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TWO GEOMETRIC CONSTANTS RELATED TO ISOSCELES ORTHOGONALITY ON BANACH SPACE

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ABSTRACT. In this paper, we introduce new geometric constant $C(X, a_i, b_i, c_i, 2)$ to measure the difference between isosceles orthogonality and special Carlsson orthogonalities. At the same time, we also present the geometric constant $C(X, a_i, b_i, c_i)$, which is a generalization of the rectangular constant proposed by Joly. According to the inequality on isosceles orthogonality, we give the boundary characterization of these geometric constants. Then the relationship between these geometric constants and uniformly non-square property can also be discussed. Furthermore, we show that there is a close relationship between these geometric constants and some important geometric constants.

Key words and phrases: Isosceles orthogonality; Geometric constants; Characterization of inner product space; Uniformly non-square.

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1. INTRODUCTION

Orthogonality plays a vital role in Euclidean geometry. It can be found in the fourth axiom of Euclidean geometry and the Pythagorean theorem. Banach space geometry differs from Euclidean geometry in that there is no unique notion of orthogonality. As Banach space geometry developed, many different orthogonalities have been introduced into the general normed linear space. In 1934, Roberts [20] introduced Roberts orthogonality: a vector x is said to be Roberts orthogonal to a vector y $(x \perp_R y)$ if

$$\|x + \alpha y\| = \|x - \alpha y\|, \forall \alpha \in \mathbb{R}.$$

In 1935, Birkhoff [4] introduced Birkhoff orthogonality: a vector x is said to be Birkhoff orthogonal to a vector y $(x \perp_B y)$ if

$$\|x + \alpha y\| \geq \|x\|, \forall \alpha \in \mathbb{R}.$$

Later, James [8] introduced isosceles orthogonality and Pythagorean orthogonality: a vector x is said to be isosceles orthogonal to a vector y $(x \perp_I y)$ if

$$\|x + y\| = \|x - y\|.$$

A vector x is said to be Pythagorean orthogonal to a vector y $(x \perp_P y)$ if

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2.$$

More studies on orthogonality can refer to [2, 3].

It is well known that different orthogonalities on inner product spaces are equivalent. On Banach spaces, however, this does not necessarily hold. It is therefore of great interest to study different orthogonality. In recent years, many scholars have studied the difference between different orthogonality by means of geometric constants. In [10] and [11], Ji et al. studied the difference between Birkhoff orthogonality and isosceles orthogonality by introducing geometric constants $D(X)$ and $D'(X)$. To discuss the relationship between Birkhoff orthogonality and Roberts orthogonality, Papini and Wu [18] introduced a geometric constant $BR(X)$. Following their work, Mizuguchi [17] proposed geometric constants $BI(X)$ and $IB(X)$ to describe the relationship between Birkhoff orthogonality and isosceles orthogonality. Meanwhile, he also investigated the geometric constant $IB'(X)$, which can be considered as the geometric constant $IB(X)$ on the unit sphere. Based on the parallelogram law and isosceles orthogonality, Liu et al. [16] introduced a new geometric constant $\Omega(X)$, denoted as

$$\Omega(X) = \sup\left\{\frac{\|x + 2y\|^2 + \|2x + y\|^2}{5\|x + y\|^2} : x, y \in X, x \perp_I y\right\}.$$

They gave some properties of this geometric constant, and also used it to characterized the inner product space.

In this paper, we introduce two new geometric constants $C(X, a_i, b_i, c_i, 2)$ and $C(X, a_i, b_i, c_i)$. The geometric constant $C(X, a_i, b_i, c_i, 2)$ can be used to estimate the difference between isosceles orthogonality and special Carlsson orthogonalities. The geometric constant $C(X, a_i, b_i, c_i)$ is a generalization of the rectangular constant. We will give some properties of these constants and connections to other constants.

2. PRELIMINARIES

In this paper, $(X, \|\cdot\|)$ will be a real normed space of dimension at least two. Let $B_X = \{x \in X : \|x\| \leq 1\}$ be the unit ball and $S_X = \{x \in X : \|x\| = 1\}$ be the unit sphere.

First, we give the following lemma, which is often used in later proofs.

Lemma 2.1. [8, Lemma 4.1, p. 294] *If $x \perp_I y$, then*

(1) $\|x+ky\| \leq |k|\|x \pm y\|$ *and* $\|x \pm y\| \leq \|x+ky\|$, *if* $|k| \geq 1$*;*
(2) $\|x+ky\| \leq \|x \pm y\|$ *and* $|k|\|x \pm y\| \leq \|x+ky\|$, *if* $|k| \leq 1$.

A Banach space X is called uniformly non-square [9] if there exists a $\delta \in (0,1)$ such that either $\|x+y\| \leq 2(1-\delta)$ or $\|x-y\| \leq 2(1-\delta)$ for any $x, y \in S_X$. If for any $\varepsilon > 0$ there exists $x, y \in S_X$ such that $\|x \pm y\| > 2 - \varepsilon$, then we say that X is not uniformly non-square.

Let X be a real normed space, the von Neumann-Jordan constant $C_{NJ}(X)$ is defined by

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, (x,y) \neq (0,0)\right\}.$$

The von Neumann-Jordan constant $C_{NJ}(X)$ of the space X was first considered by Jordan and von Neumann [13], and has since been studied by many scholars, yielding many properties. For instance, $1 \leq C_{NJ}(X) \leq 2$. For further details on von Neumann-Jordan constant, we refer the reader to [14, 15].

In [7], Gao and Lau introduced the James constant $J(X)$ of a Banach space X, which is defined as

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}.$$

An equivalent definition of the James constant is the non-square constant:

$$J(X) = \sup\{\|x+y\| : x, y \in S_X, x \perp_I y\}.$$

The basis properties of $J(X)$ are given in [7, 14]:
(i)$\sqrt{2} \leq J(X) \leq 2$.
(ii)If X is an inner product space, then $J(X) = \sqrt{2}$; the converse is not necessarily correct.
(ii)$J(X) < 2$ if and only if X is uniformly non-square.

3. THE CONSTANT $C(X, a_i, b_i, c_i, 2)$

In [5], Carlsson introduced Carlsson-orthogonalities: a vector x is said to C-orthogonal to a vector $y (x \perp_C y)$ if

$$\sum_{i=1}^{n} a_i\|b_i x + c_i y\|^2 = 0$$

where $a_i, b_i, c_i \in \mathbb{R}, i = 1, \ldots, n$, are such that

$$\sum_{i=1}^{n} a_i b_i^2 = \sum_{i=1}^{n} a_i c_i^2 = 0 \text{ and } \sum_{i=1}^{n} a_i b_i c_i \neq 0.$$

This generalizes isosceles and Pythagoream orthogonalities. He gave a characterization of the inner product space by Carlsson-orthogonalities. Later, Alonso et al. [1] proved that Carlsson-orthogonalities and Birkhoff orthogonality are equivalent in inner product space. Motivated by their work, we introduced a new geometric constant to measure the difference between special Carlsson-orthogonalities and isosceles orthogonality, defined as follows:

$$C(X, a_i, b_i, c_i, 2) = \sup\left\{\frac{\sum_{i=1}^{n} a_i\|b_i x + c_i y\|^2}{-a_j\|b_j x + c_j y\|^2}, x \perp_I y, (x,y) \neq (0,0)\right\}$$

where $a_i, b_i, c_i \in \mathbb{R}, i = 1, \ldots, j-1, j+1, \ldots, n$, are such that $\sum_{i=1}^{n} a_i b_i^2 + a_j b_j^2 = \sum_{i=1}^{n} a_i c_i^2 + a_j c_j^2 = 0$, $\sum_{i=1}^{n} a_i b_i c_i + a_j b_j c_j \neq 0$ and $a_j, b_j = c_j \in \mathbb{R} - \{0\}, j = 1, \ldots, n$.

Theorem 3.1. *Let X be a normed space, then*

$$1 \leq C(X, a_i, b_i, c_i, 2) \leq \sum_{i=1}^{n} \max\{\frac{a_i b_i^2}{-a_j b_j^2}, \frac{a_i c_i^2}{-a_j b_j^2}\}.$$

Proof. Without loss of generality, we assume $b_i \neq 0$. If $|c_i| \leq |b_i|$ for any $i = 1, \ldots, j-1, j+1, \ldots, n-1$, by Lemma 2.1, we have

$$\begin{aligned}\|b_i x + c_i y\|^2 &= b_i^2 \|x + \frac{c_i}{b_i} y\|^2 \\ &\leq b_i^2 \|x + y\|^2.\end{aligned}$$

If $|b_i| \leq |c_i|$, then we get

$$\begin{aligned}\|b_i x + c_i y\|^2 &\leq b_i^2 \left(\frac{c_i}{b_i}\right)^2 \|x + y\|^2 \\ &= c_i^2 \|x + y\|^2.\end{aligned}$$

Hence

$$\|b_i x + c_i y\|^2 \leq \max\{b_i^2, c_i^2\} \|x + y\|$$

where $i = 1, \ldots, j-1, j+1, \ldots, n-1$. It follows that

$$\begin{aligned}\frac{\sum_{i=1}^n a_i \|b_i x + c_i y\|^2}{-a_j \|b_j x + c_j y\|^2} &\leq -\frac{1}{a_j b_j^2} \sum_{i=1}^n \max\{a_i b_i^2, a_i c_i^2\} \\ &= \sum_{i=1}^n \max\{\frac{a_i b_i^2}{-a_j b_j^2}, \frac{a_i c_i^2}{-a_j b_j^2}\}.\end{aligned}$$

This shows that

$$C(X, a_i, b_i, c_i, 2) \leq \sum_{i=1}^n \max\{\frac{a_i b_i^2}{-a_j b_j^2}, \frac{a_i c_i^2}{-a_j b_j^2}\}.$$

On the other hand, we let $x \neq 0, y = 0$ in $C(X, a_i, b_i, c_i, 2)$. Then we obtain

$$C(X, a_i, b_i, c_i, 2) \geq \frac{\sum_{i=1}^n a_i b_i^2}{-a_j b_j^2} = 1,$$

which completes the proof of the theorem. ∎

We provide the following examples to illustrate that the geometric constant $C(X, a_i, b_i, c_i, 2)$ can reach the upper bound and lower bound.

Example 3.1. *Let* $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, *then* $C(X, a_i, b_i, c_i, 2) = \sum_{i=1}^n \max\{\frac{a_i b_i^2}{-a_j b_j^2}, \frac{a_i c_i^2}{-a_j b_j^2}\}$.

Proof. Taking $x = (1, 0), y = (0, 1)$, then $\|x + y\|_\infty = \|x - y\|_\infty = 1$. Thus we get

$$\begin{aligned}C(X, a_i, b_i, c_i, 2) &\geq \frac{\sum_{i=1}^n a_i \|b_i x + c_i y\|_\infty^2}{-a_j \|b_j x + c_j y\|_\infty^2} \\ &= -\frac{1}{a_j b_j^2} \sum_{i=1}^n \max\{a_i b_i^2, a_i c_i^2\} \\ &= \sum_{i=1}^n \max\{\frac{a_i b_i^2}{-a_j b_j^2}, \frac{a_i c_i^2}{-a_j b_j^2}\}.\end{aligned}$$

According to Theorem 3.1, we deduce that

$$C(X, a_i, b_i, c_i, 2) = \sum_{i=1}^n \max\{\frac{a_i b_i^2}{-a_j b_j^2}, \frac{a_i c_i^2}{-a_j b_j^2}\}.$$

∎

Example 3.2. *Let X is an inner product space, then* $C(X, a_i, b_i, c_i, 2) = 1$.

Proof. If X is an inner product space, by parallelogram rule, we have

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

Therefore

$$\begin{aligned}\frac{\sum_{i=1}^n a_i\|b_i x + c_i y\|^2}{-a_j\|b_j x + c_j y\|^2} =& \frac{\sum_{i=1}^n a_i(b_i^2\|x\|^2 + c_j\|y\|^2)}{-a_j(b_j^2\|x\|^2 + c_j^2\|y\|^2)}\\ =& \frac{-a_j(b_j^2\|x\|^2 + c_j^2\|y\|^2)}{-a_j(b_j^2\|x\|^2 + c_j^2\|y\|^2)}\\ =& 1.\end{aligned}$$

This implies that $C(X, a_i, b_i, c_i, 2) \leq 1$. By Theorem 3.1, we get

$$C(X, a_i, b_i, c_i, 2) = 1.$$

∎

By following the ideas in [16], we can prove the next theorem, which gives the relationship between the geometric constant $C(X, a_i, b_i, c_i, 2)$ and not uniformly non-square.

Theorem 3.2. *Let X be a finite-dimensional normed space. Then X is not uniformly non-square if and only if*

$$C(X, a_i, b_i, c_i, 2) = \sum_{i=1}^n \max\{\frac{a_i b_i^2}{-a_j b_j^2}, \frac{a_i c_i^2}{-a_j b_j^2}\}.$$

Proof. Suppose that X is not uniformly non-square, then there exists $x_n, y_n \in S_X$ such that

$$\|x_n + y_n\| \to 2, \|x_n - y_n\| \to 2$$

when $n \to \infty$. Letting

$$u = \frac{x_n + y_n}{2} \text{ and } v = \frac{x_n - y_n}{2},$$

then we see $\|u+v\| = \|u-v\| = 1$. Thus we get

$$\begin{aligned}\|b_i u + c_i v\|^2 =& \left\|\frac{b_i + c_i}{2}x_n + \frac{b_i - c_i}{2}y_n\right\|^2\\ \leq& \left(\left|\frac{b_i + c_i}{2}\right| + \left|\frac{b_i - c_i}{2}\right|\right)^2\\ =& \max\{b_i^2, c_i^2\}\end{aligned}$$

for any $i = 1, \ldots, j-1, j+1, \ldots, n$.

Next we want to conclude that $\|b_i u + c_i v\|^2 \to \max\{b_i^2, c_i^2\}$ for any $i = 1, \ldots, j-1, j+1, \ldots, n$, it is best to split the argument into two cases.

Case 1. $b_i c_i \geq 0$. *By triangle inequality, we have*

$$\begin{aligned}\|b_i u + c_i v\|^2 =& \left\|\frac{b_i + c_i}{2}x_n + \frac{b_i + c_i}{2}y_n - c_i y_n\right\|^2\\ \geq& \left(\left|\frac{b_i + c_i}{2}\right|\|x_n + y_n\| - |c_i|\|y_n\|\right)^2\\ \to& b_i^2 (n \to \infty)\end{aligned}$$

and

$$\begin{aligned}\|b_iu+c_iv\|^2 &= \left\|\frac{b_i+c_i}{2}x_n-\frac{b_i+c_i}{2}y_n+b_iy_n\right\|^2\\ &\geq \left(\left|\frac{b_i+c_i}{2}\right|\|x_n-y_n\|-|b_i|\|y_n\|\right)^2\\ &\to c_i^2(n\to\infty),\end{aligned}$$

which gives $\|b_iu+c_iv\|^2\to\max\{b_i^2,c_i^2\}$.

Case 2. $b_ic_i\leq 0$. *It is easily seen that*

$$\begin{aligned}\|b_iu+c_iv\|^2 &= \left\|\frac{b_i-c_i}{2}x_n+\frac{b_i-c_i}{2}y_n+c_ix_n\right\|^2\\ &\geq \left(\left|\frac{b_i-c_i}{2}\right|\|x_n+y_n\|-|c_i|\|x_n\|\right)^2\\ &\to b_i^2(n\to\infty)\end{aligned}$$

and

$$\begin{aligned}\|b_iu+c_iv\|^2 &= \left\|\frac{b_i-c_i}{2}y_n-\frac{b_i-c_i}{2}x_n+b_ix_n\right\|^2\\ &\geq \left(\left|\frac{b_i-c_i}{2}\right|\|x_n-y_n\|-|b_i|\|x_n\|\right)^2\\ &\to c_i^2(n\to\infty).\end{aligned}$$

This implies $\|b_iu+c_iv\|^2\to\max\{b_i^2,c_i^2\}$.

Hence we have

$$C(X,a_i,b_i,c_i,2)\geq\sum_{i=1}^n\max\{\frac{a_ib_i^2}{-a_jb_j^2},\frac{a_ic_i^2}{-a_jb_j^2}\}.$$

It follows from Theorem 3.1 that

$$C(X,a_i,b_i,c_i,2)=\sum_{i=1}^n\max\{\frac{a_ib_i^2}{-a_jb_j^2},\frac{a_ic_i^2}{-a_jb_j^2}\}.$$

Conversely, without loss of generality we can assume there exists $k\in\{1,\ldots,j-1,j+1,\ldots,n-1\}$ such that $b_i^2>c_i^2$ for any $i=1,\ldots,k$. If

$$C(X,a_i,b_i,c_i,2)=\sum_{i=1}^n\max\{\frac{a_ib_i^2}{-a_jb_j^2},\frac{a_ic_i^2}{-a_jb_j^2}\},$$

then we see that there exists x_1,y_1 on a finite dimensional normed space X with $x_1\perp y_1$ satisfying

$$\frac{\sum_{i=1}^n a_i\|b_ix_1+c_iy_1\|^2}{-a_j\|b_jx_1+c_jy_1\|^2}=\frac{\sum_{i=1}^k a_ib_i^2+\sum_{i=k+1}^n a_ic_i^2}{-a_jb_j^2}.$$

By Lemma 2.1, we have

$$\|b_ix_1+c_iy_1\|^2\leq b_i^2\|x_1+y_1\|^2(i=1,\ldots,k)$$

and

$$\|b_ix_1+c_iy_1\|^2\leq c_i^2\|x_1+y_1\|^2(i=k+1,\ldots,n).$$

Since

$$\frac{\sum_{i=1}^{k} a_i b_i^2 \|x_1+y_1\|^2 + \sum_{i=k+1}^{n} a_i c_i^2 \|x_1+y_1\|^2}{-a_j \|b_j x_1 + c_j y_1\|^2} = \frac{\sum_{i=1}^{k} a_i b_i^2 + \sum_{i=k+1}^{n} a_i c_i^2}{-a_j b_j^2},$$

then we get

$$\|b_i x_1 + c_i y_1\|^2 = b_i^2 \|x_1+y_1\|^2 (i = 1, \dots, k)$$

and

$$\|b_i x_1 + c_i y_1\|^2 = c_i^2 \|x_1+y_1\|^2 (i = k+1, \dots, n).$$

By triangle inequality, we have

$$\begin{cases} \|b_i x_1 + c_i y_1\| \le |b_i - c_i| \|x_1\| + |c_i| \|x_1 + y_1\|, & i = 1, \dots, k \\ \|b_i x_1 + c_i y_1\| \le |c_i - b_i| \|y_1\| + |b_i| \|x_1 + y_1\|, & i = k+1, \dots, n \end{cases}$$

and

$$\begin{cases} \|b_i x_1 + c_i y_1\| \le |b_i + c_i| \|x_1\| + |c_i| \|x_1 - y_1\|, & i = 1, \dots, k \\ \|b_i x_1 + c_i y_1\| \le |b_i + c_i| \|y_1\| + |b_i| \|x_1 - y_1\|, & i = k+1, \dots, n \end{cases}$$

If $\alpha\beta \ge 0$, then we see

$$\begin{aligned} (|b_i| - |c_i|)\|x_1 + y_1\| &\le |b_i - c_i| \|x_1\| \\ &= (|b_i| - |c_i|)\|x_1\| (i = 1, \dots, k), \end{aligned}$$

and

$$\begin{aligned} (|c_i| - |b_i|)\|x_1 + y_1\| &\le |c_i - b_i| \|y_1\| \\ &= (|c_i| - |b_i|)\|y_1\| (i = k+1, \dots, n). \end{aligned}$$

If $\alpha\beta \le 0$, then

$$\begin{aligned} (|b_i| - |c_i|)\|x_1 + y_1\| &\le |b_i + c_i| \|x_1\| \\ &= (|b_i| - |c_i|)\|x_1\| (i = 1, \dots, k), \end{aligned}$$

and

$$\begin{aligned} (|c_i| - |b_i|)\|x_1 + y_1\| &\le |b_i + c_i| \|y_1\| \\ &= (|c_i| - |b_i|)\|y_1\| (i = k+1, \dots, n). \end{aligned}$$

Hence we obtain

$$\max \|x_1 + y_1\| \le \min\{\|x_1\|, \|y_1\|\}.$$

This gives

$$\max\{\|x_1 + y_1\|, \|x_1 - y_1\|\} = \max\{\|x_1 + y_1\|, \|x_1 + y_1\|\} \le \min\{\|x_1\|, \|y_1\|\}.$$

To prove that X is not uniformly non-square, we may assume that X is uniformly non-square, then there exists $\delta \in (0, 2)$ such that

$$\begin{aligned} \min \Bigg\{ &\left\| \frac{x+y}{\max\{\|x+y\|, \|x-y\|\}} + \frac{x-y}{\max\{\|x+y\|, \|x-y\|\}} \right\|, \\ &\left\| \frac{x+y}{\max\{\|x+y\|, \|x-y\|\}} - \frac{x-y}{\max\{\|x+y\|, \|x-y\|\}} \right\| \Bigg\} \\ < 2 - \delta & \end{aligned}$$

for any $x, y \in X$. This shows

$$\min \left\{ \left\| \frac{2x}{\max\{\|x+y\|, \|x-y\|\}} \right\|, \left\| \frac{2y}{\max\{\|x+y\|, \|x-y\|\}} \right\| \right\} < 2 - \delta.$$

Accordingly,

$$\max\{\|x+y\|,\|x-y\|\} > \frac{2}{2-\delta}\min\{\|x\|,\|y\|\},$$

which contradicts the fact that

$$\max\{\|x_1+y_1\|,\|x_1-y_1\|\} \leq \min\{\|x_1\|,\|y_1\|\}.$$

Therefore, we deduce that X is not uniformly non-square. ∎

Corollary 3.3. *Let X be a finite-dimensional normed space. Then X is uniformly non-square if and only if*

$$C(X,a_i,b_i,c_i,2) < \sum_{i=1}^{n}\max\{\frac{a_ib_i^2}{-a_jb_j^2},\frac{a_ic_i^2}{-a_jb_j^2}\}.$$

The geometric constant $C(X,a_i,b_i,c_i,2)$ is closely related to other geometric constants. Below we show that the relationship between $C(X,a_i,b_i,c_i,2)$ and $C_{NJ}(X)$.

Theorem 3.4. *Let X be a normed space, then*

$$C(X,a_i,b_i,c_i,2) \leq \sum_{i=1}^{n}\min\{\frac{2a_ib_i^2}{-a_jb_j^2},\frac{2a_ic_i^2}{-a_jb_j^2}\}C_{NJ}(X)+\sum_{i=1}^{n}\frac{2a_i|b_i^2-c_i^2|}{-a_jb_j^2}$$

Proof. Note that, the geometric constant $C_{NJ}(X)$ can be written in the following equivalent form:

$$C_{NJ}(X)=\sup\left\{\frac{2(\|x\|^2+\|y\|^2)}{\|x+y\|^2+\|x-y\|^2} : x,y\in X,(x,y)\neq(0,0)\right\}$$

For all $x,y\in X$ such that $x\perp_I y$, by triangle inequality, we have

$$\begin{aligned}
\frac{\sum_{i=1}^{n}a_i\|b_ix+c_iy\|^2}{-a_j\|b_jx+c_jy\|^2} &\leq \frac{\sum_{i=1}^{n}a_i(\|b_ix\|+\|c_iy\|)^2}{-a_jb_j^2\|x+y\|^2}\\
&\leq \frac{\sum_{i=1}^{n}2a_i(\|b_ix\|^2+\|c_iy\|^2)}{-a_jb_j^2\|x+y\|^2}\\
&\leq 2\frac{\sum_{i=1}^{n}\min\{a_ib_i^2,a_ic_i^2\}(2\|x\|^2+2\|y\|^2)+2a_i|b_i^2-c_i^2|\|x+y\|^2}{-a_jb_j^2(\|x+y\|^2+\|x-y\|^2)}\\
&\leq \sum_{i=1}^{n}\min\{\frac{2a_ib_i^2}{-a_jb_j^2},\frac{2a_ic_i^2}{-a_jb_j^2}\}C_{NJ}(X)+\sum_{i=1}^{n}\frac{2a_i|b_i^2-c_i^2|}{-a_jb_j^2}.
\end{aligned}$$

This completes the proof. ∎

In the next theorem, we give the relation between between $C(X,a_i,b_i,c_i,2)$ and the James constant.

Theorem 3.5. *Let X be a normed space, then*

$$\frac{\sum_{i=1}^{n}a_i(b_i-c_i)^2}{-4a_jb_j^2}J(x) \leq C(X,a_i,b_i,c_i,2)$$

Proof. For any $x,y\in S_X$, let

$$u=\frac{x+y}{2} \text{ and } v=\frac{x-y}{2}.$$

Clearly, $u \perp_I v$. Therefore

$$\begin{aligned} C(X, a_i, b_i, c_i, 2) &\geq \frac{\sum_{i=1}^{n} a_i \|b_i u + c_i v\|^2}{-a_j \|b_j u + c_j v\|^2} \\ &= \frac{\sum_{i=1}^{n} a_i \|b_i(x+y) + c_i(x-y)\|^2}{-4a_j b_j^2} \\ &\geq \frac{\sum_{i=1}^{n} a_i (b_i^2 \|x+y\|^2 + c_i^2 \|x-y\|^2 - 2|b_i c_i| \|x+y\| \|x-y\|)}{-4a_j b_j^2} \\ &\geq \frac{\sum_{i=1}^{n} a_i (b_i - c_i)^2}{-4a_j b_j^2} J(x). \end{aligned}$$

This completes of the proof. ∎

4. THE CONSTANT $C(X, a_i, b_i, c_i)$

The rectangular constant $\mu(X)$ of a real normed linear space X was introduced by Joly [12] as follows:

$$\mu(X) = \sup_{\lambda \in \mathbb{R}} \left\{ \frac{\|x\| + \|\lambda y\|}{\|x + \lambda y\|}, x \perp_B y \right\}.$$

Joly gave upper and lower bounds for the geometric constant, 3 and $\sqrt{2}$, respectively. It is also demonstrated that in normed linear spaces of dimensions ≥ 3, $\mu(X) = \sqrt{2}$ if and only if X is inner product space. Later, $\mu(X) = \sqrt{2}$ and inner product space are also equivalent in two-dimensional spaces was proved by Del Río and Benítez [6].

In normed space, there is a difference between Birkhoff orthogonality and isosceles orthogonality. Inspired by the work of Joly and Carlsson, we propose a generalized rectangular constant on isosceles orthogonality, defined as follows:

$$C(X, a_i, b_i, c_i) = \sup \left\{ \frac{\sum_{i=1}^{n} a_i \|b_i x + c_i y\|}{\|x + y\|}, x \perp_I y, (x, y) \neq (0, 0) \right\}$$

where $a_i, b_i, c_i \in \mathbb{R}, i = 1, \ldots, n$ such that

$$\sum_{i=1}^{n} a_i^2 b_i^2 = \sum_{i=1}^{n} a_i^2 c_i^2 \ \text{ and } \ a_i^2 b_i^2 + a_i^2 c_i^2 = a_j^2 b_j^2 + a_j^2 c_j^2 \ \text{ for any } \ i \neq j.$$

We give the characterization of the geometric constant $C(X, a_i, b_i, c_i)$ on normed space in the following theorem. To complete the proof easily, we first give a lemma, which is the well-known Dvoretzky's theorem.

Lemma 4.1. [19, Theorem 10.43, p. 410] *For any $\varepsilon > 0$, any infinite-dimensional Banach space X contain $\ell_2^n s(1+\varepsilon)-$ uniformly.*

Theorem 4.2. *Let X be a infinite-dimensional normed space, then*

$$\sqrt{n(\sum_{i=1}^{n} a_i^2 b_i^2)} \leq C(X, a_i, b_i, c_i) \leq \sum_{i=1}^{n} \max\{a_i|b_i|, a_i|c_i|\}.$$

Proof. If $x \perp_I y$, by the proof in Theorem 3.1, we know

$$\begin{aligned}\frac{\sum_{i=1}^n a_i\|b_i x + c_i y\|}{\|x+y\|} &\leq \frac{\sum_{i=1}^n \max\{a_i|b_i|, a_i|c_i|\}\|x+y\|}{\|x+y\|}\\ &= \sum_{i=1}^n \max\{a_i|b_i|, a_i|c_i|\}.\end{aligned}$$

This gives

$$C(X, a_i, b_i, c_i) \leq \sum_{i=1}^n \max\{a_i|b_i|, a_i|c_i|\}.$$

Actually, for any Hilbert space H, we have $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ for all $x, y \in X$. Then we see

$$\begin{aligned}\frac{\sum_{i=1}^n a_i\|b_i x + c_i y\|}{\|x+y\|} =&\frac{\sum_{i=1}^n a_i\sqrt{\|b_i x\|^2 + \|c_i y\|^2}}{\sqrt{\|x\|^2+\|y\|^2}}\\ =&\frac{\sum_{i=1}^n \sqrt{a_i^2 b_i^2 + a_i^2 c_i^2}}{\sqrt{2}}\\ =&\frac{n\sqrt{a_1^2 b_1^2 + a_1^2 c_1^2}}{\sqrt{2}}\\ =&\sqrt{n}\sqrt{\frac{n(a_1^2 b_1^2 + a_1^2 c_1^2)}{2}}\\ =&\sqrt{n(\sum_{i=1}^n a_i^2 b_i^2)}\end{aligned}$$

for any $x, y \in S_X$. This shows $C(H, a_i, b_i, c_i) \geq \sqrt{n\left(\sum_{i=1}^n a_i^2 b_i^2\right)}$. Furthermore, by parallelogram rule, we have

$$\begin{aligned}&\left(\frac{\sum_{i=1}^n a_i\|b_i x + c_i y\|}{\|x+y\|}\right)^2\\ \leq&\frac{n\sum_{i=1}^n a_i^2\|b_i x + c_i y\|^2}{\|x+y\|^2}\\ =&\frac{n\sum_{i=1}^n (a_i^2 b_i^2\|x\|^2 + a_i^2 c_i^2\|y\|^2)}{\|x\|^2+\|y\|^2}\\ =&n\left(\sum_{i=1}^n a_i^2 b_i^2\right),\end{aligned}$$

which gives $C(H, a_i, b_i, c_i) \leq \sqrt{n(\sum_{i=1}^n a_i^2 b_i^2)}$. Thus we obtain

$$C(H, a_i, b_i, c_i) = \sqrt{n(\sum_{i=1}^n a_i^2 b_i^2)}.$$

By Lemma 4.1, for any infinite-dimensional Banach space X, we must have

$$C(X, a_i, b_i, c_i) \geq C(H, a_i, b_i, c_i) = \sqrt{n(\sum_{i=1}^n a_i^2 b_i^2)}.$$

This completes of the proof. ∎

The following example gives the case of the upper bound for the geometric constant $C(X, a_i, b_i, c_i)$.

Example 4.1. *Let* $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, *then* $C(X, a_i, b_i, c_i) = \sum_{i=1}^{n} \max\{a_i|b_i|, a_i|c_i|\}$.

We give the connection between $C(X, a_i, b_i, c_i)$ and no uniformly non-square in the following theorem.

Theorem 4.3. *Let X be a finite-dimensional normed space. Then X is not uniformly non-square if and only if*

$$C(X, a_i, b_i, c_i) = \sum_{i=1}^{n} \max\{a_i|b_i|, a_i|c_i|\}.$$

Proof. The proof is similar to that of Theorem 3.2 by some minor modifications, thus we omit it. ∎

Corollary 4.4. *Let X be a finite-dimensional normed space. Then X is uniformly non-square if and only if*

$$C(X, a_i, b_i, c_i) < \sum_{i=1}^{n} \max\{a_i|b_i|, a_i|c_i|\}.$$

5. CONCLUSION

In this article, we introduce two new geometric constants $C(X, a_i, b_i, c_i, 2)$ and $C(X, a_i, b_i, c_i)$. The constant $C(X, a_i, b_i, c_i, 2)$ can be used to measure the difference between isosceles orthogonality and special Carlsson orthogonalities. The geometric constant $C(X, a_i, b_i, c_i)$ can be considered as a generalization of the rectangular constant. First, we give upper and lower bounds for these constants. Then we characterize uniformly non-square spaces by using these constants. In addition, we show the relationship between these geometric constants and some important geometric constants.

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