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SEVERAL NEW CLOSED-FORM EVALUATIONS OF THE GENERALIZED HYPERGEOMETRIC FUNCTION WITH ARGUMENT $\frac{1}{16}$

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ABSTRACT. The main objective of this paper is to establish as many as thirty new closed-form evaluations of the generalized hypergeometric function ${}_{q+1}F_q(z)$ for $q = 2, 3, 4$. This is achieved by means of separating the generalized hypergeometric function ${}_{q+1}F_q(z)$ for $q = 1, 2, 3, 4, 5$ into even and odd components together with the use of several known infinite series involving central binomial coefficients obtained earlier by Ji and Hei & Ji and Zhang.

Key words and phrases: Generalized hypergeometric function; Central binomial coefficients; Combinatorial sum; Reciprocals.

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1. INTRODUCTION

The generalized hypergeometric function ${}_pF_q(z)$ with p numerator and q denominator parameters is defined by [13]

$$ {}_pF_q\left[\begin{array}{cc} a_1, a_2, \ldots, a_p & \\ & ; z \\ b_1, b_2, \ldots, b_q & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots (b_q)_n} \frac{z^n}{n!}, \tag{1.1} $$

where $(a)_n$ is the well-known Pochhammer's symbol defined by

$$ (a)_n = \begin{cases} a(a+1)\ldots(a+n-1) & ;\ n \in \mathbb{N} \\ 1 & ;\ n = 0. \end{cases} $$

In terms of gamma function, we have

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. $$

Here, as usual, p and q are non-negative integers and the parameters a_j $(1 \le j \le p)$ and b_j $(1 \le j \le q)$ can have arbitrary complex values with zero or negative integer values of b_j excluded. The generalized hypergeometric function ${}_pF_q(z)$ converges for $|z| < \infty, (p \le q)$, $|z| < 1 (p = q+1)$ and $|z| = 1$ $(p = q+1$ and $Re(s) > 0)$, where s is the parametric excess defined by

$$ s = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j. $$

It is not out of place to mention here that the generalized hypergeometric function occurs in many theoretical and practical applications such as mathematics, theoretical physics, engineering and statistics. For more details about this function, we refer [1, 2, 3, 10, 12, 17].
Further, it is well-known that the process of resolving a generalized hypergeometric function into even and odd components can lead to new results. This composition is facilitated by use of the identities

$$ (a)_{2n} = 2^{2n} \left(\frac{a}{2}\right)_n \left(\frac{a}{2} + \frac{1}{2}\right)_n $$

and

$$ (a)_{2n+1} = a 2^{2n} \left(\frac{a}{2} + \frac{1}{2}\right)_n \left(\frac{a}{2} + 1\right)_n. $$

Then for the generalized hypergeometric function

$$ {}_{q+1}F_q\left[\begin{array}{cc} a_1, a_2, \ldots, a_{q+1} & \\ & ; \pm z \\ b_1, b_2, \ldots, b_q & \end{array}\right], $$

it is not difficult to obtain the following two general results:

$$ \begin{aligned} & {}_{q+1}F_q\left[\begin{array}{cc} a_1, a_2, \ldots, a_{q+1} & \\ & ; z \\ b_1, b_2, \ldots, b_q & \end{array}\right] + {}_{q+1}F_q\left[\begin{array}{cc} a_1, a_2, \ldots, a_{q+1} & \\ & ; -z \\ b_1, b_2, \ldots, b_q & \end{array}\right] \\ & = 2\, {}_{2q+2}F_{2q+1}\left[\begin{array}{cc} \frac{a_1}{2}, \frac{a_1}{2} + \frac{1}{2}, \ldots, \frac{a_{q+1}}{2}, \frac{a_{q+1}}{2} + \frac{1}{2} & \\ & ; z^2 \\ \frac{1}{2}, \frac{b_1}{2}, \frac{b_1}{2} + \frac{1}{2}, \ldots, \frac{b_q}{2}, \frac{b_q}{2} + \frac{1}{2} & \end{array}\right] \end{aligned} \tag{1.2} $$

and

$$
\begin{aligned}
&{}_{q+1}F_q\left[\begin{matrix} a_1, a_2, \ldots, a_{q+1} \\ b_1, b_2, \ldots, b_q \end{matrix}; z\right] - {}_{q+1}F_q\left[\begin{matrix} a_1, a_2, \ldots, a_{q+1} \\ b_1, b_2, \ldots, b_q \end{matrix}; -z\right] \\
&= \frac{2za_1a_2\ldots a_{q+1}}{b_1b_2\ldots b_q}\, {}_{2q+2}F_{2q+1}\left[\begin{matrix} \frac{a_1}{2}+\frac{1}{2}, \frac{a_1}{2}+1, \ldots, \frac{a_{q+1}}{2}+\frac{1}{2}, \frac{a_{q+1}}{2}+1 \\ \frac{3}{2}, \frac{b_1}{2}+\frac{1}{2}, \frac{b_1}{2}+1, \ldots, \frac{b_{q+1}}{2}+\frac{1}{2}, \frac{b_q}{2}+1 \end{matrix}; z^2\right]. \qquad (1.3)
\end{aligned}
$$

The convergence condition for the series ${}_{q+1}F_q(z)$ should be

$$\sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j > 0.$$

On the other hand, the binomial coefficients are defined by

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & ;\ n \geq k \\ 0 & ;\ n < k, \end{cases} \qquad (1.4)$$

for nonnegative integers n and k. The central binomial coefficients are defined by

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \qquad (n = 0, 1, 2, \ldots). \qquad (1.5)$$

It is well-known that the binomial and reciprocal of binomial coefficients play an important role in many areas of mathematics (including number theory, probability and statistics). Actually the sums containing the central binomial coefficients and reciprocals of the central binomial coefficients have been studied for a long time. A large number of very interesting results can be seen in the research papers by Lehmer [8], Mansour [9], Pla [11], Rockett [15], Sprugnoli [18, 19], Sury [20], Sury et al. [21], Trif [22], Wheelon [23] and Zhao and Wang [24]. Many facts about the central binomial coefficients and the reciprocals of the central binomial coefficients can be found in the book of Koshy [7]. Gould [4] has collected numerous identities involving central binomial coefficients. Riordan [14] is also a good reference.

In 2012, using one known infinite series, Ji and Zhang [6] obtained the following fifteen new alternated series (containing positive and negative terms) of reciprocals of binomial coefficients by splitting items involving Golden ratio $\varphi = \frac{\sqrt{5}+1}{2}$ viz.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{2n}{n}(2n+1)} = \frac{4\sqrt{5}}{5}\ln\varphi, \qquad (1.6)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{2n}{n}(2n+3)} = -\frac{36\sqrt{5}}{5}\ln\varphi + 8, \qquad (1.7)$$

(1.8) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n}{n}(2n+5)}=\frac{572\sqrt{5}}{15}\ln\varphi-\frac{368}{9},$$

(1.9) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n}{n}(2n+7)}=-\frac{916\sqrt{5}}{5}\ln\varphi-\frac{14792}{75},$$

(1.10) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n}{n}(2n+9)}=\frac{29308\sqrt{5}}{35}\ln\varphi-\frac{3311008}{3675},$$

(1.11) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n}{n}(2n+1)(2n+3)}=4\sqrt{5}\ln\varphi-4,$$

(1.12) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n}{n}(2n+1)(2n+5)}=-\frac{28\sqrt{5}}{3}\ln\varphi+\frac{92}{9},$$

(1.13) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n}{n}(2n+3)(2n+5)}=-\frac{68\sqrt{5}}{3}\ln\varphi+\frac{220}{9},$$

(1.14) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n}{n}(2n+1)(2n+7)}=\frac{92\sqrt{5}}{3}\ln\varphi-\frac{7396}{225},$$

(1.15) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n}{n}(2n+3)(2n+7)}=44\sqrt{5}\ln\varphi-\frac{3548}{75},$$

(1.16) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n}{n}(2n+5)(2n+7)}=\frac{332\sqrt{5}}{3}\ln\varphi-\frac{26788}{225},$$

(1.17) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n}{n}(2n+1)(2n+9)}=-\frac{732\sqrt{5}}{21}\ln\varphi+\frac{413876}{3675},$$

(1.18) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n}{n}(2n+3)(2n+9)}=-\frac{2956\sqrt{5}}{21}\ln\varphi+\frac{1670204}{11025},$$

(1.19) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n}{n}(2n+5)(2n+9)}=-\frac{4196\sqrt{5}}{21}\ln\varphi+\frac{2370556}{11025},$$

(1.20) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n}{n}(2n+7)(2n+9)}=-\frac{3572\sqrt{5}}{7}\ln\varphi+\frac{2017908}{3675}.$$

Remark: The results (1.6) to (1.10) are first recorded in Sherman [16].

In addition to this, by the same technique, in 2013, Ji and Hei [5] obtained fifteen new series of reciprocals of binomial coefficients

(1.21) $$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+1)}=\frac{2\sqrt{3}\pi}{9},$$

(1.22) $$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+3)}=\frac{14\sqrt{3}\pi}{9}-8,$$

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+5)}=\frac{74\sqrt{3}\pi}{9}-\frac{400}{9}, \tag{1.23}$$

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+7)}=\frac{1774\sqrt{3}\pi}{45}-\frac{16072}{75}, \tag{1.24}$$

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+9)}=\frac{56758\sqrt{3}\pi}{315}-\frac{3602528}{3675}, \tag{1.25}$$

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+1)(2n+3)}=-\frac{2\sqrt{3}\pi}{3}+4, \tag{1.26}$$

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+1)(2n+5)}=-2\sqrt{3}\pi+\frac{100}{9}, \tag{1.27}$$

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+3)(2n+5)}=-\frac{10\sqrt{3}\pi}{3}+\frac{164}{9}, \tag{1.28}$$

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+1)(2n+7)}=-\frac{98\sqrt{3}\pi}{15}+\frac{8036}{225}, \tag{1.29}$$

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+3)(2n+7)}=-\frac{142\sqrt{3}\pi}{15}+\frac{3868}{75}, \tag{1.30}$$

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+5)(2n+7)}=-\frac{78\sqrt{3}\pi}{5}+\frac{19108}{225}, \tag{1.31}$$

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+1)(2n+9)}=-\frac{2362\sqrt{3}\pi}{105}+\frac{450316}{3675}, \tag{1.32}$$

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+3)(2n+9)}=-\frac{1042\sqrt{3}\pi}{35}+\frac{1786564}{11025}, \tag{1.33}$$

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+5)(2n+9)}=-\frac{4514\sqrt{3}\pi}{105}+\frac{2579396}{11025}, \tag{1.34}$$

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+7)(2n+9)}=-\frac{1478\sqrt{3}\pi}{21}+\frac{56300}{147}. \tag{1.35}$$

Remark: The results (1.21) to (1.25) are first recorded in Sherman [16].

The rest of the paper is organized as follows. In Section 2, the results (1.6) to (1.35) have been expressed in terms of the generalized hypergeometric functions that will be required in our present investigations. In Section 3, we shall establish as many as thirty new closed-forms evaluations of the generalized hypergeometric functions ${}_{q+1}F_q(z)$ for $q = 2, 3, 4$. This is achieved by means of separating the generalized hypergeometric functions ${}_{q+1}F_q(z)$ for $q = 1, 2, 3, 4, 5$ in to even and odd components together with the use of the results (2.1) to (2.30) given in Section 2.

2. RESULTS (1.6) TO (1.35) IN TERMS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

In this section, we shall express the results (1.6) to (1.35) in terms of generalized hypergeometric functions as follows:

$$ {}_2F_1\left[\begin{matrix}1,1\\ \frac{3}{2}\end{matrix};-\frac{1}{4}\right]=\frac{4\sqrt{5}}{5}\ln\varphi, \tag{2.1}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{3}{2}\\ \frac{1}{2},\frac{5}{2}\end{matrix};-\frac{1}{4}\right]=\frac{12}{5}\left(10-9\sqrt{5}\ln\varphi\right), \tag{2.2}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{5}{2}\\ \frac{1}{2},\frac{7}{2}\end{matrix};-\frac{1}{4}\right]=\frac{20}{45}\left(429\sqrt{5}\ln\varphi-460\right), \tag{2.3}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{7}{2}\\ \frac{1}{2},\frac{9}{2}\end{matrix};-\frac{1}{4}\right]=\frac{28}{75}\left(3698-3435\sqrt{5}\ln\varphi\right), \tag{2.4}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{9}{2}\\ \frac{1}{2},\frac{11}{2}\end{matrix};-\frac{1}{4}\right]=\frac{12}{1225}\left(769335\sqrt{5}\ln\varphi-827752\right), \tag{2.5}$$

$$ {}_2F_1\left[\begin{matrix}1,1\\ \frac{5}{2}\end{matrix};-\frac{1}{4}\right]=12\left(\sqrt{5}\ln\varphi-1\right), \tag{2.6}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{5}{2}\\ \frac{3}{2},\frac{7}{2}\end{matrix};-\frac{1}{4}\right]=\frac{20}{9}\left(23-21\sqrt{5}\ln\varphi\right), \tag{2.7}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{3}{2}\\ \frac{1}{2},\frac{7}{2}\end{matrix};-\frac{1}{4}\right]=\frac{20}{3}\left(55-51\sqrt{5}\ln\varphi\right), \tag{2.8}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{7}{2}\\ \frac{3}{2},\frac{9}{2}\end{matrix};-\frac{1}{4}\right]=\frac{28}{225}\left(1725\sqrt{5}\ln\varphi-1849\right), \tag{2.9}$$

$$ {}_4F_3\left[\begin{matrix}1,1,\frac{3}{2},\frac{7}{2}\\ \frac{1}{2},\frac{5}{2},\frac{9}{2}\end{matrix};-\frac{1}{4}\right]=\frac{28}{25}\left(825\sqrt{5}\ln\varphi-887\right), \tag{2.10}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{5}{2}\\ \frac{1}{2},\frac{9}{2}\end{matrix};-\frac{1}{4}\right]=\frac{140}{225}\left(6225\sqrt{5}\ln\varphi-6697\right), \tag{2.11}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{9}{2}\\ \frac{3}{2},\frac{11}{2}\end{matrix};-\frac{1}{4}\right]=\frac{12}{1225}\left(103469-96075\sqrt{5}\ln\varphi\right), \tag{2.12}$$

$$ {}_4F_3\left[\begin{matrix}1,1,\frac{3}{2},\frac{9}{2}\\ \frac{1}{2},\frac{5}{2},\frac{11}{2}\end{matrix};-\frac{1}{4}\right]=\frac{36}{7175}\left(757475\sqrt{5}\ln\varphi-35074284\right), \tag{2.13}$$

$$ {}_4F_3\left[\begin{matrix}1,1,\frac{5}{2},\frac{9}{2}\\ \frac{1}{2},\frac{7}{2},\frac{11}{2}\end{matrix};-\frac{1}{4}\right]=\frac{4}{245}\left(592639-550725\sqrt{5}\ln\varphi\right), \tag{2.14}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{7}{2},\\ \frac{1}{2},\frac{11}{2}\end{matrix};-\frac{1}{4}\right]=\frac{12}{175}\left(504477-468825\sqrt{5}\ln\varphi\right), \tag{2.15}$$

$$ {}_2F_1\left[\begin{matrix}1,1\\ \frac{3}{2}\end{matrix};\frac{1}{4}\right]=\frac{2\sqrt{3}\pi}{9}, \tag{2.16}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{3}{2}\\ \frac{1}{2},\frac{5}{2}\end{matrix};\frac{1}{4}\right]=\frac{14\sqrt{3}\pi}{3}-24, \tag{2.17}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{5}{2}\\ \frac{1}{2},\frac{7}{2}\end{matrix};\frac{1}{4}\right]=\frac{5}{9}\left(74\sqrt{3}\pi-400\right), \tag{2.18}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{7}{2}\\ \frac{1}{2},\frac{9}{2}\end{matrix};\frac{1}{4}\right]=\frac{14}{225}\left(4435\sqrt{3}\pi-24108\right), \tag{2.19}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{9}{2}\\ \frac{1}{2},\frac{11}{2}\end{matrix};\frac{1}{4}\right]=\frac{2}{1225}\left(993265\sqrt{3}\pi-5403792\right), \tag{2.20}$$

$$ {}_2F_1\left[\begin{matrix}1,1\\ \frac{5}{2}\end{matrix};\frac{1}{4}\right]=2(6-\sqrt{3}\pi), \tag{2.21}$$

$$(2.22)\qquad {}_3F_2\left[\begin{matrix}1,1,\frac{5}{2}\\ \frac{3}{2},\frac{7}{2}\end{matrix};\frac{1}{4}\right]=\frac{10}{9}\left(50-9\sqrt{3}\pi\right),$$

$$(2.23)\qquad {}_3F_2\left[\begin{matrix}1,1,\frac{3}{2}\\ \frac{1}{2},\frac{7}{2}\end{matrix};\frac{1}{4}\right]=\frac{5}{3}\left(164-30\sqrt{3}\pi\right),$$

$$(2.24)\qquad {}_3F_2\left[\begin{matrix}1,1,\frac{7}{2}\\ \frac{3}{2},\frac{9}{2}\end{matrix};\frac{1}{4}\right]=\frac{14}{225}\left(4018-735\sqrt{3}\pi\right),$$

$$(2.25)\qquad {}_4F_3\left[\begin{matrix}1,1,\frac{3}{2},\frac{7}{2}\\ \frac{1}{2},\frac{5}{2},\frac{9}{2}\end{matrix};\frac{1}{4}\right]=\frac{14}{25}\left(1934-355\sqrt{3}\pi\right),$$

$$(2.26)\qquad {}_3F_2\left[\begin{matrix}1,1,\frac{5}{2}\\ \frac{1}{2},\frac{9}{2}\end{matrix};\frac{1}{4}\right]=\frac{14}{75}\left(9554-2925\sqrt{3}\pi\right),$$

$$(2.27)\qquad {}_3F_2\left[\begin{matrix}1,1,\frac{9}{2}\\ \frac{3}{2},\frac{11}{2}\end{matrix};\frac{1}{4}\right]=\frac{24}{1225}\left(56277-11515\sqrt{3}\pi\right),$$

$$(2.28)\qquad {}_4F_3\left[\begin{matrix}1,1,\frac{3}{2},\frac{9}{2}\\ \frac{1}{2},\frac{5}{2},\frac{11}{2}\end{matrix};\frac{1}{4}\right]=\frac{27}{35}\left(-1042\sqrt{3}\pi+\frac{1786564}{315}\right),$$

$$(2.29)\qquad {}_4F_3\left[\begin{matrix}1,1,\frac{5}{2},\frac{9}{2}\\ \frac{1}{2},\frac{7}{2},\frac{11}{2}\end{matrix};\frac{1}{4}\right]=\frac{6}{7}\left(-2257\sqrt{3}\pi+\frac{1289698}{105}\right),$$

$$(2.30)\qquad {}_3F_2\left[\begin{matrix}1,1,\frac{7}{2}\\ \frac{1}{2},\frac{11}{2}\end{matrix};\frac{1}{4}\right]=\frac{6}{7}\left(-5173\sqrt{3}\pi+28150\right).$$

3. MAIN RESULTS

In this section, we shall establish the following thirty closed-form evaluations of the generalized hypergeometric function with argument $\frac{1}{16}$.

$$(3.1)\qquad {}_3F_2\left[\begin{matrix}\frac{1}{2},1,1\\ \frac{3}{4},\frac{5}{4}\end{matrix};\frac{1}{16}\right]=\frac{1}{45}\left(5\sqrt{3}\pi+18\sqrt{5}\ln\varphi\right),$$

$$(3.2)\qquad {}_3F_2\left[\begin{matrix}1,1,\frac{3}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};\frac{1}{16}\right]=\frac{2}{15}\left(5\sqrt{3}\pi-18\sqrt{5}\ln\varphi\right),$$

$$(3.3)\qquad {}_3F_2\left[\begin{matrix}\frac{1}{2},1,1\\ \frac{1}{4},\frac{7}{4}\end{matrix};\frac{1}{16}\right]=\frac{1}{15}\left(35\sqrt{3}\pi-162\sqrt{5}\ln\varphi\right),$$

$$(3.4)\qquad {}_3F_2\left[\begin{matrix}1,1,\frac{3}{2}\\ \frac{3}{4},\frac{9}{4}\end{matrix};\frac{1}{16}\right]=\frac{2}{9}\left(35\sqrt{3}\pi-360+162\sqrt{5}\ln\varphi\right),$$

$$(3.5)\qquad {}_4F_3\left[\begin{matrix}\frac{1}{2},1,1,\frac{5}{4}\\ \frac{1}{4},\frac{3}{4},\frac{9}{4}\end{matrix};\frac{1}{16}\right]=\frac{1}{9}\left(185\sqrt{3}\pi-1920+858\sqrt{5}\ln\varphi\right),$$

$$(3.6)\qquad {}_4F_3\left[\begin{matrix}1,1,\frac{3}{2},\frac{7}{4}\\ \frac{3}{4},\frac{5}{4},\frac{11}{4}\end{matrix};\frac{1}{16}\right]=\frac{14}{45}\left(185\sqrt{3}\pi-80-858\sqrt{5}\ln\varphi\right),$$

$$(3.7)\qquad {}_4F_3\left[\begin{matrix}\frac{1}{2},1,1,\frac{7}{4}\\ \frac{1}{4},\frac{3}{4},\frac{11}{4}\end{matrix};\frac{1}{16}\right]=\frac{1}{45}\left(6209\sqrt{3}\pi-2688-28854\sqrt{5}\ln\varphi\right),$$

$$(3.8)\qquad {}_4F_3\left[\begin{matrix}1,1,\frac{3}{2},\frac{9}{4}\\ \frac{3}{4},\frac{5}{4},\frac{13}{4}\end{matrix};\frac{1}{16}\right]=\frac{2}{25}\left(4435\sqrt{3}\pi-46296+20610\sqrt{5}\ln\varphi\right),$$

$$(3.9)\qquad {}_4F_3\left[\begin{matrix}\frac{1}{2},1,1,\frac{9}{4}\\ \frac{1}{4},\frac{3}{4},\frac{13}{4}\end{matrix};\frac{1}{16}\right]=\frac{1}{175}\left(141895\sqrt{3}\pi-1481472+659430\sqrt{5}\ln\varphi\right),$$

$$(3.10)\qquad {}_4F_3\left[\begin{matrix}1,1,\frac{3}{2},\frac{11}{4}\\ \frac{3}{4},\frac{5}{4},\frac{15}{4}\end{matrix};\frac{1}{16}\right]=\frac{11}{2205}\left(397306\sqrt{3}\pi-174912-1846404\sqrt{5}\ln\varphi\right),$$

$$(3.11)\qquad {}_3F_2\left[\begin{matrix}\frac{1}{2},1,1\\ \frac{3}{4},\frac{9}{4}\end{matrix};\frac{1}{16}\right]=6\sqrt{5}\ln\varphi-\sqrt{3}\pi,$$

$$(3.12)\qquad {}_3F_2\left[\begin{matrix}1,1,\frac{3}{2},\\ \frac{11}{4},\frac{13}{4}\end{matrix};\frac{1}{16}\right]=10\left(12-\sqrt{3}\pi-6\sqrt{5}\ln\varphi\right),$$

$$(3.13)\quad {}_3F_2\left[\begin{matrix}\frac{1}{2},1,1\\ \frac{3}{4},\frac{9}{4}\end{matrix};\frac{1}{16}\right]=\frac{5}{3}\left(32-3\sqrt{3}\pi-14\sqrt{5}\ln\varphi\right),$$

$$(3.14)\quad {}_3F_2\left[\begin{matrix}1,1,\frac{3}{2}\\ \frac{5}{4},\frac{11}{4}\end{matrix};\frac{1}{16}\right]=\frac{42}{9}\left(4-9\sqrt{3}\pi+42\sqrt{5}\ln\varphi\right),$$

$$(3.15)\quad {}_4F_3\left[\begin{matrix}\frac{1}{2},1,1,\frac{5}{4}\\ \frac{1}{4},\frac{7}{4},\frac{9}{4}\end{matrix};\frac{1}{16}\right]=5\left(64-5\sqrt{3}\pi-34\sqrt{5}\ln\varphi\right),$$

$$(3.16)\quad {}_4F_3\left[\begin{matrix}1,1,\frac{3}{2},\frac{7}{4}\\ \frac{3}{4},\frac{9}{4},\frac{11}{4}\end{matrix};\frac{1}{16}\right]=\frac{70}{9}\left(102\sqrt{5}\ln\varphi-28-15\sqrt{3}\pi\right),$$

$$(3.17)\quad {}_4F_3\left[\begin{matrix}\frac{1}{2},1,1,\frac{7}{4}\\ \frac{3}{4},\frac{5}{4},\frac{11}{4}\end{matrix};\frac{1}{16}\right]=\frac{7}{45}\left(690\sqrt{5}\ln\varphi+64-147\sqrt{3}\pi\right),$$

$$(3.18)\quad {}_4F_3\left[\begin{matrix}1,1,\frac{3}{2},\frac{9}{4}\\ \frac{5}{4},\frac{7}{4},\frac{13}{4}\end{matrix};\frac{1}{16}\right]=\frac{18}{25}\left(2572-245\sqrt{3}\pi-1150\sqrt{5}\ln\varphi\right),$$

$$(3.19)\quad {}_3F_2\left[\begin{matrix}\frac{1}{2},1,1\\ \frac{1}{4},\frac{11}{4}\end{matrix};\frac{1}{16}\right]=\frac{7}{5}\left(330\sqrt{5}\ln\varphi+32-71\sqrt{3}\pi\right),$$

$$(3.20)\quad {}_3F_2\left[\begin{matrix}1,1,\frac{3}{2}\\ \frac{3}{4},\frac{13}{4}\end{matrix};\frac{1}{16}\right]=\frac{6}{5}\left(3708-355\sqrt{3}\pi-1650\sqrt{5}\ln\varphi\right),$$

$$(3.21)\quad {}_5F_4\left[\begin{matrix}\frac{1}{2},1,1,\frac{5}{4},\frac{7}{4}\\ \frac{1}{4},\frac{3}{4},\frac{9}{4},\frac{11}{4}\end{matrix};\frac{1}{16}\right]=\frac{7}{3}\left(830\sqrt{5}\ln\varphi-117\sqrt{3}\pi-256\right),$$

$$(3.22)\quad {}_5F_4\left[\begin{matrix}1,1,\frac{3}{2},\frac{7}{4},\frac{9}{4}\\ \frac{3}{4},\frac{5}{4},\frac{11}{4},\frac{13}{4}\end{matrix};\frac{1}{16}\right]=\frac{126}{225}\left(22948-1755\sqrt{3}\pi-12450\sqrt{5}\ln\varphi\right),$$

$$(3.23)\quad {}_4F_3\left[\begin{matrix}\frac{1}{2},1,1,\frac{9}{4}\\ \frac{3}{4},\frac{5}{4},\frac{13}{4}\end{matrix};\frac{1}{16}\right]=\frac{3}{175}\left(61728-5905\sqrt{3}\pi-27450\sqrt{5}\ln\varphi\right),$$

$$(3.24)\quad {}_4F_3\left[\begin{matrix}1,1,\frac{3}{2},\frac{11}{4}\\ \frac{5}{4},\frac{7}{4},\frac{15}{4}\end{matrix};\frac{1}{16}\right]=\frac{22}{245}\left(3644-8267\sqrt{3}\pi+38430\sqrt{5}\ln\varphi\right),$$

$$(3.25)\quad {}_4F_3\left[\begin{matrix}\frac{1}{2},1,1,\frac{9}{4}\\ \frac{1}{4},\frac{7}{4},\frac{13}{4}\end{matrix};\frac{1}{16}\right]=\frac{9}{175}\left(82304-7815\sqrt{3}\pi-36950\sqrt{5}\ln\varphi\right),$$

$$(3.26)\quad {}_4F_3\left[\begin{matrix}1,1,\frac{3}{2},\frac{11}{4}\\ \frac{3}{4},\frac{9}{4},\frac{15}{4}\end{matrix};\frac{1}{16}\right]=\frac{22}{441}\left(11636-32823\sqrt{3}\pi+155190\sqrt{5}\ln\varphi\right),$$

$$(3.27)\quad {}_4F_3\left[\begin{matrix}\frac{1}{2},1,1,\frac{5}{4}\\ \frac{1}{4},\frac{3}{4},\frac{13}{4}\end{matrix};\frac{1}{16}\right]=\frac{3}{35}\left(11285\sqrt{3}\pi+117856-52450\sqrt{5}\ln\varphi\right),$$

$$(3.28)\quad {}_4F_3\left[\begin{matrix}1,1,\frac{3}{2},\frac{7}{4}\\ \frac{3}{4},\frac{7}{4},\frac{15}{4}\end{matrix};\frac{1}{16}\right]=\frac{22}{315}\left(20884-47397\sqrt{3}\pi+220290\sqrt{5}\ln\varphi\right),$$

$$(3.29)\quad {}_5F_4\left[\begin{matrix}\frac{1}{2},1,1,\frac{7}{4},\frac{9}{4}\\ \frac{1}{4},\frac{3}{4},\frac{11}{4},\frac{13}{4}\end{matrix};\frac{1}{16}\right]=\frac{3}{25}\left(244672-18475\sqrt{3}\pi-133950\sqrt{5}\ln\varphi\right),$$

$$(3.30)\quad {}_5F_4\left[\begin{matrix}1,1,\frac{3}{2},\frac{9}{4},\frac{11}{4}\\ \frac{3}{4},\frac{5}{4},\frac{13}{4},\frac{15}{4}\end{matrix};\frac{1}{16}\right]=\frac{66}{1225}\left(937650\sqrt{5}\ln\varphi-305204-129325\sqrt{3}\pi\right).$$

Proof. The derivations of the results (3.1) to (3.30) are quite straight forward. Thus in order to establish the results (3.1) and (3.2), we proceed as follows. In (1.2) and (1.3) if we set $q=2, a_1=a_2=1$ and $b_1=3/2$ and making use of the results (2.1) and (2.16) and (1.2) in the left-hand side (1.2) and (1.3), we at once arrive at the results (3.1) and (3.2) respectively. The remaining results (3.3) to (3.30) can be proven on similar lines by taking appropriate values of q, a_{q+1} and b_q, for $q=1,2,3,4,5$. So we left this as an exercise to the interest of the reader.

We conclude this section by remarking that the results (3.1) to (3.30) have been verified by using MAPLE. ∎

4. CONCLUDING REMARK

In this paper, thirty new-closed form evaluations of the generalized hypergeometric functions ${}_{q+1}F_q(z)$ for $q=2,3,4$ with arguments $\frac{1}{16}$ have been established. This is achieved by means of separating the generalized hypergeometric function ${}_{q+1}F_q(z)$ into even and odd components together with the use of the several known results of interesting series involving central binomial coefficients obtained earlier by Ji and Hei [5] & Ji and Zhang [6]. We believe that the results established in this paper have not been appeared in the literature and represent a definite contribution to the theory of generalized hypergeometric function. It is hoped that the results could be of potential use in the area of mathematics, statistics and mathematical physics.

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