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PSEUDOMONOTONICITY AND QUASIMONOTONICITY BY TRANSLATIONS VERSUS MONOTONICITY IN HILBERT SPACES

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ABSTRACT. Let $F : U \to H$ be a Gâteaux differentiable mapping on an open convex subset U of a Hilbert space H. If there exists a straight line $S \subset H$ such that $F(\cdot) - u$ is pseudomonotone for any $u \in S$, then F is monotone. Related results using a regularity condition are given.

Key words and phrases: Pseudomonotonicity (in the Karamardian's sense), Quasimonotonicity, Monotonicity, Orthogonality, Differentiability.

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1. INTRODUCTION

The notion of monotone operator, introduced by R. I. Kachurovskii [11], [12] and G. Minty [17]-[20], became very fast a fundamental concept in nonlinear analysis. Many papers and books have been dedicated to it, as for example [1], [11], [12], [17]-[20], [23]. However, notably in optimization and economics, it is important to proceed beyond monotonicity. We refer to the papers [2]-[8], [13]-[15], [22] for generalizations of monotonicity. These extensions are now currently used in complementarity problems [5]-[10], [13], variational inequalities [2], [6], [8], [16], equilibrium of economical systems [7], different topics in nonlinear analysis [23].

In the present paper we deal with the notion of pseudomonotone operator which extends the monotonicity. This concept has been introduced by S. Karamardian in [13] as a new tool in the theory of complementarity problems. It is known that a monotone mapping is pseudomonotone, but the converse is not generally true. A natural question is under what conditions a pseudomonotone operator is monotone. The paper addresses to this question.

Until now the problem has been studied in the Euclidean space $\mathbb{R}^n$ for affine mappings (i.e., mappings of the form $x \mapsto Mx + q$, where M is a matrix and q is an element of $\mathbb{R}^n$) by M. S. Gowda [5], and J. P. Crouzeix and S. Schaible [4]. For instance, it is proved in [5] that a matrix M is positive semidefinite if and only if, for every $q \in \mathbb{R}^n$, the mapping $x \mapsto Mx + q$ is pseudomonotone with respect to the convex cone $\mathbb{R}^n_+$. It is also worth to mention that Y. He supposed in several results of [9] (e.g., Lemma 5.9, Proposition 5.3, conditions (A_5), (A_6)) that, for a given $F : \mathbb{R}^n \to \mathbb{R}^n$, the mapping $F(\cdot) - u$ is pseudomonotone for any $u \in \mathbb{R}^n$. The facts presented above suggest to study when an operator possessing pseudomonotone translations is monotone. Certainly, to have an optimal result, the set of such translations must be required to be minimal in an appropriate sense. This is a significant mathematical question with a potentially large applicability in optimization.

Here we give a positive answer to the problem for Gâteaux differentiable mappings defined on an open convex subset of a Hilbert space. Namely, our main result establishes that the pseudomonotonicity for the translations of the operator that are defined by elements of a straight line ensures the monotonicity of the operator. The essence of our result is that it suffices to check for pseudomonotone translations of the operator only along a single straight line. Moreover, we prove that this is true if the translations of the operator along a straight line are quasimonotone. In our approach we point out a characterization of pseudomonotone operators, as well as of quasimonotone operators, which are continuously differentiable mappings and whose differential is invertible at the points where the mapping vanishes. This type of nonlinear operators have been studied in [21] on an Euclidean space $\mathbb{R}^n$. As a byproduct of our work we extend the basic result in [21] from finite dimensional spaces to Hilbert spaces.

The rest of the paper is organized as follows. Section 2 presents some useful results on the pseudomonotonicity and quasimonotonicity of differentiable mappings in Hilbert spaces. Section 3 is devoted to our main results.

2. PSEUDOMONOTONICITY AND QUASIMONOTONICITY OF DIFFERENTIABLE MAPPINGS IN HILBERT SPACES

Throughout the paper we denote by $(H, \langle \cdot, \cdot \rangle)$ an arbitrary real Hilbert space. Recall some important definitions.

Definition 2.1. We say that a mapping $F : U \to H$ defined on a nonempty subset U of H is monotone if $\langle F(y) - F(x), y - x \rangle \geq 0$, for any $x, y \in U$. The mapping F is called pseudomonotone (in Karamardian's sense) if whenever $x, y \in U$, we have

$$\langle F(x), y - x \rangle \geq 0 \Rightarrow \langle F(y), y - x \rangle \geq 0.$$

We say that F is quasimonotone if whenever $x, y \in U$,

$$\langle F(x), y - x\rangle > 0 \Rightarrow \langle F(y), y - x\rangle \geq 0.$$

Any pseudomonotone operator is quasimonotone, but the converse is not true [14].

In the sequel we suppose that U is an open subset of H and the mapping $F : U \to H$ is Gâteaux differentiable. The notation $DF(u)$ will stand for the Gâteaux differential of F at the point $u \in U$. Denoting by $L(H)$ the Banach space of linear bounded operators from H to H, one has $DF(u) \in L(H)$.

The next two conditions are needed in the following.

$(SD^{\perp})$ For each $u \in U$, the linear operator $DF(u) : H \to H$ is positive semidefinite on $F(u)^{\perp} := \{w \in H : \langle F(u), w\rangle = 0\}$, i.e.,

$$\langle DF(u)v, v\rangle \geq 0 \text{ whenever } v \in H \text{ is such that } \langle F(u), v\rangle = 0.$$

(CF) For each $u \in U$ satisfying $F(u) = 0$ and for each $v \in H$ with $\langle DF(u)v, v\rangle = 0$, there are no $\overline{\lambda} > 0$ so that

$$\langle F(u + \lambda v), v\rangle < 0, \quad \forall \lambda \in]0, \overline{\lambda}].$$

Remark 2.1. Conditions $(SD^{\perp})$ and (CF) have been introduced in [3] for $H = \mathbb{R}^N$.

The following result is an extension of [3, Theorem 3 (ii)] or [21, Theorem 1] from finite dimensional spaces to Hilbert spaces.

Theorem 2.1. *If $F : U \to H$ is a continuously differentiable mapping on an open convex subset U of a Hilbert space H such that conditions $(SD^{\perp})$ and (CF) are fulfilled, then the mapping F is pseudomonotone.*

Proof. Arguing by contradiction, let us suppose that F is not pseudomonotone. Then there exist $x, y \in U$ such that

(2.1) $$\langle F(x), y - x\rangle \geq 0 \text{ and } \langle F(y), y - x\rangle < 0.$$

We introduce the function $f : [0, 1] \to \mathbb{R}$ by

$$f(t) = \langle F((1 - t)x + ty), y - x\rangle, \quad \forall t \in [0, 1].$$

It is clear that f is continuously differentiable on $[0, 1]$, and by (2.1) it is seen $f(0) \geq 0$ and $f(1) < 0$. It follows that there is $t_0 \in [0, 1[$ such that

(2.2) $$f(t_0) = 0 \text{ and } f(t) < 0, \quad \forall t \in]t_0, 1].$$

We claim

(2.3) $$F(x + t_0(y - x)) \neq 0.$$

Let us admit on the contrary $F(x + t_0(y - x)) = 0$. Then one obtains from (2.2) that

$$\begin{aligned}&\langle DF(x + t_0(y - x))(y - x), y - x\rangle = f'(t_0)\\ &= \lim_{t \to 0^+} \frac{1}{t}\langle F(x + (t_0 + t)(y - x)), y - x\rangle = \lim_{t \to 0^+} \frac{1}{t} f(t_0 + t) \leq 0.\end{aligned}$$

On the other hand, by condition $(SD^{\perp})$ and the first relation in (2.2) we have

$$\langle DF(x + t_0(y - x))(y - x), y - x\rangle \geq 0.$$

So we conclude

$$\langle DF(x + t_0(y - x))(y - x), y - x\rangle = 0.$$

This enables us to apply assumption (CF) with $u = x + t_0(y - x)$ and $v = y - x$. Setting $\overline{\lambda} = 1 - t_0$ we get a contradiction between (CF) and the second relation in (2.2). This contradiction justifies (2.3).

Since F is continuous, we infer from (2.3) that

$$\langle F(x + (t_0 + t)(y - x)), F(x + t_0(y - x))\rangle > 0$$

for $t > 0$ close to 0. Then from the second relation in (2.2) we see

$$\alpha(t) := \frac{\langle F(x + (t_0 + t)(y - x)), y - x\rangle}{\langle F(x + (t_0 + t)(y - x)), F(x + t_0(y - x))\rangle} < 0 \tag{2.4}$$

whenever $0 < t \leq \bar{t}$ with $\bar{t} > 0$ near 0. A direct computation yields

$$\langle y - x - \alpha(t)F(x + t_0(y - x)), F(x + (t_0 + t)(y - x))\rangle = 0, \ \ t \in [0, \bar{t}], \tag{2.5}$$

and

(2.6)

$$\begin{aligned}
&\langle y - x - \alpha(t)F(x + t_0(y - x)), DF(x + (t_0 + t)(y - x))(y - x - \alpha(t)F(x + t_0(y - x))\rangle \\
&= \alpha(t)[\frac{\langle y - x, DF(x + (t_0 + t)(y - x))(y - x)\rangle}{\alpha(t)} \\
&+ \alpha(t)\langle F(x + t_0(y - x)), DF(x + (t_0 + t)(y - x))F(x + t_0(y - x))\rangle \\
&- \langle F(x + t_0(y - x)), DF(x + (t_0 + t)(y - x))(y - x)\rangle \\
&- \langle y - x, DF(x + (t_0 + t)(y - x))F(x + t_0(y - x))\rangle], \ \ t \in [0, \bar{t}].
\end{aligned}$$

Using the expressions of $\alpha(t)$ and $f(t)$ we derive

$$\begin{aligned}
&\frac{\langle y - x, DF(x + (t_0 + t)(y - x))(y - x)\rangle}{\alpha(t)} \\
&= \frac{f'(t_0 + t)}{f(t_0 + t)}\langle F(x + t_0(y - x)), F(x + (t_0 + t)(y - x))\rangle, \ \ t \in [0, \bar{t}].
\end{aligned} \tag{2.7}$$

In view of (2.2) we have $\ln|f(t)| \to -\infty$ as $t \to t_0^+$. Thus the function

$$\frac{d}{dt}(\ln|f(t)|) = \frac{f'(t)}{f(t)}$$

is unbounded from above as $t \to t_0^+$. Notice, by (2.7), (2.3) and the above property, that

$$\frac{\langle y - x, DF(x + (t_0 + t)(y - x))(y - x)\rangle}{\alpha(t)} \to +\infty \quad \text{as } t \to 0^+.$$

Then relation (2.6), in conjunction with (2.4) and the continuity of DF, implies

$$\begin{aligned}
&\langle y - x - \alpha(t)F(x + t_0(y - x)), DF(x + (t_0 + t)(y - x))(y - x - \alpha(t)F(x + t_0(y - x))\rangle \\
&< 0 \text{ if } t > 0 \text{ is sufficiently small.}
\end{aligned}$$

Making use of (2.5), the above inequality leads to a contradiction with assumption $(SD^{\perp})$ applied for $u = x + (t_0 + t)(y - x)$ and $v = y - x - \alpha(t)F(x + t_0(y - x))$ when $t > 0$ is small enough. The obtained contradiction ensures the situation in (2.1) is not possible. This completes the proof. ∎

The next result points out a useful necessary condition to have quasimonotonicity and a fortiori pseudomonotonocity.

Theorem 2.2. *Assume that $F : U \to H$ is a quasimonotone and Gâteaux differentiable mapping on an open subset U of a Hilbert space H. Then F verifies condition $(SD^{\perp})$.*

Proof. Suppose by contradiction that exist $u \in U$ and $v \in H$ such that

$$\langle F(u), v\rangle = 0 \text{ and } \langle DF(u)v, v\rangle < 0.$$

It turns out

$$\frac{d}{dt}\langle F(u+tv), v\rangle|_{t=0} = \lim_{t\to 0}\frac{1}{t}\langle F(u+tv), v\rangle = \langle DF(u)v, v\rangle < 0.$$

As $\langle F(u), v\rangle = 0$ and the differentiable function $t \mapsto \langle F(u+tv), v\rangle$ is decreasing around 0, there exist two numbers $t_1 > 0$ and $t_2 > 0$ such that

$$\langle F(u-t_2v), v\rangle > 0 \text{ and } \langle F(u+t_1v), v\rangle < 0. \tag{2.8}$$

Knowing by the first inequality in (2.8) that $\langle F(u-t_2v), (t_1+t_2)v\rangle > 0$, the quasimonotonicity of F guarantees $\langle F(u+t_1v), (t_1+t_2)v\rangle \geq 0$. This contradicts the second inequality in (2.8). The conclusion is achieved. ∎

Corollary 2.3. *Assume that $F : U \to H$ is a pseudomonotone and Gâteaux differentiable mapping on an open subset U of a Hilbert space H. Then F verifies condition $(SD^{\perp})$.*

Proof. Since a pseudomonotone operator is quasimonotone, the result follows readily from Theorem 2.2. ∎

The below theorem discusses an important class of pseudomonotone and continuously differentiable mappings. In particular, Theorem 2 in [21] is extended from finite dimensional spaces to Hilbert spaces.

Theorem 2.4. *Assume that $F : U \to H$ is a continuously differentiable mapping on an open subset U of a Hilbert space H such that the following regularity condition holds*

$$F(u) = 0 \Rightarrow DF(u) \text{ is invertible }. \tag{2.9}$$

Then the following assertions are equivalent:

- (i) F *is pseudomonotone;*
- (ii) F *is quasimonotone;*
- (iii) F *verifies condition* $(SD^{\perp})$.

Proof. The implication $(i) \Rightarrow (ii)$ is obvious, while Theorem 2.2 shows $(ii) \Rightarrow (iii)$.

We prove $(iii) \Rightarrow (i)$. According to Theorem 2.1 it is sufficient to check that condition (CF) holds true. Towards this let $u \in U$ and $v \in H$ satisfy

$$F(u) = 0 \text{ and } \langle DF(u)v, v\rangle = 0. \tag{2.10}$$

Fix any $\overline{\lambda} > 0$. Hypothesis (2.9), the first equality in (2.10) and the inverse function theorem yield the existence of an open convex neighborhood U_0 of u, with $U_0 \subset U$, such that

$$F(x) \neq 0, \ \ \forall x \in U_0 \setminus \{u\}. \tag{2.11}$$

Consider a number $\overline{\lambda}_0 \in]0, \overline{\lambda}]$ for which one has $u + \lambda v \in U_0$ whenever $\lambda \in [0, \overline{\lambda}_0]$. We claim that one can find $\lambda_0 \in]0, \overline{\lambda}_0]$ with

$$\langle F(u+tv), v\rangle \geq 0, \ \ \forall t \in [0, \lambda_0]. \tag{2.12}$$

On the contrary there would exist a sequence $\lambda_n \in]0, \overline{\lambda}_0[$ converging to 0 that satisfies

$$\langle F(u+\lambda_n v), v\rangle < 0. \tag{2.13}$$

We show that the property in (2.13) along a sequence $\lambda_n \to 0^+$ implies

$$\langle DF(u)w, v\rangle \leq 0, \ \ \forall w \in H. \tag{2.14}$$

Let $w \in H$. By the second equality in (2.10) it is sufficient to assume that w is not collinear with v (in particular, $w \neq 0$). For any λ_n in (2.13) there is a constant $c = c(\lambda_n) > 0$ small enough which verifies

$$\langle F(u + \lambda_n v), \mu w - \lambda_n v\rangle > 0, \ \ \forall \mu \in [0, c], \tag{2.15}$$

and $u + \mu w \in U_0$ for all $\mu \in [0, c]$. We note that, if n is sufficiently large and for any small enough $\mu > 0$, the segment $[u + \lambda_n v, u + \mu w]$ is contained in $U_0 \setminus \{u\}$. Here we use essentially the fact that, due to the linear independence of v and w, the segment $[u + \lambda_n v, u + \mu w]$ does not include the point u. Choose an open convex subset $U_1 \subset U_0 \setminus \{u\}$ containing the segment $[u+\lambda_n v, u+\mu w]$. Then relation (2.11) assures that F fulfills condition (CF) on U_1. Thanks to condition $(SD^{\perp})$ which holds for F on $U \supset U_1$, we may invoke Theorem 2.1 to obtain that F is pseudomonotone on U_1. Taking into account (2.15), this yields $\langle F(u+\mu w), \mu w - \lambda_n v\rangle \geq 0$. Then, as $\mu > 0$ is arbitrarily small and using the first relation in (2.10), we get

$$\begin{aligned} 0 &\leq \lim_{\mu \to 0^+} \frac{1}{\mu} \langle F(u + \mu w), \mu w - \lambda_n v\rangle \\ &= -\lambda_n \lim_{\mu \to 0^+} \frac{1}{\mu} \langle F(u + \mu w), v\rangle = -\lambda_n \langle DF(u)w, v\rangle. \end{aligned}$$

Since λ_n is positive, the claim in (2.14) is proved.

We know by (2.9) and the first equality in (2.10) that $DF(u)$ is an isomorphism of H. Then we deduce from (2.14) that $v = 0$. This is impossible because we supposed that (2.13) holds true. Thus property (2.12) is valid, which ensures that condition (CF) is verified. Now it suffices to apply Theorem 2.1 for completing the proof. ∎

3. MAIN RESULTS

Our first main result concerns the monotonicity via quasimonotonicity by translations, so pseudomonotonicity by translations.

Theorem 3.1. *Let U be an open convex subset of a Hilbert space H and $F : U \to H$ a Gâteaux differentiable mapping. Assume there exists a straight line $S \subset H$ such that the mapping $F(\cdot) - u$ is quasimonotone on H for any $u \in S$. Then the mapping F is monotone.*

Proof. In view of [17, Theorem 6] it is sufficient to show that $DF(x)$ is positive semidefinite for every $x \in U$. Let $x_0 \in U$. Using the straight line S given in the statement we introduce the set

$$S(x_0) = \{x \in H : \text{ there exists } w \in S \text{ such that } \langle F(x_0) - w, x\rangle = 0\}.$$

Recall that, by hypothesis, the mapping $F(\cdot) - w$ is quasimonotone for every $w \in S$. It is thus permitted to apply Theorem 2.2 ensuring that the operator $F(\cdot) - w$ satisfies condition $(SD^{\perp})$ for every $w \in S$. It turns out

$$\langle DF(x_0)v, v\rangle \geq 0, \ \ \forall v \in S(x_0). \tag{3.1}$$

We show the set $S(x_0)$ is dense in H. Towards this we notice, because $F(x_0) - S$ is a straight line, there exist $\alpha, \beta \in H$, with $\beta \neq 0$, such that

$$F(x_0) - S = \{\alpha - t\beta \in H : t \in \mathbb{R}\}.$$

In view of the definition of $S(x_0)$, this leads to

$$S(x_0) = \bigcup_{t \in \mathbb{R}} \{x \in H : \ \langle \alpha - t\beta, x\rangle = 0\}. \tag{3.2}$$

In order to prove the density of $S(x_0)$ in H let an arbitrary $x \in H$. If $\langle \beta, x\rangle \neq 0$, then $\langle \alpha - t\beta, x\rangle = 0$ for $t = \langle \alpha, x\rangle / \langle \beta, x\rangle$, which implies by (3.2) that $x \in S(x_0)$. If $\langle \beta, x\rangle = 0$, then for any $\lambda \neq 0$ we have $\langle \beta, x + \lambda\beta \rangle = \lambda \|\beta\|^2 \neq 0$. According to the previous situation it results $x + \lambda\beta \in S(x_0)$ whenever $\lambda \neq 0$. Letting $\lambda \to 0$ it follows that x is in the closure of $S(x_0)$. This proves the density of $S(x_0)$ in H.

Combining relation (3.1) with the density of $S(x_0)$ in H, we obtain that $DF(x_0)$ is positive semidefinite at any point $x_0 \in U$. By [17, Theorem 6], the proof is complete. ∎

Since a pseudomonotone operator is quasimonotone, the next statement follows directly from Theorem 3.1.

Corollary 3.2. *Let U be an open convex subset of a Hilbert space H and $F : U \to H$ a Gâteaux differentiable mapping. Assume there exists a straight line $S \subset H$ such that the mapping $F(\cdot) - u$ is pseudomonotone on H for any $u \in S$. Then the mapping F is monotone.*

As another consequence of Theorem 3.1 we have the following result.

Corollary 3.3. *Let V be a convex closed subset with non-empty interior $\mathrm{int}(V)$ of a Hilbert space H and $F : V \to H$ a continuous mapping which is Gâteaux differentiable on $\mathrm{int}(V)$. Then the following assertions are equivalent:*

(a) *there exists a straight line $S \subset H$ such that the mapping $F(\cdot) - u$ is pseudomonotone on $\mathrm{int}(V)$ for any $u \in S$;*

(b) *there exists a straight line $S \subset H$ such that the mapping $F(\cdot) - u$ is quasimonotone on $\mathrm{int}(V)$ for any $u \in S$;*

(c) *F is monotone on V.*

Proof. $(a) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (c)$. Theorem 3.1 can be applied to the restriction of F to $\mathrm{int}(V)$. We obtain that for every $x, y \in \mathrm{int}(V)$, the inequality $\langle F(x) - F(y), x - y \rangle \geq 0$ is satisfied. Because V is equal to the closure of $\mathrm{int}(V)$ in H, the continuity of F implies that F is monotone on V.

$(c) \Rightarrow (a)$. This is a straightforward consequence of Definition 2.1. ∎

We now show that the quasimonotonicity (or pseudomonotonicity) of the translations in Theorem 3.1 can be replaced by condition $(SD^{\perp})$ provided a regularity assumption holds.

Theorem 3.4. *Let U be an open convex subset of a Hilbert space H and $F : U \to H$ a continuously differentiable mapping. Assume there exists a straight line $S \subset H$ such that*

$$F(x) \in S \Rightarrow DF(x) \text{ is invertible}$$

and the mapping F satisfies condition $(SD^{\perp})$. Then F is monotone on U.

Proof. Notice that the imposed regularity assumption ensures that property (2.9) is verified for $F(\cdot) - u$ in place of F whenever $u \in S$. Consequently, Theorem 2.4 may be applied replacing F by any $F(\cdot) - u$ with $u \in S$. We thus derive that for all $u \in S$ the mapping $F(\cdot) - u$ is quasimonotone on H. Then Theorem 3.1 implies the monotonicity of F on U, which completes the proof. ∎

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