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GENERALIZED FUGLEDE-PUTNAM THEOREM AND ORTHOGONALITY

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ABSTRACT. An asymmetric Fuglede-Putnam's theorem for dominant operators and p-hyponormal operators is proved, as a consequence of this result, we obtain that the range of the generalized derivation induced by the above classes of operators is orthogonal to its kernel.

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1. INTRODUCTION

Let $L(H)$ be the class of all bounded operators acting on a complex Hilbert space H. An operator $T \in L(H)$ is said to be $p-$hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$, for $0 < p \leq 1)$. If $p = 1$, T is called hyponormal and if $p = 1/2$, T is called semi-hyponormal. It is well known that a $p-$hyponormal operator is $q-$hyponormal operator for $q \leq p$. Hyponormal operators have been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators. Semi-hyponormal operators were first introduced by D. Xia [13], p-hyponormal operators have been studied by A. Aluthge [1], M. Cho [4], [5] and Uchiyama [11]. The set of all p-hyponormal is denoted by $p - H$. According to [7], a bounded operator T is called dominant if

$$(T - zI)H \subseteq (T - zI)^*H, \text{ for all } z \in \sigma(T),$$

where $\sigma(T)$ denote the spectrum of T. This condition is equivalent to existence of a positive constant M_z for every $z \in \sigma(T)$ such that

$$(T - zI)(T - zI)^* \leq M_z^2(T - zI)^*(T - zI).$$

If there exists a constant M such that $M_z \leq M$ for all $z \in \sigma(T)$ then T is called $M-$hyponormal, and if $M = 1$, T is hyponormal. Easily we see the following inclusion relations:

$$Normal \subset Hyponormal \subset M - hyponormal \subset Dominant.$$

Given $A, B \in L(H)$, we define the generalized derivation

$$\delta_{A,B} : L(H) \to L(H) \qquad by \quad \delta_{A,B}(X) = AX - XB.$$

J. Anderson and C. Foias [3] proved that if A and B are normal operators, then $R(\delta_{A,B})$ is orthogonal to $Ker(\delta_{A,B})$, where $R(\delta_{A,B})$ and $Ker(\delta_{A,B})$ denotes the range of $\delta_{A,B}$ and the kernel of $\delta_{A,B}$ respectively. The orthogonality here is understood to be in the sense of definition in [2].

In this paper, our purpose is to prove the following results:

Theorem 1.1. *Let $A, B \in L(H)$ be such that A is dominant and B^* is p-hyponormal ($0 < p \leq 1$) . If $AC = CB$ for some $C \in L(H)$, then $A^*C = CB^*$.*

This result is known as Fuglede-Putnam-Rosemblum's theorem.

Theorem 1.2. *If $A, B \in L(H)$ are such that A is dominant and B^* is p-hyponormal ($0 < p \leq 1$), then $R(\delta_{A,B})$ is orthogonal to $Ker(\delta_{A,B})$.*

2. PRELIMINARIES

In this section, we recall some results which will be used in the sequel.

Definition 2.1. Given $A, B \in L(H)$. We say that the pair (A, B) has $(FP)_{L(H)}$ the Fuglede-Putnam property if $AC = CB$ for some $C \in L(H)$, implies $A^*C = CB^*$.

Theorem 2.1. ([1]) *If $T \in p - H$ and $T = U \mid T \mid$ the polar decomposition of T, then $\mid T \mid^{1/2} U \mid T \mid^{1/2}$ is hyponormal for $1/2 \leq p \leq 1$.*

The next theorem is due to Duggal [6]. This theorem plays important role in our arguments.

Theorem 2.2. ([6]) *Let $A, B \in L(H)$. The following assertions are equivalent*
(i) The pair (A, B) has the property $(FP)_{L(H)}$.
(ii) If $AC = CB$ for some $C \in L(H)$, then $\overline{R(C)}$ reduces A, $(KerC)^{\perp}$ reduces B and $A \mid_{\overline{R(C)}}$ and $B \mid_{(KerC)^{\perp}}$ are normal operators.

Theorem 2.3. ([12]) *If $T \in p - H$ and M be an invariant subspace of T for which $T \mid_M$ is normal, then M reduces T.*

Let's now give the well-known result.

Lemma 2.4. *Let $T \in L(H)$ and $T = U \mid T \mid$ be the polar decomposition of T, then $T^* = U^* \mid T^* \mid$ is the polar decomposition of T^*.*

Lemma 2.5. *Let A be a dominat operator and L be an invariant subspace of A, then $A \mid_L$ is a dominant operator.*

Proof. Let P be the orthogonal projection on L. Then for all $z \in \mathbb{C}$ and for all $x \in L$,

$$\begin{aligned} \| (A \mid_L - zI)^* x \| &= \| P(A - zI)^* x) \| \leq \| (A - zI)^* x) \| \\ &\leq M_z \| (A - zI)x) \| \leq M_z \| (A \mid_L - zI)x) \| . \end{aligned}$$

∎

3. MAIN RESULTS

In this section, we prove that the Fuglede-Putnam theorem holds when A is dominant and $B^* \in p - H$ for $(0 < p \leq 1)$.

Theorem 3.1. *Let $A, B \in L(H)$ be such that A is dominant and $B^* \in p - H$ for ($0 < p \leq 1$). Then the pair (A, B) has $(FP)_{L(H)}$ the Fuglede-Putnam property.*

Proof. (Case 1. $1/2 \leq p \leq 1$). Suppose that $AC = CB$ for some $C \in L(H)$. Since $KerA$ reduces A and $KerB^*$ reduces B^* by [5], we can write A, B and C as follows:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix},$$

on the following decompositions of H:

$$H = (KerA)^{\perp} \oplus (KerA) = (KerB^*)^{\perp} \oplus (KerB^*).$$

From $AC = CB$, it follows that $A_1C_1 = C_1B_1$ and $A_1C_2 = C_3B_1 = 0$. Since A_1 and B_1^* are one-to-one mapping, we obtain $C_2 = C_3 = 0$.

Let's consider the equality

$$A_1C_1 = C_1B_1. \tag{3.1}$$

Since $\overline{R(C_1)}$ and $Ker(C_1)$ are invariant subspaces of A_1 and B_1 respectively, by the decompositions

$$(KerA_1)^{\perp} = \overline{R(C_1)} \oplus [R(C_1)]^{\perp} \quad \text{and} \quad [Ker(B_1)]^{\perp} = [Ker(C_1)]^{\perp} \oplus Ker(C_1),$$

we have

$$A_1 = \begin{bmatrix} A_{11} & S \\ 0 & T \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{11} & 0 \\ E & G \end{bmatrix}, \quad C_1 = \begin{bmatrix} C_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

From equality (3.1), we obtain

$$A_{11}C_{11} = C_{11}B_{11}. \tag{3.2}$$

Let $B_{11} = U \mid B_{11} \mid$ be the polar decomposition of B_{11}. Since $U \mid B_{11} \mid = \mid B_{11}^* \mid U$ by Lemma 2.4. Hence the equality (3.2) becomes

$$A_{11}C_{11} = C_{11} \mid B_{11}^* \mid U. \tag{3.3}$$

Let's multiply the two members of (3.3) by $\mid B_{11}^* \mid^{1/2}$ in right. Hence

$$A_{11}(C_{11} \mid B_{11}^* \mid^{1/2}) = (C_{11} \mid B_{11}^* \mid^{1/2}) \mid B_{11}^* \mid^{1/2} U \mid B_{11}^* \mid^{1/2} .$$

Since the Aluthge transform $\widetilde{B_{11}^*} =\mid B_{11}^* \mid^{1/2} U^* \mid B_{11}^* \mid^{1/2}$ is hyponormal for $(1/2 < p \leq 1)$ by Theorem 2.1 and A_{11} is dominant by Lemma 2.5. Hence the pair $(A_{11}, \widetilde{B_{11}^*})$ has the Fuglede-Putnam property by [7].

Therefore the restrictions

$$A_{11} \mid_{\overline{R(C_{11}|B_{11}^*|^{1/2})}} \quad \text{and} \quad \widetilde{B_{11}^*} \mid_{[Ker(C_{11}|B_{11}^*|^{1/2})]^\perp}$$

are normal operators by Theorem 2.2.

Since C_{11} is a one-to-one mapping with dense range and $\mid B_{11}^* \mid^{1/2}$ is a one-to-one mapping, it's follows that

$$\overline{R(C_{11} \mid B_{11}^* \mid^{1/2})} = \overline{R(C_{11})} = \overline{R(C_1)}$$

and

$$Ker(C_{11} \mid B_{11}^* \mid^{1/2}) = Ker(C_{11}) = KerC_1.$$

Hence $\widetilde{B_{11}^*}$ is normal by [11]. Therefore B_{11} is normal by [7], [10].

Since A_1 is dominant, by Lemma 2.5 and the restriction A_{11} is normal, then $\overline{R(C_1)}$ reduces A_1 by [9], similarly, since $B_1^* \in p-H$ and the restriction B_{11} is normal, then $[KerC_1]^\perp$ reduces B_1^* by Theorem 2.3. Since the pair (A_{11}, B_{11}) has the Fuglede-Putnam property, then

$$A_{11}^* C_{11} = C_{11} B_{11}^*.$$

This implies that

$$A_1^* C_1 = C_1 B_1^*.$$

Since

$$A^*C = \begin{bmatrix} A_1^*C_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad CB^* = \begin{bmatrix} C_1B_1^* & 0 \\ 0 & 0 \end{bmatrix},$$

we obtain

$$A^*C = CB^*.$$

(Case 2. $0 < p \leq 1/2$). We put $p' = p + 1/2$, where $p' \in (1/2, 1]$. It comes back that $\widetilde{B_{11}^*}$ is $p'-$hyponormal. The rest of the proof is similar to the proof of the first case. ∎

Theorem 3.2. *If A is dominant and $B^* \in p - H$, then $R(\delta_{A,B})$ is orthogonal to $Ker(\delta_{A,B})$.*

Proof. The pair (A, B) has the $(FP)_{L(H)}$ property by Theorem 3.1. Let $C \in L(H)$ be such that $AC = CB$. According to the following decompositions of H:

$$H = H_1 = \overline{R(C)} \oplus \overline{R(C)}^\perp, \qquad H = H_2 = (KerC)^\perp \oplus KerC.$$

We can write A, B, C and X

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix},$$

where A_1 and B_1 are normal operators and X is an operator on H_1 into H_2. Since $AC = CB$, we obtain $A_1C_1 = C_1B_1$. Hence

$$AX - XB - C = \begin{bmatrix} A_1X_1 - X_1B_1 - C_1 & A_2X_2 - X_2B_2 \\ A_1X_3 - X_3B_1 & A_2X_4 - X_4B_2 \end{bmatrix}.$$

Since $C_1 \in Ker(\delta_{A_1,B_1})$ and A_1 and B_1 are normal operators. Hence by [3]

$$\| AX - XB - C \| \geq \| A_1X_1 - X_1B_1 - C_1 \| \geq \| C_1 \| = \| C \| \qquad \forall X \in L(H).$$

This implies that $R(\delta_{A,B})$ is orthogonal to $Ker(\delta_{A,B})$. ∎

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