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IMPROVED OSCILLATION CRITERIA OF SECOND-ORDER ADVANCED NON-CANONICAL DIFFERENCE EQUATIONS

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ABSTRACT. Employing monotonic properties of nonoscillatory solutions, we derive some new oscillation criteria for the second-order advanced non-canonical difference equation

$$\Delta(p(\ell)\Delta u(\ell)) + q(\ell)u(\sigma(\ell)) = 0.$$

Our results extend and improve the earlier ones. The outcome is illustrated via some particular difference equations.

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1. INTRODUCTION

In this paper, we study the second order advanced non-canonical difference equation

$$\Delta(p(\ell)\Delta u(\ell)) + q(\ell)u(\sigma(\ell)) = 0, \ \ell \in N(\ell_0), \tag{1.1}$$

where ℓ_0 is a positive integer and $\mathbb{N}(\ell_0) = \{\ell_0, \ell_0 + 1, ..., \}$,

(H_1) $\{p(\ell)\}$ and $\{q(\ell)\}$ are positive real sequences with

$$D(\ell_0) = \sum_{\ell=\ell_0}^{\infty} \frac{1}{p(\ell)} < \infty; \tag{1.2}$$

(H_2) $\{\sigma(\ell)\}$ is a monotone increasing sequence of integers with $\sigma(\ell) \geq \ell+1$ for all $\ell \in \mathbb{N}(\ell_0)$.

By a solution of (1.1), we mean a nontrivial sequence $\{u(\ell)\}$ that satisfies (1.1) for all $\ell \in \mathbb{N}(\ell_0)$. A *solution* $\{u(\ell)\}$ of (1.1) is called *oscillatory* if it is neither eventually negative nor eventually positive, otherwise, it is said to be *nonoscillatory*. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In recent years, many criteria have been reported in the literature on the oscillation of (1.1) for the retarded case. See for example [1, 2, 3, 7, 10, 11, 21, 6, 14, 15, 18, 19] and the references therein. However for the equation (1.1) with $D(\ell_0) = \infty$, few oscillation results available in the literature, see for example [2, 8, 4, 16, 17, 22, 23].

From the review of literature, we see that very few results are available for the oscillation of (1.1) when (1.2) holds, see [3, 12, 13, 9, 10]. Therefore, our aim in this paper is to contribute to the underdeveloped oscillation theory of second-order advanced non-canonical difference equations. Furthermore, the results obtained in this paper improve and complement those in [12, 7, 13, 9, 10].

2. MAIN RESULTS

It follows from Lemma 2.1 in [20] that the set of positive solutions of (1.1) has the following structure.

Lemma 2.1. *Let* $\{u(\ell)\}$ *be an eventually positive solution of* (1.1). *Then* $\{u(\ell)\}$ *satisfies one of the following conditions :*

(S_1) $p(\ell)\Delta u(\ell) > 0$, $\Delta(p(\ell)\Delta u(\ell)) < 0$;
(S_*) $p(\ell)\Delta u(\ell) < 0$, $\Delta(p(\ell)\Delta u(\ell)) < 0$

for all $\ell \geq \ell_1 \in \mathbb{N}(\ell_0)$.

Lemma 2.2. *If*

$$\sum_{\ell=\ell_0}^{\infty} D(\sigma(\ell))q(\ell) = \infty, \tag{2.1}$$

then the positive solution $\{u(\ell)\}$ *of* (1.1) *satisfies* (S_*) *and,*

(i) $\lim_{\ell\to\infty} u(\ell) = 0$;
(ii) $u(\ell) + p(\ell)D(\ell)\Delta u(\ell) \geq 0$;
(iii) $\{\frac{u(\ell)}{D(\ell)}\}$ *is eventually increasing.*

Proof. The proof is similar to Lemma 2.2 of [7] since (2.1) implies that $\sum_{\ell=\ell_0}^{\infty} D(\ell+1)q(\ell) = \infty$ and so the details are omitted. ∎

Lemma 2.3. *Let* (2.1) *holds. Assume that there exists a* $\delta > 0$ *such that*

$$\min_{\ell\geq\ell_0}\{p(\ell)D(\ell+1)D(\sigma(\ell))q(\ell), p(\ell+1)D(\sigma(\ell))D(\ell+2)q(\ell)\}\geq\delta \tag{2.2}$$

eventually. If $\{u(\ell)\}$ *is a positive solution of* (1.1)*, then*

(i) $\{\frac{u(\ell)}{D^{\delta}(\ell)}\}$ *is decreasing;*
(ii) $\lim_{\ell\to\infty}\frac{u(\ell)}{D^{\delta}(\ell)}=0$;
(iii) $\{\frac{u(\ell)}{D^{1-\delta}(\ell)}\}$ *is increasing.*

Proof. Let us assume that $\{u(\ell)\}$ is an eventually positive solution of (1.1). Then (2.1) implies that $\{u(\ell)\}$ satisfies (S_*) for $\ell\geq\ell_1\in\mathbb{N}(\ell_0)$. From Lemma 2.2(iii), we have

$$u(\sigma(\ell))\geq\frac{D(\sigma(\ell))}{D(\ell+1)}u(\ell+1)\geq\frac{D(\sigma(\ell))}{D(\ell)}u(\ell).$$

Summing up (1.1) from ℓ_1 to $\ell-1$, we have

$$\begin{aligned}-p(\ell)\Delta u(\ell) &= -p(\ell_1)\Delta u(\ell_1)+\sum_{s=\ell_1}^{\ell-1}q(s)u(\sigma(s))\\ &\geq -p(\ell_1)\Delta u(\ell_1)+u(\ell)\sum_{s=\ell_1}^{\ell-1}q(s)\frac{D(\sigma(s))}{D(\ell+1)},\end{aligned}$$

which in view of (2.2) yields

$$\begin{aligned}-p(\ell)\Delta u(\ell) &\geq -p(\ell_1)\Delta u(\ell_1)+\delta u(\ell)\sum_{s=\ell_1}^{\ell-1}\frac{1}{p(s+1)D(s+1)D(s+2)}\\ &= -p(\ell_1)\Delta u(\ell_1)+\delta u(\ell)\sum_{s=\ell_1}^{\ell-1}\Delta\left(\frac{1}{D(s+1)}\right)\\ &= -p(\ell_1)\Delta u(\ell_1)+\delta u(\ell)\left(\frac{1}{D(\ell+1)}-\frac{1}{D(\ell_1+1)}\right)\\ &\geq \frac{\delta u(\ell)}{D(\ell+1)}, \qquad (2.3)\end{aligned}$$

where we have used $u(\ell)\to 0$ as $\ell\to\infty$. Hence

$$\Delta\left(\frac{u(\ell)}{D^{\delta}(\ell)}\right)=\frac{D^{\delta}(\ell)\Delta u(\ell)-u(\ell)\Delta(D^{\delta}(\ell))}{D^{\delta}(\ell)D^{\delta}(\ell+1)}. \tag{2.4}$$

By Mean-value theorem, we have

$$\Delta\left(D^{\delta}(\ell)\right)\geq\frac{-\delta D^{\delta}(\ell)}{p(\ell)D(\ell+1)}. \tag{2.5}$$

Using (2.5) in (2.4) and, in view of (2.3), we see that

$$\Delta\left(\frac{u(\ell)}{D^{\delta}(\ell)}\right)\leq\frac{D(\ell+1)p(\ell)\Delta u(\ell)+\delta u(\ell)}{p(\ell)D^{\delta+1}(\ell+1)}\leq 0.$$

That is, $\frac{u(\ell)}{D^{\delta}(\ell)}$ is decreasing, and therefore there exists $\lim_{\ell\to\infty}\frac{u(\ell)}{D^{\delta}(\ell)}=M\geq 0$.

We claim that $M=0$. Indeed, if $M>0$, then $u(\ell)\geq MD^{\delta}(\ell)>0$ eventually. Now define the companion sequence

$$v(\ell)=(p(\ell)D(\ell)\Delta u(\ell)+u(\ell))D^{-\delta}(\ell).$$

In view of Lemma 2.2, it is obvious that $v(\ell) \geq 0$ and

$$\begin{aligned} \Delta v(\ell) &= \Delta(p(\ell)\Delta u(\ell))D^{1-\delta}(\ell+1) + p(\ell)\Delta u(\ell)\Delta(D^{1-\delta}(\ell)) \\ &\quad + D^{-\delta}(\ell+1)\Delta u(\ell) + u(\ell)\Delta(D^{-\delta}(\ell)). \end{aligned} \tag{2.6}$$

By Mean-value theorem, we have

$$\Delta(D^{1-\delta}(\ell)) \geq \frac{-(1-\delta)}{p(\ell)} D^{-\delta}(\ell+1), \tag{2.7}$$

and

$$\Delta(D^{-\delta}(\ell)) \leq \frac{\delta}{p(\ell)} D^{-1-\delta}(\ell+1). \tag{2.8}$$

Using (2.7) and (2.8) in (2.6), we get

$$\begin{aligned} \Delta v(\ell) &\leq -q(\ell)u(\sigma(\ell))D^{1-\delta}(\ell+1) + \delta\Delta u(\ell)D^{-\delta}(\ell+1) + \delta\frac{u(\ell)}{p(\ell)}D^{-1-\delta}(\ell+1) \\ &\leq \frac{-q(\ell)D(\sigma(\ell))u(\ell)}{D(\ell)}D^{1-\delta}(\ell+1) + \delta\Delta u(\ell)D^{-\delta}(\ell+1) + \delta\frac{u(\ell)}{p(\ell)}D^{-1-\delta}(\ell+1) \\ &\leq \frac{-\delta u(\ell)D^{1-\delta}(\ell+1)}{p(\ell)D^2(\ell+1)} + \delta\Delta u(\ell)D^{-\delta}(\ell+1) + \frac{\delta u(\ell)}{p(\ell)}D^{-1-\delta}(\ell+1) \\ &= \delta\Delta u(\ell)D^{-\delta}(\ell+1). \end{aligned} \tag{2.9}$$

Since $u(\ell) \geq MD^{\delta}(\ell) \geq MD^{\delta}(\ell+1)$ and using (2.3), we obtain from (2.9) that

$$\Delta v(\ell) \leq \frac{-M\delta^2}{D(\ell+1)p(\ell)} < 0.$$

Summing up the last inequality from ℓ_1 to $\ell-1$, we obtain

$$\begin{aligned} v(\ell_1) &\geq M\delta^2 \sum_{s=\ell_1}^{\ell-1} \frac{1}{p(s)D(s+1)} \geq M\delta^2 \sum_{s=\ell_1}^{\ell-1} \int_{D(s+1)}^{D(s)} \frac{1}{t}dt \\ &= M\delta^2 \ln\frac{D(\ell_1)}{D(\ell)} \to \infty \;\; as \;\; \ell \to \infty \end{aligned}$$

which is a contradiction. Thus

$$\lim_{\ell\to\infty} \frac{u(\ell)}{D^{\delta}(\ell)} = 0.$$

Finally, we prove (iii). Equation (1.1) can be written in the equivalent form

$$\Delta(D(\ell)p(\ell)\Delta u(\ell) + u(\ell)) + D(\ell+1)q(\ell)u(\sigma(\ell)) = 0. \tag{2.10}$$

Summing up (2.10) from ℓ to ∞ and taking into account the fact that $\frac{u(\ell)}{D(\ell)}$ is increasing and (2.2), we obtain

$$\begin{aligned} D(\ell)p(\ell)\Delta u(\ell) + u(\ell) &\geq \sum_{s=\ell}^{\infty} D(s+1)q(s)u(\sigma(s)) \geq \sum_{s=\ell}^{\infty} \frac{\delta u(\sigma(s))}{p(s)D(\sigma(s))} \\ &\geq \frac{\delta u(\ell)}{D(\ell)} \sum_{s=\ell}^{\infty} \frac{1}{p(s)} = \delta u(\ell), \end{aligned}$$

that is,

$$D(\ell)p(\ell)\Delta u(\ell) + (1-\delta)u(\ell) \;\; \geq \;\; 0. \tag{2.11}$$

Now

$$\Delta\left(\frac{u(\ell)}{D^{1-\delta}(\ell)}\right) = \frac{D^{1-\delta}(\ell)\Delta u(\ell) - u(\ell)\Delta(D^{1-\delta}(\ell))}{D^{1-\delta}(\ell)D^{1-\delta}(\ell+1)}. \tag{2.12}$$

By Mean-value theorem, we get

$$-\Delta(D^{1-\delta}(\ell)) \geq \frac{(1-\delta)}{p(\ell)}D^{-\delta}(\ell). \tag{2.13}$$

Using (2.13) in (2.12) and, in view of (2.11), we obtain

$$\Delta\left(\frac{u(\ell)}{D^{1-\delta}(\ell)}\right) \geq \frac{D(\ell)p(\ell)\Delta u(\ell) + (1-\delta)u(\ell)}{D(\ell)p(\ell)D^{1-\delta}(\ell+1)} \geq 0.$$

The proof of the lemma is complete. ∎

Based on Lemma 2.3(i) and (iii), we immediately obtain the following oscillatory criteria for (1.1).

Theorem 2.4. *Assume that* (2.1) *and* (2.2) *hold. If*

$$\delta > \frac{1}{2}, \tag{2.14}$$

then (1.1) *is oscillatory.*

If $\delta \leq \frac{1}{2}$, then one can improve the results given in Lemma 2.3. Since $D(\ell)$ is decreasing, there exists a constant $\beta \geq 1$ such that

$$\frac{D(\ell)}{D(\sigma(\ell))} \geq \beta,$$

we introduce the constant $\delta_1 > \delta$ as

$$\delta_1 = \frac{\beta^\delta \delta}{1-\delta}. \tag{2.15}$$

Lemma 2.5. *Assume that* (2.1) *and* (2.2) *hold. If* $\{u(\ell)\}$ *is a positive solution of* (1.1)*, then*

$$\delta_1 u(\ell) + D(\ell)p(\ell)\Delta u(\ell) \leq 0, \tag{2.16}$$

for all $\ell \geq \ell_1 \in \mathbb{N}(\ell_0)$.

Proof. Let $\{u(\ell)\}$ be an eventually positive solution of (1.1). From (2.1) of Lemma 2.1, we see that $u(\ell)$ satisfies (S_*) for all $\ell \geq \ell_1 \in \mathbb{N}(\ell_0)$. Summing up (1.1) from ℓ_1 to $\ell - 1$ and using the fact that $\{\frac{u(\ell)}{D^{\delta}(\ell)}\}$ is decreasing, and $\{\frac{u(\ell)}{D^{1-\delta}(\ell)}\}$ is increasing, we obtain

$$\begin{aligned}
-p(\ell)\Delta u(\ell) &\geq -p(\ell_1)\Delta u(\ell_1)+\sum_{s=\ell_1}^{\ell-1}\frac{q(s)u(\sigma(s))}{D^{1-\delta}(\sigma(s))}D^{1-\delta}(\sigma(s))\\
&\geq -p(\ell_1)\Delta u(\ell_1)+\sum_{s=\ell_1}^{\ell-1}\frac{q(s)u(s)}{D^{1-\delta}(s)}D^{1-\delta}(\sigma(s))\\
&\geq -p(\ell_1)\Delta u(\ell_1)+\sum_{s=\ell_1}^{\ell-1}\frac{\beta^{\delta}q(s)u(s)D(\sigma(s))D^{\delta}(s)}{D^{\delta}(s)D(s)}\\
&\geq -p(\ell_1)\Delta u(\ell_1)+\frac{\delta\beta^{\delta}u(\ell)}{D^{\delta}(\ell)}\sum_{s=\ell_1}^{\ell-1}\frac{D^{\delta}(s+1)}{p(s)D^{2}(s+1)}\\
&\geq -p(\ell_1)\Delta u(\ell_1)+\frac{\delta\beta^{\delta}u(\ell)}{D^{\delta}(\ell)}\sum_{s=\ell_1}^{\ell-1}\int_{D(s+1)}^{D(s)}\frac{dt}{t^{2-\delta}}\\
&= -p(\ell_1)\Delta u(\ell_1)-\frac{\delta_1 u(\ell)}{D^{\delta}(\ell)}D^{\delta-1}(\ell)+\delta_1\frac{u(\ell)}{D(\ell)}.
\end{aligned}$$

Since $\frac{u(\ell)}{D^{\delta}(\ell)}\to 0$ as $\ell\to\infty$, we get

$$-D(\ell)p(\ell)\Delta u(\ell)\geq \delta_1 u(\ell)$$

which completes the proof of the lemma. ∎

Next, we present another criteria for the oscillation of (1.1).

Theorem 2.6. *Assume that* (2.1)*,* (2.2) *and* (2.15) *hold. If*

$$\delta+\delta_1>1 \tag{2.17}$$

then (1.1) *is oscillatory.*

Proof. Assume that (1.1) has an eventually positive solution $\{u(\ell)\}$. Condition (2.1) implies that $\{u(\ell)\}$ satisfies condition (S_*). From Lemma 2.3, we see that (2.11) implies

$$(1-\delta)u(\ell)\geq -D(\ell)p(\ell)\Delta u(\ell)$$

and from (2.16), we have

$$(1-\delta)u(\ell)\geq \delta_1 u(\ell).$$

That is,

$$\delta_1+\delta\leq 1,$$

which contradicts (2.17). The proof of the theorem is complete. ∎

Theorem 2.7. *Assume that* (2.2) *holds. If*

$$\liminf_{\ell\to\infty}\sum_{s=\ell+1}^{\sigma(\ell)-1}D(\sigma(s))q(s)>\frac{1-\delta}{e} \tag{2.18}$$

then (1.1) *is oscillatory.*

Proof. Assume that $\{u(\ell)\}$ is a positive solution (1.1). First note that (2.18) along with (1.2) imply (2.1). To see this, it suffices to note that

$$\sum_{\ell=\ell_0}^{\infty} D(\sigma(\ell))q(\ell) = \infty \tag{2.19}$$

is necessary for (2.18) to be valid. Then from (2.19) and D being decreasing, (2.1) immediately follows. By Lemma 2.1, $\{u(\ell)\}$ satisfies condition (S_*) for all $\ell \geq \ell_1$. Moreover from (2.2), we have (2.11).

Now, from (1.1) and (2.11), we see that $w(\ell) = -p(\ell)(\Delta u(\ell))$ is a positive solution of the first order advanced difference inequality

$$\Delta w(\ell) - \frac{q(\ell)}{1-\delta} D(\sigma(\ell))w(\sigma(\ell)) \geq 0. \tag{2.20}$$

However, it is well-known (see, e.g., [Theorem 2.1,[3]]) that condition (2.18) implies oscillation of (2.20). This is a contradiction and this completes the proof of the theorem. ∎

Corollary 2.8. *Assume that* (2.2) *holds. If* $\sigma(\ell) = \ell + \tau$ *where* $\tau \geq 2$ *is an integer such that*

$$\liminf_{\ell\to\infty} \sum_{s=\ell+1}^{\ell+\tau-1} D(s+1)q(s) > \left(\frac{\tau-1}{\tau}\right)^{\tau} \tag{2.21}$$

then (1.1) *is oscillatory.*

Proof. The proof follows by applying Theorem 6.1.7 of [2] instead of Theorem 2.1 of [3]. This completes the proof of the corollary. ∎

Our final oscillation result obtained without using condition (2.2).

Theorem 2.9. *If*

$$\limsup_{\ell\to\infty} \left\{ D(\sigma(\ell)) \sum_{s=\ell_0}^{\ell-1} q(s) + \sum_{s=\ell}^{\sigma(\ell)-1} D(s+1)q(s) \right.$$
$$\left. + \frac{1}{D(\sigma(\ell))} \sum_{s=\sigma(\ell)}^{\infty} D(s+1)q(s)D(\sigma(s)) \right\} > 1 \tag{2.22}$$

then (1.1) *is oscillatory.*

Proof. Assume that $\{u(\ell)\}$ is a positive solution of (1.1). It follows from (2.22) that there exists a constant $M > 0$ such that

$$\lim_{\ell\to\infty} \left\{ D(\sigma(\ell)) \sum_{s=\ell_0}^{\ell-1} q(s) + \sum_{s=\ell}^{\sigma(\ell)-1} D(\sigma(s))q(s) \right.$$
$$\left. + \frac{1}{D(\sigma(\ell))} \sum_{s=\sigma(\ell)}^{\infty} D(s)q(s)D(\sigma(s)) \right\} \geq M. \tag{2.23}$$

We claim that (2.23) implies (2.1). Indeed, if not, then $\sum_{\ell=\ell_0}^{\infty} D(\sigma(\ell))q(\ell) < \infty$, which means that there exists an integer $\ell_* \geq \ell_1 \in \mathbb{N}(\ell_0)$ such that

$$\sum_{\ell=\ell_0}^{\infty} D(\sigma(\ell))q(\ell) < \frac{M}{6}. \tag{2.24}$$

That is, for $\ell \geq \ell_1$

$$\begin{aligned} D(\sigma(\ell)) \sum_{s=\ell_1}^{\ell-1} q(s) &= D(\sigma(\ell)) \sum_{s=\ell_1}^{\ell_*-1} q(s) + D(\sigma(\ell)) \sum_{s=\ell_*}^{\ell-1} q(s) \\ &\leq D(\sigma(\ell)) \sum_{s=\ell_1}^{\ell_*-1} q(s) + \sum_{s=\ell_*}^{\ell-1} D(\sigma(s))q(s) \\ &\leq D(\sigma(\ell)) \sum_{s=\ell_1}^{\ell_*-1} q(s) + \frac{M}{6}. \end{aligned}$$

Hence, for $\ell \geq \ell_*$

$$\sum_{s=\ell}^{\sigma(\ell)-1} D(\sigma(s))q(s) \leq \frac{M}{6}.$$

On the other hand, for $\ell \geq \ell_*$

$$\frac{1}{D(\sigma(\ell))} \sum_{s=\sigma(\ell)}^{\infty} D(s+1)q(s)D(\sigma(s)) \leq \sum_{s=\sigma(\ell)}^{\infty} D(\sigma(s))q(s) < \frac{M}{6}.$$

Considering the above inequalities, we see that

$$\begin{aligned} \limsup_{\ell \to \infty} \Bigg\{ & D(\sigma(\ell)) \sum_{s=\ell_1}^{\ell-1} q(s) + \sum_{s=\ell}^{\sigma(\ell)-1} D(\sigma(s))q(s) \\ & + \frac{1}{D(\sigma(\ell))} \sum_{s=\sigma(\ell)}^{\infty} D(s+1)q(s)D(\sigma(s)) \Bigg\} \leq \frac{M}{2}, \end{aligned}$$

which contradicts (2.23) and therefore (2.1) holds. Thus $\{u(\ell)\}$ satisfies the conditions of Lemma 2.2. Simple computation shows that (1.1) can be rewritten as follows:

$$\Delta(D(\ell)p(\ell)\Delta u(\ell) + u(\ell)) + D(\ell+1)q(\ell)u(\sigma(\ell)) = 0.$$

Summing up the last equation from ℓ to ∞, we have

$$D(\ell)p(\ell)\Delta u(\ell) + u(\ell) \geq \sum_{s=\ell}^{\infty} D(s+1)q(s)u(\sigma(s)). \tag{2.25}$$

On the other hand, summing (1.1) from ℓ_1 to $\ell - 1$, we get

$$-p(\ell)\Delta u(\ell) \geq \sum_{s=\ell_1}^{\ell-1} q(s)u(\sigma(s)). \tag{2.26}$$

Combining (2.26) in (2.25), we obtain

$$u(\ell) \geq D(\ell) \sum_{s=\ell_1}^{\ell-1} q(s)u(\sigma(s)) + \sum_{s=\ell}^{\infty} D(s+1)q(s)u(\sigma(s)).$$

Therefore

$$\begin{aligned} u(\sigma(\ell)) \;&\geq\; D(\sigma(\ell))\sum_{s=\ell_1}^{\ell-1} q(s)u(\sigma(s)) + D(\sigma(\ell))\sum_{s=\ell}^{\sigma(\ell)-1} q(s)u(\sigma(s)) \\ &\quad + \sum_{s=\sigma(\ell)}^{\infty} D(s+1)q(s)u(\sigma(s)). \end{aligned}$$

Since $\{u(\ell)\}$ is decreasing and $\{\frac{u(\ell)}{D(\ell)}\}$ is increasing, we obtain that

$$\begin{aligned} 1 = \frac{u(\sigma(\ell))}{u(\sigma(\ell))} \;&\geq\; \Bigg\{ D(\sigma(\ell))\sum_{s=\ell_1}^{\ell-1} q(s) + \sum_{s=\ell}^{\sigma(\ell)-1} q(s)D(\sigma(s)) \\ &\quad + \frac{1}{D(\sigma(\ell))}\sum_{s=\ell}^{\infty} D(s+1)q(s)D(\sigma(s)) \Bigg\}. \end{aligned}$$

Taking $\limsup$ as $\ell \to \infty$ on both sides of the last inequality we get a contradiction. This completes the proof of the theorem. ∎

3. EXAMPLES

In this section, we provide two examples to illustrate the importance of the main results.

Example 3.1. *Consider the second-order noncanonical advanced difference equation*

$$\Delta(\ell(\ell+1)\Delta u(\ell)) + \gamma u(2\ell) = 0,\ \ell \geq 2. \tag{3.1}$$

Here $p(\ell) = \ell(\ell+1)$, $q(\ell) = \gamma > 0$, $\sigma(\ell) = 2\ell$. Now $D(\ell) = \frac{1}{\ell}$ and condition (2.1) holds. By taking $\delta = \frac{\gamma}{2}$, we see that (2.2) holds. By Theorem 2.4, equation (3.1) is oscillatory if $\gamma > 1$.

By simple calculation, we see that $\beta = 2$ and

$$\delta_1 = \frac{\gamma}{2-\gamma}2^{\gamma/2}.$$

By taking $\gamma = 3/4$, we have

$$\delta + \delta_1 = 2^{\frac{3}{8}}\frac{3}{5} + \frac{3}{8} > 1.$$

Therefore by Theorem 2.6, equation (3.1) is oscillatory for $\gamma \geq \frac{3}{4}$.

Note that the equation (3.1) was considered in [9] and it was shown that (3.1) is oscillatory if $\gamma > 2$; in [13] it is shown that (3.1) is oscillatory if $\gamma = 2$ and in [12], it is shown that (3.1) is oscillatory if $\gamma \geq 1$. Therefore Theorem 2.6 improves Theorem 4 in [9], Theorem 3.3 in [13] and Theorem 2.4 in [12].

Example 3.2. *Consider again the equation* (3.1). *For this equation* (2.1) *holds. The condition* (2.22) *becomes*

$$\limsup_{\ell\to\infty}\left\{\frac{\gamma}{2\ell}(\ell-2) + \frac{\gamma}{4} + \frac{\gamma}{2}\right\} = \frac{5\gamma}{4} > 1.$$

Hence, by Theorem 2.9, equation (3.1) is oscillatory if $\gamma > \frac{4}{5}$.

Note that, Theorem 2.9 improves Theorem 4 in [9] and Theorem 3.3 in [13].

4. CONCLUSION

In this paper we have established some new oscillation criteria which have improved some of the results already reported, and this is illustrated through two examples.

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