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NEW REFINEMENTS FOR INTEGRAL AND SUM FORMS OF GENERALIZED HÖLDER INEQUALITY FOR N TERM

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ABSTRACT. We know that in the field of functional analysis, Hölder inequality is very well known, important, and very applicable. So many researchers are interested in discussing these inequalities. Many world mathematicians try to improve these inequalities. In general, the Hölder inequality has two forms, namely the integral form and the sum form. In this paper, we will introduce a new refinement of the generalization of Hölder inequalities in both integral and addition forms. Especially in the sum form, improvements will be introduced that are better than the previous improvements that have been published by Jing-feng Tian, Ming-hu Ha, and Chao Wang.

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1. INTRODUCTION

The classical Hölder's inequalities are usually defined as follows.

Theorem 1.1. *(Hölder's inequality for integrals* [1]*) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[u, v]$ such that $|f|^p$ and $|g|^q$ are integrable functions on $[u, v]$, then*

$$\int_u^v |f(x)g(x)|dx \leq \Big(\int_u^v |f(x)|^p dx\Big)^{\frac{1}{p}} \Big(\int_u^v |g(x)|^q dx\Big)^{\frac{1}{q}} \tag{1.1}$$

with equality if and only if $a|f(x)|^p = b|g(x)|^q$ almost everywhere for some real numbers a and b (not both of them zero).

Theorem 1.2. *(Hölder's inequality for sums* [1]*) Let $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_n)$ be two positive n-tuples, and let $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$\sum_{n=1}^{m} |u_n v_n| \leq \Big(\sum_{n=1}^{m} |u_n|^p\Big)^{\frac{1}{p}} \Big(\sum_{n=1}^{m} |v_n|^q\Big)^{\frac{1}{q}} \tag{1.2}$$

with equality if and only if $a|u_n|^p = b|v_n|^q$ for some real numbers a and b (not both of them zero).

By utilizing the generalization of Young's inequality for n numbers [2], mathematicians obtain the Generalized Hölder's Inequality for n terms.

Theorem 1.3. *(Generalized Hölder's inequality for integrals and n term* [2]*) Let $p_i > 1$ and $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. If f_i are real functions defined on $[u, v]$ such that $|f_i|^{p_i}$ are integrable functions on $[u, v]$, then*

$$\int_u^v \Big|\prod_{i=1}^{n} f_i(x)\Big| dx \leq \prod_{i=1}^{n} \Big(\int_u^v |f_i(x)|^{p_i} dx\Big)^{\frac{1}{p_i}} \tag{1.3}$$

with equality if and only if there exist real numbers a and b (not both of them zero) such that $a|f_i(x)|^{p_i} = b|f_j(x)|^{p_j}$ almost everywhere for $i \neq j$.

Theorem 1.4. *(Generalized Hölder's inequality for sums and n term* [2]*) Let $u_i = (u_{i_1}, \ldots, u_{i_m})$ be a n-tuples for every $i \in \{1, \ldots, n\}$, and let $p_i > 1$ and $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. Then we have*

$$\sum_{k=1}^{m} \Big|\prod_{i=1}^{n} u_{i_k}\Big| \leq \prod_{i=1}^{n} \Big(\sum_{k=1}^{m} |u_{i_k}|^{p_i}\Big)^{\frac{1}{p_i}} \tag{1.4}$$

with equality if and only if there exist real numbers a and b (not both of them zero) such that $a|u_{i_n}|^{p_i} = b|u_{j_n}|^{p_j}$ for $i \neq j$.

Of course, the Hölder's inequality has been extensively explored and tested to a new situation by a number of scientists. Many its generalizations and refinements have been obtained so far. See, for example, [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], and [13] and the references therein. In this paper, we obtain some new refinements for integral and sum forms of Generalized Hölder's Inequality for n terms, and these refinements are better than before, especially the ones that Jing-feng Tian, Ming-hu Ha, and Chao Wang has published [11].

2. RESULT

2.1. **Refined Generalized Hölder's Inequality for integral form.**

Theorem 2.1. *Let* $p_i > 1$ *and* $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. *If* f_i *are real functions defined on* $[u, v]$ *such that* $|f_i|^{p_i}$ *are integrable functions on* $[u, v]$, *then*

$$\int_u^v \big|\prod_{i=1}^n f_i(x)\big|dx \leq \frac{1}{v-u}\Big[\prod_{i=1}^n \Big(\int_u^v (v-x)|f_i(x)|^{p_i}dx\Big)^{\frac{1}{p_i}} + \prod_{i=1}^n \Big(\int_u^v (x-u)|f_i(x)|^{p_i}dx\Big)^{\frac{1}{p_i}}\Big] \tag{2.1}$$

and

$$\frac{1}{v-u}\Big[\prod_{i=1}^n \Big(\int_u^v (v-x)|f_i(x)|^{p_i}dx\Big)^{\frac{1}{p_i}} + \prod_{i=1}^n \Big(\int_u^v (x-u)|f_i(x)|^{p_i}dx\Big)^{\frac{1}{p_i}}\Big] \leq \prod_{i=1}^n \Big(\int_u^v |f_i(x)|^{p_i}dx\Big)^{\frac{1}{p_i}} \tag{2.2}$$

with equality if and only if there exist real numbers a *and* b *(not both of them zero) such that* $a|f_i(x)|^{p_i} = b|f_j(x)|^{p_j}$ *almost everywhere for* $i \neq j$.

Proof. Using the Generized Hölders's Inequality (1.3), we easily see that

$$\begin{aligned}
\int_u^v \big|\prod_{i=1}^n f_i(x)\big|dx &= \frac{1}{v-u}\int_u^v (v-x+x-u)\big|\prod_{i=1}^n f_i(x)\big|dx \\
&= \frac{1}{v-u}\Big(\int_u^v (v-x)\big|\prod_{i=1}^n f_i(x)\big|dx + \int_u^v (x-u)\big|\prod_{i=1}^n f_i(x)\big|dx\Big) \\
&= \frac{1}{v-u}\Big(\int_u^v \Big|\Big(\prod_{i=1}^n (v-x)^{1/p_i}\Big)\Big(\prod_{i=1}^n f_i(x)\Big)\Big|dx \\
&\quad + \int_u^v \Big|\Big(\prod_{i=1}^n (x-u)^{1/p_i}\Big)\Big(\prod_{i=1}^n f_i(x)\Big)\Big|dx\Big) \\
&= \frac{1}{v-u}\Big(\int_u^v \Big|\Big(\prod_{i=1}^n (v-x)^{\frac{1}{p_i}}|f_i(x)|\Big)\Big|dx + \int_u^v \Big|\Big(\prod_{i=1}^n (x-u)^{\frac{1}{p_i}}|f_i(x)|\Big)\Big|dx\Big) \\
&\leq \frac{1}{v-u}\Big(\prod_{i=1}^n \Big(\int_u^v (v-x)|f_i(x)|^{p_i}dx\Big)^{\frac{1}{p_i}} + \prod_{i=1}^n \Big(\int_u^v (x-u)|f_i(x)|^{p_i}dx\Big)^{\frac{1}{p_i}}\Big)
\end{aligned}$$

First, Let us consider the case

$$\prod_{i=1}^n \Big(\int_u^v |f_i(x)|^{p_i}dx\Big)^{\frac{1}{p_i}} = 0$$

Then there is i_0 such that $f_{i_0}(x) = 0$ for almost every $x \in [u, v]$.
Thus we have

$$\int_u^v \big|\prod_{i=1}^n f_i(x)\big|dx = 0$$

Therefore inequality is trivial in this case.
Finally, we consider the case

$$I = \prod_{i=1}^n \Big(\int_u^v |f_i(x)|^{p_i}dx\Big)^{\frac{1}{p_i}} \neq 0$$

Then using generalized Young's inequality we get

$$\frac{1}{(v-u)I}\Big(\prod_{i=1}^n \Big(\int_u^v (v-x)|f_i(x)|^{p_i}dx\Big)^{\frac{1}{p_i}} + \prod_{i=1}^n \Big(\int_u^v (x-u)|f_i(x)|^{p_i}dx\Big)^{\frac{1}{p_i}}\Big)$$

$$
\begin{aligned}
&= \frac{1}{(v-u)}\Big(\prod_{i=1}^{n}\big(\frac{\int_u^v (v-x)|f_i(x)|^{p_i}dx}{\int_u^v |f_i(x)|^{p_i}dx}\big)^{\frac{1}{p_i}} + \prod_{i=1}^{n}\big(\frac{\int_u^v (x-u)|f_i(x)|^{p_i}dx}{\int_u^v |f_i(x)|^{p_i}dx}\big)^{\frac{1}{p_i}}\Big)\\
&\le \frac{1}{(v-u)}\Big(\sum_{i=1}^{n}\big(\frac{\int_u^v (v-x)|f_i(x)|^{p_i}dx}{p_i\int_u^v |f_i(x)|^{p_i}dx}\big) + \sum_{i=1}^{n}\big(\frac{\int_u^v (x-u)|f_i(x)|^{p_i}dx}{p_i\int_u^v |f_i(x)|^{p_i}dx}\big)\Big)\\
&= \frac{1}{(v-u)}\Big(\sum_{i=1}^{n}\big(\frac{\int_u^v (v-x)|f_i(x)|^{p_i}dx + \int_u^v (x-u)|f_i(x)|^{p_i}dx}{p_i\int_u^v |f_i(x)|^{p_i}dx}\big)\Big)\\
&= \sum_{i=1}^{n}\frac{1}{p_i}\\
&= 1.
\end{aligned}
$$

This completes the proof. ∎

More general versions of Theorem 2.1 are given in the following:

Theorem 2.2. *Let* $p_i > 1$ *and* $\sum_{i=1}^{n}\frac{1}{p_i} = 1$*. If* f_i *are real functions defined on* $[u,v]$ *such that* $|f_i|^{p_i}$ *are integrable functions on* $[u,v]$*, then*

$$\int_u^v \big|\prod_{i=1}^{n} f_i(x)\big|dx \le \prod_{i=1}^{n}\big(\int_u^v \alpha(x)|f_i(x)|^{p_i}dx\big)^{\frac{1}{p_i}} + \prod_{i=1}^{n}\big(\int_u^v \beta(x)|f_i(x)|^{p_i}dx\big)^{\frac{1}{p_i}} \tag{2.3}$$

where $\alpha,\beta : [u,v] \to [0,\infty)$ *are continuous functions such that* $\alpha(x)+\beta(x) = 1$*,* $x \in [u,v]$*.*

Proof. The proof of the theorem is easily seen by using a similar method as in the proof of Theorem 2.1. ∎

More general versions of Theorem 2.2 are given in the following:

Theorem 2.3. *Let* $p_i > 1$ *and* $\sum_{i=1}^{n}\frac{1}{p_i} = 1$*. If* f_i *are real functions defined on* $[u,v]$ *such that* $|f_i|^{p_i}$ *are integrable functions on* $[u,v]$*, then*

$$\int_u^v \big|\prod_{i=1}^{n} f_i(x)\big|dx \le \sum_{k=1}^{m}\big[\prod_{i=1}^{n}\big(\int_u^v \alpha_k(x)|f_i(x)|^{p_i}dx\big)^{\frac{1}{p_i}}\big] \le \prod_{i=1}^{n}\big(\int_u^v |f_i(x)|^{p_i}dx\big)^{\frac{1}{p_i}} \tag{2.4}$$

where $\alpha_k : [u,v] \to [0,\infty)$ *are continuous functions such that* $\sum_{k=1}^{m}\alpha_k(x) = 1$*,* $x \in [u,v]$*.*

Proof. The proof of the theorem is easily seen by using a similar method as in the proof of Theorem 2.1. Using the Generized Hölders's Inequality (1.3), we easily see that

$$\begin{aligned}
\int_u^v \big|\prod_{i=1}^n f_i(x)\big|dx &= \int_u^v \big(\sum_{k=1}^m \alpha_k(x)\big)\big|\prod_{i=1}^n f_i(x)\big|dx \\
&= \sum_{k=1}^m \int_u^v \big(\alpha_k(x)\big)\big|\prod_{i=1}^n f_i(x)\big|dx \\
&= \sum_{k=1}^m \int_u^v \big|\big(\alpha_k(x)\big)\prod_{i=1}^n f_i(x)\big|dx \\
&= \sum_{k=1}^m \int_u^v \big|\big(\prod_{i=1}^n \big(\alpha_k(x)\big)^{\frac{1}{p_i}}\big)\big(\prod_{i=1}^n f_i(x)\big)\big|dx \\
&= \sum_{k=1}^m \int_u^v \big|\big(\prod_{i=1}^n \big(\alpha_k(x)\big)^{\frac{1}{p_i}} f_i(x)\big)\big|dx \\
&\leq \sum_{k=1}^m \Big[\prod_{i=1}^n \big(\int_u^v \alpha_k(x)|f_i(x)|^{p_i}\big)^{\frac{1}{p_i}} dx\Big]
\end{aligned}$$

First, Let us consider the case

$$\prod_{i=1}^n \Big(\int_u^v |f_i(x)|^{p_i} dx\Big)^{\frac{1}{p_i}} = 0$$

Then there is i_0 such that $f_{i_0}(x) = 0$ for almost every $x \in [u, v]$.
Thus we have

$$\int_u^v \big|\prod_{i=1}^n f_i(x)\big|dx = 0$$

Therefore inequality is trivial in this case.
Finally, we consider the case

$$I = \prod_{i=1}^n \Big(\int_u^v |f_i(x)|^{p_i} dx\Big)^{\frac{1}{p_i}} \neq 0$$

Then using generalized Young's inequality we get

$$\begin{aligned}
\frac{1}{I}\sum_{k=1}^m \Big[\prod_{i=1}^n \big(\int_u^v \alpha_k(x)|f_i(x)|^{p_i} dx\big)^{\frac{1}{p_i}}\Big] &= \sum_{k=1}^m \Big[\prod_{i=1}^n \big(\frac{\int_u^v \alpha_k(x)|f_i(x)|^{p_i} dx}{\int_u^v |f_i(x)|^{p_i} dx}\big)^{\frac{1}{p_i}}\Big] \\
&\leq \sum_{k=1}^m \Big[\sum_{i=1}^n \big(\frac{\int_u^v \alpha_k(x)|f_i(x)|^{p_i} dx}{p_i \int_u^v |f_i(x)|^{p_i} dx}\big)\Big] \\
&= \sum_{i=1}^n \Big[\sum_{k=1}^m \big(\frac{\int_u^v \alpha_k(x)|f_i(x)|^{p_i} dx}{p_i \int_u^v |f_i(x)|^{p_i} dx}\big)\Big] \\
&= \sum_{i=1}^n \Big[\frac{\sum_{k=1}^m \big(\int_u^v \alpha_k(x)|f_i(x)|^{p_i} dx\big)}{p_i \int_u^v |f_i(x)|^{p_i} dx}\Big] \\
&= \sum_{i=1}^n \frac{1}{p_i} \\
&= 1.
\end{aligned}$$

This completes the proof. ∎

Remark 2.1. We easily see that the inequalities obtained in Theorem 2.2 are better than inequality (1.3).

Remark 2.2. Let $k \in \{1, 2, 3, \ldots, m\}$. By taking

$$\alpha_k(x) \begin{cases} \prod_{k=1}^{m-1} \sin^2 x, & \text{If } x = m \\ \cos^2 x \prod_{k=1}^{m-1} \sin^2 x, & \text{If } 1 < x < m \\ \cos^2 x, \text{ If} & \text{If } x = 1 \end{cases},$$

we have

$$\begin{aligned} \int_u^v \big|\prod_{i=1}^n f_i(x)\big| dx \;\; &\leq \;\; \prod_{i=1}^n \big(\int_u^v \cos^2 x |f_i(x)|^{p_i} dx\big)^{\frac{1}{p_i}} \\ &\quad + \sum_{k=2}^{m-1} \Big(\prod_{i=1}^n \big(\int_u^v \big(\cos^2 x \prod_{k=1}^{m-1} \sin^2 x\big) |f_i(x)|^{p_i} dx\big)^{\frac{1}{p_i}} \Big) \\ &\quad + \prod_{i=1}^n \big(\int_u^v \big(\prod_{k=1}^{m-1} \sin^2 x\big) |f_i(x)|^{p_i} dx\big)^{\frac{1}{p_i}}. \end{aligned}$$

2.2. **Refined Generalized Hölder's Inequality for sums form.** There is a refinement of Generalized Hölder's inequality introduced by Jing-feng Tian, Ming-hu Ha, and Chao Wang in [11] as follows.
Let $u_i = (u_{i_1}, \ldots, u_{i_m})$ be a n-tuples for every $i \in 1, \ldots, n$, let s be any given natural number $(1 \leq n \leq m)$, and let $p_i > 1$ and $\sum_{i=1}^n \frac{1}{p_i} = 1$. Then we have

$$\sum_{k=1}^m \big|\prod_{i=1}^n u_{i_k}\big| \leq \big[\prod_{i=1}^n \big(\sum_{k=1}^m |u_{i_k}|^{p_i}\big)^{\frac{1}{p_i}}\big] \Big[\prod_{i=1}^n \Big(1 - \big(\frac{|u_{i_s}|^{p_i}}{\sum_{j=1}^m |u_{i_j}|^{p_i}} - \frac{|u_{(i+1)_s}|^{p_{i+1}}}{\sum_{j=1}^m |u_{(i+1)_j}|^{p_{i+1}}}\big)^2\Big)^{\frac{1}{2p_i}} \Big] \tag{2.5}$$

Now, we will introduce another refinement of the generalization of the Hölder inequality.

Theorem 2.4. *Let $u_i = (u_{i_1}, \ldots, u_{i_m})$ be a n-tuples for every $i \in \{1, \ldots, n\}$, let s be any given natural number $(1 \leq n \leq m)$, and let $p_i > 1$ and $\sum_{i=1}^n \frac{1}{p_i} = 1$. Then we have*
(i)

$$\begin{aligned} \sum_{k=1}^m \big|\prod_{i=1}^n u_{i_k}\big| \leq \frac{1}{m}\Big(\big[\prod_{i=1}^n \big(\sum_{k=1}^m k|u_{i_k}|^{p_i}\big)^{\frac{1}{p_i}} + \prod_{i=1}^n \big(\sum_{k=1}^m (m-k)|u_{i_k}|^{p_i}\big)^{\frac{1}{p_i}} \big] \\ \Big[\prod_{i=1}^n \Big(1 - \big(\frac{|u_{i_s}|^{p_i}}{\sum_{j=1}^m |u_{i_j}|^{p_i}} - \frac{|u_{(i+1)_s}|^{p_{i+1}}}{\sum_{j=1}^m |u_{(i+1)_j}|^{p_{i+1}}}\big)^2\Big)^{\frac{1}{2p_i}} \Big]\Big) \end{aligned} \tag{2.6}$$

(ii)

$$\begin{aligned} \frac{1}{m}\Big(\big[\prod_{i=1}^n \big(\sum_{k=1}^m k|u_{i_k}|^{p_i}\big)^{\frac{1}{p_i}} + \prod_{i=1}^n \big(\sum_{k=1}^m (m-k)|u_{i_k}|^{p_i}\big)^{\frac{1}{p_i}} \big] \\ \Big[\prod_{i=1}^n \Big(1 - \big(\frac{|u_{i_s}|^{p_i}}{\sum_{j=1}^m |u_{i_j}|^{p_i}} - \frac{|u_{(i+1)_s}|^{p_{i+1}}}{\sum_{j=1}^m |u_{(i+1)_j}|^{p_{i+1}}}\big)^2\Big)^{\frac{1}{2p_i}} \Big]\Big) \leq \Big[\prod_{i=1}^n \big(\sum_{k=1}^m |u_{i_k}|^{p_i}\big)^{\frac{1}{p_i}} \Big] \\ \Big[\prod_{i=1}^n \Big(1 - \big(\frac{|u_{i_s}|^{p_i}}{\sum_{j=1}^m |u_{i_j}|^{p_i}} - \frac{|u_{(i+1)_s}|^{p_{i+1}}}{\sum_{j=1}^m |u_{(i+1)_j}|^{p_{i+1}}}\big)^2\Big)^{\frac{1}{2p_i}} \Big] \end{aligned} \tag{2.7}$$

(iii)

$$\frac{1}{m}\Big([\prod_{i=1}^{n}(\sum_{k=1}^{m}k|u_{i_k}|^{p_i})^{\frac{1}{p_i}}+\prod_{i=1}^{n}(\sum_{k=1}^{m}(m-k)|u_{i_k}|^{p_i})^{\frac{1}{p_i}}]$$

$$\Big[\prod_{i=1}^{n}\Big(1-(\frac{|u_{i_s}|^{p_i}}{\sum_{j=1}^{m}|u_{i_j}|^{p_i}}-\frac{|u_{(i+1)_s}|^{p_{i+1}}}{\sum_{j=1}^{m}|u_{(i+1)_j}|^{p_{i+1}}})^2\Big)^{\frac{1}{2p_i}}\Big]\Big)\leq\Big[\prod_{i=1}^{n}(\sum_{k=1}^{m}|u_{i_k}|^{p_i})^{\frac{1}{p_i}}\Big] \tag{2.8}$$

Proof. (i) Using the refinement of Generalized Hölder's inequality (2.5), we see that

$$\begin{aligned}
\sum_{k=1}^{m}|\prod_{i=1}^{n}u_{i_k}| &= \sum_{k=1}^{m}(\frac{k}{m}+\frac{m-k}{m})|\prod_{i=1}^{n}u_{i_k}| \\
&= \frac{1}{m}\Big[\sum_{k=1}^{m}k|\prod_{i=1}^{n}u_{i_k}|+\sum_{k=1}^{m}(m-k)|\prod_{i=1}^{n}u_{i_k}|\Big] \\
&= \frac{1}{m}\Big[\sum_{k=1}^{m}|\prod_{i=1}^{n}k^{\frac{1}{p_i}}u_{i_k}|+\sum_{k=1}^{m}(m-k)|\prod_{i=1}^{n}(m-k)^{\frac{1}{p_i}}u_{i_k}|\Big] \\
&\leq \frac{1}{m}\Big([\prod_{i=1}^{n}(\sum_{k=1}^{m}k|u_{i_k}|^{p_i})^{\frac{1}{p_i}}+\prod_{i=1}^{n}(\sum_{k=1}^{m}(m-k)|u_{i_k}|^{p_i})^{\frac{1}{p_i}}] \\
&\quad \Big[\prod_{i=1}^{n}\Big(1-(\frac{|u_{i_s}|^{p_i}}{\sum_{j=1}^{m}|u_{i_j}|^{p_i}}-\frac{|u_{(i+1)_s}|^{p_{i+1}}}{\sum_{j=1}^{m}|u_{(i+1)_j}|^{p_{i+1}}})^2\Big)^{\frac{1}{2p_i}}\Big]\Big)
\end{aligned}$$

(ii) Let us first consider the case

$$\prod_{i=1}^{n}\Big(\sum_{k=1}^{m}|u_{i_k}|^{p_i}\Big)^{\frac{1}{p_i}}=0$$

Then $u_{i_k}=0$ for $k=1,2,3,\dots,n$.
Finally, we consider the case

$$S=\prod_{i=1}^{n}\Big(\sum_{k=1}^{m}|u_{i_k}|^{p_i}\Big)^{\frac{1}{p_i}}\neq 0$$

Then

$$\begin{aligned}
&\frac{1}{mS}\Big(\prod_{i=1}^{n}(\sum_{k=1}^{n}k|u_{i_k}|^{p_i})^{\frac{1}{p_i}}+\prod_{i=1}^{n}(\sum_{k=1}^{n}(m-k)|u_{i_k}|^{p_i})^{\frac{1}{p_i}}\Big) \\
= {} & \frac{1}{m}\Big(\prod_{i=1}^{n}(\frac{\sum_{k=1}^{n}k|u_{i_k}|^{p_i}}{\sum_{k=1}^{n}|u_{i_k}|^{p_i}})^{\frac{1}{p_i}}+\prod_{i=1}^{n}(\frac{\sum_{k=1}^{n}(m-k)|u_{i_k}|^{p_i}}{\sum_{k=1}^{n}|u_{i_k}|^{p_i}})^{\frac{1}{p_i}}\Big) \\
\leq {} & \frac{1}{m}\Big(\sum_{i=1}^{n}(\frac{\sum_{k=1}^{n}k|u_{i_k}|^{p_i}}{p_i\sum_{k=1}^{n}|u_{i_k}|^{p_i}})+\sum_{i=1}^{n}(\frac{\sum_{k=1}^{n}(m-k)|u_{i_k}|^{p_i}}{p_i\sum_{k=1}^{n}|u_{i_k}|^{p_i}})\Big) \\
= {} & \sum_{i=1}^{n}\frac{1}{p_i} \\
= {} & 1.
\end{aligned}$$

Thus we get

$$\frac{1}{m}\Big(\prod_{i=1}^{n}(\sum_{k=1}^{m}k|u_{i_k}|^{p_i})^{\frac{1}{p_i}}+\prod_{i=1}^{n}(\sum_{k=1}^{m}(m-k)|u_{i_k}|^{p_i})^{\frac{1}{p_i}}\Big)T\leq\Big[\prod_{i=1}^{n}(\sum_{k=1}^{m}|u_{i_k}|^{p_i})^{\frac{1}{p_i}}\Big]T$$

with

$$T = \Big[\prod_{i=1}^{n}\Big(1-\big(\frac{|u_{i_s}|^{p_i}}{\sum_{j=1}^{m}|u_{i_j}|^{p_i}}-\frac{|u_{(i+1)_s}|^{p_{i+1}}}{\sum_{j=1}^{m}|u_{(i+1)_j}|^{p_{i+1}}}\big)^2\Big)^{\frac{1}{2p_i}}\Big]\Big)$$

(iii) Base on the inequality point (ii) and the fact that

$$\Big[\prod_{i=1}^{n}\Big(1-\big(\frac{|u_{i_s}|^{p_i}}{\sum_{j=1}^{m}|u_{i_j}|^{p_i}}-\frac{|u_{(i+1)_s}|^{p_{i+1}}}{\sum_{j=1}^{m}|u_{(i+1)_j}|^{p_{i+1}}}\big)^2\Big)^{\frac{1}{2p_i}}\Big] < 1,$$

we directly have

$$\frac{1}{m}\Big(\big[\prod_{i=1}^{n}\big(\sum_{k=1}^{m}k|u_{i_k}|^{p_i}\big)^{\frac{1}{p_i}}+\prod_{i=1}^{n}\big(\sum_{k=1}^{m}(m-k)|u_{i_k}|^{p_i}\big)^{\frac{1}{p_i}}\big]$$

$$\Big[\prod_{i=1}^{n}\Big(1-\big(\frac{|u_{i_s}|^{p_i}}{\sum_{j=1}^{m}|u_{i_j}|^{p_i}}-\frac{|u_{(i+1)_s}|^{p_{i+1}}}{\sum_{j=1}^{m}|u_{(i+1)_j}|^{p_{i+1}}}\big)^2\Big)^{\frac{1}{2p_i}}\Big]\Big) \le \Big[\prod_{i=1}^{n}\big(\sum_{k=1}^{m}|u_{i_k}|^{p_i}\big)^{\frac{1}{p_i}}\Big]$$

∎

Remark 2.3. In Theorem 2.4, the Inequality 2.7 and 2.8 shows that Inequality 2.6 is better than Inequality 2.5 and 1.4.

More general versions of Theorem 2.4 are given in the following:

Theorem 2.5. *Let $u_i = (u_{i_1}, \dots, u_{i_m})$ be n-tuples for every $i \in \{1, \dots, n\}$, and let $p_i > 1$ and $\sum_{i=1}^{n}\frac{1}{p_i} = 1$. If $c = (c_1, \dots, c_m)$ and $d = (d_1, \dots, d_n)$ be two n-tuples such that $c_k + d_k = 1$ for $k = 1, 2, 3, \dots, m$, then*

$$\begin{aligned}\sum_{k=1}^{m}\Big|\prod_{i=1}^{n}u_{i_k}\Big| \;\le\; & \Big[\prod_{i=1}^{n}\big(\sum_{k=1}^{m}|c_k||u_{i_k}|^{p_i}\big)^{\frac{1}{p_i}}+\prod_{i=1}^{n}\big(\sum_{k=1}^{m}|d_k||u_{i_k}|^{p_i}\big)^{\frac{1}{p_i}}\Big]\\ & \Big[\prod_{i=1}^{n}\Big(1-\big(\frac{|u_{i_s}|^{p_i}}{\sum_{j=1}^{m}|u_{i_j}|^{p_i}}-\frac{|u_{(i+1)_s}|^{p_{i+1}}}{\sum_{j=1}^{m}|u_{(i+1)_j}|^{p_{i+1}}}\big)^2\Big)^{\frac{1}{2p_i}}\Big]\end{aligned}$$

Theorem 2.6. *Let $u_i = (u_{i_1}, \dots, u_{i_m})$ be n-tuples for every $i \in \{1, \dots, n\}$, and let $p_i > 1$ and $\sum_{i=1}^{n}\frac{1}{p_i} = 1$. If $c_j = (c_{j_1}, \dots, c_{j_m})$ be n-tuples for every $j \in \{1, \dots, r\}$ such that $\sum_{j=1}^{r}c_{j_k} = 1$ for $k = 1, 2, 3, \dots, m$, then*

$$\begin{aligned}\sum_{k=1}^{m}\Big|\prod_{i=1}^{n}u_{i_k}\Big| \;\le\; & \Big[\sum_{k=1}^{r}\Big(\prod_{i=1}^{n}\big(\sum_{k=1}^{m}|c_{j_k}||u_{i_k}|^{p_i}\big)^{\frac{1}{p_i}}\Big)\Big]\\ & \Big[\prod_{i=1}^{n}\Big(1-\big(\frac{|u_{i_s}|^{p_i}}{\sum_{j=1}^{m}|u_{i_j}|^{p_i}}-\frac{|u_{(i+1)_s}|^{p_{i+1}}}{\sum_{j=1}^{m}|u_{(i+1)_j}|^{p_{i+1}}}\big)^2\Big)^{\frac{1}{2p_i}}\Big]\end{aligned}$$

Proof. The proof of the Theorem 2.5 and 2.6 are easily seen by using a similar method as in the proof of Theorem 2.4. ∎

Remark 2.4. We easily see that the inequalities obtained in Theorem 2.5 and Theorem 2.6 are better than Inequality (1.4).

3. CONCLUDING REMARKS

We have have successfully constructed several new refinements of the generalization of Hölder inequalities in both integral and addition forms. Especially in the sum form, our improvements are better than the previous ones that have been published by Jing-feng Tian, Ming-hu Ha, and Chao Wang.

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