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APPLICATION OF CHEBYSHEV POLYNOMIALS TO VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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ABSTRACT. The goal of this work is to examine the numerical solution of linear Volterra-Fredholm integral equations of the second kind using the first, second, third and fourth Chebyshev polynomials. Noting that, the approximate solution is given in the form of series which converges to the exact one. Numerical examples are compared with other methods, in order to prove the applicability and the efficiency of this technical.

Key words and phrases: Chebyshev polynomials; Volterra-Fredholm integral equation; Collocation method; Numerical method.

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1. INTRODUCTION

Integral equations, leads to appear some phenomenon in many areas of scientific fields such as mathematical biology, chemical kinetics and fluid dynamics. Also we can transform equations occur of scattering and radiation of surface water wave based on ordinary differential equation of the second order with boundary conditions into a Volterra-Fredholm integral equations of the form.

$$\varphi(x) - \int_a^x k_1(x,t)\varphi(t)dt - \int_a^b k_2(x,t)\varphi(t)dt = f(x), \tag{1.1}$$

with a given function $k(x,t)$ and a function $f(x)$, the kernel $k(x,t)$ is bounded in $a \leq x, t \leq b$, and value 1 is not an eigenvalue of (1.1), the function $\varphi(x)$ is the unknown function to be determined. Many authors are launched to solve this kind of equations by different methods, where we find a moving least square method and Chebyshev polynomials in [2] and an Adomian decomposition using maple in [3], the authors in [7, 8, 9] use the Chebyshev, Euler series and quadratic numerical methods to solve the Fredholm integral equations. In [4, 10] the authors estimate the density function $\varphi(x)$ by means of Legendre and the first Chebyshev polynomials.

For this study we replace the function $\varphi(x)$ by the four Chebyshev polynomials and compare the accuracy of the estimation of the unknown function with many numerical examples.

2. CHEBYSHEV POLYNOMIALS

1- **The first-kind polynomial** T_n

The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n; defined by the relation

$$T_n(x) = \cos n\theta \quad \text{when } \; x = \cos\theta, \tag{2.1}$$

where $x \in [-1,1]$, this involves that the corresponding variable $\theta \in [0,\pi]$. It is easy to see that $T_0(x) = 1, \;\; T_1(x) = x$ and by the recurrence formula satisfied by Chebyshev polynomials

$$\cos n\theta + \cos(n-2)\theta = 2\cos\theta\cos(n-1)\theta,$$

we obtain the fundamental relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, ...$$

Noting that the functions $\{T_n(x), \;\; n = 0, 1, 2, ...\}$ form an orthogonal system on the interval $[-1,1]$ with respect to the weight $w(x) = \dfrac{1}{\sqrt{1-x^2}}$ and so the polynomial system $S_n(x)$ given by

$$\left\{ S_0(x) = \sqrt{\frac{1}{\pi}}T_0(x),\; S_1(x) = \sqrt{\frac{2}{\pi}}T_1(x),\; S_2(x) = \sqrt{\frac{2}{\pi}}T_2(x), ...S_n(x) = \sqrt{\frac{2}{\pi}}T_n(x)... \right\},$$

form an orthonormal system on the interval $[-1,1]$ with respect to the weight $w(x) = \dfrac{1}{\sqrt{1-x^2}}$. In other words

$$\langle S_k(x), S_l(x)\rangle = \int_{-1}^{1} \frac{S_k(x)S_l(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0 & \text{if} \quad k \neq l \\ 1 & \text{if} \quad k = l \end{cases}$$

2- **The second-kind polynomial** U_n

The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial in x of degree n; defined by the relation

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta} \quad \text{when} \quad x = \cos\theta \tag{2.2}$$

The terms recurrence formula satisfied by Chebyshev polynomials is the translation of the elementary trigonometric identity

$$\sin(n+1)\theta + \sin(n-1)\theta = 2\cos\theta\sin n\theta,$$

which gives

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, ...$$

With

$$U_0(x) = 1, \;\; U_1(x) = 2x$$

Noting that the functions $\{U_n(x), \;\; n = 0, 1, 2, ...\}$ form an orthogonal system on the interval $[-1, 1]$ with respect to the weight $w(x) = \sqrt{1-x^2}$ and so the polynomial system $S_n(x)$ given by

$$\left\{ S_0(x) = \sqrt{\frac{2}{\pi}}U_0(x), \; S_1(x) = \sqrt{\frac{2}{\pi}}U_1(x), \; S_2(x) = \sqrt{\frac{2}{\pi}}U_2(x), ...S_n(x) = \sqrt{\frac{2}{\pi}}U_n(x)... \right\},$$

form an orthonormal system on the interval $[-1, 1]$ with respect to the weight $w(x) = \sqrt{1-x^2}$. In other words

$$\langle S_k(x), S_l(x)\rangle = \int_{-1}^{1} S_k(x)S_l(x)\sqrt{1-x^2}dx = \begin{cases} 0 & \text{if} \quad k \neq l \\ 1 & \text{if} \quad k = l \end{cases}$$

3- **The third-kind polynomial** V_n

The Chebyshev polynomial $V_n(x)$ of the third kind is a polynomial in x of degree n; defined by the relation

$$V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta} \quad \text{when} \quad x = \cos\theta \tag{2.3}$$

The three term recurrence formula satisfied by Chebyshev polynomials is the translation of the elementary trigonometric identity

$$\cos(n+\frac{1}{2})\theta + \cos(n-2+\frac{1}{2})\theta = 2\cos\theta\cos(n-1+\frac{1}{2})\theta,$$

which becomes

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x), \quad n = 2, 3, ...$$

With

$$V_0(x) = 1, \;\; V_1(x) = 2x - 1$$

Noting that the functions $\{V_n(x), \;\; n = 0, 1, 2, ...\}$ form an orthogonal system on the interval $[-1, 1]$ with respect to the weight $w(x) = \sqrt{\frac{1+x}{1-x}}$ and so the polynomial system $S_n(x)$ given by

$$\left\{S_0(x)=\sqrt{\frac{1}{\pi}}V_0(x),\ S_1(x)=\sqrt{\frac{1}{\pi}}V_1(x),\ S_2(x)=\sqrt{\frac{1}{\pi}}V_2(x),...S_n(x)=\sqrt{\frac{1}{\pi}}V_n(x)...\right\},$$

form an orthonormal system on the interval $[-1,1]$ with respect to the weight $w(x)=\sqrt{\dfrac{1+x}{1-x}}$.
In other words

$$\langle S_k(x),S_l(x)\rangle=\int_{-1}^{1}S_k(x)S_l(x)\sqrt{\frac{1+x}{1-x}}dx=\begin{cases}0 & \text{if} \quad k\neq l\\ 1 & \text{if} \quad k=l\end{cases}$$

4- **The fourth-kind polynomial** W_n

The Chebyshev polynomial $W_n(x)$ of the fourth kind is a polynomial in x of degree n; defined by the relation

$$W_n(x)=\frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta} \quad \text{when} \quad x=\cos\theta \tag{2.4}$$

The three term recurrence formula satisfied by Chebyshev polynomials is the translation of the elementary trigonometric identity

$$\sin(n+\frac{1}{2})\theta+\sin(n-2+\frac{1}{2})\theta=2\cos\theta\sin(n-1+\frac{1}{2})\theta,$$

which becomes

$$W_n(x)=2xW_{n-1}(x)-W_{n-2}(x),\quad n=2,3,....$$

With

$$W_0(x)=1,\ \ W_1(x)=2x+1.$$

Noting that the functions $\{W_n(x),\ \ n=0,1,2,....\}$ form an orthogonal system on the interval $[-1,1]$ with respect to the weight $w(x)=\sqrt{\dfrac{1+x}{1-x}}$ and so the polynomial system $S_n(x)$ given by

$$\left\{S_0(x)=\sqrt{\frac{1}{\pi}}W_0(x),\ S_1(x)=\sqrt{\frac{1}{\pi}}W_1(x),\ S_2(x)=\sqrt{\frac{1}{\pi}}W_2(x),...S_n(x)=\sqrt{\frac{1}{\pi}}W_n(x)...\right\},$$

form an orthonormal system on the interval $[-1,1]$ with respect to the weight $w(x)=\sqrt{\dfrac{1-x}{1+x}}$.
In other words

$$\langle S_k(x),S_l(x)\rangle=\int_{-1}^{1}S_k(x)S_l(x)\sqrt{\frac{1-x}{1+x}}dx=\begin{cases}0 & \text{if} \quad k\neq l\\ 1 & \text{if} \quad k=l\end{cases}$$

3. DISCRETIZATION OF INTEGRAL EQUATION

Applying a collocation method to the equation (1.1) in order to discredit and convert this equation to a system of linear equations. For this latter, supposing that $a=-1$ and $b=1$ and approximate the unknown function $\varphi(x)$ by a finite sum of the form

$$\varphi(x)=\sum_{k=0}^{N}c_kS_k(x), \tag{3.1}$$

where $S_n(x)$ denotes the nth Chebyshev polynomial of the first, second, third or fourth kind. After substitution of the expansion (3.1) into the equation (1.1) this latter becomes an approximate equation as

$$\sum_{k=0}^{N} \alpha_k S_k(x) - \int_a^x k_1(x,t) \sum_{k=0}^{N} \alpha_k S_k(t) - \int_a^b k_2(x,t) \sum_{k=0}^{N} \alpha_k S_k(t) = f(x). \tag{3.2}$$

Choosing the Fourier's coefficients α_k such that (3.2) is satisfied on the interval $[-1,1]$. For this technical we take the equidistant collocation points as follows

$$t_j = -1 + \frac{2j}{N}, \ \ j = 0, 1, ...N, \tag{3.3}$$

and define the residual as

$$R_N(x) = \sum_{k=0}^{N} \alpha_k S_k(x) - \int_a^x k_1(x,t) \sum_{k=0}^{N} \alpha_k S_k(t) - \int_a^b k_2(x,t) \sum_{k=0}^{N} \alpha_k S_k(t) - f(x) \tag{3.4}$$

Then, by imposing conditions at collocation points

$$R_N(x_j) = 0, \quad j = 0, 1, ...N, \tag{3.5}$$

the integral equation (3.2) is converted to a system of linear equations.

Theorem 3.1. *Suppose that for the equation* (1.1)

we have

(1) $f \in C([a,b]), \ k_1(x,t) \in C(D_1)$ with $D_1 = \{(x,t) \in \mathbb{R}^2; \ a \le t \le x \le b\}$
(2) $\varphi \in C([a,b]), \ k_2(x,t) \in C(D_2)$ with $D_2 = [a,b] \times [a,b]$
(3) $M_1 = \max_{D_1} k_1(x,t); \ \ M_2 = \max_{D_2} k_2(x,t)$
(4) *There exists a contant* $c > 0$ *such that*

$$\frac{1}{c}\left[M_1 + M_2 e^{c(b-a)}\right] < 1.$$

Then the equation (1) admits a unique solution $\varphi \in C([a,b])$.

Proof. Application of the fixed point theory. See [6] ∎

Theorem 3.2. *Let* $A : X \to X$ *be compact operator and suppose that the equation*

$$(I - A)\varphi = f, \tag{3.6}$$

admits a unique solution.For the projections $P_n \ X \to X_n$ *such that* $\|P_n A - A\| \to 0, \ n \to \infty$.*The approximate equation*

$$\varphi_n - P_n A \varphi_n = P_n f, \tag{3.7}$$

has a unique solution for all $f \in X$ *with sufficiently large* n,*besides*

$$\|\varphi - \varphi_n\| \le M \|\varphi - P_n \varphi\|, \tag{3.8}$$

with some positive constant M *depending on* A.

Proof

As it is known for all sufficiently large n the inverse operators $(I - P_nA)^{-1}$ exist and are uniformly bounded, see [1, 5]. To verify the error bound, we apply the projection operator P_n to the equation (3.6) and get

$$P_n\varphi - P_nA\varphi = P_nf, \tag{3.9}$$

or again

$$\varphi - P_nA\varphi = P_nf + \varphi - P_n\varphi. \tag{3.10}$$

Subtracting (3.10) from (3.7) we find

$$(I - P_nA)(\varphi - \varphi_n) = (I - P_n)\varphi.$$

Hence the estimate (3.8) follows.

4. Numerical examples

Example 1

Consider the Fredholm integral equation

$$\varphi(x) - \int_0^x (x+t)\varphi(t)dt - \int_0^1 (x-t)\varphi(t)dt = f(x),$$

where the function $f(x)$ is chosen so that the solution $\varphi(x)$ is given by

$$\varphi(x) = x^3$$

Applying the second Chebyshev polynomial $T_n(x)$ to approximate the solution $\varphi(x)$, that is to say $\varphi_N(x)$ solution of the system of linear equations for $N = 20$

Points of x	Exact sol	Approx sol	Error	Error [3]
0.1000	1.0000e-003	9.9997e-04	2.7733e-08	2.2180e-04
0.2000	8.0000e-03	7.9999e-03	2.7743e-08	3.4990e-04
0.4000	6.4000e-02	6.3999e-02	3.0414e-08	1.9947e-03
0.6000	2.1600e-01	2.1600e-01	3.7607e-08	4.2426e-03
0.8000	5.1200e-01	5.1199e-01	5.1990e-08	6.4507e-03
1.0000	1.0000e+00	9.9999e-01	7.9374e-08	6.2804e-03

Table 1. The exact and approximate solutions of example 1 in some arbitrary points, using the first Chebyshev polynomial $T_n(x)$

Example 2

consider the linear Volterra–Fredholm integral equation,

$$\varphi(x) - \int_0^x (x-t)\varphi(t)dt - \int_0^1 x\varphi(t)dt = f(x),$$

where the function $f(x)$ is chosen so that the solution $\varphi(x)$ is given by

$$\varphi(x) = xe^x.$$

Applying the second Chebyshev polynomial $U_n(x)$ to approximate the solution $\varphi(x)$, that is to say $\varphi_N(x)$ solution of the system of linear equations for $N = 20$

Points of x	Exact sol	Approx sol	Error	Error [3]
0.1000	1.1051e-01	1.1051e-01	3.4955e-09	1.1984e-03
0.2000	2.4428e-01	2.4428e-01	7.3223e-09	2.4176e-03
0.4000	5.9672e-01	5.9672e-01	1.6285e-08	5.0039e-03
0.6000	1.0932e+00	1.0932e+00	2.7624e-08	7.9393e-03
0.8000	1.7804e+00	1.7804e+00	4.2272e-08	1.1428e-02
1.0000	2.7182e+00	2.7182e+00	6.1421e-08	1.5715e-02

Table 2. The exact and approximate solutions of example 2
in some arbitrary points, using the second Chebyshev polynomial $U_n(x)$

Example 3
Consider the Volterra-Fredholm integral equation

$$\varphi(x) - \int_0^x \cos(x-t)\varphi(t)dt - \int_0^1 \sin(x-t)\varphi(t)dt = f(x),$$

where the function $f(x)$ is chosen so that the solution $\varphi(x)$ is given by

$$\varphi(x) = e^x.$$

Applying the third Chebyshev polynomial $V_n(x)$ to approximate the solution $\varphi(x)$, say $\varphi_N(x)$ solution of the system of linear equations for $N = 20$

Points of x	Exact sol	Approx sol	Error	Error [2]
0.0000	1.0000e+00	1.0000e+00	1.0215e-08	1.0000e-04
0.2000	1.2214e+00	1.2214e+00	7.6344e-09	1.0000e-04
0.4000	1.4918e+00	1.4918e+00	3.6767e-09	1.0000e-04
0.6000	1.8221e+00	1.8221e+00	1.8910e-09	1.0000e-04
0.8000	2.2255e+00	2.2255e+00	9.3059e-09	1.0000e-04
1.0000	2.7182e+00	2.7182e+00	1.8804e-08	1.0000e-04

Table 3. The exact and approximate solutions of example 3
in some arbitrary points, using the third Chebyshev polynomial $V_n(x)$

Example 4
Consider the Fredholm integral equation

$$\varphi(x) - \int_{-1}^x (xt)\,\varphi(t)dt - \int_{-1}^1 2\cosh(x+t)\varphi(t)dt = f(x),$$

where the function $f(x)$ is chosen so that the solution $\varphi(x)$ is given by

$$\varphi(x) = \frac{\cosh x}{\sinh 2 + 1}.$$

Applying the fourth Chebyshev polynomial $W_n(x)$ to approximate the solution $\varphi(x)$, say $\varphi_N(x)$ solution of the system of linear equations for $N = 20$

Points of x	Exact sol	Approx sol	Error
-1.0000	3.3350e-01	3.3350e-01	1.1299e-07
-0.6000	2.5621e-01	2.5621e-01	1.0667e-07
-0.2000	2.2046e-01	2.2046e-01	9.6093e-08
0.0000	2.1612e-01	2.1612e-01	9.3194e-08
0.4000	2.3365e-01	2.3365e-01	1.0029e-07
0.8000	2.8905e-01	2.8905e-01	1.4034e-07
1.0000	3.3350e-01	3.3350e-01	1.8618e-07

Table 4. The exact and approximate solutions of example 4 in some arbitrary points, using the fourth Chebyshev polynomial $W_n(x)$

5. CONCLUSION

In this work, we assume that the unknown the function $\varphi(x)$ may be approximated by a finite sum of four Chebyshev polynomials. Substituting this finite sum into the Volterra-Fredholm integral equation in order to obtain a system of linear equations with $N+1$ unknowns. The comparison of examples with other methods shows its efficiency of this technical

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