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FEKETE SZEGÖ PROBLEM ON THE CLASS OF BAZILEVIČ FUNCTIONS $\mathcal{B}_1(\alpha)$ RELATED TO THE LEMNISCATE BERNOULLI

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ABSTRACT. We provide a sharp boundaries inequalities for Fekete Szegö problem $|a_3 - \mu a_2^2|$, the coefficients of logarithmic function $log\ f(z)/z$, and the coefficients of the inverse function $f(f'(w))$ on the Bazilevič functions $\mathcal{B}_1(\alpha)$ related to the Lemniscate Bernoulli on the unit disk $D = \{z : |z| < 1\}$. We obtained the result by using some properties of function with positive real part relates to coefficients problems.

Key words and phrases: Univalent; Bazilevič functions $\mathcal{B}_1(\alpha)$; Lemniscate Bernoulli; Subordination; Fekete Szegö problem; Logarithmic function; Inverse function.

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1. INTRODUCTION

Let S denote the class of analytic normalized univalent function f defined on the unit disk $D = \{z : |z| < 1\}$, and normalized by $f(0) = 0$ and $f^{'}(0) = 1$, and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

One of the classes of univalent function is Bazilevič function $B(\alpha)$. The class $B_1(\alpha) \subset B(\alpha)$ defined as follow was introduce in [6] and [12]. Then for $\alpha \geq 0, f \in B_1(\alpha) \subset S$ if and only if for $z \in D$ the equation gives,

$$Re\left[f^{'}(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}\right] \geq 0. \tag{1.2}$$

The previous research on Bazilevič function was investigated by Singh [11]. The class Bazilevič function $B_1(\alpha)$ research by [13], [6], and [8]. While research on subordination was studied by Marjono [9] and Starlike function related Bernoulli Lemniscate studied by Thomas [10]. This research studied the Bazilevič functions $\mathcal{B}_1(\alpha)$ related to the Lemniscate Bernoulli.

We say that an analytic function f is subordinate to an analytic function g, and write $f(z) \prec g(z)$, if and only if there exists a function ω, analytic in D, such that $\omega(0) = 0, |\omega(z)| < 1$ for $|z| < 1$ and $f(z) = g(\omega(z))$.

Lets $B_1(\alpha)$ denote the class of function f, analytic in the unit disk D, normalized by $f(0) = 0$ and $f^{'}(0) = 1$ and satisfying subordination condition

$$\left[\frac{f^{'}(z)\ f(z)^{\alpha-1}}{z^{\alpha-1}}\right] \prec \sqrt{1+z} =: \xi(z),\ z \in D. \tag{1.3}$$

Where the branch of the square root is chosen to be $\xi(0) = 1$ the set $\xi(D)$ lies in the region bounded the right loop of the Bernoulli Lemniscate function is $(x^2 + y^2)^2 - a^2(x^2 - y^2)) = 0$. In the Bernoulli Lemniscate function, for the positive real part it is multiplied by δ. When $\delta \geq 0$ and the class Bazilevič function $B_1(\alpha)$ subordinate with Bernoulli Lemniscate function we have,

$$\left[\frac{f^{'}(z)\ f(z)^{\alpha-1}}{z^{\alpha-1}}\right] \prec \sqrt{1 + \frac{\delta p(z) - 1}{\delta p(z) + 1}},\ z \in D. \tag{1.4}$$

The form of the Lemniscate Bernoulli will be depended on the value of positive real δ. Previous research Fekete Szegö problem for sharp boundaries on univalent function in [6].
Here, we carry out research carried out according to (1.4) and we have initial coefficients, so we determine sharp boundaries for Fekete Szegö problem, and we also work on the Logarithmic and Inverse function.
A research about Fekete Szegö problem for sharp boundaries, let $f \in B_1(\alpha)$ for $\alpha \geq 0$ and $\delta \geq 0$, gives

$$|a_3 - \mu a_2^2|. \tag{1.5}$$

For Logarithmic function let $f \in B_1(\alpha)$, $\alpha \geq 0$ and $\delta \geq 0$, gives

$$log\frac{f(z)}{z} = 2\sum_{n=2}^{\infty} \gamma_n z^n. \tag{1.6}$$

The coefficients γ_n is an important part of the univalent function. For example $f \in S$ the inequality obtained gives,

$$\sum_{m=1}^{n}\sum_{k=1}^{m}(k|\gamma_k|^2 - \frac{1}{k}) \leq 0. \tag{1.7}$$

Produce the conjecture Bieberbach for $f \in S$ and $n \geq 2$. Proof the conjecture Bieberbach by Branges [3].

Function in 1.1, we have $f \in S$ has inverse function f^{-1} analytic on $|\omega| < r_0(f)$ where $r_0(f) \geqslant \frac{1}{4}$. Let $f \in B_1(\alpha)$, $\alpha \geq 0$ and $\delta \geq 0$, gives

$$f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} A_n \omega^n < r_0(f). \tag{1.8}$$

All functions work on the Bazilevič functions $\mathcal{B}_1(\alpha)$ related to the Lemniscate Bernoulli on the unit disk $D = \{z : |z| < 1\}$.

2. PRELIMINARY LEMMAS

There are, we have some lemmas and definitions used to solve the problem of determining sharp boundaries in this research.

Denote by $\mathcal{P}$ the class f function p satisfying $Re(p(z)) > 0$ for $z \in \mathbb{D} = \{z : |z| < 1\}$ with Taylor series [1],

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{2.1}$$

We need some Lemmas which can be seen in e.g [6], [7], [9], [10], and [13].

Lemma 2.1. *If $p \in \mathcal{P}$ analytic in $\mathbb{D}$ with $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ for $n \geq 1$ then*

$$|p_n| \leq 2, \tag{2.2}$$

with $p(z) = (1+z)/(1-z)$. This Lemma is known as inequality Caratheodory-Toeplitz.

Lemma 2.2. *If $p \in \mathcal{P}$, with coefficient p_n then $|p_n| \leq 2$ for $n \geq 1$ and*

$$|p_2 - \frac{\mu}{2}p_1^2| \leq max\{2, 2|\mu - 1|\} = \begin{cases} 2 & , \quad 0 \leq \mu \leq 2 \\ 2|\mu - 1| & ; \quad \textit{elsewhere}. \end{cases} \tag{2.3}$$

Lemma 2.3. *If $p \in \mathcal{P}$, $0 \leq B \leq 1$ and $B(2B-1) \leq D \leq B$ then*

$$|p_1 - 2Bp_1p_2 + Dp_1^3| \leq 2. \tag{2.4}$$

We also have Theorem 2.4 about modulus coefficients inequality in 1.3. The proof this theorem can be seen in [11].

Theorem 2.4. *If $f \in B_1(\alpha)$ where $\alpha \geq 0$ and $\delta \geq 0$ by 1.4 then,*

$$\begin{aligned}
|a_2| &\leq \frac{2\sqrt{\alpha}}{\sqrt{2}(1+\delta)^{3/2}},\ if\ \alpha \geq 0\ and\ \delta \geq 0, \\
|a_3| &\leq \frac{\sqrt{\delta}\sqrt{2}}{(2+\alpha)(1+\delta)^3/2},\ if\ 0 \leq \alpha \leq 1,596\ and\ \delta \geq 0, \\
&\leq \frac{\sqrt{\delta}\sqrt{2}}{(2+\alpha)(1+\delta)^3/2},\ if\ \alpha > 1,596\ and\ 0 \leq \delta \leq \frac{9}{-1-8\alpha-6\alpha^2+4\alpha^3+2\alpha^4}, \\
&\leq \frac{2\sqrt{\delta}\, 4\sqrt{2}(1+\delta)^2}{8(2+\alpha)(1+\delta)^{7/2}},\ if\ \alpha > 1,596\ and\ \delta \geq \frac{9}{-1-8\alpha-6\alpha^2+4\alpha^3+2\alpha^4}. \quad (2.5)
\end{aligned}$$

All inequalities are sharp.

3. FEKETE SZEGÖ PROBLEM

Theorem 3.1. *If $f \in B_1(\alpha)$ where $\alpha \geq 0$, $\delta \geq 0$ and $T \in \mathbb{R}$,*

$$|a_3 - Ta_2^2| \leq \frac{\sqrt{2}\sqrt{\delta}}{(2+\alpha)(1+\delta)^{3/2}}\ for\ some\ conditions\ of\ T,$$

(3.1)

$$\begin{array}{ll}
T < \frac{1}{8}(4-3\sqrt{6}), & ,\ if\ 0 \leq \alpha < \alpha_1(T,\alpha)\ and\ 0 < \delta \leq \delta_1(T,\alpha),\ \delta \geq \delta_2(T,\alpha) \\
 & ,\ if\ \alpha_1(T,\alpha) \leq \alpha \leq \alpha_2(T)\ and\ \delta > 0 \\
 & ,\ if\ \alpha > \alpha_2(T)\ and\ 0 < \delta \leq \delta_3(T,\alpha), \\
T = \frac{1}{8}(4-3\sqrt{6}), & ,\ if\ \alpha = 0\ and\ 0 < \delta \leq \delta_4(T)\ and\ \delta \geq \delta_5 \\
 & ,\ if\ 0 < \alpha < \alpha_1(T)\ and\ 0 < \delta \leq \delta_1(T,\alpha)\ and\ \delta \geq \delta_2(T,\alpha) \\
 & ,\ if\ \alpha_1(T) \leq \alpha \leq \alpha_2(T)\ and\ \delta > 0 \\
 & ,\ if\ \alpha > \alpha_2(T)\ and\ 0 < \delta \leq \delta_3(T,\alpha), \\
\frac{1}{8}(4-3\sqrt{6}) < T < -0,27, & ,\ if\ 0 \leq \alpha < \alpha_1(T)\ and\ 0 < \delta \leq \delta_1(T),\ and\ \delta \geq \delta_2(T,\alpha) \\
 & ,\ if\ \alpha_1(T) \leq \alpha \leq \alpha_2(T)\ and\ \delta > 0 \\
 & ,\ if\ \alpha > \alpha_2(T)\ and\ 0 < \delta \leq \delta_3(T,\alpha), \\
-0,27 \leq T \leq 0,0435, & ,\ if\ 0 \leq \alpha \leq \alpha_2(T)\ and\ \delta > 0 \\
 & ,\ if\ \alpha > \alpha_2(T)\ and\ 0 < \delta \leq \delta_3(T,\alpha), \\
0,0435 < T < \frac{1}{8}(4+3\sqrt{2}), & ,\ if\ 0 \leq \alpha \leq \alpha_2(T)\ and\ \delta > 0 \\
 & ,\ if\ \alpha > \alpha_2(T)\ and\ 0 < \delta \leq \delta_3(T,\alpha), \\
T = \frac{1}{8}(4+3\sqrt{2}), & ,\ if\ \alpha = 0\ and\ \delta > 0 \\
 & ,\ if\ \alpha > 0\ and\ 0 < \delta \leq \delta_6(\alpha), \\
T > \frac{1}{8}(4+3\sqrt{2}), & ,\ if\ \alpha \geq 0\ and\ 0 < \delta \leq \delta_3(T,\alpha),
\end{array}$$

where,
$\alpha_1(T)$ *is the smallest positive root of*

$$[-107+64T+960T^2-2048T^3+1024T^4+(16+512T-576T^2-2048T^3 \\ +2048T^4)x+(76+336T-2448T^2+1024T^3+1536T^4)x^2+(88-656T-768T^2 \\ -2048T^3+512T^4)x^3+(-32-544T+864T^2-896T^3+64T^4)x^4 \\ +(-80+96T+576T^2-2048128T^3)x^5 \\ +(-8+160T+96T^2)x^6+(16+32T)x^7+4x^8]=0,$$

$\alpha_2(T)$ *is the smallest positive root of*

$$[-1-32T+32T^2+(-8+32T^2)x+(-6+24T+8T^2)x^2+(4+8T)x^3+2x^4]=0,$$

$\delta_1(T,\alpha)$ *is the smallest positive root of*

$$[1+(1+32T-32T^2+8\alpha-32T^2\alpha+6\alpha^2-24T\alpha^2-8T^2\alpha^2-4\alpha^3-8T\alpha^3-2\alpha^4)x \\ +24x^2+16x^3]=0,$$

$\delta_2(T,\alpha)$ *is the largest positive root of*

$$[1+(1+32T-32T^2+8\alpha-32T^2\alpha+6\alpha^2-24T\alpha^2-8T^2\alpha^2-4\alpha^3-8T\alpha^3-2\alpha^4)x \\ +24x^2+16x^3]=0,$$

$$\delta_3(T,\alpha)=\frac{9}{-1-32T+32T^2-8\alpha+32T^2\alpha-6\alpha^2+24T\alpha^2+8T^2\alpha^2+4\alpha^3+8T\alpha^3+2\alpha^4},$$

$\delta_4(T)$ *is the smallest positive root of*

$$[1+(1+32T-32T^2)x+24x^2+16x^3]=0,$$

$\delta_5(T)=9/(-1-32T+32T^2)$,

$$\delta_6(\alpha) = 9/[-1-4(4+\sqrt{2})+\frac{1}{2}(4+\sqrt{2})^2-8\alpha+\frac{1}{2}(4+\sqrt{2})^2\alpha-6\alpha^2+3(4+\sqrt{2})^2 \\ +\frac{1}{8}(4+\sqrt{2})^2\alpha^2+4\alpha^3+(4+\sqrt{2})\alpha^3+2\alpha^4],$$

and,

$$|a_3-Ta_2^2|\leq\frac{4(\sqrt{\delta}+5\delta^{5/2}+4\delta^{5/2}+2(2+\alpha)(-1+2T+\alpha\delta\sqrt{1+\delta})}{(2+\alpha)(1+\delta)^{3/2}},\ elsewhere.$$

All inequalities are sharp.

Proof. We have initial coefficients from 1.4 and can determine the boundaries for Fekete Szego Problem from 1.5, gives,

$$|a_3-Ta_2^2| = \frac{\sqrt{\delta}\,4\sqrt{2}(1+\delta)^2}{8(2+\alpha)(1+\delta)^{7/2}} \\ \left|p_2-\frac{\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2+2(2+\alpha)(-1+2T+\alpha)\sqrt{\delta}\sqrt{1+\delta}}{4\sqrt{2}(1+\delta)^2}\,p_1^2\right|,$$

We aplly Lemma 2.3 with,

$$\mu = \frac{\sqrt{2} + 5\sqrt{2}\delta + 4\sqrt{2}\delta^2 + 2(2+\alpha)(-1+2T+\alpha)\sqrt{\delta}\sqrt{1+\delta}}{2\sqrt{2}(1+\delta)^2}.$$

For some conditions of T in Theorem 3.1 is proof and the inequality for $|a_3 - Ta_2^2|$ easily follow. We have two conditions for boundaries inequality of T. The first inequality is sharp when $p_1 = 0$ and $p_2 = 2$. The second inequality is sharp when $p_1 = p_2 = 2$.
The proof is completed. ∎

4. INVERSE FUNCTION

Theorem 4.1. *If $f \in B_1(\alpha)$ for $\alpha \geq 0$ and $\delta \geq 0$ then,*

$$\begin{aligned}
|A_2| &\leq -\frac{2\sqrt{\delta}}{\sqrt{2}(1+\delta)^{3/2}}, \; for\ \alpha \geq 0\ and\ \delta \geq 0, \\
|A_3| &\leq -\frac{\sqrt{\delta} - 4\sqrt{2}(1+\delta)^2}{4(2+\alpha)(1+\delta)^{7/2}}, \; for\ some\ conditions\ of\ \alpha\ and\ \delta,
\end{aligned}$$

$$\begin{array}{l}
0 \leq \alpha \leq 0,140266\ and\ \delta_1(\alpha) \leq \delta \leq \delta_2(\alpha), \\
0,140 \leq \alpha \leq 0,504\ and\ \delta_1(\alpha) \leq \delta \leq \delta_3(\alpha),\ and\ \delta_4(\alpha) \leq \delta \leq \delta_2(\alpha), \\
0,504 \leq \alpha \leq 2,496\ and\ \delta_1(\alpha) \leq \delta \leq \delta_3(\alpha),\ and\ \delta_5(\alpha) \leq \delta \leq \delta_2(\alpha), \\
2,496 \leq \alpha \leq 2,859\ and\ \delta_1(\alpha) \leq \delta \leq \delta_3(\alpha),\ and\ \delta_4(\alpha) \leq \delta \leq \delta_2(\alpha), \\
2,859 \leq \alpha \leq 4,523\ and\ \delta_1(\alpha) \leq \delta \leq \delta_2(\alpha),
\end{array} \tag{4.1}$$

and

$$|A_3| \leq \frac{-6\sqrt{2} + \sqrt{\delta}(1 + 40\sqrt{1+\delta} - 4(-3+\alpha)\alpha\sqrt{1+\alpha} - 6\sqrt{2}\sqrt{\delta}(3+2\delta))}{4(2+\alpha)(1+\delta)^{7/2}}, \; elsewhere,$$

where,

$\delta_1(\alpha)$ *is the smallest positive root of*

$$[1 + (-191 - 120\alpha + 22\alpha^2 + 12\alpha^3 - 2\alpha^4)x + 24x^2 + 16x^3] = 0,$$

$\delta_2(\alpha)$ *is the largest positive root of*

$$[1 + (-191 - 120\alpha + 22\alpha^2 + 12\alpha^3 - 2\alpha^4)x + 24x^2 + 16x^3] = 0,$$

$\delta_3(\alpha)$ *is the smallest positive root of*

$$\begin{aligned}
&[2500 + (-14100 - 2400\alpha + 4400\alpha^2 + 2400\alpha^3 - 400\alpha^4)x + (28081 - 42720\alpha \\
&+4768\alpha^2 - 25392\alpha^3 - 8872\alpha^4 + 4032\alpha^5 + 224\alpha^6 - 192\alpha^7 + 16\alpha^8)x^2 + (12364 \\
&-19680\alpha + 11592\alpha^2 - 44208\alpha^3 - 18840\alpha^4 + 8064\alpha^5 + 448\alpha^6 - 384\alpha^7 + 32\alpha^8)x^3 \\
&+(-94908 = 246720\alpha + 102832\alpha^2 + 3552\alpha^3 - 13696\alpha^4 + 4032\alpha^5 + 224\alpha^6 - 192\alpha^7 \\
&+16\alpha^8)x^4 + (45440 - 261120\alpha + 47872\alpha^2 + 26112\alpha^3 - 4352\alpha^4+)x^5 \\
&+(198144 - 61440\alpha + 11624\alpha^2 + 6144\alpha^3 - 1024\alpha^4)x^6 + 106496x^7 + 16384x^8] = 0,
\end{aligned}$$

$\delta_4(\alpha)$ *is the smallest positive root of*

$$[2500 + (-14100 - 2400\alpha + 4400\alpha^2 + 2400\alpha^3 - 400\alpha^4)x + (28081 - 42720\alpha$$
$$+4768\alpha^2 - 25392\alpha^3 - 8872\alpha^4 + 4032\alpha^5 + 224\alpha^6 - 192\alpha^7 + 16\alpha^8)x^2 + (12364$$
$$-19680\alpha + 11592\alpha^2 - 44208\alpha^3 - 18840\alpha^4 + 8064\alpha^5 + 448\alpha^6 - 384\alpha^7 + 32\alpha^8)x^3$$
$$+(-94908 + 246720\alpha + 102832\alpha^2 + 3552\alpha^3 - 13696\alpha^4 + 4032\alpha^5 + 224\alpha^6 - 192\alpha^7$$
$$+16\alpha^8)x^4 + (45440 - 261120\alpha + 47872\alpha^2 + 26112\alpha^3 - 4352\alpha^4+)x^5$$
$$+(198144 - 61440\alpha + 11624\alpha^2 + 6144\alpha^3 - 1024\alpha^4)x^6 + 106496x^7 + 16384x^8] = 0,$$

$\delta_5(\alpha)$ *is the smallest positive root of*

$$[2500 + (-14100 - 2400\alpha + 4400\alpha^2 + 2400\alpha^3 - 400\alpha^4)x + (28081 - 42720\alpha$$
$$+4768\alpha^2 - 25392\alpha^3 - 8872\alpha^4 + 4032\alpha^5 + 224\alpha^6 - 192\alpha^7 + 16\alpha^8)x^2 + (12364$$
$$-19680\alpha + 11592\alpha^2 - 44208\alpha^3 - 18840\alpha^4 + 8064\alpha^5 + 448\alpha^6 - 384\alpha^7 + 32\alpha^8)x^3$$
$$+(-94908 + 246720\alpha + 102832\alpha^2 + 3552\alpha^3 - 13696\alpha^4 + 4032\alpha^5 + 224\alpha^6$$
$$-192\alpha^7 + 16\alpha^8)x^4 + (45440 - 261120\alpha + 47872\alpha^2 + 26112\alpha^3 - 4352\alpha^4+)x^5$$
$$+(198144 - 61440\alpha + 11624\alpha^2 + 6144\alpha^3 - 1024\alpha^4)x^6 + 106496x^7 + 16384x^8] = 0.$$

All inequalities are sharp.

Proof. From 1.8 we equating inverse coefficient. It is known that $f \in B_1(\alpha)$ and $f^{-1} \in B_1(\alpha)$, $f(f^{-1}(\omega)) = \omega$. We have initial coefficients from 1.4 , gives

$$A_2 = -a_2,$$

and

$$A_3 = 2a_2^2 - a_3,$$

Following initial coefficients a_2 and a_3, next we have A_2 and A_3.

For

$$A_2 = -\frac{p_1\sqrt{\delta}}{\sqrt{2}(1+\delta)^{3/2}}, \tag{4.2}$$

we used Lemma 2.1 determine sharp bound for $|A_2|$. Since $|p_n| \leq 2$ for $\alpha \geq 0$ gives,

$$|A_2| = -\frac{2\sqrt{\delta}}{\sqrt{2}(1+\delta)^{3/2}}. \tag{4.3}$$

The inequality for $|A_2|$ is sharp when $p_1 = 2$.

Next for A_3 let,

$$\begin{aligned} A_3 &= \frac{1}{8(2+\alpha)(1+\delta)^{7/2}}\Bigg[(\sqrt{\delta} - 4\sqrt{2}\,(1+\delta)^2\,p_2 \\ &\quad +(\sqrt{2} + 5\sqrt{2}\,\delta + 4\sqrt{2}\delta^2) + 2(-10 - 3\alpha + \alpha^2)\sqrt{\delta}\sqrt{1+\delta}\,p_1^2)\Bigg]. \end{aligned} \tag{4.4}$$

By using Lemma 2.2 we determined sharp boundaries for $|A_3|$, gives,

$$|A_3| = \frac{\sqrt{\delta}-4\sqrt{2}(1+\delta)^2}{8(2+\alpha)(1+\delta)^{7/2}}\,|p_2 - \frac{\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2+2(-10-3\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta}}{\sqrt{\delta}-4\sqrt{2}(1+\delta)^2}p_1^2|, \tag{4.5}$$

with,

$$\mu = \frac{2(\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2+2(-10-3\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta})}{\sqrt{\delta}-4\sqrt{2}(1+\delta)^2}.$$

So that provider for some conditions of α and δ in Theorem 4.1, we apply Lemma 2.2 to get the bound of $|A_3|$. For $|A_3|$ the first boundaries inequality is sharp when $p_1 = 0$ and $p_2 = 2$. The second boundaries inequality for $|A_3|$ is sharp when $p_1 = p_2 = 2$.
The proof is completed. ∎

5. LOGARITHMIC FUNCTION

Theorem 5.1. *If $f \in B_1(\alpha)$ where $\alpha \geq 0$ and $\delta \geq 0$ then,*

$$\begin{aligned}
|\gamma_1| &\leq \frac{2\sqrt{\delta}}{\sqrt{2}(1+\delta)^{3/2}},\ for\ \alpha \geq 0\ and\ \delta \geq 0,\\
|\gamma_2| &\leq \frac{\sqrt{\delta}}{\sqrt{2}(2+\alpha)(1+\delta)^{3/2}},\ for\ 0 \leq \alpha \leq 0,7667\ and\ \delta > 0,\\
&\leq \frac{\sqrt{\delta}}{\sqrt{2}(2+\alpha)(1+\delta)^{3/2}}, for\ \alpha \geq 0,7667\ and\ 0 < \delta \leq \frac{9}{-9+8\alpha_2+8\alpha_3+2\alpha_4},\\
&\leq \frac{-\sqrt{2}-5\sqrt{2}\delta-4\sqrt{2}\delta^2-2(1+\alpha)(2+\alpha)^2\sqrt{\delta}\sqrt{1+\delta}+4\sqrt{2}\sqrt{\delta}(1+\delta)^2}{8(2+\alpha)(1+\delta)^{7/2}},\ elsewhere.
\end{aligned} \tag{5.1}$$

All inequalities are sharp.

Proof. From 1.7 with the initial coefficient we gives,

$$\begin{aligned}
\gamma_1 &= \frac{a_2}{2},\\
\gamma_2 &= \frac{1}{2}(a_3 - \frac{1}{2}a_2^2).
\end{aligned}$$

For $|\gamma_1|$, follow from 2.1, $|p_n| \leq 2$ we obtain,

$$|\gamma_1| \leq \frac{2\sqrt{\delta}}{\sqrt{2}(1+\delta)^{3/2}}. \tag{5.2}$$

For $|\gamma_2|$ following 2.2 we have,

$$|\gamma_2| = \frac{\sqrt{\delta}\,4\sqrt{2}(1+\delta)^2}{16(2+\alpha)(1+\delta)^{7/2}}\,|p_2 - \frac{\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2+2+2(2+3\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta}}{4\sqrt{2}(1+\delta)^2}p_1^2|,$$

with,

$$\mu = \frac{\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2+2(2+3\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta}}{2\sqrt{2}(1+\delta)^2}.$$

So that, it is provided when $0 \le \alpha \le 0,7667$ and $\delta > 0$, and when $\alpha \ge 0,7667$ and

$$0 < \delta \le \frac{9}{-9+8\alpha_2+8\alpha_3+2\alpha_4}$$

in 5.1. For $|\gamma_2|$ we have the first boundaries inequality sharp when $p_1 = 0$ and $p_2 = 2$. The second boundaries inequality sharp when $p_1 = p_2 = 2$.
The proof is completed. ∎

6. DISCUSSION

The best from of $|a_4|$ still be discussed to have precisely simpler form and to obtain the upper bound of determinant matrix's Hankel, let $H_2(2)$.

If $f \in B_1(\alpha)$ for $\alpha \ge 0$ and $\delta \ge 0$ given by 1.4, gives,

$$\begin{aligned} a_4 &= \left(\frac{1}{48(2+\alpha)(1+\alpha)^{9/2}}\right)\Big[\sqrt{\alpha}(24\sqrt{2}p_3(2+\alpha)(1+\delta)^3 - 12p_1p_2(1+\delta) \\ &\quad (2\alpha^2\sqrt{\delta}\sqrt{1+\delta}+2(\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2-3\sqrt{\delta}\sqrt{1+\delta})+\alpha(\sqrt{2}+5\sqrt{2}\delta \\ &\quad +4\sqrt{2}\delta^2-4\sqrt{\delta}\sqrt{1+\delta}))+p_1^3(14\sqrt{2}\alpha^3\delta+4\sqrt{2}\alpha^4\delta \\ &\quad +6(\sqrt{2}+7\sqrt{2}\delta+12\sqrt{2}\delta^2+8\sqrt{2}\delta^3-3\sqrt{\delta}\sqrt{1+\delta}-12\delta^{3/2}\sqrt{1+\delta}) \\ &\quad +\delta^2(-4\sqrt{2}\delta+6\sqrt{\delta}\sqrt{1+\delta}+24\delta^{3/2}\sqrt{1+\delta})+\alpha(3\sqrt{2}-11\sqrt{2}\delta \\ &\quad +36\sqrt{2}\delta^2+24\sqrt{2}\delta^3+12\delta\sqrt{1+\delta}+48\delta^{3/2}\sqrt{1+\delta})))\Big]. \end{aligned}$$

Further simplified according Lemma 2.3 we give,

$$\begin{aligned} |a_4| &= \frac{\sqrt{\delta}(24\sqrt{2}(2+\alpha)(1+\alpha)^3)}{48(2+\alpha)(1+\alpha)^{9/2}}\Big|p_3 - (((24\alpha\delta(1+\delta)^{3/2}+2(\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2 \\ &\quad -3\sqrt{\delta}\sqrt{1+\delta})+\alpha(\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2-4\sqrt{\delta}\sqrt{1+\delta}))/\sqrt{\delta}(24\sqrt{2}(2+\alpha) \\ &\quad (1+\alpha)^3))\,p_1p_2 + \frac{1}{24\sqrt{2}(2+\alpha)(1+\alpha)^3}((14\sqrt{2}\alpha^3\delta+4\sqrt{2}\alpha^4\delta+6(\sqrt{2}+7\sqrt{2}\delta \\ &\quad +12\sqrt{2}\delta^2+8\sqrt{2}\delta^3-3\sqrt{\delta}\sqrt{1+\delta}-12\delta^{3/2}\sqrt{1+\delta})+\delta^2(-4\sqrt{2}\delta+6\sqrt{\delta}\sqrt{1+\delta} \\ &\quad +24\delta^{3/2}\sqrt{1+\delta})+\alpha(3\sqrt{2}-11\sqrt{2}\delta+36\sqrt{2}\delta^2 \\ &\quad +24\sqrt{2}\delta^3+12\delta\sqrt{1+\delta}+48\delta^{3/2}\sqrt{1+\delta}))\,p_1^3\Big|, \end{aligned} \tag{6.1}$$

with

$$\begin{aligned} B &= \frac{1}{12\sqrt{2}\sqrt{\delta}(2+\alpha)(1+\alpha)^3}\Big[24\alpha\delta(1+\delta)^{3/2}+2(\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2-3\sqrt{\delta}\sqrt{1+\delta}) \\ &\quad +\alpha(\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2-4\sqrt{\delta}\sqrt{1+\delta})\Big], \end{aligned}$$

and

$$\begin{aligned}
D \;=\; & \frac{1}{24\sqrt{2}(2+\alpha)(1+\alpha)^3}\Bigg[(14\sqrt{2}\alpha^3\delta + 4\sqrt{2}\alpha^4\delta + 6(\sqrt{2} + 7\sqrt{2}\delta + 12\sqrt{2}\delta^2 + 8\sqrt{2}\delta^3 \\
& -3\sqrt{\delta}\sqrt{1+\delta} - 12\delta^{3/2}\sqrt{1+\delta}) + \delta^2(-4\sqrt{2}\delta + 6\sqrt{\delta}\sqrt{1+\delta} + 24\delta^{3/2}\sqrt{1+\delta}) \\
& +\alpha(3\sqrt{2} - 11\sqrt{2}\delta + 36\sqrt{2}\delta^2 + 24\sqrt{2}\delta^3 + 12\delta\sqrt{1+\delta} + 48\delta^{3/2}\sqrt{1+\delta}))\, p_1^3\Bigg].
\end{aligned}$$

So that, the condition of $0 \le B \le 1$ for $0 \le \alpha \le 1$ and $0,0017 \le \delta \le 1$. Next, the condition for $B(2B-1) \le D \le B$ still open problem. The application of Lemma 2.3 on $|p_3 - 2Bp_1p_2 + Dp_1^3|$ under discussion.

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