Aust. J. Math. Anal. Appl.
Vol. **19** (2022), No. 2, Art. 7, 8 pp.
AJMAA

PRESERVER OF LOCAL SPECTRUM OF SKEW-PRODUCT OPERATORS

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Received 17 May, 2022; accepted 7 September, 2022; published 23 September, 2022.

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ABSTRACT. Let H and K be infinite-dimensional complex Hilbert spaces, and $B(H)$ (resp. $B(K)$) be the algebra of all bounded linear operators on H (resp. on K). For an operator $T \in B(H)$ and a vector $h \in H$, let $\sigma_T(h)$ denote the local spectrum of T at h. For two nonzero vectors $h_0 \in H$ and $k_0 \in K$, we show that if two maps φ_1 and φ_2 from $B(H)$ into $B(K)$ satisfy

$$\sigma_{\varphi_1(T)\varphi_2(S)^*}(k_0) = \sigma_{TS^*}(h_0)$$

for all $T, S \in B(H)$, and their range containing all operators of rank at most two, then there exist bijective linear maps $P : H \to K$ and $Q : K \to H$ such that $\varphi_1(T) = PTQ$ and $\varphi_2(T)^* = Q^{-1}T^*P^{-1}$ for all $T \in B(H)$. Also, we obtain some interesting results in this direction.

Key words and phrases: Local spectrum; Nonlinear preservers; Rank-one operators.

2010 *Mathematics Subject Classification.* Primary 47B49: Secondary 47A10, 47A11.

ISSN (electronic): 1449-5910

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1. INTRODUCTION

Throughout this paper, H and K are two infinite-dimensional complex Hilbert spaces. As usual $B(H,K)$ denotes the space of all bounded linear operators from H into K. When $H=K$ we simply write $B(H)$ instead of $B(H,H)$, and its unit will be denoted by I. The inner product of H or K will be denoted by $\langle , \rangle$ if there is no confusion. For an operator $T\in B(H,K)$, let T^* denote as usual its adjoint. Linear preserver problems, in the most general setting, demand the characterization of linear maps between algebras that leave a certain property, a particular relation, or even a subset invariant. This subject is very old and goes back well over a century to the so-called first linear preserver problem, due to Frobenius [9], that determines linear maps preserving the determinant of matrices. The local resolvent set, $\rho_T(x)$, of an operator $T\in B(H)$ at a point $x\in H$ is the union of all open subsets U of the complex plane $\mathbb{C}$ for which there is an analytic function $f:U\longrightarrow H$ such that $(\mu I-T)f(\mu)=x$ for all $\mu\in U$. The complement of local resolvent set is called the local spectrum of T at x, denoted by $\sigma_T(x)$, and is obviously a closed subset (possibly empty) of $\sigma(T)$, the spectrum of T. Recall that an operator $T\in B(H)$ is said to have the single-valued extension property (henceforth abbreviated to SVEP) if, for every open subset U of $\mathbb{C}$, there exists no nonzero analytic solution, $f:U\longrightarrow H$, of the equation

$$(\mu I-T)f(\mu)=0, \quad \forall\, \mu\in U.$$

Every operator $T\in B(H)$ for which the interior of its point spectrum, $\sigma_p(T)$, is empty enjoys this property. For more information about these notions one can see the books [1, 11].

The study of linear and nonlinear local spectra preserver problems attracted the attention of a number of authors. Bourhim and Ransford were the first ones to consider this type of preserver problem, characterizing in [8] additive maps on the algebra of all linear bounded operators on a complex Banach space X that preserve the local spectrum of operators at each vector of X. Their results cleared the way for several authors to describe maps on matrices or operators that preserve local spectrum, local spectral radius, and local inner spectral radius; see, for instance, the survey articles [4, 12] and the references introduced in them. Gonzalez and Mbekhta [10] characterized linear maps on $M_n(\mathbb{C})$ that preserving the local spectrum at only a fixed nonzero vector $x_0\in\mathbb{C}^n$. They proved that a linear map φ preserves the local spectrum at x_0 if and only if there exists an invertible matrix A in $M_n(\mathbb{C})$ such that $Ax_0=x_0$ and $\varphi(T)=ATA^{-1}$ for all $T\in M_n(C)$. Bourhim and Miller [6] described linear maps on $M_n(C)$ preserving the local spectral radius at a fixed nonzero vector in $\mathbb{C}^n$. Bracic and Muller [7] extended the both main results of [6, 10] to infinite dimensional Banach space by characterizing surjective continuous linear maps φ on $B(X)$ that preserve the local spectrum and the local spectral radius at a fixed nonzero vector in X. Bourhim and Mashreghi [5] characterized surjective maps on $B(X)$ that preserve the local spectrum of product operators at fixed nonzero vector. Abdelali et al. [2] characterized maps $\varphi:B(H)\to B(K)$ that preserve the local spectrum at fixed nonzero vector of the skew-product operators. Bourhim and Lee [3] investigated the form of all surjective maps φ_1 and φ_2 on $B(X)$ such that, for every T and S in $B(X)$, the local spectra of ST and $\varphi_1(T)\varphi_2(S)$ are the same at a nonzero fixed vector x_0 of X. In this paper, we follow the same path of studies by considering general local spectra preservers, and characterize the form of all maps φ_1 and φ_2 of $B(H)$ into $B(K)$ such that, for every T and S in $B(H)$, the local spectrum of TS^* and $\varphi_1(T)\varphi_2(S)^*$ are the same at a nonzero fixed vector.

2. PRELIMINARIES

The first lemma summarizes some known basic properties of the local spectrum.

Lemma 2.1. [1, 11] *Let X be a Banach space and $T \in B(X)$. For every $x, y \in X$ and a scalar $\alpha \in \mathbb{C}$ the following statements hold.*
(a) $\sigma_T(\alpha x) = \sigma_T(x)$ if $\alpha \neq 0$, and $\sigma_{\alpha T}(x) = \alpha\sigma_T(x)$.
(b) If $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then $\sigma_T(x) \subseteq \{\lambda\}$. Further, if $x \neq 0$ and T has SVEP, then $\sigma_T(x) = \{\lambda\}$.

For a nonzero $h \in H$ and $T \in B(H)$, we use a useful notation defined by A. Bourhim and J. Mashreghi in [5] by

$$\sigma^*_T(h) := \begin{cases} \{0\} & if\ \sigma_T(h) = \{0\}, \\ \sigma_T(h) \setminus \{0\} & if\ \sigma_T(h) \neq \{0\}. \end{cases}$$

For any $x, y \in H$, let $x \otimes y$ denote the operator of rank at most one on H defined by

$$(x \otimes y)z = \langle z, y\rangle\, x, \qquad \forall\ z \in H.$$

Note that every rank one operator in $B(H)$ can be written in this form, and that every finite rank operator $T \in B(H)$ can be written as a finite sum of rank one operators; i.e., $T = \sum_{i=1}^{n} x_i \otimes y_i$ for some $x_i, y_i \in H$ and $i = 1, 2, ..., n$. We denote by $F(H)$ the set of all finite rank operators in $B(H)$ and $F_n(H)$ the set of all operators of rank at most n, n is a positive integer.

The following lemma is an elementary observation that gives the nonzero local spectrum of any rank one operator.

Lemma 2.2. *(See* [5, Lemma 2.2]*) Let h_0 be a nonzero vector in H. For every vectors $x, y \in H$, the following statements hold.*
(a)

$$\sigma^*_{x\otimes y}(h_0) := \begin{cases} \{0\} & if\ \ \langle h_0, y\rangle = 0, \\ \langle x, y\rangle & if\ \ \langle h_0, y\rangle \neq 0. \end{cases}$$

(b) For all rank one operators $R \in B(H)$ and all $T, S \in B(H)$, we have

$$\sigma^*_{(T+S)R}(h_0) = \sigma^*_{TR}(h_0) + \sigma^*_{SR}(h_0)$$

The following theorem, which may be of independent interest, gives a spectral characterization of rank one operators in term of local spectrum.

Theorem 2.3. *(See* [5, Theorem 4.1]*) For a nonzero vector $h \in H$ and a nonzero operator $R \in B(H)$, the following statements are equivalent.*
(a) R has rank one.
(b) $\sigma^*_{RT}(h)$ contains at most one element for all $T \in B(H)$.
(c) $\sigma^*_{RT}(h)$ contains at most one element for all $T \in F_2(H)$.

The following result characterizes in term of the local spectrum when two operators are the same.

Lemma 2.4. *(See* [5, Theorem 3.2]*) For a nonzero vector h in H and two operators A and B in $B(H)$, the following statements are equivalent.*
(a) $A = B$.
(b) $\sigma_{AT}(h) = \sigma_{BT}(h)$ for all operators $T \in B(H)$.
(c) $\sigma_{AT}(h) = \sigma_{BT}(h)$ for all rank one operators $T \in B(H)$.
(d) $\sigma^*_{AT}(h) = \sigma^*_{BT}(h)$ for all rank one operators $T \in B(H)$.

The following theorem will be useful in the proof of the main results. We recall that if $h : \mathbb{C} \to \mathbb{C}$ is a ring homomorphism, then an additive map $A : H \to H$ satisfying $A(\alpha x) = h(\alpha)x, (x \in H, \alpha \in \mathbb{C})$ is called an h-quasilinear operator.

Theorem 2.5. *(See* [13, Theorem 3.3]*.) Let $\varphi : F(H) \to F(H)$ be a bijective additive map preserving rank one operators in both directions. Then there exists a ring automorphism $h : \mathbb{C} \to \mathbb{C}$, and either there are h-quasilinear bijective mappings $A : H \to H$ and $B : H \to H$ such that*

$$\varphi(x \otimes y) = Ax \otimes By,\ x, y \in H,$$

or there are h-quasilinear bijective mappings $C : H \to H$ and $D : H \to H$ such that

$$\varphi(x \otimes y) = Cy \otimes Dx,\ x, y \in H.$$

Note that, if in Theorem 2.5 the map φ is linear, then h is the identity map on $\mathbb{C}$ and so the maps A, B, C and D are linear.

3. MAIN RESULTS

In the following theorem, we investigate the form of all maps φ_1 and φ_2 of $B(H)$ into $B(K)$ such that, for every T and S in $B(H)$, the local spectrum of TS^* and $\varphi_1(T)\varphi_2(S)^*$ are the same at a nonzero fixed vector.

Theorem 3.1. *Let $h_0 \in H$ and $k_0 \in K$ be two nonzero vectors. Suppose that φ_1 and φ_2 be maps from $B(H)$ into $B(K)$ which satisfy*

$$\sigma_{\varphi_1(T)\varphi_2(S)^*}(k_0) = \sigma_{TS^*}(h_0),\ (T, S \in B(H)). \tag{3.1}$$

If the range of φ_1 and φ_2 contain $F_2(K)$, then there exist bijective linear maps $P : H \to K$ and $Q : K \to H$ such that $\varphi_1(T) = PTQ$ and $\varphi_2(T)^* = Q^{-1}T^*P^{-1}$ for all $T \in B(H)$.

Proof. The proof is long and we break it into several claims.

Claim 1. φ_1 is injective and $\varphi_1(0) = 0$.

If $\varphi_1(T) = \varphi_1(S)$ for some $T, S \in B(H)$, we get that

$$\begin{aligned}\sigma_{TR^*}(h_0) &= \sigma_{\varphi_1(T)\varphi_2(R)^*}(k_0)\\ &= \sigma_{\varphi_1(S)\varphi_2(R)^*}(k_0)\\ &= \sigma_{SR^*}(h_0)\end{aligned}$$

for all $R \in B(H)$. By Lemma 2.4, we see that $T = S$ and φ_1 is injective. For the second part of this claim, $\sigma_{\varphi_1(0)\varphi_2(T)^*}(k_0) = \sigma_{0T^*}(h_0) = \{0\} = \sigma_{0\varphi_2(T)^*}(k_0)$ for all $T \in B(H)$. As the range of φ_1 contains all rank one operators, Lemma 2.4 entails that $\varphi_1(0) = 0$.

Claim 2. φ_1 preserves rank one operators in both directions.

Let R be a rank one operator, and note that $\varphi_1(R) \neq 0$, since $\varphi_1(0) = 0$ and φ_1 is injective. Let $T \in B(H)$ be an arbitrary operator, then $\sigma_{RT^*}(h_0)$ has at most one element for all $T \in B(H)$, and so is $\sigma^*_{\varphi_1(R)\varphi_2(T)^*}(k_0)$. As the range of φ_1 contains $F_2(K)$, we see that $\sigma^*_{\varphi_1(R)S^*}(k_0)$ has at most one element for all operators $S \in F_2(K)$. By Lemma 2.3, we see that $\varphi_1(R)$ is rank one. Conversely, assume that $\varphi_1(R)$ is rank one for some operator $R \in B(H)$, and note that $R \neq 0$ and that $\sigma^*_{\varphi_1(R)\varphi_2(T)^*}(k_0)$ has at most one element for all $T \in B(H)$. Therefore, $\sigma^*_{RT^*}(h_0)$ has at most one element for all $T \in B(H)$. Again Lemma 2.3 tells us that R is rank one.

Claim 3. φ_1 is linear.

First we show that φ_1 is additive. Let R be a rank one operator, and note that, by the previous claim, $\varphi_1(R)$ is a rank one operator too. Let T and S be two operators in $B(H)$, and note that, by applying the statement (b) of Lemma 2.2, we have

$$\begin{aligned}\sigma^*_{\varphi_1(T+S)\varphi_2(R)^*}(k_0) &= \sigma^*_{(T+S)R^*}(h_0)\\ &= \sigma^*_{TR^*}(h_0) + \sigma^*_{SR^*}(h_0)\\ &= \sigma^*_{\varphi_1(T)\varphi_2(R)^*}(k_0) + \sigma^*_{\varphi_1(S)\varphi_2(R)^*}(k_0)\\ &= \sigma^*_{(\varphi_1(T)+\varphi_1(S))\varphi_2(R)^*}(k_0).\end{aligned}$$

for all rank one operators $R \in B(H)$. By Lemma 2.4, we conclude that $\varphi_1(T+S) = \varphi_1(T) + \varphi_1(S)$ for all $T, S \in B(H)$, and φ_1 is additive; as desired.

Now, let us show that φ_1 is homogeneous. For every $\alpha \in \mathbb{C}$ and $T \in B(X)$, we have

$$\begin{aligned}\sigma_{\alpha\varphi_1(T)\varphi_2(R)^*}(k_0) &= \alpha\sigma_{\varphi_1(T)\varphi_2(R)^*}(k_0)\\ &= \alpha\sigma_{TR^*}(h_0)\\ &= \sigma_{(\alpha T)R^*}(h_0)\\ &= \sigma_{\varphi_1(\alpha T)\varphi_2(S)^*}(k_0).\end{aligned}$$

Lemma 2.4 shows that $\varphi_1(\alpha T) = \alpha\varphi_1(T)$. So φ_1 is linear.

Claim 4. There are bijective linear mappings $A : H \to K$ and $B : H \to K$ such that $\varphi_1(x \otimes y) = Ax \otimes By$ for all $x, y \in H$.

By the previous claim $\varphi_1 : F(H) \to F(K)$ is a bijective linear map which preserves rank one operators in both directions. Thus by Theorem 2.5, φ_1 has one of the following forms.
(1) There exist bijective linear maps $A : H \to K$ and $B : H \to K$ such that

$$\varphi_1(x \otimes y) = Ax \otimes By,\ (x, y \in H). \tag{3.2}$$

(2) There exist bijective linear maps $C : H \to K$ and $D : H \to K$ such that

$$\varphi_1(x \otimes y) = Cy \otimes Dx,\ (x, y \in H). \tag{3.3}$$

Assume that φ_1 takes the form (3.3). Let $v \in K$ be a nonzero vector such that $\langle \varphi_2(I)^*(k_0), v\rangle = 0$, since $x = D^{-1}v$ and h_0 are nonzero vectors in H, there exist a nonzero vector $y \in H$ such that $\langle h_0, y\rangle \neq 0$ and $\langle x, y\rangle \neq 0$, then

$$\begin{aligned}\{0\} &= \sigma^*_{(Cy\otimes v)\varphi_2(I)^*}(k_0)\\ &= \sigma^*_{(Cy\otimes Dx)\varphi_2(I)^*}(k_0)\\ &= \sigma^*_{x\otimes y}(h_0).\end{aligned}$$

But Lemma 2.2 implies that

$$\sigma^*_{x\otimes y}(h_0) = \{\langle x, y\rangle\} \neq \{0\}.$$

This contradiction shows that φ_1 only takes the form (3.2).

Claim 5. For every $x, y \in H$, $\langle x, y\rangle = \langle Ax, \varphi_2(I)(By)\rangle$.

Let x and y be arbitrary vector in H. The previous claim and (3.1) entail that

$$\begin{aligned}\sigma_{x\otimes y}(h_0) &= \sigma_{\varphi_1(x\otimes y)\varphi_2(I)^*}(k_0)\\ &= \sigma_{(Ax\otimes By)\varphi_2(I)^*}(k_0).\end{aligned}$$

Assume first that $\langle h_0, y\rangle \neq 0$, using Lemma 2.2,

$$\begin{aligned}\{0\} \neq \{\langle h_0, y\rangle\} &= \sigma_{h_0\otimes y}(h_0)\\ &= \sigma_{\varphi_1(h_0\otimes y)\varphi_2(I)^*}(k_0)\\ &= \sigma_{(Ah_0\otimes By)\varphi_2(I)^*}(k_0).\end{aligned}$$

Which means that $\langle k_0, \varphi_2(I)(By)\rangle \neq 0$. Then Lemma 2.2 implies that

$$\begin{aligned}\{\langle x, y\rangle\} &= \sigma_{x\otimes y}(h_0)\\ &= \sigma_{(Ax\otimes By)\varphi_2(I)^*}(k_0)\\ &= \{\langle Ax, \varphi_2(I)(By)\rangle\}.\end{aligned}$$

Now, if $\langle h_0, y\rangle = 0$, we choose a vector $u \in H$ such that $\langle h_0, u\rangle \neq 0$. By what has been shown lastly applied to both u and $x + u$, we have
$\langle x, u\rangle = \langle Ax, \varphi_2(I)(Bu)\rangle$ and $\langle x, y + u\rangle = \langle Ax, \varphi_2(I)(B(u + y))\rangle$. Then

$$\begin{aligned}\langle x, y\rangle + \langle x, u\rangle &= \langle x, y + u\rangle\\ &= \langle Ax, \varphi_2(I)(B(y + u))\rangle\\ &= \langle Ax, \varphi_2(I)(By)\rangle + \langle Ax, \varphi_2(I)(Bu)\rangle\\ &= \langle Ax, \varphi_2(I)(By)\rangle + \langle x, u\rangle\,.\end{aligned}$$

This shows that $\langle x, y\rangle = \langle Ax, \varphi_2(I)(By)\rangle$ in this case too.

Claim 6. $\varphi_2(I)^*$ is invertible and $Ah_0 = \alpha k_0$ for some nonzero scalar $\alpha \in \mathbb{C}$.

It is clear that $\varphi_2(I)^*$ is injective, if not, there is a nonzero vector $y \in H$ such that $\varphi_2(I)^*y = 0$. Take $x = A^{-1}y$, and let $u \in H$ be a vector such that $\langle x, u\rangle \neq 0$. By the previous claim, we have

$$\begin{aligned}0 \neq \langle x, u\rangle &= \langle Ax, \varphi_2(I)(Bu)\rangle\\ &= \langle y, \varphi_2(I)(Bu)\rangle\\ &= \langle \varphi_2(I)^*y, Bu\rangle = 0.\end{aligned}$$

This contradiction tells us that $\varphi_2(I)^*$ is injective. Now, we show that A is continuous. Assume that $(x_n)_n$ is a sequence in H such that $\lim_{n\longrightarrow\infty} x_n = x \in H$ and $\lim_{n\longrightarrow\infty} Ax_n = y \in H$. Then, for every $u \in H$, we have

$$\begin{aligned}\langle y, \varphi_2(I)(Bu)\rangle &= \lim_{n\longrightarrow\infty} \langle Ax_n, \varphi_2(I)(Bu)\rangle\\ &= \lim_{n\longrightarrow\infty} \langle x_n, u)\rangle = \langle x, u\rangle\\ &= \langle Ax, \varphi_2(I)(By)\rangle\,.\end{aligned}$$

Since B is bijective and $u \in H$ is an arbitrary vector, the closed graph theorem shows that A is continuous. Moreover, we have $\langle x, y\rangle = \langle Ax, \varphi_2(I)(By)\rangle = \langle x, A^*\varphi_2(I)(By)\rangle$ for all $x, y \in H$, and thus $I = A^*\varphi_2(I)B$. It follows that $\varphi_2(I)^*$ is invertible.
For the second part of this claim, suppose, by the way of contradiction, let y be a nonzero vector in H such that $\langle h_0, y\rangle = 1$ and $\langle A^{-1}k_0, y\rangle = 0$. We have $(h_0 \otimes y)h_0 = h_0$ and $A(x \otimes y)(A^{-1})^*k_0 = 0$, then $\sigma_{h_0\otimes y}(h_0) = \{1\}$ and $\sigma_{A(h_0\otimes y)A^{-1}}(k_0) = \{0\}$. The previous claim

implies that

$$\begin{aligned}\{1\} = \sigma_{h_0\otimes y}(h_0) &= \sigma_{\varphi_1(h_0\otimes y)\varphi_2(I)^*}(k_0)\\ &= \sigma_{(Ah_0\otimes By)\varphi_2(I)^*}(k_0) = \sigma_{(Ah_0\otimes \varphi_2(I)By)}(k_0)\\ &= \sigma_{A(h_0\otimes y)A^{-1}}(k_0) = \{0\}.\end{aligned}$$

This contradiction shows that there is a nonzero scalar $\alpha \in \mathbb{C}$ such that $Ah_0 = \alpha k_0$.

Claim 7. $\varphi_1(T) = PTQ$ and $\varphi_2(T)^* = Q^{-1}T^*P^{-1}$ for every $T \in B(H)$, where $P = \alpha^{-1}A$ for some nonzero scalar $\alpha \in \mathbb{C}$ and $Q = (\varphi_2(I)^*P)^{-1}$.

By the previous claim, we see that $B = \varphi_2(I)^{-1}(A^*)^{-1}$, and thus

$$\begin{aligned}\varphi_1(x\otimes y) &= Ax\otimes By = A(x\otimes y)B^*\\ &= A(x\otimes y)(\varphi_2(I)^*A)^{-1} = P(x\otimes y)Q\end{aligned}$$

for all $x, y \in H$. Therefore, for any $x, y \in H$ and $T \in B(H)$, we have

$$\begin{aligned}\sigma_{P(x\otimes y)Q\varphi_2(T)^*}(k_0) &= \sigma_{\varphi_1(x\otimes y)\varphi_2(T)^*}(k_0)\\ &= \sigma_{(x\otimes y)T^*}(h_0)\\ &= \sigma_{P(x\otimes y)QQ^{-1}T^*P^{-1}}(Ph_0)\\ &= \sigma_{P(x\otimes y)QQ^{-1}T^*P^{-1}}(k_0).\end{aligned}$$

By Lemma 2.4, we conclude that $\varphi_2(T)^* = Q^{-1}T^*P^{-1}$ for all $T \in B(H)$.
For the second part of this claim, Observe that

$$\begin{aligned}\sigma_{PTQ\varphi_2(S)^*}(k_0) &= \sigma_{PTQQ^{-1}S^*P^{-1}}(k_0)\\ &= \sigma_{TS^*}(h_0) = \sigma_{\varphi_1(T)\varphi_2(S)^*}(k_0).\end{aligned}$$

for all $T \in B(H)$. By Lemma 2.4, we have $\varphi_1(T) = PTQ$, for every $T \in B(H)$. ∎

Theorem 3.1 leads to the following corollary.

Corollary 3.2. *Suppose $U \in B(H, K)$ be an unitary operator and $h_0 \in H$ be nonzero vector. Let φ_1 and φ_2 be maps from $B(H)$ into $B(K)$ which satisfy*

$$\sigma_{\varphi_1(T)\varphi_2(S)^*}(Uh_0) = \sigma_{TS^*}(h_0)$$

for all $T, S \in B(H)$. If the range of φ_1 and φ_2 contain $F_2(K)$, then there exist bijective linear maps $P : H \to H$ and $Q : H \to H$ such that $\varphi_1(T) = UPTQU^*$ and $\varphi_2(T)^* = UQ^{-1}T^*P^{-1}U^*$ for all $T \in B(H)$.

Proof. We consider the maps $\psi_1 : B(H) \to B(H)$ defined by $\psi_1(T) = U^*\varphi_1(T)U$ and $\psi_2 : B(H) \to B(H)$ defined by $\psi_2(T) = U^*\varphi_2(T)U$ for all $T \in B(H)$. We have,

$$\begin{aligned}\sigma_{\psi_1(T)\psi_2(S)^*}(h_0) &= \sigma_{U^*\varphi_1(T)UU^*\varphi_2(S)^*U}(h_0)\\ &= \sigma_{U^*\varphi_1(T)\varphi_2(S)^*U}(h_0)\\ &= \sigma_{\varphi_1(T)\varphi_2(S)^*}(Uh_0)\\ &= \sigma_{TS^*}(h_0)\end{aligned}$$

for every $T, S \in B(H)$. So by Theorem 3.1, there exist bijective linear maps $P : H \to H$ and $Q : H \to H$ such that $\psi_1(T) = PTQ$ and $\psi_2(T)^* = Q^{-1}T^*P^{-1}$ for all $T \in B(H)$. The result follows. ∎

4. CONCLUSION

In this paper, we have followed the same path of studies by considering general local spectra preservers, and characterized the form of all maps $\varphi_1 : B(H) \to B(K)$ and $\varphi_2 : B(H) \to B(K)$ that preserve the local spectrum at fixed nonzero vector of the skew-double product TS^*, and under the mild assumption on the ranges of the maps contain operators with rank at most two.

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