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THREE INEQUALITIES ASSOCIATED WITH RADO INEQUALITY

RIN MIYAO, YUSUKE NISHIZAWA, AND KEIGO TAKAMURA

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FACULTY OF EDUCATION, SAITAMA UNIVERSITY, SHIMO-OKUBO 255, SAKURA-KU, SAITAMA-CITY, SAITAMA, JAPAN.
r.miyao.242@ms.saitama-u.ac.jp

FACULTY OF EDUCATION, SAITAMA UNIVERSITY, SHIMO-OKUBO 255, SAKURA-KU, SAITAMA-CITY, SAITAMA, JAPAN.
ynishizawa@mail.saitama-u.ac.jp

FACULTY OF EDUCATION, SAITAMA UNIVERSITY, SHIMO-OKUBO 255, SAKURA-KU, SAITAMA-CITY, SAITAMA, JAPAN.
k.takamura.442@ms.saitama-u.ac.jp

ABSTRACT. In this short note we estimate three inequalities associated with Rado inequality and show the refinement and reverse of Arithmetic mean- Geometric mean inequality.

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1. INTRODUCTION

We define the arithmetic mean A_n and the geometric mean G_n by

$$A_n = \frac{1}{n}\sum_{k=1}^{n} a_k \quad and \quad G_n = \left(\prod_{k=1}^{n} a_k\right)^{\frac{1}{n}}$$

for positive real numbers $a_1, a_2, \cdots, a_n$. It is known that the inequality

$$A_n - G_n \geq \frac{n-1}{n}\left(A_{n-1} - G_{n-1}\right) \tag{1.1}$$

holds for the integer $n \geq 2$. The inequality (1.1) is known as Rado inequality, see [1] in pp. 94–105 and [5] in pp. 94. From the inequality (1.1), we can get Arithmetic mean-Geometric mean inequality $A_n - G_n \geq 0$, immediately. We show the refinement and reverse of the inequality (1.1) as follows.

Theorem 1.1. *Let $a_1, a_2, \cdots, a_n$ be positive real numbers, then the inequality*

$$A_n - G_n \geq \frac{n-1}{n}\left(A_{n-1} - G_{n-1}\right) + \frac{n-1}{n} \cdot \frac{(G_n - G_{n-1})^2}{G_{n-1}} \tag{1.2}$$

holds for the integer $n \geq 2$.

Theorem 1.2. *Let l and n be integers with $2 \leq l \leq n$ and let $a_{l-1}, a_l, \cdots, a_n$ be real numbers with $0 < a_{l-1}, a_l, \cdots, a_n$, then the inequality*

$$A_n - G_n \leq \frac{n-1}{n}\left(A_{n-1} - G_{n-1}\right) + \frac{a_n \mathcal{R}_l}{n} \tag{1.3}$$

holds for the integer $n \geq l$, where $\mathcal{R}_l = 1 + (l-1)\left(\frac{G_n^n}{a_n^n}\right)^{\frac{1}{l-1}} - l\left(\frac{G_n^n}{a_n^n}\right)^{\frac{1}{l}}$.

Moreover, we obtain the following refinement and reverse of Arithmetic mean-Geometric mean inequality.

Corollary 1.3. *Let $a_1, a_2, \cdots, a_n$ be positive real numbers, then the inequality*

$$A_n - G_n \geq \frac{1}{n}\sum_{k=1}^{n-1} \frac{(G_{k+1} - G_k)^2}{G_k}$$

holds for the integer $n \geq 2$.

Corollary 1.4. *Let $a_1, a_2, \cdots, a_n$ be real numbers with $0 < a_1 \leq a_2 \leq \cdots \leq a_n$, then the inequality*

$$A_n - G_n \leq \frac{\mathcal{R}_2}{n}\sum_{k=2}^{n} a_k$$

holds for the integer $n \geq 2$, where $\mathcal{R}_2 = \left(1 - \frac{G_n^{\frac{n}{2}}}{a_n^{\frac{n}{2}}}\right)^2$.

If A_n and G_n are the arithmetic and geometric means of the integers $1, 2, \cdots, n$, respectively, then McCartin [4] asserts that $\lim_{n\to\infty} A_n/G_n = e/2$. Recently, Hassani [2] proved that the rational expression A_n/G_n is strictly increasing for the integer $n \geq 1$ and the inequality $A_n/G_n < e/2$ holds for the integer $n \geq 1$. As mentioned above, A_n and G_n have interesting relations for the integers $1, 2, \cdots, n$. We show the following Rado type inequality for positive integers $1, 2, \cdots, n$.

Theorem 1.5. *Let A_n and G_n be the arithmetic and geometric means of the integers $1, 2, \cdots, n$, respectively, then the inequality*

$$A_n - G_n > \left(\frac{n-1}{n}\right)^{-1} (A_{n-1} - G_{n-1}) \tag{1.4}$$

holds for the integer $n \geq 2$.

2. PROOF OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1. We set

$$f(x, y) = x^y - yx + y - 1 - (y-1)(x-1)^2 \tag{2.1}$$

for the real numbers $x > 0$ and $y \geq 2$. Since the derivatives of $f(x, y)$ are

$$\frac{\partial f}{\partial y}(x, y) = x^y(\ln x) - x^2 + x$$

and

$$\frac{\partial^2 f}{\partial y^2}(x, y) = x^y(\ln x)^2 > 0,$$

$\partial f/\partial y(x, y)$ is strictly increasing for $y > 2$. We have $\partial f/\partial y(x, 2) = xg(x)$, where $g(x) = x(\ln x) - x + 1$. Since the derivative of $g(x)$ is $g'(x) = \ln x$, we have $g(x) \geq g(1) = 0$ and $\partial f/\partial y(x, 2) > 0$ for $x \neq 1$. Therefore, $f(x, y)$ is strictly increasing for $y > 2$ and $f(x, 2) = 0$, also, $f(1, y) = 0$ for $y \geq 2$. Thus, we have $f(x, y) \geq 0$ with equality if and only if $y = 2$ or $x = 1$. Put $x = G_n/G_{n-1}$ and $y = n$ in (2.1), then we have

$$\left(\frac{G_n}{G_{n-1}}\right)^n \geq n\frac{G_n}{G_{n-1}} - n + 1 + (n-1)\left(\frac{G_n}{G_{n-1}} - 1\right)^2 \tag{2.2}$$

with equality if and only if $n = 2$ or $G_n = G_{n-1}$. By the inequality (2.2) and

$$A_n = \frac{G_{n-1}}{n}\left((n-1)\frac{A_{n-1}}{G_{n-1}} + \left(\frac{G_n}{G_{n-1}}\right)^n\right),$$

we can get

$$\begin{aligned} A_n &= \frac{G_{n-1}}{n}\left((n-1)\frac{A_{n-1}}{G_{n-1}} + \left(\frac{G_n}{G_{n-1}}\right)^n\right) \\ &\geq \frac{G_{n-1}}{n}\left((n-1)\frac{A_{n-1}}{G_{n-1}} + n\frac{G_n}{G_{n-1}} - n + 1 + (n-1)\left(\frac{G_n}{G_{n-1}} - 1\right)^2\right) \\ &= \frac{n-1}{n}A_{n-1} + G_n - \frac{n-1}{n}G_{n-1} + \frac{n-1}{n} \cdot \frac{(G_n - G_{n-1})^2}{G_{n-1}}. \end{aligned}$$

Therefore, we have

$$A_n - G_n \geq \frac{n-1}{n}(A_{n-1} - G_{n-1}) + \frac{n-1}{n} \cdot \frac{(G_n - G_{n-1})^2}{G_{n-1}}$$

and the proof of Theorem 1.1 is complete. ∎

Proof of Theorem 1.2. We set

$$f(x, y) = 1 + (y-1)x^{\frac{1}{y-1}} - yx^{\frac{1}{y}} \tag{2.3}$$

for the real numbers $0 < x \leq 1$ and $y \geq 2$. The derivative of $f(x,y)$ is

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= -x^{\frac{1}{y}} + \frac{x^{\frac{1}{y}}\ln x}{y} + x^{\frac{1}{y-1}} - \frac{x^{\frac{1}{y-1}}\ln x}{y-1} \\ &= \frac{x^{\frac{1}{y-1}}(-\ln x + y - 1)}{y-1} - \frac{x^{\frac{1}{y}}(y - \ln x)}{y}.\end{aligned}$$

From $-\ln x + y - 1 > 0$ and $y - \ln x > 0$ for $0 < x \leq 1$ and $y \geq 2$, we can take the logarithm. Hence, we consider the function

$$\begin{aligned}g(x,y) &= \ln\left(\frac{x^{\frac{1}{y-1}}(-\ln x + y - 1)}{y-1}\right) - \ln\left(\frac{x^{\frac{1}{y}}(y - \ln x)}{y}\right) \\ &= \frac{\ln x}{y-1} - \frac{\ln x}{y} + \ln(-\ln x + y - 1) - \ln(y - \ln x) - \ln(y-1) + \ln y.\end{aligned}$$

The derivative of $g(x,y)$ is

$$\frac{\partial g}{\partial y}(x,y) = \frac{(\ln x)^2(-2y\ln x + \ln x + 3y^2 - 3y + 1)}{(y-1)^2 y^2(-\ln x + y - 1)(y - \ln x)}.$$

By $-2y\ln x + \ln x > 0$ and $3y^2 - 3y + 1 > 0$ for $0 < x \leq 1$ and $y \geq 2$, $\partial g/\partial y(x,y) > 0$. Hence, $g(x,y)$ is strictly increasing for $y > 2$. From $\lim_{y\to\infty} g(x,y) = 0$ for $0 < x \leq 1$, we can get $g(x,y) < 0$ for $0 < x \leq 1$ and $y \geq 2$. Therefore, $f(x,y)$ is strictly decreasing for $y > 2$. Put $x = G_n^n/a_n^n$ and $y = n$ in (2.3), then we have the following inequality for $n \geq l$.

$$1 + (n-1)\left(\frac{G_n^n}{a_n^n}\right)^{\frac{1}{n-1}} - n\left(\frac{G_n^n}{a_n^n}\right)^{\frac{1}{n}} \leq 1 + (l-1)\left(\frac{G_n^n}{a_n^n}\right)^{\frac{1}{l-1}} - l\left(\frac{G_n^n}{a_n^n}\right)^{\frac{1}{l}},$$

$$1 + (n-1)\frac{G_{n-1}}{a_n} - n\frac{G_n}{a_n} \leq \mathcal{R}_l$$

and

$$a_n + (n-1)G_{n-1} - nG_n \leq a_n\mathcal{R}_l,$$

where $\mathcal{R}_l = 1 + (l-1)\left(\frac{G_n^n}{a_n^n}\right)^{\frac{1}{l-1}} - l\left(\frac{G_n^n}{a_n^n}\right)^{\frac{1}{l}}$. From $a_n = nA_n - (n-1)A_{n-1}$, we have

$$A_n - A_{n-1} \leq \frac{n-1}{n}(A_{n-1} - G_{n-1}) + \frac{a_n\mathcal{R}_l}{n}.$$

Therefore, the proof of Theorem 1.2 is complete. ∎

3. PROOF OF COROLLARIES 1.3 AND 1.4

Proof of Corollary 1.3. From the refinement of Rado type inequality (1.2), we have the following inequality.

$$\begin{aligned}
A_n - G_n &\geq \frac{n-1}{n}(A_{n-1} - G_{n-1}) + \frac{n-1}{n} \cdot \frac{(G_n - G_{n-1})^2}{G_{n-1}} \\
&\geq \frac{n-2}{n}(A_{n-2} - G_{n-2}) + \frac{1}{n}\sum_{k=n-2}^{n-1} k\frac{(G_{k+1} - G_k)^2}{G_k} \\
&\geq \cdots\cdots \\
&\geq \frac{2}{n}(A_2 - G_2) + \frac{1}{n}\sum_{k=2}^{n-1} k\frac{(G_{k+1} - G_k)^2}{G_k} \\
&\geq \frac{1}{n} \cdot \frac{(G_2 - G_1)^2}{G_1} + \frac{1}{n}\sum_{k=2}^{n-1} k\frac{(G_{k+1} - G_k)^2}{G_k} = \frac{1}{n}\sum_{k=1}^{n-1} k\frac{(G_{k+1} - G_k)^2}{G_k}.
\end{aligned}$$

Hence, the proof of Corollary 1.3 is complete. ∎

Proof of Corollary 1.4. From the refinement of Rado type inequality (1.3), we assume $l = 2$, then we have the following inequality.

$$\begin{aligned}
A_n - G_n &\leq \frac{n-1}{n}(A_{n-1} - G_{n-1}) + \frac{a_n}{n}\mathcal{R}_2 \\
&\leq \frac{n-2}{n}(A_{n-2} - G_{n-2}) + \frac{a_{n-1}}{n}\mathcal{R}_2 + \frac{a_n}{n}\mathcal{R}_2 \\
&\leq \cdots\cdots \\
&\leq \frac{2}{n}(A_2 - G_2) + \frac{\mathcal{R}_2}{n}\sum_{k=3}^{n} a_k \\
&\leq \frac{1}{n}(A_1 - G_1) + \frac{\mathcal{R}_2}{n}\sum_{k=2}^{n} a_k = \frac{\mathcal{R}_2}{n}\sum_{k=2}^{n} a_k.
\end{aligned}$$

Hence, the proof of Corollary 1.4 is complete. ∎

4. PROOF OF THEOREM 1.5

We need some lemmas to prove theorem 1.5.

Lemma 4.1. *The inequality*

$$e^{\frac{11}{12}-n}n^{n+\frac{1}{2}} < n! < e^{1-n}n^{n+\frac{1}{2}}$$

holds for the integer $n > 1$.

The Lemma 4.1 is proved by Hummel in [3].

Lemma 4.2. *The inequality*

$$\begin{aligned}
\frac{11}{12} - x + \frac{\ln x}{2} &+ (1-x)x\Big(- \ln(2 - \sqrt{2} - x + \sqrt{2}x) \\
&+ \ln(3 - 2\sqrt{2} - x + \sqrt{2}x)\Big) > 0
\end{aligned}$$

holds for the real number $x \geq 3$.

Proof. We set

$$f(x) = \frac{11}{12} - x + \frac{\ln x}{2} + (1-x)x\Big(- \ln(2-\sqrt{2}-x+\sqrt{2}x) + \ln(3-2\sqrt{2}-x+\sqrt{2}x)\Big)$$

and the derivative of $f(x)$ is $f'(x) = (1-2x)g(x)$, where

$$g(x) = \frac{10-7\sqrt{2}-31x+22\sqrt{2}x+31x^2-22\sqrt{2}x^2-12x^3+8\sqrt{2}x^3}{2x(3-2\sqrt{2}-x+\sqrt{2}x)(2-\sqrt{2}-x+\sqrt{2}x)(1-2x)} + \ln(3-2\sqrt{2}-x+\sqrt{2}x) - \ln(2-\sqrt{2}-x+\sqrt{2}x).$$

The derivative of $g(x)$ is

$$g'(x) = \frac{h(x)}{2x^2(-1+2x)^2(3-2\sqrt{2}-x+\sqrt{2}x)^2(2-\sqrt{2}-x+\sqrt{2}x)^2},$$

where $h(x) = -198+140\sqrt{2}+1236x-874\sqrt{2}x-2701x^2+1910\sqrt{2}x^2+2540x^3-1796\sqrt{2}x^3-911x^4+644\sqrt{2}x^4+68x^5-48\sqrt{2}x^5$. The derivatives of $h(x)$ are

$$h'(x) = 2(618-437\sqrt{2}-2701x+1910\sqrt{2}x+3810x^2-2694\sqrt{2}x^2-1822x^3 + 1288\sqrt{2}x^3+170x^4-120\sqrt{2}x^4),$$

$$h''(x) = 2(-2701+1910\sqrt{2}+7620x-5388\sqrt{2}x-5466x^2 + 3864\sqrt{2}x^2+680x^3-480\sqrt{2}x^3),$$

$$h^{(3)}(x) = 24(635-449\sqrt{2}-911x+644\sqrt{2}x+170x^2-120\sqrt{2}x^2)$$

and

$$h^{(4)}(x) = 24(-240\sqrt{2}x+340x+644\sqrt{2}-911).$$

From $340-240\sqrt{2} \cong 0.588745 > 0$, $h^{(4)}(x)$ is strictly increasing for $x>1$ and $h^{(4)}(x) > h^{(4)}(1) = 24(-571+404\sqrt{2}) \cong 8.2147 > 0$. Thus, $h^{(3)}(x)$ is strictly increasing for $x>1$ and $h^{(3)}(x) > h^{(3)}(1) = 24(-106+75\sqrt{2}) \cong 1.58441 > 0$. By $h''(x)$ is strictly increasing for $x>1$ and $h''(x) > h''(1) = 2(133-94\sqrt{2}) \cong 0.12785 > 0$, $h'(x)$ is strictly increasing for $x>1$. From $h'(x) > h'(1) = 2(75-53\sqrt{2}) \cong 0.0933624 > 0$, $h(x)$ is strictly increasing for $x>1$ and $h(x) > h(1) = 34-24\sqrt{2} \cong 0.0588745 > 0$. Hence, we have $g'(x) > 0$ and $g(x)$ is strictly increasing for $x>1$. Since we have $\lim_{x\to\infty} g(x) = 0$, $g(x) < 0$ and $f'(x) > 0$. Thus, $f(x)$ is strictly increasing for $x>1$. From $f(3) = -25/12 - 3\ln 2 + \ln 3/2 + 6\ln(-1+2\sqrt{2}) \cong 0.00726788 > 0$, we can get $f(x) > 0$ for $x \geq 3$. ∎

Lemma 4.3. *The inequality*

$$((n-1)!)^{\frac{1}{n-1}} \geq \frac{3-2\sqrt{2}-n+\sqrt{2}n}{2-\sqrt{2}-n+\sqrt{2}n}(n!)^{\frac{1}{n}} \tag{4.1}$$

holds for the integer $n \geq 2$.

Proof. If $n=2$, then we have

$$((n-1)!)^{\frac{1}{n-1}} = 1 \quad \text{and} \quad \frac{3-2\sqrt{2}-n+\sqrt{2}n}{2-\sqrt{2}-n+\sqrt{2}n}(n!)^{\frac{1}{n}} = 1.$$

Hence, we consider the case of $n \geq 3$. The inequality (4.1) is equivalent to the following inequality.

$$\frac{\ln((n-1)!)}{n(n-1)} - \frac{\ln n}{n} - \ln(3-2\sqrt{2}-n+\sqrt{2}n) + \ln(2-\sqrt{2}-n+\sqrt{2}n) \geq 0.$$

From Lemma 4.1, it is suffices to show that

$$\begin{aligned}&\frac{\frac{11}{12}-n+(n-\frac{1}{2})\ln n}{n(n-1)} - \frac{\ln n}{n} - \ln(3-2\sqrt{2}-n+\sqrt{2}n) + \ln(2-\sqrt{2}-n+\sqrt{2}n)\\ &= \frac{1}{n(n-1)}\Big(\frac{11}{12}-n+\frac{\ln n}{2}+(1-n)n\Big(-\ln(2-\sqrt{2}-n+\sqrt{2}n)\\ &\quad + \ln(3-2\sqrt{2}-n+\sqrt{2}n)\Big)\Big)\end{aligned}$$

for $n \geq 3$. By Lemma 4.2, we can prove the inequality (4.1). ∎

Proof of Theorem 1.5. From $A_n = (n+1)/2$ and $G_n = (n!)^{\frac{1}{n}}$, the inequality (1.4) is equivalent to

$$n\left((n-1)!\right)^{\frac{1}{n-1}} - (n-1)(n!)^{\frac{1}{n}} > \frac{1}{2}.$$

By Lemmas 4.1 and 4.3, we have

$$n((n-1)!)^{\frac{1}{n-1}} - (n-1)(n!)^{\frac{1}{n}} \geq \frac{(2-\sqrt{2})(n!)^{\frac{1}{n}}}{2-\sqrt{2}-n+\sqrt{2}n} > \frac{(2-\sqrt{2})(e^{\frac{11}{12}-n}n^{n+\frac{1}{2}})^{\frac{1}{n}}}{2-\sqrt{2}-n+\sqrt{2}n}.$$

Hence, it is suffices to show that

$$\frac{(2-\sqrt{2})(e^{\frac{11}{12}-n}n^{n+\frac{1}{2}})^{\frac{1}{n}}}{2-\sqrt{2}-n+\sqrt{2}n} > \frac{1}{2},$$

so, we may show that

$$n\ln 2 + n\ln(2-\sqrt{2}) + \frac{11}{12} - n + \left(\frac{1}{2}+n\right)\ln n - n\ln(2-\sqrt{2}-n+\sqrt{2}n) > 0$$

for the integer $n \geq 2$. We set

$$f(x) = x\ln 2 + x\ln(2-\sqrt{2}) + \frac{11}{12} - x + \left(\frac{1}{2}+x\right)\ln x - x\ln(2-\sqrt{2}-x+\sqrt{2}x)$$

for the real number $x \geq 2$. The derivatives of $f'(x)$ are

$$\begin{aligned}f'(x) =& \frac{2-\sqrt{2}-x+\sqrt{2}x+2x^2-2\sqrt{2}x^2}{2x(2-\sqrt{2}-x+\sqrt{2}x)} + \ln 2 + \ln(2-\sqrt{2})\\ &+ \ln x - \ln(2-\sqrt{2}-x+\sqrt{2}x)\end{aligned}$$

and

$$f''(x) = \frac{-6+4\sqrt{2}+20x-14\sqrt{2}x-3x^2+2\sqrt{2}x^2}{2x^2(2-\sqrt{2}-x+\sqrt{2}x)^2}$$

By $20-14\sqrt{2}+2(-3+2\sqrt{2})\cong -0.142136<0$, we have

$$\begin{aligned}
-6+&4\sqrt{2}+20x-14\sqrt{2}x-3x^2+2\sqrt{2}x^2\\
&=-6+4\sqrt{2}+x(20-14\sqrt{2}-3x+2\sqrt{2}x)\\
&=-6+4\sqrt{2}+x\left(20-14\sqrt{2}+x(-3+2\sqrt{2})\right)\\
&<-6+4\sqrt{2}+x\left(20-14\sqrt{2}+2(-3+2\sqrt{2})\right)\\
&<-6+4\sqrt{2}+2\left(20-14\sqrt{2}+2(-3+2\sqrt{2})\right)\\
&=2(11-8\sqrt{2})\cong -0.627417.
\end{aligned}$$

Therefore, $f'(x)$ is strictly decreasing for $x>2$ and $f'(x)>\lim_{x\to\infty}f'(x)=-1+\ln 2+\ln(2-\sqrt{2})-\ln(-1+\sqrt{2})\cong 0.0397208$. Since $f(x)$ is strictly increasing for $x>2$ and $f(x)\geq f(2)=-13/12+(7\ln 2)/2+2\ln(2-\sqrt{2})\cong 0.273082$ for $x\geq 2$, we obtain $f(x)>0$ for $x\geq 2$ and the proof of Theorem 1.5 is complete. ∎

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