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SIMPLE INTEGRAL REPRESENTATIONS FOR THE FIBONACCI AND LUCAS NUMBERS

SEÁN M. STEWART

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PHYSICAL SCIENCE AND ENGINEERING DIVISION, KING ABDULLAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, THUWAL 23955-6900, SAUDI ARABIA.
sean.stewart@kaust.edu.sa

ABSTRACT. Integral representations of the Fibonacci numbers F_{kn+r} and the Lucas numbers L_{kn+r} are presented. Each is established using methods that rely on nothing beyond elementary integral calculus.

Key words and phrases: Integral representation; Fibonacci number; Lucas number.

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1. INTRODUCTION

The representation of special numbers obtained from various counting sequences using an integral is one of a number of important tools available in their analysis. Integral representations for a range of classical counting numbers are known. Perhaps the greatest variety in integral representations is to be found among the Catalan numbers [12, 2, 3, 4, 5, 11, 15, 13, 14, 10]. To a lesser extent is the number of integral representations found for the Fibonacci numbers while there is a distinct paucity of such representations for the Lucas numbers. Integral representations for the Fibonacci numbers have been given recently by Glasser and Zhou [8] and Andrica and Bagdasar [1, p. 132] while another for the even Fibonacci numbers was given two decades ago by Dilcher [6]. Andrica and Bagdasar [1, p. 133] have given the only integral representation I have managed to find in the literature for the Lucas numbers.

In this note we give new integral representations of the Fibonacci numbers F_{kn+r} and the Lucas numbers L_{kn+r}. Here $n \in \mathbb{Z}_{\geqslant 0} = \{0, 1, 2, \dots\}$ is a non-negative integer, $k \in \mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$ is an arbitrary but fixed positive integer, while $r \in \mathbb{Z}_{\geqslant 0}$ is an arbitrary but fixed non-negative integer. In the past various integral representations for the Fibonacci numbers have usually been established using advanced techniques such as using a complex analytic approach based on a Fourier integral representation [8], using a method that relied on the (Gaussian) hypergeometric function [6], or using a method that employs the Cauchy integral formula [1]. The integral representations to be given here will be established using nothing beyond elementary integral calculus.

Before proceeding we briefly recall some definitions and results we are going to have a need for. As usual the nth Fibonacci number F_n is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$ with $F_0 = 0$ and $F_1 = 1$ while the nth Lucas number L_n is defined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$ with $L_0 = 2$ and $L_1 = 1$. Binet's formula for the Fibonacci numbers is [9, Thm. 5.5, p. 90]

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^n - \frac{(-1)^n}{\varphi^n} \right), \tag{1.1}$$

while Binet's formula for the Lucas numbers is [9, Thm. 5.7, p. 93]

$$L_n = \varphi^n + \frac{(-1)^n}{\varphi^n}. \tag{1.2}$$

Here φ denotes the golden ratio $(1 + \sqrt{5})/2$. From (1.1) and (1.2) one can see the connection between the Fibonacci numbers, the Lucas numbers, and the golden ratio is

$$\varphi^n = \frac{L_n + F_n \sqrt{5}}{2}. \tag{1.3}$$

An identity we need is [9, p. 117, Ex. 5.37]

$$L_n^2 - 5F_n^2 = 4(-1)^n. \tag{1.4}$$

Finally, for $m, r \in \mathbb{Z}_{\geqslant 0}$ the Fibonacci and Lucas index addition formulae are [7, 16]

$$2F_{m+r} = F_r L_m + L_r F_m, \tag{1.5}$$

and

$$2L_{m+r} = L_m L_r + 5F_m F_r, \tag{1.6}$$

respectively.

2. INTEGRAL REPRESENTATIONS

We first give integral representations for the Fibonacci numbers F_{kn} and the Lucas numbers L_{kn}. These are then used to establish integral representations for the Fibonacci numbers F_{kn+r} and the Lucas numbers L_{kn+r}.

Theorem 2.1. *For $n \in \mathbb{Z}_{\geqslant 0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$, the Fibonacci numbers F_{kn} can be represented by the integral*

$$F_{kn} = \frac{nF_k}{2^n} \int_{-1}^{1} \left(L_k + F_k x\sqrt{5}\right)^{n-1} dx. \tag{2.1}$$

Proof. Direct elementary integration yields

$$\begin{aligned} \frac{nF_k}{2^n} \int_{-1}^{1} \left(L_k + F_k x\sqrt{5}\right)^{n-1} dx &= \frac{1}{\sqrt{5}} \left[\left(\frac{L_k + F_k x\sqrt{5}}{2}\right)^n\right]_{-1}^{1} \\ &= \frac{1}{\sqrt{5}} \left[\left(\frac{L_k + F_k\sqrt{5}}{2}\right)^n - \left(\frac{L_k - F_k\sqrt{5}}{2}\right)^n\right]. \end{aligned}$$

Noting that

$$\frac{1}{\varphi^m} = \frac{2}{L_m + F_m\sqrt{5}} = \frac{2(L_m - F_m\sqrt{5})}{L_m^2 - 5F_m^2} = \frac{(-1)^m}{2}\left(L_m - F_m\sqrt{5}\right), \tag{2.2}$$

where $m \in \mathbb{Z}_{\geqslant 0}$ and the result given in (1.4) has been used, allows one to write

$$\frac{nF_k}{2^n} \int_{-1}^{1} \left(L_k + F_k x\sqrt{5}\right)^{n-1} dx = \frac{1}{\sqrt{5}} \left(\varphi^{kn} - \frac{(-1)^{kn}}{\varphi^{kn}}\right) = F_{kn},$$

where (1.3) together with the result for Binet's formula for the Fibonacci numbers given by (1.1) with n replaced with kn have been used, and completes the proof. ∎

Remark 2.1. Setting $k = 2$ in (2.1) gives

$$F_{2n} = \frac{n}{2^n} \int_{-1}^{1} \left(3 + x\sqrt{5}\right)^{n-1} dx, \tag{2.3}$$

an integral representation of the even Fibonacci numbers and corresponds to the result found by Dilcher using an approach that relied on the (Gaussian) hypergeometric function [6]. We should note Dilcher's result is presented as a trigonometric integral that is obtained by making a substitution of $x = \cos t$ in (2.3). An integral representation of the odd Fibonacci numbers can be readily found. Using the obvious identity $F_{2n+1} = F_{2n+2} - F_{2n}$, a reindexing of $n \mapsto n+1$ in (2.3), the difference between this result and (2.3) produces

$$F_{2n+1} = \frac{1}{2^{n+1}} \int_{-1}^{1} \left(n + 3 + (n+1)x\sqrt{5}\right) \left(3 + x\sqrt{5}\right)^{n-1} dx. \tag{2.4}$$

Remark 2.2. Indeed, (2.1) can be seen as a thinly disguised form of Binet's formula for F_{kn} with the connection becoming obvious if we write

$$F_{kn} = \frac{n}{\sqrt{5}} \int_{\frac{1}{(-\varphi)^k}}^{\varphi^k} t^{n-1}\, dt,$$

and substitute $t = (L_k + F_k x\sqrt{5})/2$.

Theorem 2.2. *For $n \in \mathbb{Z}_{\geqslant 0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$, the Lucas numbers L_{kn} can be represented by the integral*

$$L_{kn} = \frac{1}{2^n} \int_{-1}^{1} \left(L_k + F_k(n+1)x\sqrt{5}\right) \left(L_k + F_k x\sqrt{5}\right)^{n-1} dx. \tag{2.5}$$

Proof. Denote the integral to be found by I. Integrating by parts, we have

$$\begin{aligned} I &= \frac{1}{nF_k\sqrt{5}} \left[\left(\frac{L_k + F_k x\sqrt{5}}{2}\right)^n \left(L_k + F_k(n+1)x\sqrt{5}\right)\right]_{-1}^{1} - \frac{n+1}{n2^n} \int_{-1}^{1} \left(L_k + F_k x\sqrt{5}\right)^n dx \\ &= \frac{1}{nF_k\sqrt{5}} \left(\frac{L_k + F_k\sqrt{5}}{2}\right)^n \left(L_k + F_k(n+1)\sqrt{5}\right) \\ &\quad - \frac{1}{nF_k\sqrt{5}} \left(\frac{L_k - F_k\sqrt{5}}{2}\right)^n \left(L_k - F_k(n+1)\sqrt{5}\right) - \frac{2}{nF_k} F_{kn+k}, \end{aligned}$$

where in the last line the result for the integral given by (2.1) with $n \mapsto n+1$ has been used. Applying (1.3) and (2.2) before expanding, followed by the application of both Binet formulae (1.1) and (1.2) with n replaced with kn leads to

$$I = \frac{1}{nF_k} \left(L_k F_{kn} + (n+1) F_k L_{kn} - 2F_{kn+k}\right) = L_{kn}.$$

Here application of the Fibonacci index addition formula given in (1.5) has been made with m replaced with kn, and completes the proof. ∎

Theorem 2.3. *For $n \in \mathbb{Z}_{\geqslant 0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}_{\geqslant 0}$, the Fibonacci numbers F_{kn+r} can be represented by the integral*

$$F_{kn+r} = \frac{1}{2^{n+1}} \int_{-1}^{1} \left(nF_k L_r + F_r L_k + F_k F_r (n+1)x\sqrt{5}\right) \left(L_k + F_k x\sqrt{5}\right)^{n-1} dx. \tag{2.6}$$

Proof. The Fibonacci index addition formula given by (1.5) with m replaced with kn produces $2F_{kn+r} = F_{kn}L_r + F_r L_{kn}$. The result immediately follows on substituting for the integral representations of F_{kn} and L_{kn} given by (2.1) and (2.5) respectively into the given index addition formula, and completes the proof. ∎

Remark 2.3. Notice the result for the integral representation of the odd Fibonacci numbers given by (2.4) is recovered from (2.6) on setting $(k, r) = (2, 1)$.

Theorem 2.4. *For $n \in \mathbb{Z}_{\geqslant 0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}_{\geqslant 0}$, the Lucas numbers L_{kn+r} can be represented by the integral*

$$L_{kn+r} = \frac{1}{2^{n+1}} \int_{-1}^{1} \left(5nF_k F_r + L_k L_r + F_r L_r (n+1)x\sqrt{5}\right) \left(L_k + F_k x\sqrt{5}\right)^{n-1} dx. \tag{2.7}$$

Proof. The Lucas index addition formula given by (1.6) with m replaced with kn produces $2L_{kn+r} = L_{kn}L_r + 5F_{kn}F_r$. The result immediately follows on substituting for the integral representations of F_{kn} and L_{kn} given by (2.1) and (2.5) respectively into the given index addition formula, and completes the proof. ∎

Remark 2.4. As a consequence of (2.7), integral representations for the Lucas numbers L_n and the even and odd Lucas numbers L_{2n} and L_{2n+1} immediately follow. These are obtained on setting (k, r) equal to $(1, 0)$, $(2, 0)$, and $(2, 1)$ respectively. They are:

$$L_n = \frac{1}{2^n} \int_{-1}^{1} \left(1 + (n+1)x\sqrt{5}\right) \left(1 + x\sqrt{5}\right)^{n-1} dx,$$

$$L_{2n} = \frac{1}{2^n} \int_{-1}^{1} \left(3 + (n+1)x\sqrt{5}\right) \left(3 + x\sqrt{5}\right)^{n-1} \, dx,$$

and

$$L_{2n+1} = \frac{1}{2^{n+1}} \int_{-1}^{1} \left(5n + 3 + (n+1)x\sqrt{5}\right) \left(3 + x\sqrt{5}\right)^{n-1} \, dx.$$

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