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A FUZZY SOFT QUOTIENT TOPOLOGY AND ITS PROPERTIES

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ABSTRACT. This research is to construct a new topology on fuzzy soft set by using the concept of quotient topology. Then we study the concept of quotient map to define the fuzzy soft quotient map and provide some relevant properties of fuzzy soft quotient map. Furthermore, we give some examples related to fuzzy soft quotient topology and fuzzy soft quotient map to apply some properties of fuzzy soft quotient map.

Key words and phrases: Fuzzy soft quotient topology; Fuzzy soft quotient map; Fuzzy soft topology; Fuzzy soft set.

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1. INTRODUCTION

The notion of fuzzy set theory was introduced by Zadeh in 1965 [9] as generalization of classical set theory to solve the problem of uncertainty. In 1968, Chang [6] initiated the notion of fuzzy topological space and studied more basic concepts of fuzzy topology. Then in 1974, Wong [5] introduced the new concept of fuzzy topology called fuzzy quotient topology and discussed some properties of fuzzy quotient topology.

Molodtsov [7] introduced a new concept called "Soft Set Theory", a mathematical tool for dealing with uncertainty problems. The soft set theory had many applications in several directions such as Riemann integration, probability, game theory, and measurement theory. Maji et. al [13] first initiated the application of soft set theory in decision-making problems. Then Maji et al. [12] acquainted the notion of fuzzy soft sets, a combination of fuzzy sets and soft sets.

Researchers have developed many exciting applications of fuzzy soft sets. Çağman et al. [11] defined fuzzy soft sets and some operations of fuzzy soft sets, and they specified fuzzy soft aggregation operator, which allows the construction of more efficient decision processes. In 2009, Athar et. al [2] introduced a mapping on classes of fuzzy soft sets and defined properties of fuzzy soft images and inverse images. Later, Abdülkadir et. al [1] studied the concept of fuzzy soft groups and discussed some of their properties and structural characteristic.

Furthermore, the notion of topological study on fuzzy soft set was introduced by Tanay et. al [4]. They studied fuzzy soft topological space over a subset of the initial universe set. They discussed several properties of fuzzy soft topological structures. In 2012, Pazar et. al [3] familiarized the initial fuzzy soft topology and studied the fuzzy soft continuity of fuzzy soft mappings, fuzzy soft closure and fuzzy interior operators. Furthermore, Roy and Samanta [15] defined some definitions of fuzzy soft set in another form. They improved some results of fuzzy soft topology that obtained in Tanay et. al [4]. Moreover, Mahanta et. al [8] introduced the concept of neighborhood of a fuzzy soft point, fuzzy soft closure, fuzzy soft interior and investigated separation axioms and connectedness for fuzzy soft topological spaces.

Referring to the method studied by Wong [5] and Roy and Samanta [15], it is interesting to construct a new topology on fuzzy soft sets by using the concept of quotient topology. In this paper, we define fuzzy soft quotient topology by following basic concepts of quotient topology in Munkres [10]. Furthermore, we study a fuzzy soft quotient map and consider some properties of fuzzy soft quotient map.

2. PRELIMINARIES

Let U be an initial universe. A fuzzy set F in U is characterized by a membership function μ_F, where $\mu_F : U \longrightarrow [0,1]$, with the value of $\mu_F(u)$ at u representing the "grade of membership" of u in F . A fuzzy set F over U can be represented by

$$F = \{(u, \mu_F(u)) | u \in U\} \tag{2.1}$$

The collection of all fuzzy sets in U will be denoted by $\mathcal{I}^U$. The empty fuzzy set is denoted by $\bar{0}$ if and only if its membership function is identically zero or $\mu_{\bar{0}}(u) = 0$ for all $u \in U$. A fuzzy set A is contained in B, written as $A \bar{\subseteq} B$ if and only if $\mu_A(u) \leq \mu_B(u)$, for all $u \in U$. Two fuzzy sets A and B are called to be equal, writen as $A = B$ if and only if $\mu_A(u) = \mu_B(u)$, for all $u \in U$. The union of two fuzzy sets A and B is a fuzzy set C, written as $C = A \bar{\cup} B$ if and only if $\mu_C(u) = max\{\mu_A(u), \mu_B(u)\}$, for all $u \in U$. The intersection of two fuzzy sets A and B is a fuzzy set D, written as $D = A \bar{\cap} B$ if and only if $\mu_D(u) = min\{\mu_A(u), \mu_B(u)\}$, for all $u \in U$[9].

Let E be the set of all parameters for U and $A \subseteq E$. A soft set over U is a pair (L, A), where L is a mapping from A into all subsets of the set U. In other words, if $\mathcal{P}^{(U)}$ is a power set of U, then $L(e) \in \mathcal{P}^{(U)}$ for every $e \in A$. Thus a soft set F over U can be represented by

$$(L, A) = \{(e, L(e))|e \in A, L(e) \in \mathcal{P}^U\} \tag{2.2}$$

where for all $e \in A$, $L(e)$ is a subset of U. The set of all soft sets over U will be denoted by $\mathcal{S}(U)$[7].

Definition 2.1. [15] Suppose $A \subseteq E$. A fuzzy soft set f_A over (U, E) is a mapping from E to $\mathcal{I}^U$, i.e $f_A : E \longrightarrow \mathcal{I}^U$, where $f_A(e) = \bar{0}$ if $e \in E \backslash A$ and $f_A(e) \neq \bar{0}$ if $e \in A$.

As mentioned in [3], a fuzzy soft set f_A can be defined by

$$f_A = \{(e, f_A(e))|e \in E, f_A(e) \in \mathcal{I}^U\} \tag{2.3}$$

where for every $e \in E$, it is clear that $f_A(e)$ is a fuzzy set in U. The membership function of $f_A(e)$ is denoted by $\mu^e_{f_A}$. The collections of all fuzzy soft sets over (U, E) is stated by $\mathcal{FS}(U, E)$.

Example 2.1. *Suppose U is a set of bags and E is a set of parameters where each parameter is a sentence involving fuzzy words such as expensive, cheap and modern. In this case, defining a fuzzy soft set means to point expensive bags, cheap bags and modern bags. The fuzzy soft sets describe the attractiveness of bags. Suppose the universe U given by $U = \{u_1, u_2\}$ and $A = \{e_1, e_2\} \subset E$ where e_1 stands for the parameter ' expensive' and e_2 stands for the parameter 'cheap'. Then the fuzzy soft set f_A over (U, E) is defined by*

$$f_A = \{(e_1, \{u_1^{0.3}, u_2^{0.2}\}), (e_2, \{u_1^{0.5}, u_2^{0.4}\})\} \tag{2.4}$$

Definition 2.2. [15] A fuzzy soft set f_E over (U, E) is claimed to be an absolute fuzzy soft set and denoted by $\bar{I}$ if and only if for all $e \in E$, $\mu^e_{\bar{I}}(u) = 1$ for all $u \in U$.

Definition 2.3. [15] A fuzzy soft set is claimed to be a null fuzzy soft set and denoted by Φ if and only if for all $e \in E$, $\mu^e_\Phi(u) = 0$ for all $u \in U$.

Definition 2.4. [15] Suppose f_A be fuzzy soft sets over (U, E). A fuzzy soft set f_{A_l} is a fuzzy soft subset of f_A, written by $f_{A_l} \sqsubseteq f_A$ if for all $e \in E$, $\mu^e_{f_{A_l}}(u) \leq \mu^e_{f_A}(u)$ for all $u \in U$.

Definition 2.5. [3] Let $f_A, g_B \in \mathcal{FS}(U, E)$. Two fuzzy soft sets are called to be equal, denoted by $f_A = g_B$ if $f_A \sqsubseteq g_B$ and $g_B \sqsubseteq f_A$.

Definition 2.6. [15] The union of two fuzzy soft sets f_A and g_B is a fuzzy soft set over (U, E) defined by

$$f_A \sqcup g_B = \{(e, f_A(e)\bar{\cup}g_B(e))|e \in E\} \tag{2.5}$$

where $f_A(e)\bar{\cup}g_B(e)$ is the fuzzy set in U.

Definition 2.7. [15] The intersection of two fuzzy soft sets f_A and g_B is a fuzzy soft set over (U, E) defined by

$$f_A \sqcap g_B = \{(e, f_A(e)\bar{\cap}g_B(e))|e \in E\} \tag{2.6}$$

where $f_A(e)\bar{\cap}g_B(e)$ is the fuzzy set in U.

Definition 2.8. [3] Suppose $\mathcal{FS}(U, E)$ and $\mathcal{FS}(V, K)$ are the collection of all fuzzy soft sets over U and V respectively. Let $\varphi : U \longrightarrow V$ and $\psi : E \longrightarrow K$ be two maps. Then the pair (φ, ψ) is a fuzzy soft map and is denoted by $(\varphi, \psi) : \mathcal{FS}(U, E) \longrightarrow \mathcal{FS}(V, K)$.

Definition 2.9. [3] Let f_A and g_B are fuzzy soft sets over U and V, respectively. The image of $f_A \in \mathcal{FS}(U,E)$ under the fuzzy soft map (φ,ψ) denoted $(\varphi,\psi)(f_A)$ is a fuzzy soft set over V and defined by $(\varphi,\psi)(f_A) = (\varphi,\psi)f_A(k)(y)$ where if $\varphi^{-1}(y) \neq \emptyset, \psi^{-1}(k) \neq \emptyset$, then

$$(\varphi,\psi)f_A(k)(y) = sup_{\varphi(x)=y}(sup_{\psi(e)=k}f_A(e))(x). \tag{2.7}$$

Definition 2.10. [3] Let f_A and g_B are fuzzy soft sets over U and V respectively. The pre-image of g_B under the fuzzy soft map (φ,ψ) denoted by $(\varphi,\psi)^{-1}(g_B)$ is the fuzzys soft set over U and defined by

$$(\varphi,\psi)^{-1}g_B(e)(x) = g_B(\psi(e))(\varphi(x)), \forall u \in U, e \in E. \tag{2.8}$$

Definition 2.11. [3] Suppose (φ,ψ) be a fuzzy soft map. If φ and ψ are injective, then (φ,ψ) is said to be injective. Furthermore, if φ and ψ are surjective, then (φ,ψ) is said to be surjective.

Definition 2.12. [4] Let f_A be a fuzzy soft set over (U,E), $\mathcal{P}(f_A)$ be the set of all fuzzy soft subsets of f_A and τ be a subfamily of $\mathcal{P}(f_A)$. Then τ is called a fuzzy soft topology on f_A if the following conditions are satisfied :

i. $\Phi, f_A \in \tau$
ii. If $f_A, g_B \in \tau$, then $f_A \sqcap g_B \in \tau$
iii. If $f^a_{A_a} \in \tau$, $a \in \Lambda$, then $\bigsqcup_{a\in\Lambda} f^a_{A_a} \in \tau$.

The pair (f_A,τ) is called a fuzzy soft topological space and every members of τ is an open fuzzy soft set.

Example 2.2. *Let $U = \{u_1,u_2\}$ and $E = \{e_1,e_2\}$. Suppose f_E is a fuzzy soft set over (U,E) defined by*

$$f_E = \{(e_1,\{u_1^{0.5},u_2^{0.8}\}),(e_2,\{u_1^{0.1},u_2^{0.6}\})\} \tag{2.9}$$

Let $f_{E_1} = \{(e_1,\{u_1^{0.5},u_2^{0.8}\}),(e_2,\{u_1^{0},u_2^{0}\})\}$ and $f_{E_2} = \{(e_1,\{u_1^{0},u_2^{0}\}),(e_2,\{u_1^{0.1},u_2^{0.6}\})\}$ be the fuzzy soft subsets of f_E. Then $\tau = \{\Phi, f_E, f_{E_1}, f_{E_2}\}$ is a fuzzy soft topology on f_E. A pair (f_E,τ) is a fuzzy soft topological space.

Definition 2.13. [3] Let (f_E,τ_{f_E}) and (g_K,τ_{g_K}) be two fuzzy soft topological space. A fuzzy soft mapping $(\varphi,\psi) : (f_E,\tau_{f_E}) \longrightarrow (g_K,\tau_{g_K})$ is called

(1) Fuzzy soft continuous if $(\varphi,\psi)^{-1}(g_{K_k}) \in \tau_{f_E}$, for all $g_{K_k} \in \tau_{g_K}$
(2) Fuzzy soft open if $(\varphi,\psi)(f_{E_l}) \in \tau_{g_K}$, for all $f_{E_l} \in \tau_{f_E}$

Theorem 2.1. [3] *Let $(\varphi,\psi) : (f_E,\tau_1) \longrightarrow (g_K,\tau_2)$ and $(\varphi',\psi') : (g_K,\tau_2) \longrightarrow (h_C,\tau_3)$ be fuzzy soft continuous map, then $(\varphi',\psi') \circ (\varphi,\psi) : (f_E,\tau_1) \longrightarrow (h_C,\tau_3)$ is fuzzy soft continuous map.*

Proof. See [3]. ∎

3. FUZZY SOFT QUOTIENT TOPOLOGY

Many techniques are used to construct a new fuzzy soft topology based on existing topology concepts. In this paper, we construct a new topology for fuzzy soft set by using the concept of quotient topology. We define a fuzzy soft quotient topology, a fuzzy soft quotient map, and we study how fuzzy soft quotient topology is characterized by fuzzy soft quotient map. We consider and establish some properties of fuzzy soft quotient map and give some examples related to fuzzy soft quotient map. This section presents a new topology for a fuzzy soft set, a fuzzy soft quotient topology. We introduce a fuzzy soft quotient map and its properties using the concept of quotient topology in Munkres [10].

Definition 3.1. Suppose (f_E, τ_{f_E}) be fuzzy soft topological space over U and g_K be a fuzzy soft set over V. Let $(\varphi, \psi) : (f_E, \tau_{f_E}) \longrightarrow (g_K, \tau_{g_K})$ be a surjective map. The fuzzy soft quotient topology on g_K induced by (φ, ψ) is defined by declaring a fuzzy soft subset g_{K_a} of g_K to be open if and only if $(\varphi, \psi)^{-1}(g_{K_a})$ is open in (f_E, τ_{f_E}).

Example 3.1. *Let $U = \{u_1, u_2\}$ and $V = \{v_1, v_2\}$ be an initial universal, $E = \{e_1, e_2\}$ and $K = \{k_1, k_2\}$ be the set of parameters for U and V, respectively. Suppose two fuzzy soft sets f_E and g_K over U and V, respectively, are defined by*

$$f_E = \{(e_1, \{(u_1^{0.6}), (u_2^{0.8})\}), (e_2, \{u_1^{0.7}, u_2^{0.5}\})\} \tag{3.1}$$

$$g_K = \{(k_1, \{(v_1^{0.5}), (v_2^{0.7})\}), (k_2, \{v_1^{0.8}, v_2^{0.6}\})\} \tag{3.2}$$

and $\tau_{f_E} = \{\Phi, f_E, \{(e_1, \{(u_1^{0}), (u_2^{0.4})\}), (e_2, \{u_1^{0}, u_2^{0}\})\}$ be fuzzy soft topology on f_E. Show there exist fuzzy soft quotient topologi on g_K.

We want to show that there exists one fuzzy soft topology on g_K, which is called fuzzy soft quotient topology by following Definition 3.1. Suppose $\varphi : U \longrightarrow V$ and $\psi : E \longrightarrow K$, respectively, are defined by

$$\varphi(u_1) = v_2, \varphi(u_2) = v_1, \psi(e_1) = k_2, \psi(e_2) = k_1. \tag{3.3}$$

Then defined a fuzzy soft map $(\varphi, \psi) : (f_E, \tau_{f_E}) \longrightarrow g_K$ as $(\varphi, \psi)(f_{E_{e_a}}) = g_{K_{k_a}}$, where $f_{E_{e_a}} \in f_E$ and $g_{K_{k_a}} \in g_K$. It is clear that $(\varphi, \psi) : (f_E, \tau_{f_E}) \longrightarrow g_K$ is fuzzy soft surjective map. Furthermore, we choose fuzzy soft subset on g_K. If $\Phi \sqsubseteq g_K$, then $(\varphi, \psi)^{-1}(\Phi) = \Phi \in \tau_{f_E}$. So, $(\varphi, \psi)^{-1}(\Phi)$ is open fuzzy soft set in (f_E, τ_{f_E}).

If $g_{K_1} = \{(k_1, \{(v_1^0), (v_2^0)\}), (k_2, \{v_1^{0.4}, v_2^0\})\} \sqsubseteq g_K$, then we want to show that $(\varphi, \psi)^{-1}(g_{K_1})$ is open fuzzy soft set in (f_E, τ_{f_E}). For each $e \in E$ and $u \in U$, then

$$\begin{aligned}
(\varphi, \psi)^{-1}(g_{K_1})(e_1)(u_1) &= g_{K_1}(\psi(e_1))(\varphi(u_1)) \\
&= g_K(k_2)(v_2) \\
&= 0; \\
(\varphi, \psi)^{-1}(g_{K_1})(e_1)(u_2) &= g_K(\psi(e_1))(\varphi(u_2)) \\
&= g_{K_1}(k_2)(v_1) \\
&= 0.4; \\
(\varphi, \psi)^{-1}(g_{K_1})(e_2)(u_1) &= g_{K_1}(\psi(e_2))(\varphi(u_1)) \\
&= g_{K_1}(k_1)(v_2) \\
&= 0; \\
(\varphi, \psi)^{-1}(g_{K_1})(e_2)(u_2) &= g_{K_1}(\psi(e_2))(\varphi(u_2)) \\
&= g_{K_1}(k_1)(v_1) \\
&= 0.
\end{aligned}$$

We have $(\varphi,\psi)^{-1}(g_{K_1}) = \{(e_1,\{(u_1^0),(u_2^{0.4})\}),(e_2,\{u_1^0,u_2^0\})\} \in \tau_{f_E}$. Therefore, $(\varphi,\psi)^{-1}(g_{K_1})$ is open fuzzy soft set in (f_E,τ_{f_E}). The same way holds for g_K, for each $e \in E$ and $u \in U$, then

$$\begin{aligned}
(\varphi,\psi)^{-1}(g_K)(e_1)(u_1) &= g_K(\psi(e_1))(\varphi(u_1)) \\
&= g_K(k_2)(v_2) \\
&= 0.6; \\
(\varphi,\psi)^{-1}(g_K)(e_1)(u_2) &= g_K(\psi(e_1))(\varphi(u_2)) \\
&= g_K(k_2)(v_1) \\
&= 0.8; \\
(\varphi,\psi)^{-1}(g_K)(e_2)(u_1) &= g_K(\psi(e_2))(\varphi(u_1)) \\
&= g_K(k_1)(v_2) \\
&= 0.2; \\
(\varphi,\psi)^{-1}(g_K)(e_2)(u_2) &= g_K(\psi(e_2))(\varphi(u_2)) \\
&= g_K(k_1)(v_1) \\
&= 0.5.
\end{aligned}$$

We have $(\varphi,\psi)^{-1}(g_K) = f_E \in \tau_{f_E}$. So, $(\varphi,\psi)^{-1}(g_K)$ is open fuzzy soft set in (f_E,τ_{f_E}). So that, $\tau_{g_K} = \{\Phi, g_K, \{(k_1,\{(v_1^0),(v_2^0)\}),(k_2,\{v_1^{0.4},v_2^0\})\}\}$ is a fuzzy soft topology on g_K, which is called a fuzzy soft quotient topology.

Definition 3.2. Suppose (f_E,τ_{f_E}) and (g_K,τ_{g_K}) are fuzzy soft topological space over U and V, respectively. Let $(\varphi,\psi) : (f_E,\tau_{f_E}) \longrightarrow (g_K,\tau_{g_K})$ be a surjective map. The map (φ,ψ) is called to be a fuzzy soft quotient map provided a fuzzy soft subset g_{K_a} of g_K open in (g_K,τ_{g_K}) if and only if $(\varphi,\psi)^{-1}(g_{K_a})$ is open in (f_E,τ_{f_E}).

Example 3.2. *Let $U = \{u_1,u_2\}$ and $V = \{v_1,v_2\}$ be an initial universal, $E = \{e_1,e_2\}$ and $K = \{k_1,k_2\}$ be the set of parameters for U and V, respectively. Suppose two fuzzy soft sets f_E and g_K over U and V, respectively, are defined by*

$$f_E = \{(e_1,\{(u_1^{0.6}),(u_2^{0.3})\}),(e_2,\{u_1^{0.1},u_2^{0.5}\})\}, \tag{3.4}$$
$$g_K = \{(k_1,\{(v_1^{0.1}),(v_2^{0.5})\}),(k_2,\{v_1^{0.6},v_2^{0.3}\})\}. \tag{3.5}$$

Let $\tau_{f_E} = \{\Phi, f_E\}$ and $\tau_{g_K} = \{\Phi, g_K\}$ be fuzzy soft topology on f_E and g_K, respectively. Suppose $(\varphi,\psi) : (f_E,\tau_{f_E}) \longrightarrow (g_K,\tau_{g_K})$ be a surjective map, then (φ,ψ) is a fuzzy soft quotient map.

Let us check that (φ,ψ) is a fuzzy soft quotient map by following the Definition 3.2. Suppose $\varphi : U \longrightarrow V$ and $\psi : E \longrightarrow K$, respectively, are defined by

$$\varphi(u_1) = v_1, \varphi(u_2) = v_2, \psi(e_1) = k_2, \psi(e_2) = k_1. \tag{3.6}$$

Defined $(\varphi,\psi) : (f_E,\tau_{f_E}) \longrightarrow (g_K,\tau_{g_K})$ as $(\varphi,\psi)(f_{E_{e_a}}) = g_{K_{k_a}}$, where $f_{E_{e_a}} \in f_E$ and $g_{K_{k_a}} \in g_K$. Obviously, a map $(\varphi,\psi) : (f_E,\tau_{f_E}) \longrightarrow (g_K,\tau_{g_K})$ is surjective map. Furthermore, if $\Phi \sqsubseteq g_K$, then $(\varphi,\psi)^{-1}(\Phi) = \Phi \in \tau_{f_E}$. So, $(\varphi,\psi)^{-1}(\Phi)$ is open fuzzy soft set in (f_E,τ_{f_E}). Later then if g_K be open fuzzy set in (g_K,τ_{g_K}), then we can show that $(\varphi,\psi)^{-1}(g_K)$ is open fuzzy soft set in (f_E, τ_{f_E}). Moreover, for each $e \in E$ and $u \in U$, then

$$\begin{aligned}
(\varphi,\psi)^{-1}(g_K)(e_1(u_1) &= g_K(\psi(e_1))(\varphi(u_1)) \\
&= g_K(k_2)(v_1) \\
&= 0.6; \\
(\varphi,\psi)^{-1}(g_K)(e_1(u_2) &= g_K(\psi(e_1))(\varphi(u_2)) \\
&= g_K(k_2)(v_2) \\
&= 0.3; \\
(\varphi,\psi)^{-1}(g_K)(e_2(u_1) &= g_K(\psi(e_2))(\varphi(u_1)) \\
&= g_K(k_1)(v_1) \\
&= 0.1; \\
(\varphi,\psi)^{-1}(g_K)(e_2(u_2) &= g_K(\psi(e_2))(\varphi(u_2)) \\
&= g_K(k_1)(v_2) \\
&= 0.5.
\end{aligned}$$

We have $(\varphi,\psi)^{-1}(g_K) = f_E \in \tau_{f_E}$. So that, $(\varphi,\psi)^{-1}(g_K)$ is open fuzzy soft set in (f_E, τ_{f_E}). Thus, $(\varphi,\psi) : (f_E, \tau_{f_E}) \longrightarrow (g_K, \tau_{g_K})$ is a fuzzy soft quotient map.

Theorem 3.1. *Every fuzzy soft quotient map is continuous*

Proof. Based on Definition 3.2, we know that (φ,ψ) is a surjective map and if g_{K_a} is open in (g_K, τ_{g_K}), then $(\varphi,\psi)^{-1}(g_{K_a})$ is open in (f_E, τ_{f_E}). So, following Definition 2.13, this shows that the fuzzy soft quotient map is continuous. ∎

Theorem 3.2. *Let $(\varphi,\psi) : (f_E, \tau_{f_E}) \longrightarrow (g_K, \tau_{g_K})$ be a fuzzy soft quotient map and (h_C, τ_{h_C}) is fuzzy soft topological space. A fuzzy soft map $(\zeta,\eta) : (g_K, \tau_{g_K}) \longrightarrow (h_C, \tau_{h_C})$ is called fuzzy soft continuous if and only if $(\zeta,\eta) \circ (\varphi,\psi) : (f_E, \tau_{f_E}) \longrightarrow (h_C, \tau_{h_C})$ is continuous.*

Proof. If $(\zeta,\eta) : (g_K, \tau_{g_K}) \longrightarrow (h_C, \tau_{h_C})$ is fuzzy soft continuous map, then clearly that $(\zeta,\eta) \circ (\varphi,\psi) : (f_E, \tau_{f_E}) \longrightarrow (h_C, \tau_{h_C})$ is fuzzy soft continuous map. Coversely, suppose $(\zeta,\eta) \circ (\varphi,\psi) : (f_E, \tau_{f_E}) \longrightarrow (h_C, \tau_{h_C})$ is fuzzy soft continuous map. Given an open fuzzy soft subset h_{C_1} of h_C. Then $((\zeta,\eta) \circ (\varphi,\psi))^{-1}(h_{C_1}) = (\varphi,\psi)^{-1}((\zeta,\eta)^{-1}((h_{C_1}))$. Because (φ,ψ) is a fuzzy soft quotient map, it follows that $(\zeta,\eta)^{-1}(h_{C_1})$ is open in (g_K, τ_{g_K}). Hence $(\zeta,\eta) : (g_K, \tau_{g_K}) \longrightarrow (h_C, \tau_{h_C})$ is a fuzzy soft continuous map. ∎

Example 3.3. *Let $U = \{u_1, u_2\}, V = \{v_1, v_2\}$ and $W = \{w_1, w_2\}$ be an initial universe and let $E = \{e_1, e_2\}, K = \{k_1, k_2\}$ and $L = \{l_1, l_2\}$ be the set of parameter for U, V, W respectively. Suppose fuzzy soft sets f_E, g_B and h_L over U, V, W, respectively, are defined by*

$$f_E = \{(e_1, \{u_1^1, u_2^{0.2}\}), (e_2, \{u_1^{0.5}, u_2^{0.3}\})\} \tag{3.7}$$
$$g_K = \{(k_1, \{v_1^{0.2}, v_2^1\}), (k_2, \{v_1^{0.3}, v_2^{0.5}\})\} \tag{3.8}$$
$$h_L = \{(l_1, \{w_1^{0.3}, w_2^{0.5}\}), (l_2, \{w_1^{0.2}, w_2^1\})\} \tag{3.9}$$

Let τ_{f_E}, τ_{g_K}, and τ_{h_L} be topology for fuzzy fuzzy soft sets f_E, g_K, and h_L respectively and defined by

$$\tau_{f_E} = \{\Phi, f_E, \{(e_1, \{u_1^{0.9}, u_2^{0.2}\}), (e_2, \{u_1^{0.4}, u_2^0\})\}\} \tag{3.10}$$
$$\tau_{g_K} = \{\Phi, g_B, \{(k_1, \{v_1^{0.2}, v_2^{0.9}\}), (k_2, \{v_1^0, v_2^{0.4}\})\}\} \tag{3.11}$$
$$\tau_{hL} = \{\Phi, h_L, \{(l_1, \{w_1^0, w_2^{0.4}\}), (l_2, \{w_1^{0.2}, w_2^{0.9}\})\}\} \tag{3.12}$$

Suppose $(\varphi,\psi):(f_E,\tau_{f_E})\longrightarrow(g_K,\tau_{g_K})$ *is fuzzy soft quotient map. A fuzzy soft map* $(\zeta,\eta):(g_K,\tau_{g_K})\longrightarrow(h_L,\tau_{h_L})$ *is called fuzzy soft continuous if and only if* $(\zeta,\eta)\circ(\varphi,\psi):(f_E,\tau_{f_E})\longrightarrow(h_L,\tau_{h_L})$ *is continuous.*

Let us check that $(\zeta,\eta):(g_K,\tau_{g_K})\longrightarrow(h_L,\tau_{h_L})$ is fuzzy soft continuous if and only if $(\zeta,\eta)\circ(\varphi,\psi):(f_E,\tau_{f_E})\longrightarrow(h_L,\tau_{h_L})$ is continuous. Suppose (φ,ψ) be fuzzy soft quotient map. Then $\varphi:U\longrightarrow V$ and $\psi:E\longrightarrow K$,respectively, are defined by

$$\varphi(u_1)=v_2,\varphi(u_2)=v_1,\psi(e_1)=k_1,\psi(e_2)=k_2. \tag{3.13}$$

Then, define a map $(\zeta,\eta):(g_K,\tau_{g_K})\longrightarrow(h_L,\tau_{h_L})$ by $(\zeta,\eta)(g_{K_a})=h_{L_a}$ where $\zeta:V\longrightarrow W$ and $\eta:K\longrightarrow L$, respectively, are defined by

$$\zeta(v_1)=w_1,\zeta(v_2)=w_2,\eta(k_1)=l_2,\eta(k_2)=l_1. \tag{3.14}$$

Furthermore, if Φ be open fuzzy soft set in τ_{h_L}, then we can show that $((\zeta,\eta)\circ(\varphi,\psi))^{-1}(\Phi)$ is open fuzzy soft set in (f_E,τ_{f_E}). Then

$$((\zeta,\eta)\circ(\varphi,\psi))^{-1}(\Phi)=(\varphi,\psi))^{-1}((\zeta,\eta)^{-1}(\Phi))=(\varphi,\psi))^{-1}(\Phi) \tag{3.15}$$

We have $(\varphi,\psi))^{-1}(\Phi)=\Phi\in\tau_{f_E}$. So, $((\zeta,\eta)\circ(\varphi,\psi))^{-1}(\Phi)=\Phi$ is open fuzzy soft set in (f_E,τ_{f_E}). The same way holds for

$$h_{L_1}=\{(l_1,\{w_1^0,w_2^{0.4}\}),(l_2,\{w_1^{0.2},w_2^{0.9}\})\}. \tag{3.16}$$

We want to show that $((\zeta,\eta)\circ(\varphi,\psi))^{-1}(h_{L_1})=(\varphi,\psi))^{-1}((\zeta,\eta)^{-1}(h_{L_1}))\in\tau_{f_E}$. First, we show that $(\zeta,\eta)^{-1}(h_{L_1})$ is open fuzzy soft set in τ_{g_K}. For each $k\in K$ and $v\in V$, then

$$\begin{aligned}
(\zeta,\eta)^{-1}(h_{L_1})(k_1)(v_1) &= h_{L_1}(\eta(k_1))(\zeta(v_1))\\
&= h_{L_1}(l_2)(w_1)\\
&= 0.2;\\
(\zeta,\eta)^{-1}(h_{L_1})(k_1)(v_2) &= h_{L_1}(\eta(k_1))(\zeta(v_2))\\
&= h_{L_1}(l_2)(w_2)\\
&= 0.9;\\
(\zeta,\eta)^{-1}(h_{L_1})(k_2)(v_1) &= h_{L_1}(\eta(k_2))(\zeta(v_1))\\
&= h_{L_1}(l_1)(w_1)\\
&= 0;\\
(\zeta,\eta)^{-1}(h_{L_1})(k_2)(v_2) &= h_{L_1}(\eta(k_2))(\zeta(v_2))\\
&= h_{L_1}(l_1)(w_2)\\
&= 0.4.
\end{aligned}$$

We have $(\zeta,\eta)^{-1}(h_{L_1}) = g_{K_1} = \{(k_1,\{v_1^{0.2},v_2^{0.9}\}),(k_2,\{v_1^{0},v_2^{0.4}\})\}$ is open fuzzy soft subset in τ_{g_K}. Then we show $(\varphi,\psi)^{-1}(g_{K_1})$ is open fuzzy soft subset in τ_{f_E}. For $e\in E$ and $u\in U$, then

$$
\begin{aligned}
(\varphi,\psi)^{-1}(g_{K_1})(e_1)(u_1) &= g_{K_1}(\psi(e_1))(\varphi(u_1))\\
&= g_{K_1}(k_1)(v_2)\\
&= 0.9\\
(\varphi,\psi)^{-1}(g_{K_1})(e_1)(u_2) &= g_{K_1}(\psi(e_1))(\varphi(u_2))\\
&= g_{K_1}(k_1)(v_1)\\
&= 0.2\\
(\varphi,\psi)^{-1}(g_{K_1})(e_2)(u_1) &= g_{K_1}(\psi(e_2))(\varphi(u_1))\\
&= g_{K_1}(k_2)(v_2)\\
&= 0.4\\
(\varphi,\psi)^{-1}(g_{K_1})(e_2)(u_2) &= g_{K_1}(\psi(e_2))(\varphi(u_2))\\
&= g_{K_1}(k_2)(v_1)\\
&= 0
\end{aligned}
$$

We have $(\varphi,\psi)^{-1}(g_{K_1}) = \{(e_1,\{u_1^{0.9},u_2^{0.2}\}),(e_2,\{u_1^{0.4},u_2^{0}\})\}\in\tau_{f_E}$. Furthermore, if $h_L\in\tau_{h_L}$, then we can show that $((\zeta,\eta)\circ(\varphi,\psi))^{-1}(h_L)$. Note that,

$$
\begin{aligned}
((\zeta,\eta)\circ(\varphi,\psi))^{-1}(h_L) &= (\varphi,\psi)^{-1}((\zeta,\eta)^{-1}(h_L))\\
&= (\varphi,\psi)^{-1}(g_K)\\
&= f_E
\end{aligned}
$$

Thus, $(\zeta,\eta)\circ(\varphi,\psi):(f_E,\tau_{f_E})\longrightarrow(h_L,\tau_{h_L})$ is continuous.

Conversely, suppose Φ be an open fuzzy soft subset in τ_{h_L}, then

$$(\zeta,\eta)\circ(\varphi,\psi)(\Phi) = (\varphi,\psi)^{-1}((\zeta,\eta)^{-1}(\Phi)) \tag{3.17}$$

In fact, (φ,ψ) is fuzzy soft quotient map. It follows that $(\zeta,\eta)^{-1}(\Phi)$ is open fuzzy soft subset in τ_{g_K}. In a similar way to h_L and h_{L_1}, then we have $(\zeta,\eta)^{-1}(h_L)$ and $(\zeta,\eta)^{-1}(h_{L_1})$ respectively is also open fuzzy soft subset in τ_{g_K}. Therefore, $(\zeta,\eta):(g_K,\tau_{g_K})\longrightarrow(h_L,\tau_{h_L})$ is fuzzy soft continuous map.

Theorem 3.3. *Let $(\varphi,\psi):(f_E,\tau_{f_E})\longrightarrow(g_K,\tau_{g_K})$ be fuzzy soft quotient map. A fuzzy soft map $(\zeta,\eta):(g_K,\tau_{g_K})\longrightarrow(h_C,\tau_{h_C})$ is called fuzzy soft quotient map if and only if $(\zeta,\eta)\circ(\varphi,\psi):(f_E,\tau_{f_E})\longrightarrow(h_C,\tau_{h_C})$ is fuzzy soft quotient map.*

Proof. If $(\zeta,\eta):(g_K,\tau_{g_K})\longrightarrow(h_C,\tau_{h_C})$ is a fuzzy soft quotient map, then $(\zeta,\eta)\circ(\varphi,\psi)$ is the composite of two fuzzy soft quotient maps. Thus, $(\zeta,\eta)\circ(\varphi,\psi)$ is a fuzzy soft quotient map. Conversely, suppose : $(\zeta,\eta)\circ(\varphi,\psi):(f_E,\tau_{f_E})\longrightarrow(h_C,\tau_{h_C})$ is fuzzy soft quotient map. Since $(\zeta,\eta)\circ(\varphi,\psi)$ is surjective, so (φ,ψ) is also surjective. Furthermore, let h_{C_1} be fuzzy soft subset of h_C. We show that h_{C_1} is a fuzzy soft open in h_C if $(\zeta,\eta)^{-1}(h_{C_1})$ is open in $(g_K,\tau_{g_{[K]}})$. Since (φ,ψ) is continuous, then $(\varphi,\psi)^{-1}((\zeta,\eta)^{-1}(h_{C_1}))$ is open fuzzy soft set. Thus $((\zeta,\eta)\circ(\varphi,\psi))^{-1}(h_{C_1})$ is open in (f_E,τ_{f_E}). Since $(\zeta,\eta)\circ(\varphi,\psi)$ is fuzzy soft continuous map, then $((\zeta,\eta)^{-1}(h_{C_1}))$ is open in (g_K,τ_{g_K}). Then h_{C_1} is open fuzzy soft subset in h_C. ∎

4. CONCLUSION

This paper initiates the construction of a fuzzy soft quotient topology. The fuzzy soft quotient topology and fuzzy soft quotient map are introduced. Moreover, we have established several properties of a fuzzy soft quotient map with the help of some examples. We hope these results discucssed in this paper will help the researcher enhance further study on fuzzy soft topology.

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