Aust. J. Math. Anal. Appl.
Vol. **19** (2022), No. 1, Art. 9, 8 pp.
AJMAA

REVERSE HÖLDER AND MINKOWSKI TYPE INTEGRAL INEQUALITIES FOR n FUNCTIONS

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Received 12 August, 2021; accepted 3 May, 2022; published 29 June, 2022.

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ABSTRACT. We present and prove new reverse Hölder and Minkowski type integral inequalities for n functions. We compare our results with other known results from the relative literature in order to test their performance. In this respect, our theorems can be viewed as generalizations of some already known integral inequalities.

Key words and phrases: Hölder's inequality, Minkowski's inequality, Pólya-Szegö inequality, Integral inequalities.

2010 *Mathematics Subject Classification.* Primary 26D15.

ISSN (electronic): 1449-5910

1. INTRODUCTION AND PRELIMINARIES

Theory of inequalities is an interesting and active field of mathematics which has many important applications in other sciences. There are many interesting and useful inequalities and there are also many books which are devoted to their study, see e.g. [4], [7]. Among the famous inequalities there are two of them with huge importance, namely Hölder's inequality and Minkowski's inequality. These two inequalities together with their refinements and generalizations appear frequently in the existing literature, see e.g. the books [4], [7], where whole chapters are devoted to them. They both have a discrete version and a corresponding integral analogue. In this article we present and prove reverse Hölder and Minkowski type integral inequalities for n functions. To the best of our knowledge these inequalities are new and interestingly their proofs are not involved. Furthermore, we compare our results with other known results from the relative literature ([1], [2], [6]) in order to test and support their performance.

Before we go on, we need to define a useful notation. In what follows we consider real functions f defined on a finite interval $[a,b] \subset \mathbb{R}$, although our results could be applied to a general measure space (S, Σ, μ) as well. Let us denote $\mathcal{L}^p([a,b])$, $p > 0$ the class of real functions f on $[a,b]$ such that

$$\int_a^b |f(x)|^p dx < \infty. \tag{1.1}$$

We are also using the following notation for a function f on $[a,b]$ and a $p > 0$:

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}. \tag{1.2}$$

We do not claim that the above notation $\|.\|$ indicates a norm, since for $p < 1$ the triangle inequality fails. We just use it as an easy way to compactly present the integral in (1.2).

Now we can present Hölder's inequality for the functions f, g such that $f \in \mathcal{L}^p([a,b])$ and $g \in \mathcal{L}^q([a,b])$, where $1/p + 1/q = 1$ and $p, q \in (1, \infty)$:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q . \tag{1.3}$$

Let us also note that for $0 < p < 1$ and of course a $q < 0$ since we still need $1/p + 1/q = 1$, Hölder's inequality (1.3) is reversed. Note that in this case our compact notation (1.2) is still valid even if $q < 0$.

Likewise, we can present Minkowski's integral inequality for the functions f, g such that $f, g \in \mathcal{L}^p([a,b])$, $p \geq 1$:

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p . \tag{1.4}$$

Again for $p < 1$ Minkowski's integral inequality (1.4) is reversed.

Another very interesting integral inequality, which we will need later, was proved by Pólya and Szegö (see [5], page 57). Let the positive bounded functions $f, g \in \mathcal{L}^2([a,b])$ with $0 < m_1 \leq f \leq M_1$ and $0 < m_2 \leq g \leq M_2$. Then it holds

$$\int_a^b f^2(x)dx \int_a^b g^2(x)dx \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\int_a^b f(x)g(x)dx\right)^2 .$$

By taking square roots of both sides and using the compact notation (1.2), Pólya-Szegö inequality can be written as

$$\|f\|_2 \|g\|_2 \leq \frac{1}{2}\left(\sqrt{\frac{M_1M_2}{m_1m_2}}+\sqrt{\frac{m_1m_2}{M_1M_2}}\right)\|fg\|_1 . \tag{1.5}$$

We can easily observe that the Pólya-Szegö inequality is a reverse Hölder type inequality. Pólya-Szegö inequality will be used later in a proof and as a measure to test our results. Before this section ends, we also present some other inequalities from the literature that we will need to test the performance of our results.

Another reverse Hölder type inequality for two positive functions has been proved in [6]. Let two positive functions f and g such that $f \in \mathcal{L}^p([a,b])$ and $g \in \mathcal{L}^q([a,b])$, satisfying $0 < m \leq f^p/g^q \leq M$ and $1/p + 1/q = 1$, $p, q > 1$, then

$$\|f\|_p \|g\|_q \leq \left(\frac{m}{M}\right)^{-\frac{1}{pq}}\|fg\|_1 . \tag{1.6}$$

A reverse Minkowski type inequality was presented and proved in [1]. Specifically, let two positive functions f and g such that $f, g \in \mathcal{L}^p([a,b])$, $p > 0$, satisfying $0 < m \leq f/g \leq M$ for all $x \in [a,b]$, then

$$\|f\|_p + \|g\|_p \leq \frac{M(m+1)+(M+1)}{(m+1)(M+1)}\|f+g\|_p . \tag{1.7}$$

In [2], among other interesting results, a reverse Minkowski type inequality was presented, where its proof is a direct consequence of (1.7). Specifically, for two bounded positive functions f and g such that $f, g \in \mathcal{L}^p([a,b])$, $p > 0$ where $0 < m_1 \leq f \leq M_1$, $0 < m_2 \leq g \leq M_2$ for all $x \in [a,b]$, we have

$$\|f\|_p + \|g\|_p \leq \frac{M_1(m_1+M_2)+M_2(m_2+M_1)}{(m_1+M_2)(m_2+M_1)}\|f+g\|_p . \tag{1.8}$$

From the above results, it is clear that the conditions we need to get reverse inequalities are either of the form of bounded functions or are of the form of a bounded ratio f/g. This observation will be our guide when we will generalize the reverse inequalities to n functions. In the next two sections, we present and prove our results and we compare them to other reverse type inequalities in order to test their performance.

2. REVERSE HÖLDER TYPE INEQUALITIES

We will first prove a simple reverse Hölder type inequality for n positive bounded functions. This simple proof is a first stable step in order to consider later other interesting results. Thus, our first result is the next Theorem:

Theorem 2.1. *Let n positive functions $f_1, \ldots, f_n$ such that $0 < m_i \leq f_i \leq M_i, i = 1, \ldots, n$ for all $x \in [a,b]$ and let $f_i \in \mathcal{L}^{p_i}([a,b])$ for $i = 1, \ldots, n$ and $p_i \geq 1, i = 1, \ldots, n$, then*

$$\prod_{i=1}^{n}\left(\|f_i\|_{p_i}\right)^{\frac{p_i}{n}} \leq M^{\frac{1}{n}}\|f_1 \ldots f_n\|_1 , \tag{2.1}$$

where $M = \frac{\prod_{i=1}^{n} M_i^{p_i-1}}{\prod_{i=1}^{n} m_i^{n-1}}$.

Proof: The proof is easy if one observes that:

$$\frac{f_k^{p_k}}{\prod_{i=1}^n f_i} = \frac{f_k^{p_k-1}}{\prod_{\substack{i=1\\i\neq k}}^n f_i} \leq \frac{M_k^{p_k-1}}{\prod_{\substack{i=1\\i\neq k}}^n m_i},$$

or equivalently,

$$f_k^{p_k} \leq \frac{M_k^{p_k-1}}{\prod_{\substack{i=1\\i\neq k}}^n m_i} \prod_{i=1}^n f_i,$$

for $k = 1, \dots, n$. Next, we integrate the above inequalities from a to b, we multiply them and raise both sides to the power of $1/n$ to straightforwardly get (2.1). ■

Theorem 2.1 is a reverse Hölder type inequality for n positive bounded functions. An interesting version of this Theorem is available if we set $p_k = n$ for $k = 1, \dots, n$, then (2.1) becomes

$$\prod_{i=1}^n \|f_i\|_n \leq M^{\frac{1}{n}} \|f_1 \dots f_n\|_1 . \tag{2.2}$$

Let us see what we get if we consider the case $n = 2$. In this case inequality (2.2) becomes

$$\|f_1\|_2 \|f_2\|_2 \leq \left(\frac{M_1 M_2}{m_1 m_2}\right)^{\frac{1}{2}} \|f_1 f_2\|_1 .$$

We can easily observe that the above inequality is not optimal since the Pólya-Szegö inequality (1.5) has a better upper bound. Nevertheless, Theorem 2.1 is a first result which gives a reverse Hölder type inequality for $n \geq 3$ functions.

A last comment for Theorem 2.1 is to examine its conclusion if we consider $0 < p_i < 1$. So, let $0 < p_i < 1$ for $i = 1, \dots, n$ then we easily get the following inequality:

$$\prod_{i=1}^n \left(\|f_i\|_{p_i}\right)^{\frac{p_i}{n}} \leq \left(\frac{1}{\prod_{i=1}^n m_i^{n-p_i}}\right)^{\frac{1}{n}} \|f_1 \dots f_n\|_1 . \tag{2.3}$$

We will now use another condition for the n functions besides boundedness. This will be our second Theorem and is given bellow:

Theorem 2.2. *Let n positive functions $f_1, \dots, f_n$ on $[a, b]$ such that $0 < \frac{f_k^{p_k}}{\prod_{i=1}^n f_i} \leq A_k$, $k = 1, \dots, n$ where $p_k > 0$, $k = 1, \dots, n$ and $A_k > 0$ for $k = 1, ..., n$. Moreover, let $f_i \in \mathcal{L}^{p_i}([a, b])$ for $i = 1, \dots, n$, then*

$$\prod_{i=1}^n \left(\|f_i\|_{p_i}\right)^{\frac{p_i}{n}} \leq \left(\prod_{i=1}^n A_i\right)^{\frac{1}{n}} \|f_1 \dots f_n\|_1 . \tag{2.4}$$

Proof: The proof is similar to the proof of Theorem 2.1, where we use the assumption:

$$\frac{f_k^{p_k}}{\prod_{i=1}^n f_i} \leq A_k,$$

for $k = 1, \dots, n$. We can now integrate

$$f_k^{p_k} \leq A_k \prod_{i=1}^n f_i, \qquad k = 1, \dots, n$$

from a to b, multiply the resulting inequalities and raise both sides to the power of $1/n$, to easily get (2.4). ■

Although Theorem 2.2 does not seem a big improvement of Theorem 2.1, as we will see it has its own importance. To support its importance let us examine the case $n = 2$ and $p_1 = p_2 = 2$. We then get from the assumptions of Theorem 2.2 that

$$\frac{1}{A_2} \leq \frac{f_1}{f_2} \leq A_1,$$

and we have

$$\|f_1\|_2 \|f_2\|_2 \leq (A_1 A_2)^{\frac{1}{2}} \|f_1 f_2\|_1 .$$

This is exactly inequality (1.6) that was proved in [6] (Theorem 2.1) for $p = q = 2$. This fact gives us a solid indication that Theorem 2.2 has its own importance.

Now, as we have seen by examining the case $n = 2$ and $p_1 = p_2 = 2$, Theorem 2.1 is not optimal. So, it would be nice to find another Theorem which is valid for $n \geq 3$ functions and at the same time Pólya-Szegö inequality (1.5) holds for $n = 2$. In fact we can prove such a Theorem. We need the following inequality which can be easily proved by using the integral analogue of the discrete power mean inequality (see [7], pp. 127-128). So, for a positive function $f \in \mathcal{L}^2([a, b])$ we have

$$\|f\|_1 \leq \sqrt{b-a}\, \|f\|_2 . \tag{2.5}$$

Let us also note that (2.5) is a special case of a known Theorem which states that in a finite measure space (S, Σ, μ) where $0 < \mu(S) < \infty$ and $1 \leq p < q < \infty$ we have $\|f\|_p \leq \mu(S)^{1/p-1/q} \|f\|_q$ (see [3], page 240). Likewise, in our case and using the Lebesgue measure $\mu([a, b]) = b - a$, (2.5) can be written as $\|f\|_1 \leq \mu([a, b])^{1/2} \|f\|_2$.

Now we are ready to prove the following:

Theorem 2.3. *Let $n \geq 2$ positive functions $f_1, \ldots, f_n$ such that $0 < m_i \leq f_i \leq M_i, i = 1, ..., n$ for all $x \in [a, b]$ and let $\prod_{i=1}^n f_i \in \mathcal{L}^2([a, b])$, then*

$$\prod_{i=1}^{n} ||f_i||_2 \leq \frac{\left(\sqrt{b-a}\right)^{n-2}}{2^{n-1}} \left(\prod_{k=2}^{n} B_k\right) \|f_1 \ldots f_n\|_1 , \tag{2.6}$$

where $B_k = \sqrt{\prod_{i=1}^k \frac{M_i}{m_i}} + \sqrt{\prod_{i=1}^k \frac{m_i}{M_i}}$, for $k \geq 2$.

Proof: First let us observe that we use $\prod_{i=1}^n f_i \in \mathcal{L}^2([a, b])$ as an integrability condition, since this condition implies that all the functions f_k, $k = 1, \ldots, n$ and their products are also in $\mathcal{L}^2([a, b])$. This fact can be easily seen by using (2.5) and (1.5). To see it, observe that from (2.5) we get $\prod_{i=1}^n f_i \in \mathcal{L}^1([a, b])$. Then by using Pólya-Szegö inequality (1.5) we have that for any factorization $GH = \prod_{i=1}^n f_i$ where the two factors are of the form $G = \prod f_i$ and $H = \prod f_k$, it holds $\|G\|_2 \|H\|_2 < \infty$ and so $G, H \in \mathcal{L}^2([a, b])$ as we wanted.

Now for the proof we use induction:

For $n = 2$ we get the Pólya-Szegö inequality (1.5).

Let us assume that the inequality holds for $n - 1$ functions $f_1, \ldots, f_{n-1}$, i.e.

$$\prod_{i=1}^{n-1} \|f_i\|_2 \leq \frac{\left(\sqrt{b-a}\right)^{n-3}}{2^{n-2}} \left(\prod_{k=2}^{n-1} B_k\right) \|f_1 \ldots f_{n-1}\|_1 . \tag{2.7}$$

We want to prove inequality's true for n functions, i.e to prove (2.6). From the Pólya-Szegö inequality we have

$$\|f_1 \dots f_{n-1}\|_2 \, \|f_n\|_2 \leq \frac{1}{2} B_n \, \|f_1 \dots f_n\|_1 \, .$$

Now by using (2.5) we have

$$\|f_1 \dots f_{n-1}\|_1 \leq \sqrt{b-a} \, \|f_1 \dots f_{n-1}\|_2 \, .$$

Using the above two inequalities we get

$$\|f_1 \dots f_{n-1}\|_1 \, \|f_n\|_2 \leq \sqrt{b-a} \, \|f_1 \dots f_{n-1}\|_2 \, \|f_n\|_2 \leq \frac{\sqrt{b-a}}{2} B_n \, \|f_1 \dots f_n\|_1 \, .$$

Finally, we multiply both sides of (2.7) by $\|f_n\|_2$ and use the above inequality to easily get (2.6). This completes the proof. ■

Theorem 2.3 gives another reverse Hölder type inequality for n positive bounded functions and for $n = 2$ it gives the Pólya-Szegö inequality. We end this section with the following remark.

Remark 2.1. Using the assumptions of Theorem 2.3 and inequality (2.5) we can also get the following reverse Hölder type inequality:

$$\prod_{i=1}^{n} \|f_i\|_1 \leq \frac{\left(\sqrt{b-a}\right)^{2n-2}}{2^{n-1}} \left(\prod_{k=2}^{n} B_k \right) \|f_1 \dots f_n\|_1 \tag{2.8}$$

3. REVERSE MINKOWSKI TYPE INEQUALITIES

In this section we will present and prove two reverse Minkowski type integral inequalities for n functions. As we saw in Introduction a reverse Minkowski inequality can be obtained from (1.4) for $p < 1$. It is already known and easy to see that for n functions $f_1, \dots, f_n$ and a $p < 1$ (1.4) becomes

$$\|f_1 + \dots + f_{n-1}\|_p + \|f_n\|_p \leq \|f_1 + \dots + f_n\|_p \, , \tag{3.1}$$

and by repeating the above argument, we finally get

$$\|f_1\|_p + \dots + \|f_n\|_p \leq \|f_1 + \dots + f_n\|_p \, . \tag{3.2}$$

Thus, for a $p < 1$ we have already available a reverse Minkowski type inequality (3.2). In what follows, we consider a $p > 0$, but we should also remember that our results have a value for $p \geq 1$ since for $p < 1$ (3.2) holds. Let us now state and prove our first result of this section.

Theorem 3.1. *Let n positive functions $f_1, \dots, f_n$ on $[a, b]$ such that $0 < m_i \leq f_i \leq M_i$ for $i = 1, \dots, n$ and for all $x \in [a, b]$. Moreover, let $f_i \in \mathcal{L}^p([a, b])$ for $i = 1, \dots, n$ and $p > 0$, then*

$$\|f_1\|_p + \dots + \|f_n\|_p \leq \left(\sum_{k=1}^{n} \frac{M_k}{\sum_{\substack{i=1 \\ i \neq k}}^{n} m_i + M_k} \right) \|f_1 + \dots + f_n\|_p \, . \tag{3.3}$$

Proof: For a $k \in \{1, \dots, n\}$, we have that

$$\frac{f_1 + \dots + f_n}{f_k} = \frac{\sum_{\substack{i=1 \\ i \neq k}}^{n} f_i}{f_k} + 1 \geq \frac{\sum_{\substack{i=1 \\ i \neq k}}^{n} m_i}{M_k} + 1 = \frac{\sum_{\substack{i=1 \\ i \neq k}}^{n} m_i + M_k}{M_k},$$

and so,

$$\frac{f_k}{f_1+\cdots+f_n} \le \frac{M_k}{\sum_{\substack{i=1\\i\neq k}}^n m_i + M_k}.$$

or

$$f_k^p \le \left(\frac{M_k}{\sum_{\substack{i=1\\i\neq k}}^n m_i + M_k}\right)^p (f_1+\cdots+f_n)^p.$$

Integrating the above inequality from a to b and then raising both sides to the power of $1/p$ we get

$$\|f_k\|_p \le \frac{M_k}{\sum_{\substack{i=1\\i\neq k}}^n m_i + M_k} \|f_1+\cdots+f_n\|_p .$$

Lastly, by summing form $k=1$ to n we get (3.3). ■

We can test our inequality for $n=2$ by using the reverse Minkowski type inequality (1.8) that was proved in [2]. Theorem 3.1 for $n=2$ gives

$$\|f_1\|_p + \|f_2\|_p \le \left(\frac{M_1}{m_2+M_1} + \frac{M_2}{m_1+M_2}\right) \|f_1+f_2\|_p ,$$

and one sees that the bounds are identical. So, Theorem 3.1 is a generalization of (1.8) for $n>2$ functions.

If we consider n positive functions with the same bounds $0<m\le f_k\le M$, then Theorem 3.1 gives

$$\|f_1\|_p + \cdots + \|f_n\|_p \le \frac{nM}{m^{n-1}+M} \|f_1+\cdots+f_n\|_p .$$

Observe that the above inequality does not provide a better bound than (3.2) since $\frac{nM}{m^{n-1}+M} > 1$ for $M>m$. But, as we have already seen (3.2) holds only for $p<1$ and so Theorem 3.1 has a value for $p\ge 1$.

By observing Theorem 3.1 and its proof, we see that we can replace the bounding conditions to prove another result which has its own importance.

Theorem 3.2. *Let n positive functions $f_1,\ldots,f_n$ on $[a,b]$ such that $0<\frac{f_i}{f_1+\cdots+f_n}\le A_i<1$ for $i=1,\ldots,n$ and for all $x\in[a,b]$. Moreover, let $f_i\in\mathcal{L}^p([a,b])$ for $i=1,\ldots,n$ and $p>0$, then*

$$\|f_1\|_p + \cdots + \|f_n\|_p \le \left(\sum_{k=1}^n A_k\right) \|f_1+\cdots+f_n\|_p . \tag{3.4}$$

Proof: For a $k\in\{1,\ldots,n\}$, we have

$$f_k^p \le A_k^p (f_1+\cdots+f_n)^p$$

Integrating the above inequality from a to b and then raising both sides to the power of $1/p$ we get

$$\|f_k\|_p \le A_k \|f_1+\cdots+f_n\|_p .$$

Lastly, by summing form $k=1$ to n we directly get (3.4). ■

For $n=2$ Theorem 3.2 gives

$$\|f_1\|_p + \|f_2\|_p \leq (A_1 + A_2) \|f_1 + f_2\|_p \,. \tag{3.5}$$

In fact this is exactly inequality (1.7) from [1]. To see this let us rewrite the conditions of Theorem 3.2 for $n = 2$ as

$$\frac{1 - A_2}{A_2} \leq \frac{f_1}{f_2} \leq \frac{A_1}{1 - A_1}.$$

By setting $m = \frac{1-A_2}{A_2}$ and $M = \frac{A_1}{1-A_1}$ and after simple algebraic calculations we see that the quantity $\frac{M(m+1)+(M+1)}{(m+1)(M+1)}$ of (1.7) is equal to $A_1 + A_2$. So, (1.7) and (3.5) are equivalent. Consequently, Theorem 3.2 can be regarded as a generalization of (1.7) for $n > 2$ functions.

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