ideas of best approximations and generalized inverses. Furthermore, we illustrated there numerical experiments by using computers and we can realize that we were able to obtain practical real inversion formulas in [1]. There, we used the method of the Tikhonov regularization and the theory of reproducing kernels.

In this paper, by the same method, we shall examine the problem 1.1 for the wave equation on multidimensional spaces. We are, in particular, interested in their numerical experiments by using computers. Furthermore, we shall establish error estimates for our solutions $u(x,t)$, because practical data g contain errors and noises in 1.1.

We recall the m order Sobolev Hilbert space H^m comprising functions F on $\mathbf{R}^n$ with the norm

$$\|F\|_{H^m}^2$$

$$= \sum_{\nu=0}^{m} {}_mC_\nu \sum_{r_1, r_2, \ldots, r_n \geq 0} \frac{\nu!}{r_1! r_2! \cdots r_n!} \int_{\mathbf{R}^n} \left(\frac{\partial^\nu F(x)}{\partial x_1^{r_1} \partial x_2^{r_2} \cdots \partial x_n^{r_n}} \right)^2 dx. \tag{1.2}$$

Here, of course,

$$r_1 + r_2 + \cdots + r_n = \nu.$$

This Hilbert space admits the reproducing kernel

$$K(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{1}{(1+|\xi|^2)^m} e^{i(x-y)\cdot\xi} d\xi \tag{1.3}$$

as we see easily by using Fourier's transform (cf. [3], page 58). Note that the Sobolev Hilbert space H^s admitting the reproducing kernel 1.3 for $m = s$ can be defined for any positive number $s (s > n/2)$ in terms of Fourier integrals $\hat{F}$, of F

$$\hat{F}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-i\xi\cdot x} F(x) dx,$$

as follows:

$$\|F\|_{H^s}^2 = \int_{\mathbf{R}^n} |\hat{F}(\xi)|^2 (1+|\xi|^2)^s d\xi.$$

Our results are stated as follows:

Theorem 1.1. *Let $n \geq 2, s \geq 2$ and $s > n/2$. For any function $g \in L_2(\mathbf{R}^n)$ and for any $\lambda > 0$, the best approximate function $F^*_{\lambda,s,g}$ in the sense*

$$\inf_{F\in H^s} \left\{ \lambda \|F\|^2_{H^s} + \|g - \Box_c F\|^2_{L_2(\mathbf{R}^n)} \right\}$$

$$= \lambda \|F^*_{\lambda,s,g}\|^2_{H^s} + \|g - \Box_c F^*_{\lambda,s,g}\|^2_{L_2(\mathbf{R}^n)} \tag{1.4}$$

exists uniquely and $F^*_{\lambda,s,g}$ is represented by

$$F^*_{\lambda,s,g}(x) = \int_{\mathbf{R}^n} g(\xi) Q_{\lambda,s}(\xi - x) d\xi \tag{1.5}$$

for

$$Q_{\lambda,s}(\xi - x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{(-p_n^2 + c^2|p'|^2) e^{-ip\cdot(\xi - x)} dp}{\lambda(|p|^2+1)^s + (-p_n^2 + c^2|p'|^2)^2}.$$

If, for $F \in H^s$ we consider the function $(\Box_c F)(x)$ and we take it as g, then we have the result: as $\lambda \to 0$

$$F^*_{\lambda,s,g} \to F, \tag{1.6}$$

uniformly.

When the practical data g contain errors or noises, we need error estimates for the approximate solution 1.5. Following the idea of weighted convolution inequalities in [4], we obtain

Theorem 1.2. *In the representation of the approximate solution 1.5, we obtain the estimate*

$$\int_{\mathbf{R}^n} |F^*_{\lambda,s,g}(x)|^2 dx \leq \frac{\Gamma(s - n/2)}{2^{n+2}\Gamma(s)} \frac{1}{\lambda} \int_{\mathbf{R}^n} |g(\xi)|^2 e^{|\xi|^2} d\xi. \tag{1.7}$$

The estimate 1.7 implies that for g_δ containing errors and noises,

$$\int_{\mathbf{R}^n} |F^*_{\lambda,s,g}(x) - F^*_{\lambda,s,g_\delta}(x)|^2 dx \leq \frac{\Gamma(s - n/2)}{2^{n+2}\Gamma(s)} \frac{1}{\lambda} \int_{\mathbf{R}^n} |g(\xi) - g_\delta(\xi)|^2 e^{|\xi|^2} d\xi.$$

Meanwhile, in Theorem 1.1 we wish to take a small λ in order to obtain a true solution F. Therefore, for

$$\int_{\mathbf{R}^n} |g(\xi) - g_\delta(\xi)|^2 e^{|\xi|^2} d\xi \leq \delta,$$

we wish to take δ and λ as follows:

$$\delta \to 0$$

and

$$\frac{\delta}{\lambda} \to 0.$$

The integral weight $e^{|\xi|^2}$ will be acceptable, because the functions g and g_δ decay exponentially or we can assume that they have compact supports.

In terms of the Sobolev norm, we obtain

Theorem 1.3. *Let $\delta > 0$ and let g, g_δ satisfy*

$$\|g - g_\delta\|_{L_2(\mathbf{R}^n)} \leq \delta.$$

Then, we have

$$\|F^*_{\lambda,s,g_\delta} - F^*_{\lambda,s,g}\|_{H^s} \leq \frac{\delta}{2\sqrt{\lambda}}.$$

2. BACKGROUND THEOREMS

We shall use the following two general theorems.

Theorem 2.1. *([5,2]) Let H_K be a Hilbert space admitting the reproducing kernel $K(p,q)$ on a set E. Let $L : H_K \to \mathcal{H}$ be a bounded linear operator on H_K into $\mathcal{H}$. For $\lambda > 0$ introduce the inner product in H_K and call it H_{K_λ} as*

$$\langle f_1, f_2\rangle_{H_{K_\lambda}} = \lambda \langle f_1, f_2\rangle_{H_K} + \langle Lf_1, Lf_2\rangle_{\mathcal{H}}, \tag{2.1}$$

then H_{K_λ} is the Hilbert space with the reproducing kernel $K_\lambda(p,q)$ on E and satisfying the equation

$$K(\cdot, q) = (\lambda I + L^* L)K_\lambda(\cdot, q), \tag{2.2}$$

where L^* is the adjoint of $L : H_K \to \mathcal{H}$.

Theorem 2.2. *([5,2]) Let H_K, L, $\mathcal{H}$, E and K_λ be as in Theorem 2.1. Then, for any $\lambda > 0$ and for any $g \in \mathcal{H}$, the extremal function in*

$$\inf_{f \in H_K} \left(\lambda \|f\|^2_{H_K} + \|Lf - g\|^2_{\mathcal{H}} \right) \tag{2.3}$$

exists uniquely and the extremal function is represented by

$$f^*_{\lambda,g}(p) = \langle g, LK_\lambda(., p)\rangle_{\mathcal{H}} \tag{2.4}$$

which is the member of H_K attaining the infimum in 2.3.

3. PROOF OF THEOREM 1.1

First, of course, we have the inequality, for a positive constant $M > 0$,

$$\|\Box_c F\|^2_{L_2(\mathbf{R}^n)} \le M \|F\|^2_{H^s}; \tag{3.1}$$

that is, the operator $\Box_c$ is a bounded linear operator from H^s into $L_2(\mathbf{R}^n)$. Then we can see directly that

$$K_\lambda(x, y; \Box_c)$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{e^{ip\cdot(x-y)}}{\lambda(|p|^2+1)^s + (-p_n^2 + c^2|p'|^2)^2} dp \tag{3.2}$$

satisfies the functional equation 2.2 in our situation; that is, it is the reproducing kernel for the Hilbert space with the norm square

$$\lambda \|F\|^2_{H^s} + \|\Box_c F\|^2_{L_2(\mathbf{R}^n)}.$$

In particular, we thus obtain 1.5 from Theorem 2.2.

In order to prove the result 1.6, as we see from 1.3, note that any member $F \in H^s$ is represented uniquely by a function $\mathbf{F}$ in the form

$$F(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{e^{ix\cdot\eta}}{(1+|\eta|^2)^s} \mathbf{F}(\eta) d\eta \tag{3.3}$$

satisfying

$$\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{1}{(1+|\eta|^2)^s} |\mathbf{F}(\eta)|^2 d\eta < \infty$$

and

$$\|F\|^2_{H^s} = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{1}{(1+|\eta|^2)^s} |\mathbf{F}(\eta)|^2 d\eta. \tag{3.4}$$

Then, we insert this F in 1.1 and we have the function $(\Box_c F)(x)$. Then, we set it as $g(\xi)$ in 1.5 and we obtain, directly

$$F^*_{\lambda,s,g}(x)$$

$$= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \frac{(-\eta_n^2 + c^2|\eta'|^2)^2 e^{ix\cdot\eta}}{(\lambda(1+|\eta|^2)^s + (-\eta_n^2 + c^2|\eta'|^2)^2)(1+|\eta|^2)^s}\mathbf{F}(\eta)d\eta. \tag{3.5}$$

From 3.3 and 3.5 we thus obtain the desired result 1.6.

4. PROOF OF THEOREM 1.2

As in the proof of the weighted convolution inequalities in [4], we obtain directly, for $p = 2$ and $\rho_2 \equiv 1$

$$\int_{\mathbf{R}^n} |F^*_{\lambda,s,g}(x)|^2 dx$$

$$\leq \int_{\mathbf{R}^n} e^{-|\xi|^2} d\xi \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |g(\xi)|^2 e^{|\xi|^2} |Q_{\lambda,s}(\xi - x)|^2 d\xi dx$$

$$= \int_{\mathbf{R}^n} e^{-|\xi|^2} d\xi \int_{\mathbf{R}^n} |g(\xi)|^2 e^{|\xi|^2} d\xi \int_{\mathbf{R}^n} |Q_{\lambda,s}(\xi)|^2 d\xi$$

$$\leq \pi^{n/2} \int_{\mathbf{R}^n} |g(\xi)|^2 e^{|\xi|^2} d\xi \cdot \frac{1}{4\lambda(2\pi)^n}\int_{\mathbf{R}^n} \frac{dp}{(|p|^2+1)^s}$$

$$= \frac{\Gamma(s - n/2)}{2^{n+2}\Gamma(s)}\frac{1}{\lambda}\int_{\mathbf{R}^n} |g(\xi)|^2 e^{|\xi|^2} d\xi.$$

5. PROOF OF THEOREM 1.3

We first have

$$\begin{aligned} \hat{F}^*_{\lambda,s,g_\delta}(p) &= \hat{g}_\delta(p)\frac{(-p_n^2 + c^2|p'|^2)}{\lambda(|p|^2+1)^s + (-p_n^2 + c^2|p'|^2)^2}, \\ \hat{F}^*_{\lambda,s,g}(p) &= \hat{g}(p)\frac{(-p_n^2 + c^2|p'|^2)}{\lambda(|p|^2+1)^s + (-p_n^2 + c^2|p'|^2)^2}. \end{aligned}$$

It follows that

$$\hat{F}^*_{\lambda,s,g_\delta}(p) - \hat{F}^*_{\lambda,s,g}(p) = \frac{(\hat{g}_\delta(p) - \hat{g}(p))(-p_n^2 + c^2|p'|^2)}{\lambda(|p|^2+1)^s + (-p_n^2 + c^2|p'|^2)^2}.$$

Hence, we obtain

$$\begin{aligned} (1+|p|^2)^s|(\hat{F}^*_{\lambda,s,g_\delta}(p) - \hat{F}^*_{\lambda,s,g}(p))|^2 &\leq \frac{|\hat{g}_\delta(p) - \hat{g}(p)|^2(1+|p|^2)^s}{4\lambda(|p|^2+1)^s} \\ &= \frac{|\hat{g}_\delta(p) - \hat{g}(p)|^2}{4\lambda}. \end{aligned}$$

Integrating the latter inequality over $\mathbf{R}^n$ we obtain the desired result.

6. LIMITING PROPERTIES

Our solution 1.5 will give a practical formula for the wave equation. We will show experimental results by using computer. There, we will see that in order to overcome the difficulty in the equation, we must take a very small λ and we must calculate the integral 1.5 very accurately. Computer programs help us to calculate the integral for a very small λ.

Meanwhile, for any $\lambda > 0$, we shall define a linear mapping

$$M_{\lambda,s} : L_2(\mathbf{R}^n) \to H^s$$

by $M_{\lambda,s}(g) = F^*_{\lambda,s,g}$. Now, we consider the composite operators $\Box_c M_{\lambda,s}$ and $M_{\lambda,s}\Box_c$. Using Fourier's integrals, it can be shown that, for $F \in H^s$,

$$(M_{\lambda,s}\Box_c F)(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\Big\{F(\xi) \cdot \int_{\mathbf{R}^n} \frac{(-p_n^2 + c^2|p'|^2)^2 e^{-ip\cdot(\xi-x)}dp}{\lambda(|p|^2+1)^s + (-p_n^2+c^2|p'|^2)^2}\Big\}d\xi \tag{6.1}$$

and for $g \in L_2(\mathbf{R}^n)$,

$$(\Box_c M_{\lambda,s} g)(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\Big\{g(\xi) \cdot \int_{\mathbf{R}^n} \frac{(-p_n^2 + c^2|p'|^2)^2 e^{-ip\cdot(\xi-x)}dp}{\lambda(|p|^2+1)^s + (-p_n^2+c^2|p'|^2)^2}\Big\}d\xi. \tag{6.2}$$

Setting

$$\Delta_{\lambda,s}(x-\xi) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \frac{(-p_n^2 + c^2|p'|^2)^2 e^{-ip\cdot(\xi-x)}}{\lambda(|p|^2+1)^s + (-p_n^2+c^2|p'|^2)^2}dp$$

in 6.1 and 6.2, we have

$$(M_{\lambda,s}\Box_c F)(x) = \int_{\mathbf{R}^n} F(\xi)\Delta_{\lambda,s}(x-\xi)d\xi, \quad (F \in H^s) \tag{6.3}$$

and

$$(\Box_c M_{\lambda,s} g)(x) = \int_{\mathbf{R}^n} g(\xi)\Delta_{\lambda,s}(x-\xi)d\xi, \quad (g \in L_2(\mathbf{R}^n)). \tag{6.4}$$

Then we obtain that

$$\lim_{\lambda\to 0}\Delta_{\lambda,s}(x-\xi) = \delta(x-\xi), \tag{6.5}$$

$$\lim_{\lambda\to 0} M_{\lambda,s}\Box_c = I \tag{6.6}$$

and

$$\lim_{\lambda\to 0} \Box_c M_{\lambda,s} = I. \tag{6.7}$$

The precise meaning of 6.3 and 6.6 is given as follows:

For any $F \in H^s$

$$\lim_{\lambda\to 0}(M_{\lambda,s}\Box_c F)(x) = F(x) \tag{6.8}$$

uniformly on $\mathbf{R}^n$(cf. [7], Section 3). The precise meaning of 6.4 and 6.7 is given as follows:

For any $g \in \mathcal{R}(\Box_c) + \mathcal{R}(\Box_c)^{\perp}$

$$\lim_{\lambda \to 0} \Box_c M_{\lambda,s} g = g$$

in $L_2(\mathbf{R}^n)$(cf. [7]).

7. NUMERICAL EXPERIMENTS

Now we give experimental result to see the behaviour of

$$\lim_{\lambda \to 0} \Box_c M_{\lambda,s}$$

on $L_2(\mathbf{R}^2) \setminus \mathcal{R}(\Box_c)$. Here, if we consider $g(x) = \chi_{[-1,1]}(x_1) \times \chi_{[-1,1]}(x_2)$ then $g \in L_2(\mathbf{R}^2) \setminus \mathcal{R}(\Box_c)$. See Figures 1 - 5.

Similarly, if we consider $F(x) = e^{-|x|^2}(n = 2)$ then $F \in H^s$. We see from Figures 6 - 10 that

$$\lim_{\lambda \to 0} (\Box_c M_{\lambda,s} F)(x) = F(x).$$

In all cases, we assume that

$$n = 2 \quad (x = (x_1, x_2) = (x, t))$$

and

$$c = 1.$$

Further, space $x_1 = x$ is the right hand side direction and time $x_2 = t$ is the deep direction.

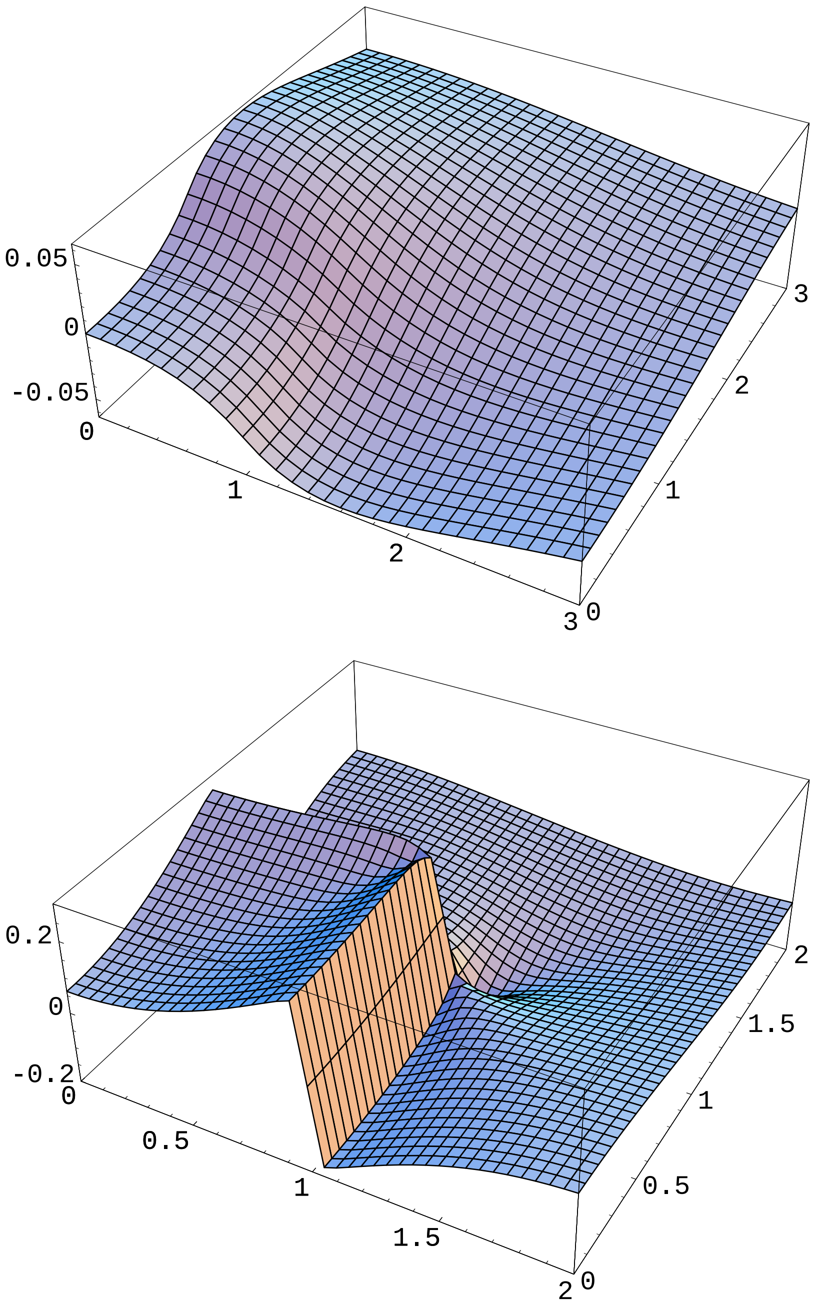


Figure 1: For $g(x_1, x_2) = \chi_{[-1,1]}(x_1) \times \chi_{[-1,1]}(x_2)$ on $\mathbf{R}^2$, the figures of $F^*_{\lambda,2,g}(x_1, x_2)$ and $\Box_c F^*_{\lambda,2,g}(x_1, x_2)$ for $\lambda = 10^0$.

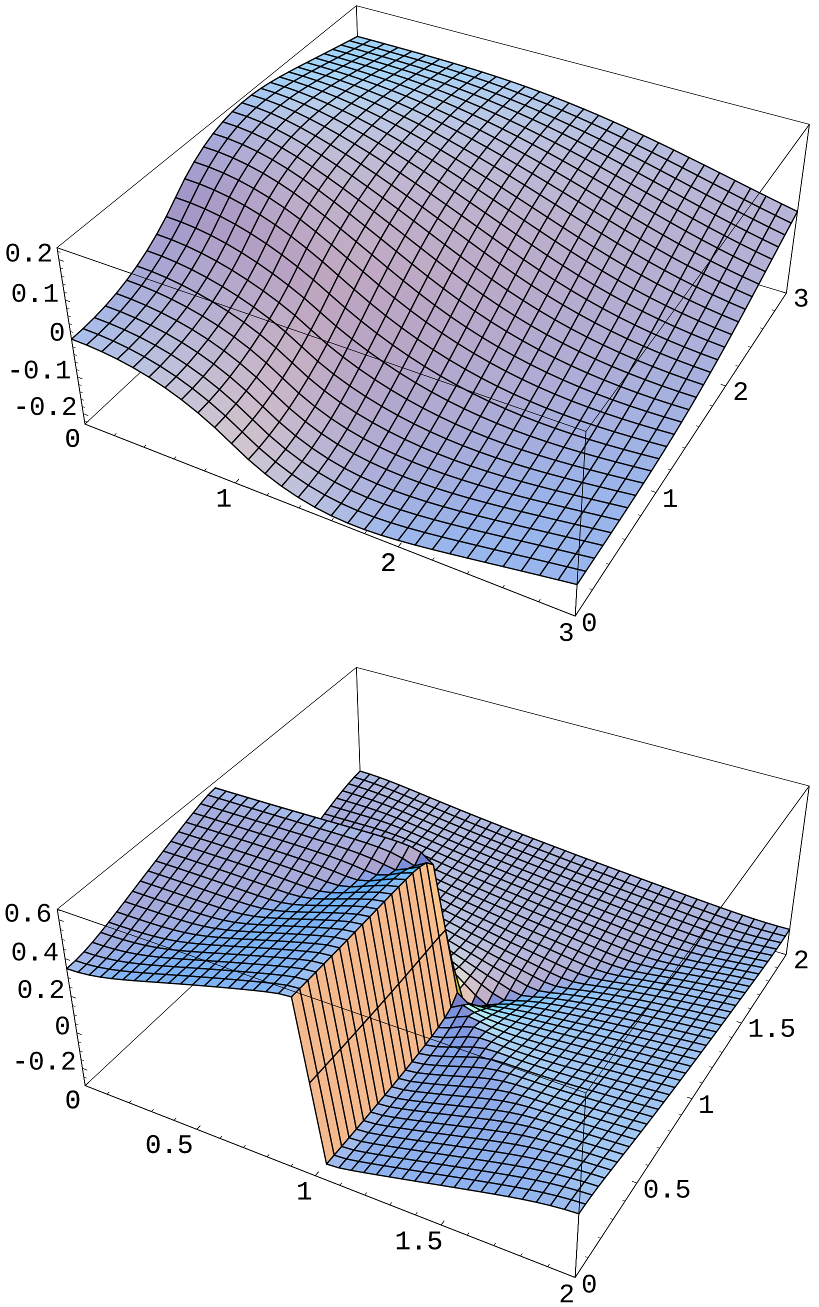
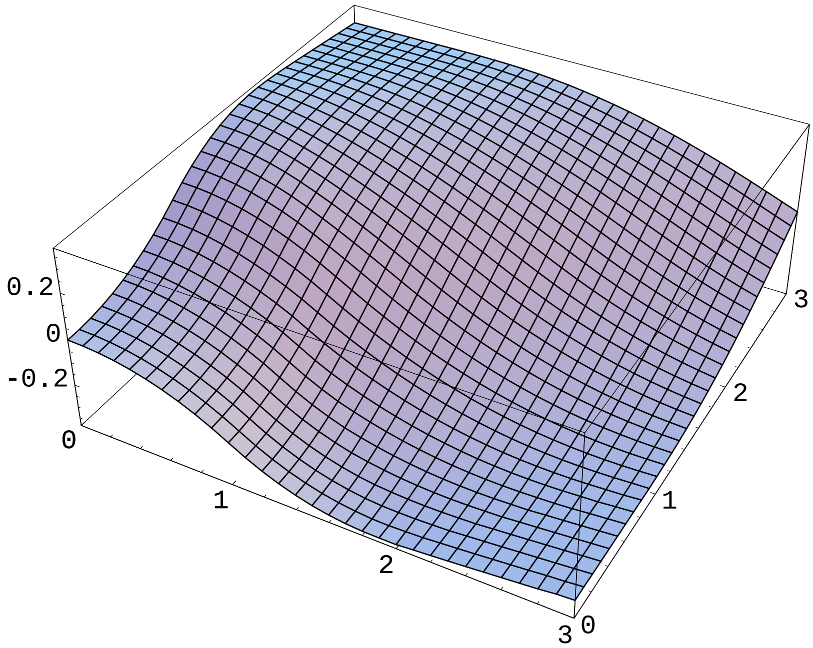


Figure 2: For $g(x_1, x_2) = \chi_{[-1,1]}(x_1) \times \chi_{[-1,1]}(x_2)$ *on* $\mathbf{R}^2$*, the figures of* $F^*_{\lambda,2,g}(x_1, x_2)$ *and* $\Box_c F^*_{\lambda,2,g}(x_1, x_2)$ *for* $\lambda = 10^{-1}$.



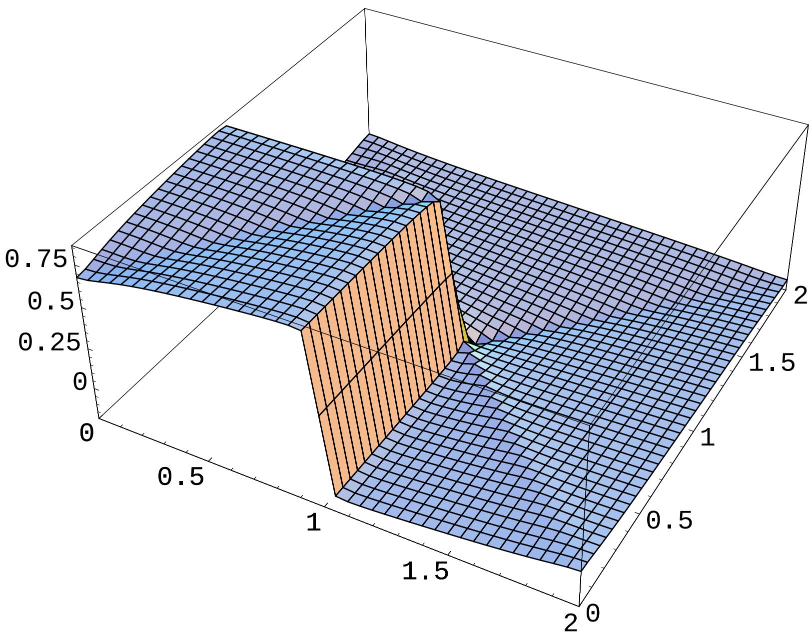


Figure 3: For $g(x_1, x_2) = \chi_{[-1,1]}(x_1) \times \chi_{[-1,1]}(x_2)$ on $\mathbf{R}^2$, the figures of $F^*_{\lambda,2,g}(x_1, x_2)$ and $\Box_c F^*_{\lambda,2,g}(x_1, x_2)$ for $\lambda = 10^{-2}$.

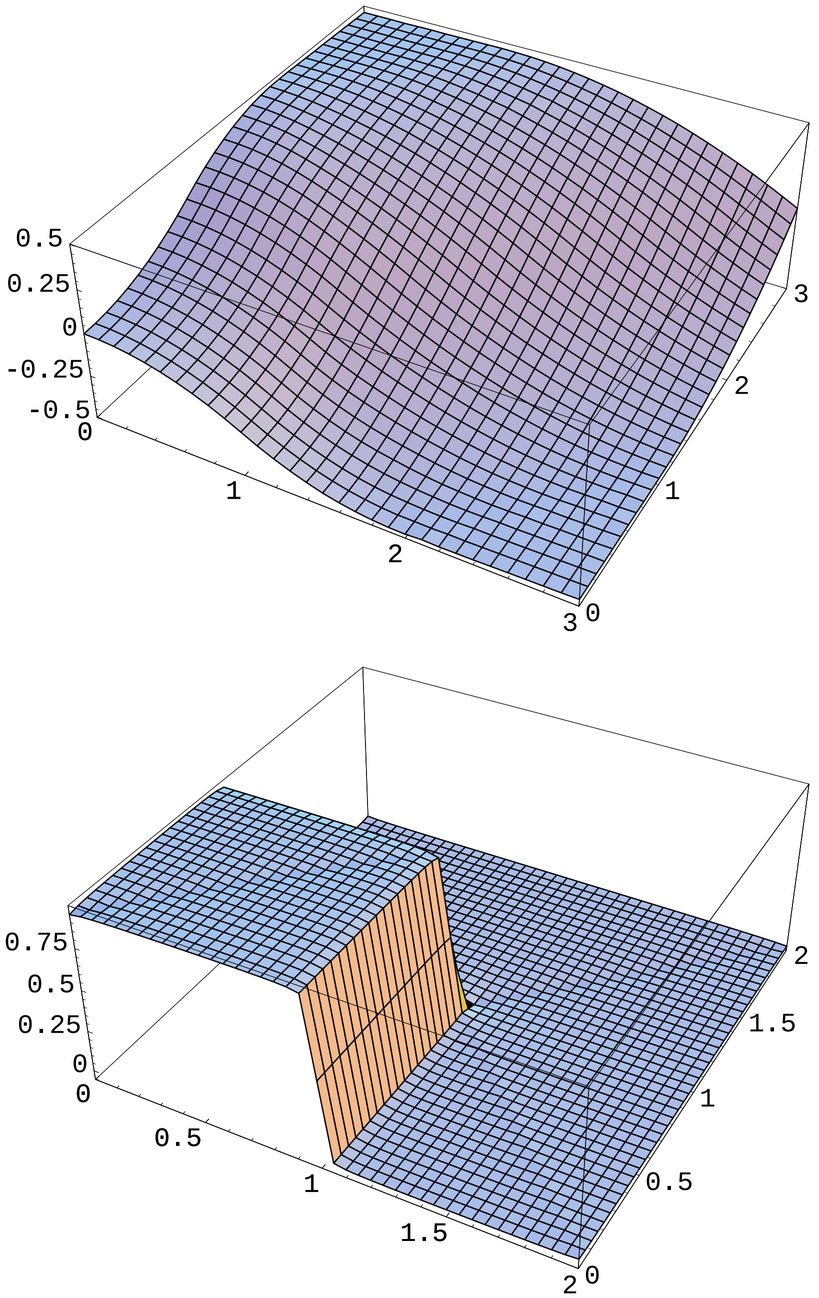


Figure 4: For $g(x_1, x_2) = \chi_{[-1,1]}(x_1) \times \chi_{[-1,1]}(x_2)$ on $\mathbf{R}^2$, the figures of $F^*_{\lambda,2,g}(x_1, x_2)$ and $\Box_c F^*_{\lambda,2,g}(x_1, x_2)$ for $\lambda = 10^{-4}$.

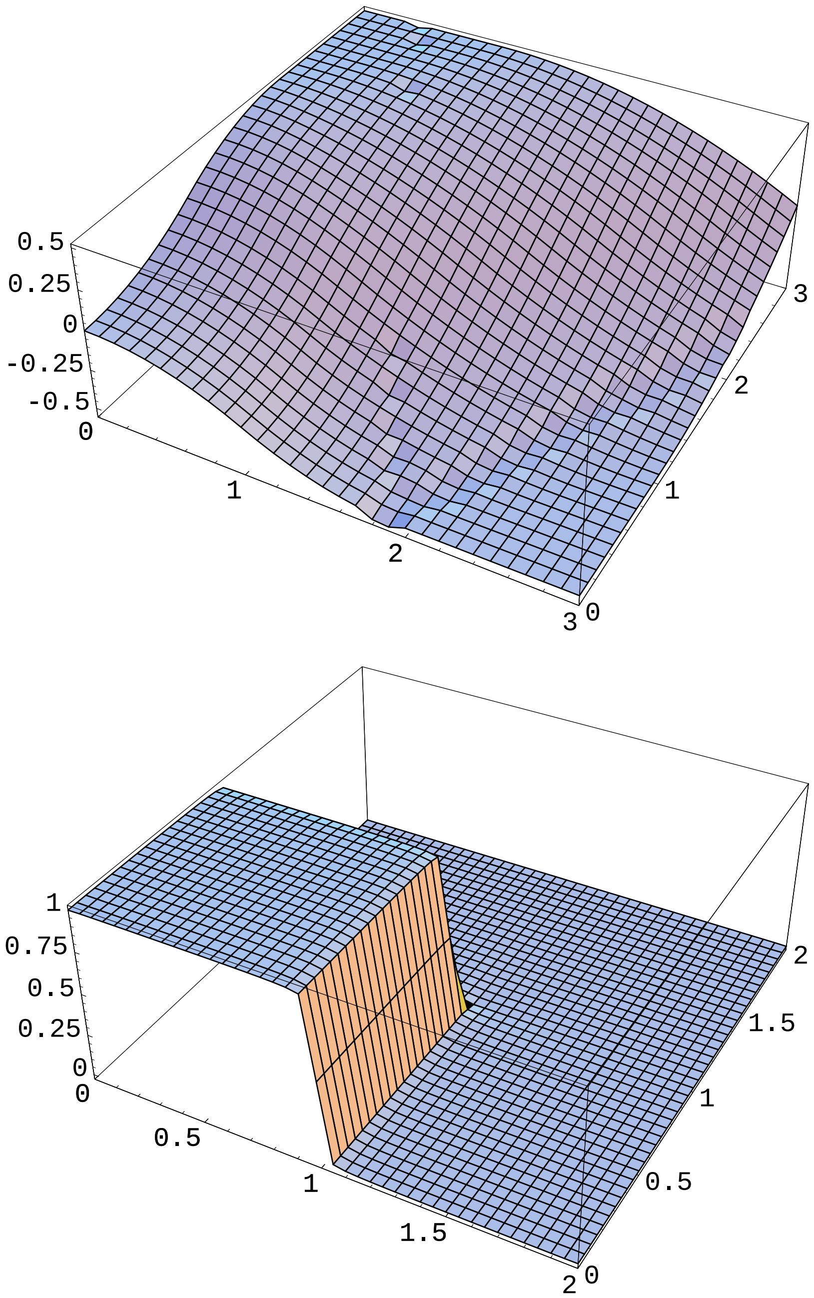


Figure 5: For $g(x_1, x_2) = \chi_{[-1,1]}(x_1) \times \chi_{[-1,1]}(x_2)$ *on* $\mathbf{R}^2$*, the figures of* $F^*_{\lambda,2,g}(x_1, x_2)$ *and* $\Box_c F^*_{\lambda,2,g}(x_1, x_2)$ *for* $\lambda = 10^{-6}$.

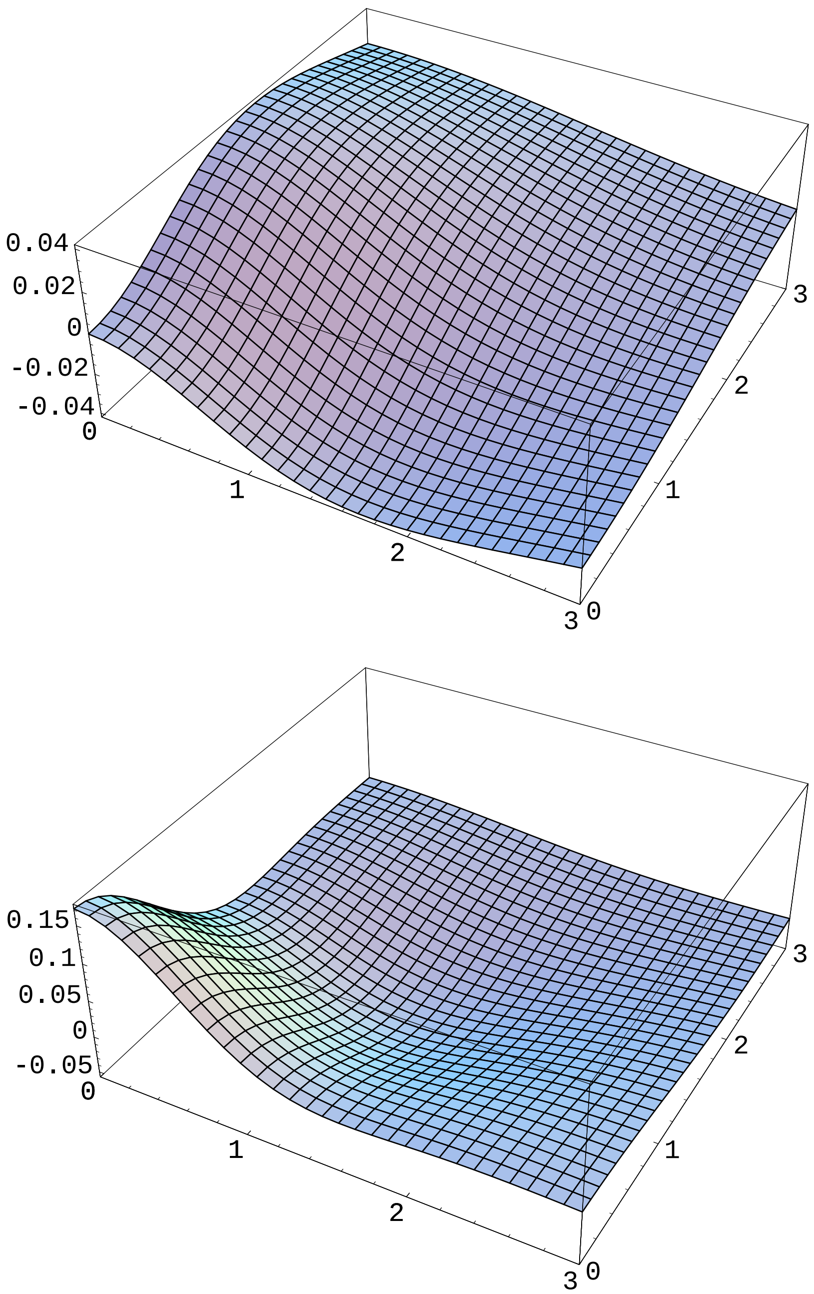


Figure 6: For $g(x_1, x_2) = e^{-(x_1^2+x_2^2)}$ on $\mathbf{R}^2$, the figures of $F^*_{\lambda,2,g}(x_1, x_2)$ and $\Box_c F^*_{\lambda,2,g}(x_1, x_2)$ for $\lambda = 10^0$.

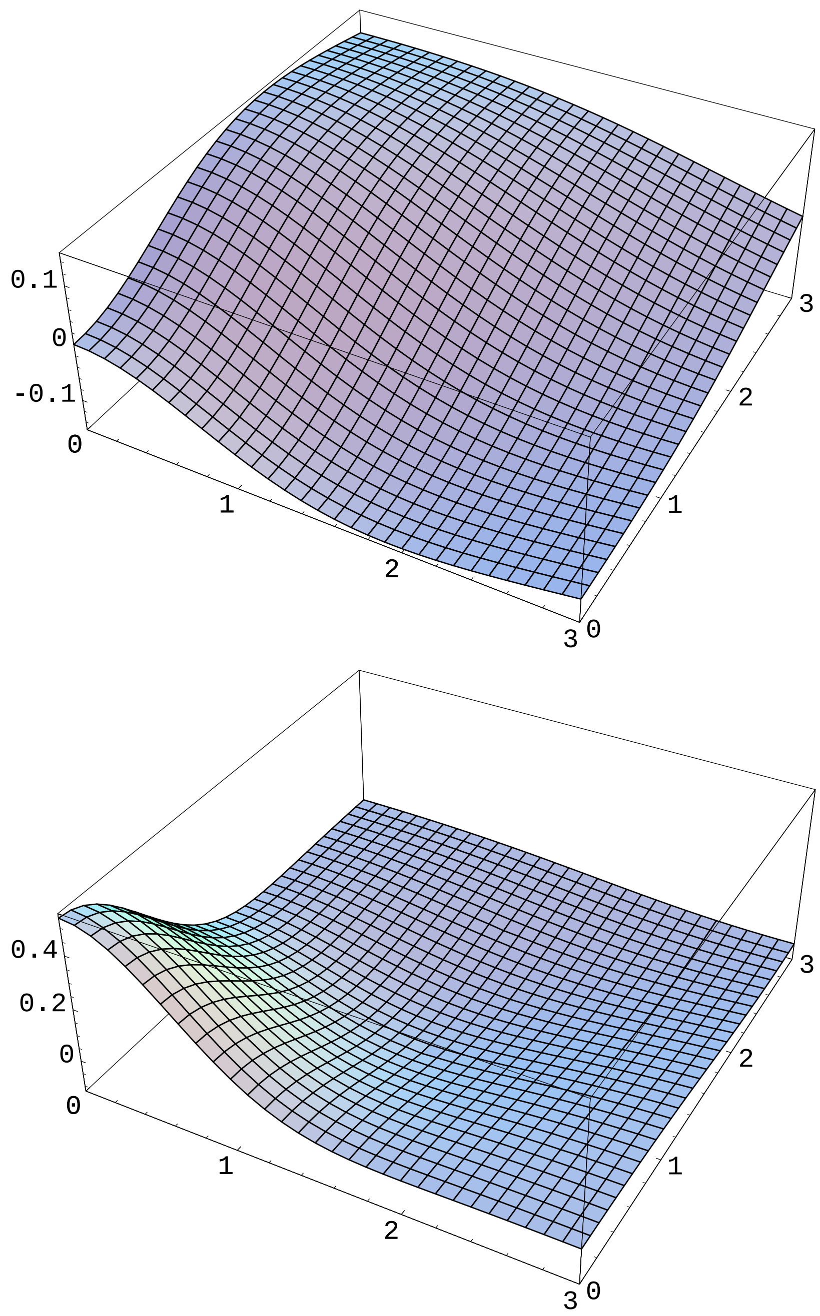


Figure 7: For $g(x_1, x_2) = e^{-(x_1^2+x_2^2)}$ on $\mathbf{R}^2$, the figures of $F^*_{\lambda,2,g}(x_1, x_2)$ and $\Box_c F^*_{\lambda,2,g}(x_1, x_2)$ for $\lambda = 10^{-1}$.

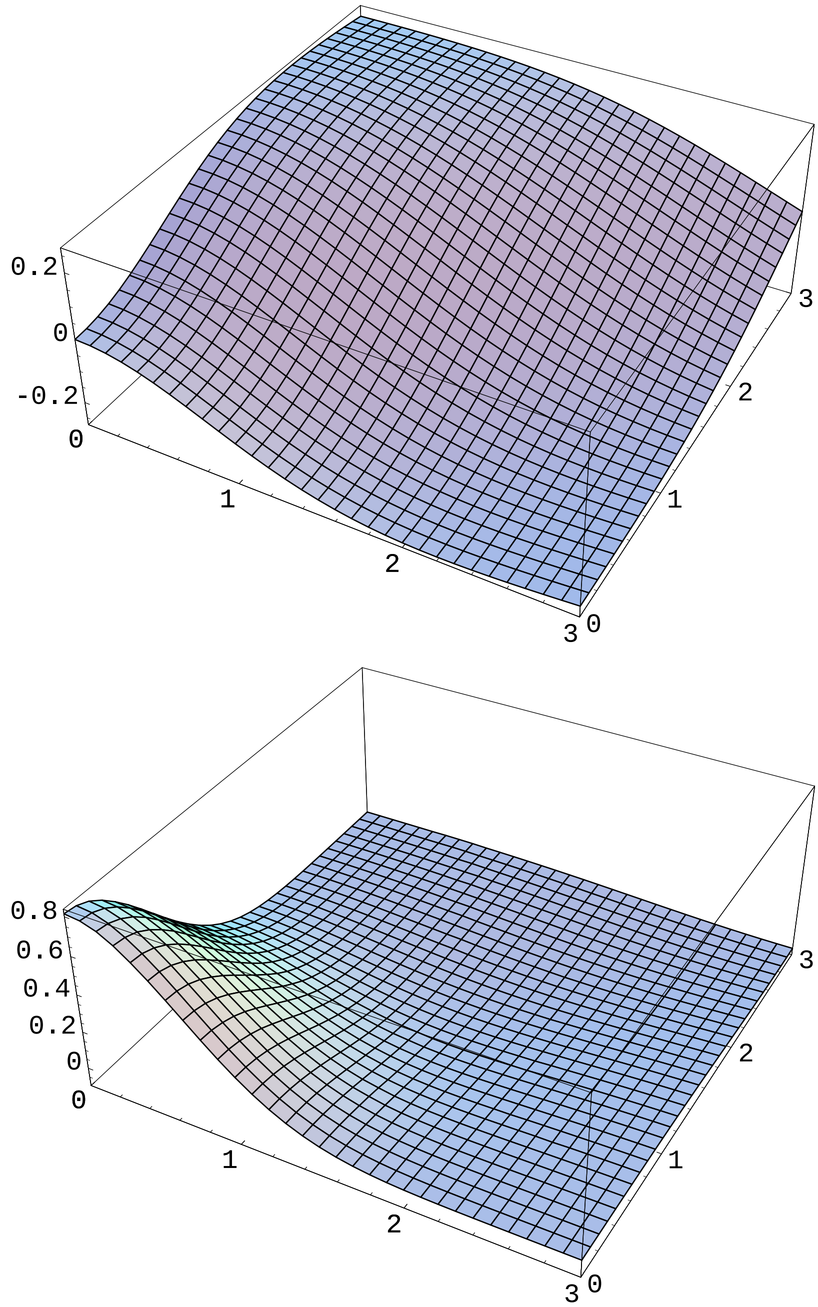


Figure 8: For $g(x_1, x_2) = e^{-(x_1^2+x_2^2)}$ on $\mathbf{R}^2$, the figures of $F^*_{\lambda,2,g}(x_1, x_2)$ and $\Box_c F^*_{\lambda,2,g}(x_1, x_2)$ for $\lambda = 10^{-2}$.

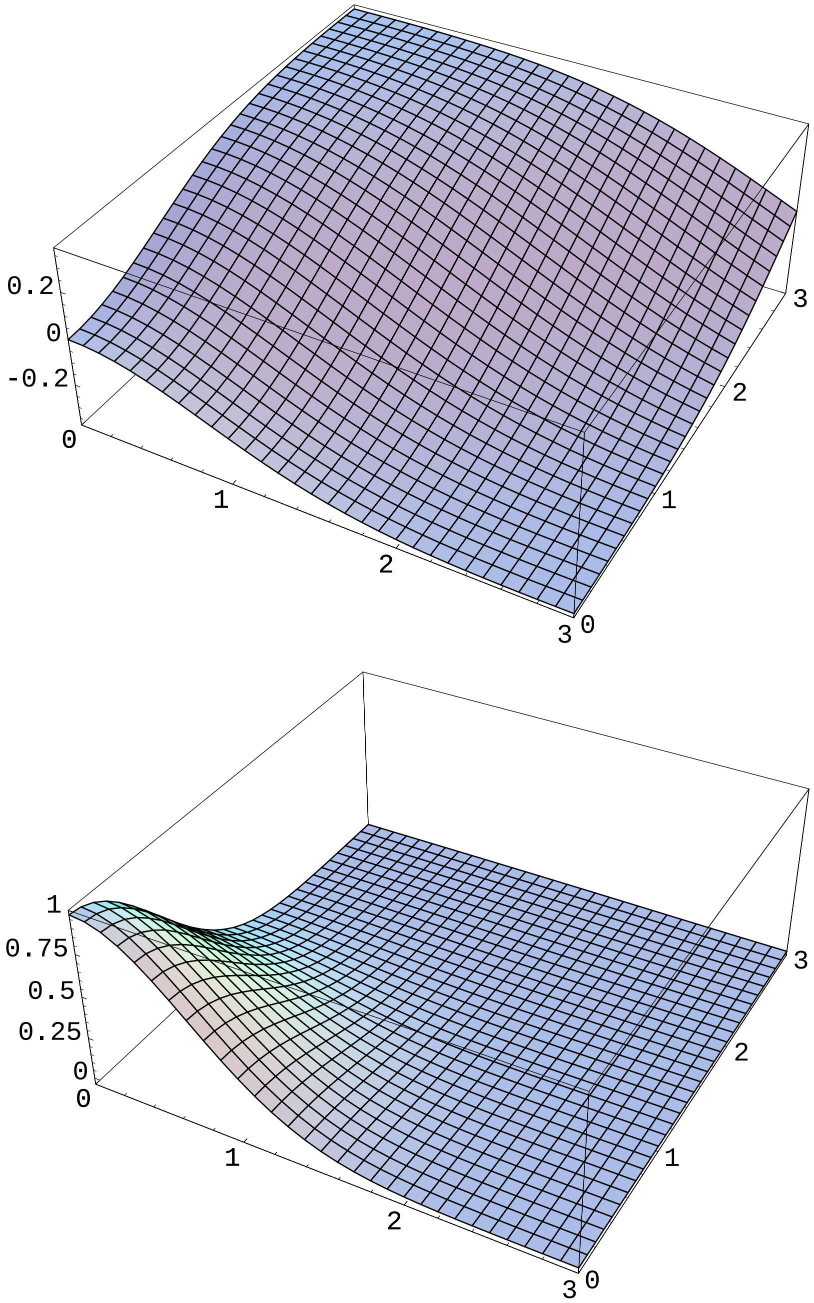


Figure 9: For $g(x_1, x_2) = e^{-(x_1^2+x_2^2)}$ *on* $\mathbf{R}^2$*, the figures of* $F^*_{\lambda,2,g}(x_1, x_2)$ *and* $\Box_c F^*_{\lambda,2,g}(x_1, x_2)$ *for* $\lambda = 10^{-4}$.

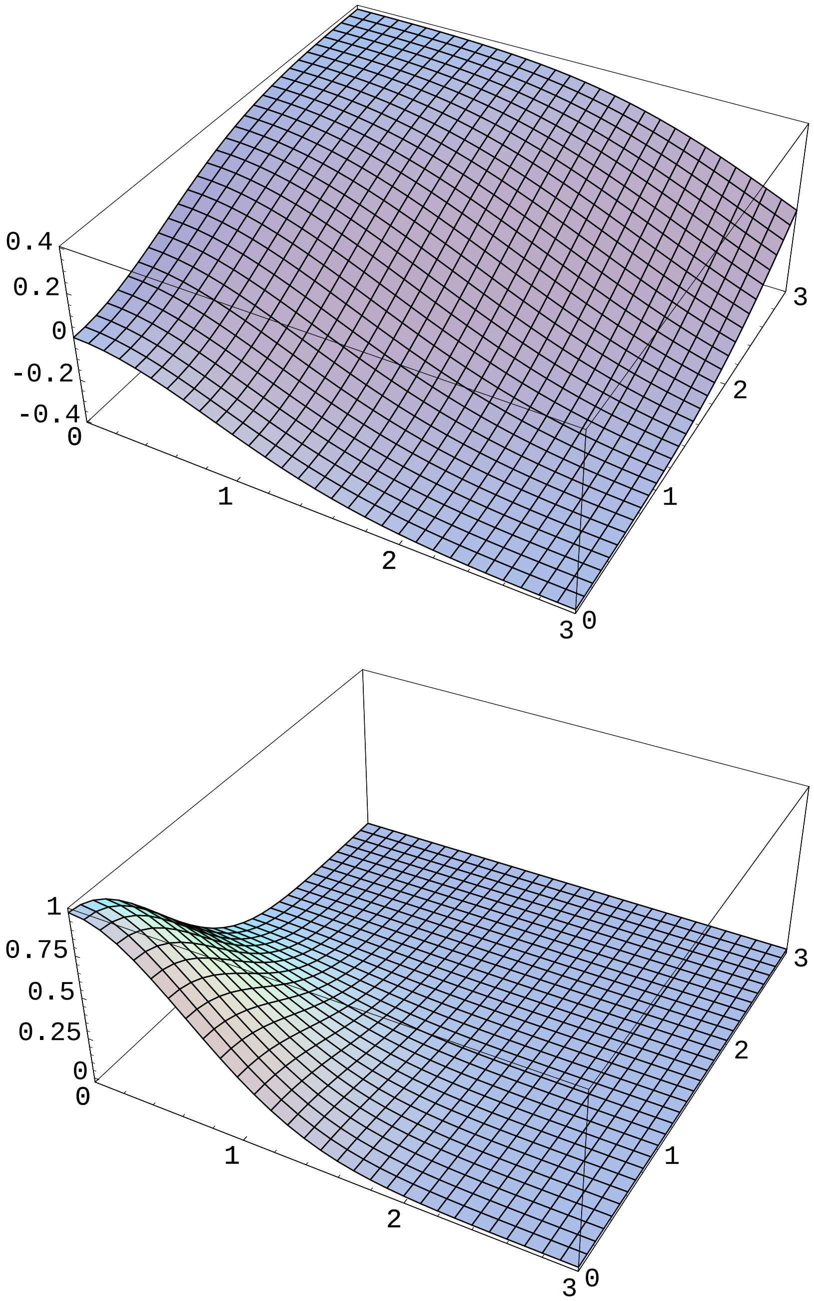


Figure 10: For $g(x_1,x_2) = e^{-(x_1^2+x_2^2)}$ on $\mathbf{R}^2$, the figures of $F^*_{\lambda,2,g}(x_1,x_2)$ and $\Box_c F^*_{\lambda,2,g}(x_1,x_2)$ for $\lambda = 10^{-6}$.

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