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WALRASIAN EQUILIBRIUM FOR SET-VALUED MAPPING

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ABSTRACT. In this article, we obtain the existence of Walras equilibrium for set-valued demand mappings in a pure exchange economy. In this case, the set-valued mappings are defined by the loss function. Therefore, we shall summarize the features describing the exchange economy system which contain the loss function.

Key words and phrases: Set-valued mapping; Consumer theory; Walras equilibrium; loss function.

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1. INTRODUCTION

The role of prices in the problem of decentralization of consumer choice is taken to mean that knowledge of prices enables each consumer to make his own choice, in accordance with his own objectives, without knowing the global state of the economy, in particular, without knowing the choice of other consumers because of scarcity constraints.

It was Adam Smith who originated this concept of decentralization more three centuries ago. In his book entitled "The Wealth of Nations", published in 1778, he did not state what the famous hand-manipulated by forwarding a rigorous argument to justify its existence.

It was a century later that Leon Walras suggested that this invisible hand acted on prices via the function of demand, which it used to provide economic agents (consumers) with sufficient information to guarantee the consistency of their actions while respecting scarcity constraint.

The study of the Walras model can now be considered as Classical. See the fundamental book of Debrue [4]in 1959. The first proof of the existence of an equilibrium in a series of alternative models are due to Wald [10] and later, by Arrow-Debrue [2], McKenzie [6], and Nikaido [7]. The most general version is in Debrue [4]. Recently in [8] be obtained the existence Walras equilibrium weaker of the result [9] by used variational inequalities. In addition, in [3] by used game Theory.

The economic activity we discuss in this article consists of the simple exchange of commodities among a finite number of consumers and it is known well as pure exchange economic. In this case, we will the attention to the behavior of set-valued demand maps of the consumer. Notice that we have left aside the problem of utility theory. We have chosen to use lose functions instead of preference orderings. For relationship between these two concept, we refer to Debrue [4]. See Also Kannai [5] for characterization of convex loss function.

2. SET-VALUED MAPPING

Let X and Y be a topological space. The family of all subset of Y is denoted by $\mathcal{P}(Y)$. The set-valued is a function F from X to $\mathcal{P}(Y)$ with the property that $F(x) \neq \emptyset$ for all x in X.

Definition 2.1. : Set-valued $F : X \longrightarrow \mathcal{P}_0(Y)$ is called to be **upper semi-continuous** at $p \in X$ if for any neighbourhood $\mathcal{U}_\epsilon(F(p))$ of $F(p)$ there exists $\delta > 0$ such that for all $x \in N_\delta(p)$, $F(x) \subset \mathcal{U}_\epsilon(F(p))$

Definition 2.2. : Set-valued $F : X \longrightarrow \mathcal{P}_0(Y)$ is called to be **lower semi-continuous** at $p \in X$ if for any neighbourhood $\mathcal{U} \subset Y$ such that $\mathcal{U} \cap F(p) \neq \emptyset$ there exists $\delta > 0$ for all $x \in N_\delta(p)$, $F(x) \cap \mathcal{U} \neq \emptyset$

The characterization of upper semi- continuous of the set–valued is described the following.

Theorem 2.1. *Suppose that*

i. *function f is a continuous on $X \times Y$,*
ii *set-valued F is a continuous on Y,*
iii *image $F(y)$ is a compact set.*

Then function $y \longmapsto \gamma(y) = \inf_{x \in B(y)} f(x, y)$ is continuous and set-valued $y \longmapsto F(y) = \{x \in B(y) : f(x, y) = \gamma(y)\}$ is an upper semi-continuous.

Proposition 2.2. *Any closed set-valued F mapping Y into a compact space X is upper semi-continuous.*

Definition 2.3. The function $f : X \longrightarrow \mathbb{R}$ is called has non satiation property if

$$\forall x \in X, \exists x^* \in X, f(x^*) < f(x).$$

Proposition 2.3. *Suppose that $f : X \longrightarrow \mathbb{R}$ is convex and that has non satiation property. If x^* minimizes f on X, then x^* belongs to the boundary of X.*

3. ECONOMIC SYSTEM

The concept of an economic system or economic may be formalized in different ways. The certain formalization chosen is a reflection of the institutional structure, of the scope, of the level of abstraction, of the static or dynamic features,or conceptual characteristic of the background structure that people want to say; of course, it depends directly on the chosen language of formalization and, indirectly, on the theoretical aims the formalization is supposed to serve.

We often found of the theory is only concerned with the economic exchange process (the markets) without regard to the production sector. Such economic systems are referred to in the literature as pure exchange economy. Structure of the pure exchange economy is presented as follows.

$$\mathcal{E}_0 = \left\{N, \mathbf{R}^\ell, D\right\} \tag{3.1}$$

Where N is a finite sets of consumer, $\mathbf{R}^\ell$ is a commodity spaces and D is a demand set-valued maps. Suppose the subset $C \subset \mathbf{R}^\ell$ is an available commodities. We know a price is a linear form $\mathbf{p} \in (\mathbf{R}^\ell)^*$, dual of $\mathbf{R}^\ell$, which associates a commodity $\mathbf{x} \in \mathbf{R}^\ell$ with its value $\langle \mathbf{p}, \mathbf{x}\rangle \in \mathbf{R}$, expressed in monetary units. Since an i-th commodity is represented by the i-th vector $e^i = (0, 0, \cdots 1, 0, \dots, 0)$ of the canonical basis of $\mathbf{R}^\ell$, the components $p^i = \langle \mathbf{p}, e^i\rangle$ of the price $\mathbf{p}$ represent what is usually called the price of the commodity i. Suppose $N = \{1, 2, \cdot, n\}$ is consumer sets. Each consumer $i \in N$ we associate its consumption subset $K_i \subset \mathbf{R}^\ell$. Where the subset K_i is assumed always be closed, convex, and bounded to below. To design a decentralized mechanism that allocates available commodities $\mathbf{c} \in C$, we will assume that the C subset of available commodities is "appropriated". This means that consumer $i \in N$ is given with a subset of $C(i)$ of commodities such that $C = \sum_{i\in N} C(i)$ where $C(i) \subset \mathbf{R}^l$. In the other words, any available commodity $\mathbf{c} = \sum_{i\in N} c_i \in C$ is assumed to be adequate in the sense that every consumer of $i \in N$ is entitled to the available commodity $c_i \in C(i)$.

Subset $C(i)$ is the initial contribution of the available commodity. Those an appropriation of C allows the profit function to be shared among consumers, i.e. $r(\mathbf{p}) = \sum_{i=1}^n r_i(\mathbf{p})$ for each $\mathbf{p} \in P$, where $r_i(\mathbf{p}) = \sup_{\mathbf{c}\in C(i)}\langle \mathbf{p}, \mathbf{c}\rangle$ denotes the maximum profit that consumer i obtains from his initial contribution of $C(i)$. This profit $r_i(\mathbf{p})$ is used as income.

Suppose $B_i(\mathbf{p}, r)$ is the budget set of consumers i that defined as

$$B_i(\mathbf{p}, r) = \{\mathbf{x} \in K_i : \langle \mathbf{p}, \mathbf{x}\rangle \le r\}. \tag{3.2}$$

The budget set of consumers i defined by the appropriation $C = \sum_{i=1}^n C(i)$ is $B_i(\mathbf{p}, r_i(\mathbf{p}))$.

After the set of available commodities has been appropriated, each consumer knows his budget set $B_i(\mathbf{p}, r_i(\mathbf{p}))$ with price $\mathbf{p}$ prevails. Furthermore assumed that each consumer i devises a selection procedure who enables him to consume goods in form the demand function $D_i(\mathbf{p}, r)$ which subset of his budget set $B_i(\mathbf{p}, r)$.

Definition 3.1. The function $D_i : P \times \mathbf{R} \to \mathbf{R}^l$ is a demand set-valued of consumer i if for each $\mathbf{p} \in P$ and $r \in \mathbf{R}$, $D_i(\mathbf{p}, r) \subset B_i(\mathbf{p}, r)$.

An economic system is therefore defined by the pairs $\{K_i, C(i), D_i\}$.

Definition 3.2. Suppose $\bar{\mathbf{x}} = \{\bar{\mathbf{x}}^1, \cdots, \bar{\mathbf{x}}^n\}$ is a Walras allocation and $\bar{\mathbf{p}}$ a Walras price . The pairs $\{\bar{\mathbf{x}}, \bar{\mathbf{p}}\}$ is called Walras equilibrium if satisfying

(i) $\forall i \in N, \quad \bar{\mathbf{x}}^i \in D_i(\bar{\mathbf{p}}, r_i(\bar{\mathbf{p}}))$,
(ii) $\sum_{i \in N} \bar{\mathbf{x}}^i \in C$,
(iii) $\forall i \in N, \quad \langle \bar{\mathbf{p}}, \bar{\mathbf{x}}_i \rangle = r_i(\bar{\mathbf{p}})$.

The notion of recession cone is essential. The following that in question. Let U be a vector spaces and X be a subset of U containing 0. The set $\cap_{\alpha \geq 0} \alpha X$ is called ***recession cone*** of X.

Furthermore, the following is the existence of Walras equilibrium theorems.

Theorem 3.1. *Let $\xi \in \boldsymbol{R}^l_+$ be a numeraire and $P = \{\boldsymbol{p} \in \boldsymbol{R}^l_+ : \langle \boldsymbol{p}, \xi \rangle = 1\}$. Suppose that*

(1) *the consumption sets K_i closed, convex and bounded below*

(2) *the C subset of the available commodities satisfy C is closed, convex and $\boldsymbol{R}^l_+$ is a recession cone, and*
(3) *there exists $\boldsymbol{c} \in C$ such that $(C - \boldsymbol{c}) \cap \boldsymbol{R}^l_+ = \{0\}$*
If the set-valued demand D_i and the income function r_i satisfy
(4) *$\forall i \in N, \boldsymbol{p} \in P \to D_i(\boldsymbol{p}, r_i(\boldsymbol{p}))$ is upper semi-continuous with a closed convex nonempty image,then there exists Walras pre-equilibrium $\{\bar{\boldsymbol{x}}, \bar{\boldsymbol{p}}\} \in \boldsymbol{R}^l_+ \times P$.*

The requirement of the existence of the Walras equilibrium in that theorem is the set-valued demand function must be the upper semi-continuous. For this purpose, we need a theorem on the surjectivity of the set-valued functions.

Definition 3.3. Let X and Y be two topological spaces and let T be a continuous map from X into Y. The mapping T is said to be proper if T is a closed map and, for each $y \in Y, T^{-1}(y)$ is compact.

Proposition 3.2. *Suppose that for all $i = 1, 2, \cdots n$, $K_i \subset \boldsymbol{R}^l$ is closed and bounded below and that C satisfies properties*
(i) *the commodity set C is convex and closed*
(ii) *there exists $\boldsymbol{c} \in C$ such that $(C - \boldsymbol{c}) \cap \boldsymbol{R}^l_+ = \{0\}$.*

Then the map $\{\boldsymbol{x}^1, \cdots, \boldsymbol{x}^n, \boldsymbol{c}\} \in \prod_{i=1}^n K_i \times C \mapsto \sum_{i=1}^n (\boldsymbol{x}^i - \boldsymbol{c}) \in \boldsymbol{R}^l$ is a proper.

In particular, the subset $X(N) = \{\mathbf{x} \in \prod_{i=1}^n K_i : \sum_{i=1}^n \mathbf{x}^i \in C\}$ is compact and its projection $\pi_i X(N)$ of $X(N)$ are compact.

If B is a closed ball of positive radius, then $\pi_i X(N) + B$ is also compact. Furthermore, we setting

$$(*) \begin{cases} \tilde{K}_i = K_i \cap (\pi_i X(N) + B); \quad \tilde{C}_0(i) = C(i) \cap (\pi_i X(N) + B); \\ \tilde{C}(i) = \tilde{C}_0(i) - \mathbf{R}^l_+; \quad \tilde{C} = \sum_{i \in N} \tilde{C}(i) \subset C \cap \sum_{i \in N} (\pi_i X(N) + B). \end{cases}$$

4. DEMAND SET-VALUED DEFINED BY LOSS FUNCTION

In this section, we investigate the case when the set-valued demand function is defined by

$$D_i(\mathbf{p}, r) = \{\mathbf{x} \in B_i(\mathbf{p}, r) : f_i(\mathbf{x}, \mathbf{p}) = \min_{\mathbf{y} \in B_i(\mathbf{p}, r)} f_i(\mathbf{y}, \mathbf{p})\}.$$

In the case, the set-valued demand function D_i is obtained by assuming that i-th consumer chooses a commodity in the budget set B_i by achieving his/him minimum loss. Specifically, we assume that each consumer chooses a commodity according to the loss function $f_i : K_i \times P \mapsto \mathbf{R}$. which can be determined by prices prevailing in the economy, then we define D_i such as above

$$D_i(\mathbf{p}, r) = \{\mathbf{x} \in B_i(\mathbf{p}, r) : f_i(\mathbf{x}, \mathbf{p}) = \min_{\mathbf{y} \in B_i(\mathbf{p}, r)} f_i(\mathbf{y}, \mathbf{p})\}. \tag{4.1}$$

In such a situation, an economy system can be expressed by the pairs $\{K_i, C(i), f_i\}_{i\in N}$
Since the set-valued demand function D_i depends on the loss function f_i, we begin by stating the existence theorem for a Walras equilibrium as in Theorem 4.5.

As in Theorem 3.1, the requirement of the upper semi-continuous for set-valued demand functions is essential. In order to be efficient, the proof of the Theorem 4.5 will utilize some of the results.

First, we begin with the result as follows.

Proposition 4.1. *Suppose that*

(1) *the income function r_i is continuous on P for each $i \in N$,*
(2) *for all $\boldsymbol{p} \in P$ there exists $\bar{\boldsymbol{x}}^i \in K_i$ such that $\langle \boldsymbol{p}, \bar{\boldsymbol{x}}^i\rangle < r_i(\boldsymbol{p})$, and*
(3) *the set-valued budget function $\boldsymbol{p} \mapsto B(\boldsymbol{p}, r_i(\boldsymbol{p}))$ has non-empty closed-convex images and is closed and lower semi continuous.*

If the consumption sets K_i is compact and the loss function $f_i : K_i \times P \mapsto \boldsymbol{R}$ is continuous for all $i \in N$, then the set-valued demand function $\boldsymbol{p} \to D_i(\boldsymbol{p}, r_i(\boldsymbol{p}))$ is upper semi-continuous.

Proof. Clear that subsets $B(\mathbf{p}, r_i(\mathbf{p}))$ are non-empty set by (2). The graph of set-valued $\mathbf{p} \mapsto B(\mathbf{p}, r_i(\mathbf{p}))$ is closed since the pairs $\{\mathbf{p}, \mathbf{x}\}$ satisfying $\langle \mathbf{p}, \mathbf{x}\rangle - r_i(\mathbf{p}) \leq 0$ and $\mathbf{p} \mapsto r_i(\mathbf{p})$ is upper semi-continuous by (1).
Now, let $\tilde{\mathbf{x}}^i \in K_i$ be defined by hypothesis (2). Then for $\mathbf{p}_0 \in P$, $\langle \mathbf{p}_0, \tilde{\mathbf{x}}^i\rangle - r_i(\mathbf{p}_0) = -k_0 < 0$. Also, there exists $\alpha > 0$ such that $\mathbf{x}_\alpha = \alpha\tilde{\mathbf{x}}^i + (1-\alpha)\mathbf{x}_0$ belongs to $N(\mathbf{x}_0)$. Since $\langle \mathbf{p}_0, \mathbf{x}_0\rangle - r_i(\mathbf{p}_0) \leq 0$, we have $\langle \mathbf{p}_0, \mathbf{x}_\alpha\rangle - r_i(\mathbf{p}_0) \leq -k_0\alpha$.
Take $\varepsilon = \frac{1}{4}k_0\alpha$. Since r_i is lower semi-continuous, there exists a neighborhood $N(\mathbf{p}_0)$ of $\mathbf{p}_0$ such that,

$$\langle \mathbf{p}, \mathbf{x}_0\rangle - r_i(\mathbf{p}_0) \leq \langle \mathbf{p}_0, \mathbf{x}_0\rangle - r_i(\mathbf{p}_0) + \varepsilon + \langle \mathbf{p} - \mathbf{p}_0, \mathbf{x}_0\rangle \leq -\frac{1}{2}k_0\alpha < 0. \tag{4.2}$$

for all $\mathbf{p} \in N(\mathbf{p}_0)$. This implies that $\mathbf{x}_0$ belongs to $B(\mathbf{p}, r_i(\mathbf{p}))$ whenever $\mathbf{p} \in N(\mathbf{p}_0)$. This show that the set-valued budget is lower semi-continuous.

Furthermore, if the consumption sets K_i is compact, then the set-valued budget function $B_i(\mathbf{p}, r_i(\mathbf{p}))$ stays in the compact set K_i and thus, is an upper semi-continuous by Proposition 2.2. Thus it is continuous. Since the loss function f_i is continuous, from Theorem 2.1 that the set-valued demand function is upper semi-continuous. ∎

Proposition 4.2. *Suppose that*

(i) *the consumption subsets K_i are compact and convex,*
(ii) *$C(i) = C_0(i) - \boldsymbol{R}^l_+$ where $C_0(i)$ are compact,*
(iii) *the loss function $f_i : K_i \times P \to \boldsymbol{R}$ are continuous and convex w.r.t $\boldsymbol{x}_i$, and*
(iv) *for each i , $0 \in K_i - \mathit{Int}(K_i)$.*

Then there exists a Walras pre -equilibrium.

Proof. This is an easy consequence from Theorem3.1 and Proposition 4.2. Since (iv) implies that there existst $\tilde{\mathbf{x}}^i \in K_i$ such that $\langle \mathbf{p}, \tilde{\mathbf{x}}^i\rangle < r_i(\mathbf{p})$ for each $\mathbf{p} \in P$. ∎

An economy $\{K_i, C(i), f_i\}$ satisfying (i) and (ii) is called a compact economy. Is each economy equivalent to a compact economy?

Definition 4.1. Suppose the map $\{\mathbf{x}, \mathbf{c}\} \in R^N \times C \mapsto \sum_{i\in N}(\mathbf{x}^i - \mathbf{c}) \in \mathbf{R}^l$ is proper. Then the pairs $\{\tilde{K}_i, \tilde{C}(i), f_i\}_{i\in N}$ is called a "compactified economy" of $\{K_i, C(i), f_i\}_{i\in N}$.

We denote the compactified budget sets by $\tilde{B}_i(\mathbf{p}, r) = \{\mathbf{x} \in \tilde{K}_i : \langle \mathbf{p}, \mathbf{x} \rangle \leq r\}$, and the compactified demand subsets by

$$\tilde{D}_i(\mathbf{p}, r) = \{\mathbf{x} \in \tilde{B}_i(\mathbf{p}, r) : f_i(\mathbf{x}, \mathbf{p}) = \min_{\mathbf{y} \in \tilde{B}_i(\mathbf{p}, r)} f_i(\mathbf{y}, \mathbf{p})\}.$$

The compactified income functions by $\tilde{r}_i(\mathbf{p}) \leq r_i(\mathbf{p})$ are continuous. It is clear that any Walras pre-equilibrium $\{\bar{\mathbf{x}}, \bar{\mathbf{p}}\} \in \prod_{i=1}^n K_i \times P$ of the economy $\{K_i, C(i), f_i\}_{i \in N}$ is a Walras pre-equilibrium of the compactified economy $\{\tilde{K}_i, \tilde{C}(i), f_i\}_{i \in N}$.

Proposition 4.3. *If the map $\{\boldsymbol{x}, \boldsymbol{c}\} \in R^N \times C \mapsto \sum_{i \in N} (\boldsymbol{x}^i - \boldsymbol{c}) \in \boldsymbol{R}^l$ is proper and the subsets $K_i, C(i)$ and loss function $\boldsymbol{x}^i \mapsto f_i(\boldsymbol{x}^i, \boldsymbol{p})$ are convex, then any Walras equilibrium of the compactified economy $\{\tilde{K}_i, \tilde{C}(i), f_i\}_{i \in N}$ is a Walras equilibrium of the initial economy $\{K_i, C(i), f_i\}_{i \in N}$.*

Proof. Let $\{\bar{\mathbf{x}}, \bar{\mathbf{p}}\}$ = $\{\bar{\mathbf{x}}^1, \cdots, \bar{\mathbf{x}}^n, \bar{\mathbf{p}}\}$ be a Walras equilibrium of the compactified economy. Then $\bar{\mathbf{x}}^i \in \tilde{K}_i \subset K_i$ for each i and $\bar{\mathbf{c}} = \sum_{i=1}^n \bar{\mathbf{x}}^i \in \tilde{C} \subset C$. Thus $\bar{\mathbf{x}} \in X(N)$. This implies that for all $i \in N, \bar{\mathbf{x}}^i \in \pi_i X(N)$.
Furthermore, we detailed the proof as follows.

(a) we proved that $\langle \bar{\mathbf{p}}, \bar{\mathbf{c}} \rangle = \tilde{r}(\bar{\mathbf{p}}) = r(\bar{\mathbf{p}}) = \langle \bar{\mathbf{p}}, \mathbf{c} \rangle$. If it is false, then there exists $\mathbf{c} \in C$ such that $\langle \bar{\mathbf{p}}, \mathbf{c} \rangle > \langle \bar{\mathbf{p}}, \bar{\mathbf{c}} \rangle$. Suppose $\mathbf{c}_\alpha = (1-\alpha)\bar{\mathbf{c}} + \alpha \mathbf{c}$. Since C is convex and $\bar{\mathbf{c}} \in C$, it follows that $\mathbf{c}_\alpha \in C$. Since $\mathbf{c} = \sum_{i \in N} \mathbf{c}^i$ where $\mathbf{c}^i \in C(i)$ and $\bar{\mathbf{x}}^i \in \pi_i X(N)$, $\bar{\mathbf{x}}^i + \alpha(\mathbf{c}^i - \bar{\mathbf{x}}^i) \in C(i) \cap (\pi_i X(N) + B) = \tilde{C}_\alpha(i)$ for α that small enough. Since $\mathbf{c}_\alpha = \sum_{i \in N} (\bar{\mathbf{x}}^i + \alpha(\mathbf{c}^i - \bar{\mathbf{x}}^i)) \in \tilde{C}$. On the other hand, $\langle \bar{\mathbf{p}}, \mathbf{c}_\alpha \rangle = (1-\alpha)\langle \bar{\mathbf{p}}, \bar{\mathbf{c}} \rangle + \alpha \langle \bar{\mathbf{p}}, \mathbf{c} \rangle > \langle \bar{\mathbf{p}}, \bar{\mathbf{c}} \rangle$. This is a contradiction of the fact that $\langle \bar{\mathbf{p}}, \bar{\mathbf{c}} \rangle$ maximizes $\langle \bar{\mathbf{p}}, \cdot \rangle$ againts to $\tilde{C}$ (i.e that $\langle \bar{\mathbf{p}}, \bar{\mathbf{c}} \rangle = \tilde{r}(\mathbf{p})$).

(b) Since $\langle \bar{\mathbf{p}}, \bar{\mathbf{x}}^i \rangle \leq \tilde{r}_i(\bar{\mathbf{p}})$ and $\langle \bar{\mathbf{p}}, \bar{\mathbf{c}} \rangle = \tilde{r}(\bar{\mathbf{p}}) = r(\bar{\mathbf{p}}) = \sum_{i=1}^n r_i(\bar{\mathbf{p}})$, we obtain that

$$\sum_{i=1}^n (\tilde{r}_i(\bar{\mathbf{p}}) - r_i(\bar{\mathbf{p}})) = 0.$$

But $\tilde{r}_i(\mathbf{p}) \leq r_i(\mathbf{p})$ for each $i \in N$. This implies that

$$\tilde{r}_i(\bar{\mathbf{p}}) = r_i(\bar{\mathbf{p}}) \quad \text{for each} \quad i \in N.$$

(c) Therefore $\bar{\mathbf{x}}^i \in \tilde{D}_i(\bar{\mathbf{p}}, r_i(\bar{\mathbf{p}})) = \tilde{D}_i(\bar{\mathbf{p}}, \tilde{r}_i(\bar{\mathbf{p}}))$. We will prove that there exists $\bar{\mathbf{x}}^i \in D_i(\bar{\mathbf{p}}, r_i(\bar{\mathbf{p}}))$. To prove this, suppose that $\bar{\mathbf{x}}^i \notin D_i(\bar{\mathbf{p}}, \tilde{r}_i(\bar{\mathbf{p}}))$. Then there exists $\mathbf{x}^i \in D_i(\bar{\mathbf{p}}, \tilde{r}_i(\bar{\mathbf{p}}))$ such that $f_i(\mathbf{x}^i, \bar{\mathbf{p}}) < f_i(\bar{\mathbf{x}}^i, \bar{\mathbf{p}})$.

Therefore, for each $\alpha \in (0,1), \mathbf{x}^i_\alpha = (1-\alpha)\bar{\mathbf{x}}^i + \alpha \mathbf{x}^i$ satisfies the budget constraint $\langle \bar{\mathbf{p}}, \mathbf{x}^i \rangle \leq r_i(\bar{\mathbf{p}})$.

For α small enough, $\mathbf{x}^i_\alpha$ belong to $\pi_i X(N) + B$ because $\bar{\mathbf{x}}^i \in \pi_i X(N)$. Since $\mathbf{x}^i$ and $\bar{\mathbf{x}}^i$ belong to the convex set K_i, $\mathbf{x}^i_\alpha$also belongs to K_i. Therefore $\mathbf{x}^i_\alpha \in \tilde{B}_i(\bar{\mathbf{p}}, r_i(\bar{\mathbf{p}}))$ for α is small enough. But, since f_i is convex w.r.t $\mathbf{x}^i$, we deduce the following contradiction

$$f_i(\mathbf{x}^i_\alpha, \mathbf{p}) \leq (1-\alpha) f_i(\bar{\mathbf{x}}^i, \bar{\mathbf{p}}) + \alpha f_i(\mathbf{x}^i, \bar{\mathbf{p}}) < f_i(\bar{\mathbf{x}}^i, \bar{\mathbf{p}}).$$

This proof is complete. ∎

Proposition 4.4. *Let the map $\{\boldsymbol{x}, \boldsymbol{c}\} \in R^N \times C \mapsto \sum_{i \in N} (\boldsymbol{x}^i - \boldsymbol{c}) \in \boldsymbol{R}^l$ be a proper. If for each $i \in N, 0 \in K_i -$ Int$(C(i))$ holds in initial economy, then also hold in compactification economy, namely for each $i \in N, 0 \in \tilde{K}_i -$ Int$(\tilde{C}(i))$.*

Proof. From hypothesis is meaning for each $i \in N$, there exists $\tilde{\mathbf{x}}^i \in (K_i \bigcap \text{Int}(C(i))) \subset (K_i \bigcap C(i))$. Of course $\sum_{i=1}^n \tilde{\mathbf{x}}^i \in C$, that is $\tilde{\mathbf{x}} = (\{\tilde{\mathbf{x}}^1, \cdots, \tilde{\mathbf{x}}^n\}) \in X(N)$. This implies

$\tilde{\mathbf{x}}^i \in \pi_i X(N)$. Therefore, $\tilde{\mathbf{x}}^i \in \tilde{K}_i$. On the other hand, $\left(\tilde{\mathbf{x}}^i + B_\rho\right) \subset C(i)$ where B_ρ is ball with ρ is radius the small enough. Since $\left(\tilde{\mathbf{x}}^i + B_\rho\right) \subset \left(\pi_i X(N) + B\right)$ if the radius ρ is an enough small, then $\left(\tilde{\mathbf{x}}^i + B_\rho\right) \subset \tilde{C}(i)$, that is $\tilde{\mathbf{x}}^i$ belong to the interior of $\tilde{C}(i)$. ∎

The following theorem is the main result of the existence of a Walras equilibrium for a set-valued demand function which depends on the loss function.

Theorem 4.5. *Suppose that*

(1) *the consumption sets K_i closed, convex and bounded below,*
(2) *the initial endowments $C(i)$ are closed, convex and $\boldsymbol{R}^l_+$ is their recession cone, and*
(3) *for each $i \in N$, $0 \in (K_i \setminus Int(C(i)))$.*

Suppose also that

(4) *the map $\{\boldsymbol{x}, \boldsymbol{c}\} \in R^N \times C \mapsto \sum_{i\in N}(\boldsymbol{x}^i - \boldsymbol{c}) \in \boldsymbol{R}^l$ is proper and*
(5) *the loss function f_i satisfy*

$$\begin{cases} (i) \quad \forall \boldsymbol{p} \in P, \boldsymbol{x}^i \mapsto f_i(\boldsymbol{x}^i, \boldsymbol{p}) \quad \text{is convex} \\ (ii) \quad f_i \quad \text{is continuous on} \quad R^l \times P. \end{cases}$$

if the loss function f_i satisfy the non-satiation property, then there exists a Walras $\{\bar{\boldsymbol{x}}, \bar{\boldsymbol{p}}\}$.

Proof. Assumptions (1),(2),(4) and (5(i)) and Proposition 4.3 imply that we can replace the initial economy $\{K_i, C(i), f_i\}$ by the compactified economy $\{\tilde{K}_i, \tilde{C}(i), f_i\}$. Assumptions (3), (5) and Proposition 4.2 imply the existence of a Walras pre-equilibrium of the compactified economy. $\{\tilde{K}_i, \tilde{C}(i), f_i\}$.

Since the loss function f_i satisfy the non-satiation property and assumptions (5) imply that $\bar{\mathbf{x}}^i \in D_i(\bar{\mathbf{x}}, \tilde{r}_i(\bar{\mathbf{p}}))$ satisfies $\langle \bar{\mathbf{p}}, \bar{\mathbf{x}}^i \rangle = r_i(\bar{\mathbf{p}})$ by Proposition 2.3.

Hence there exists a Walras equilibrium of the compactified economy, which is a Walras equilibrium of the initial economy by Proposition 4.3. ∎

As in Theorem 3.1, the requirement of the upper semi-continuous for set-valued demand functions is essential. To prove the upper semi-continuity of such a demand set-valued, we need the compactness of the consumption sets K_i. Therefore, we have to compactify the economy. For this purpose, we introduce an assumption (4) in Theorem 4.5. Hence we construct a new economy in which the consumption subsets K_i are compact (see Proposition 4.1). We check that these two economies are equivalent in the sense that their subsets of equilibria coincide (Proposition4.3).

In the theorem 4.5, we conclude that the requirement of the upper semi-continuity of the demand set-valued functions that depend on the loss function is a compactified economy of the initial economy.

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