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TIMELIKE SURFACES WITH A COMMON LINE OF CURVATURE IN MINKOWSKI 3-SPACE

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ABSTRACT. In this paper, we analyze the problem of constructing a timelike surface family from a given non-null curve line of curvature. Using the Frenet frame of the non-null curve in Minkowski space $\mathbb{E}_1^3$ we express the family of surfaces as a linear combination of the components of this frame, and derive the necessary and sufficient conditions for the coefficients to satisfy both the line of curvature and the isoparametric requirements. In addition, a necessary and sufficient condition for the given non-null curve to satisfy the line of curvature and the geodesic requirements is investigated. The extension to timelike surfaces of revolution is also outlined. Meanwhile, some representative non-null curves are chosen to construct the corresponding timelike surfaces which possessing these curves as lines of curvature. Results presented in this paper have applications in geometric modeling and the manufacturing of products. In addition, some computational examples are given and plotted.

Key words and phrases: Frenet frame; Line of curvature; Timelike surface pencil; Marching-scale functions.

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1. INTRODUCTION

Line of curvature is one of the most interesting topics in differential geometry and it is being study by many mathematicians until now, for example [1, 2]. It is an important tool in surface analysis for exhibiting variations of the principal directions. In Euclidean 3-space, surface with common line of curvature has been the subject of many studies. Li et al. [3] studied the parametric surface family which the given curve as the line of curvature. Moreover, they gave an approach to constructing the developable surface through the given curve as line of curvature [4]. Recently, due to its relationship with physical sciences in Minkowski space, the surface pencils with a common line of curvature have been studied by [5, 6].

In this work, we extend the work of Wang et al. [7] to derive the sufficient and necessary condition for a given non-null curve to be both iso-parametric and line of curvature on a timelike surface. Then, we give family of timelike surfaces with a common line of curvature. Moreover, we show with the helps of given examples that the member, having any desired property, can be choosing the appropriate coefficients. Meanwhile, the extension to timelike surfaces of revolution is also outlined. In addition, the results of being theoretical interest also have applications in geometric modeling and the manufacturing of products, for examples, designing agriculture machines' tools, development models of bulldozers moldboard by geometric modeling method (engineering design).

2. PRELIMINARIES

Let $\mathbb{E}_1^3$ be the three-dimensional Minkowski space, that is, the three-dimensional real vector space $\mathbb{R}^3$ with the metric

$$\langle d\mathbf{y}, d\mathbf{y}\rangle = dy_1^2 + dy_2^2 - dy_3^2,$$

where $(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)$ denotes the canonical coordinates in $\mathbb{R}^3$. An arbitrary vector $\mathbf{y}$ of $\mathbb{E}_1^3$ is said to be spacelike if $\langle \mathbf{y},\mathbf{y}\rangle > 0$ or $\mathbf{y}=\mathbf{0}$, timelike if $\langle \mathbf{y},\mathbf{y}\rangle < 0$ and lightlike or null if $\langle \mathbf{y},\mathbf{y}\rangle = 0$ and $\mathbf{y}=\mathbf{0}$. A timelike or light-like vector in $\mathbb{E}_1^3$ is said to be causal. For $\mathbf{y}\in\mathbb{E}_1^3$ the norm is defined by $\|\mathbf{y}\| = \sqrt{|\langle \mathbf{y}, \mathbf{y}\rangle|}$, then the vector $\mathbf{y}$ is called a spacelike unit vector if $\langle \mathbf{y},\mathbf{y}\rangle = 1$ and a timelike unit vector if $\langle \mathbf{y},\mathbf{y}\rangle = -1$. Similarly, a regular curve in $\mathbb{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [8, 9, 10, 11, 12, 13]. For any two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ of $\mathbb{E}_1^3$, the inner product is the real number $\langle \mathbf{a},\mathbf{b}\rangle = a_1b_1 + a_2b_2 - a_3b_3$ and the vector product is defined by $\mathbf{a}\times\mathbf{b} = ((a_2b_3 - a_3b_2), (a_3b_1 - a_1b_3), -(a_1b_2 - a_2b_1))$.

Let us consider two non-null vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{E}_1^3$, then there are the following cases;

i) Let $\mathbf{x}$ and $\mathbf{y}$ be spacelike vectors.

If $\mathbf{x}$ and $\mathbf{y}$ span a spacelike plane, then there is a unique real number $0 \leq \theta \leq \pi$ such that $\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos\theta$. Here θ is called the spacelike angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

If $\mathbf{x}$ and $\mathbf{y}$ span a timelike plane, then there is a unique real number $\theta \geq 0$ such that $\langle \mathbf{x}, \mathbf{y}\rangle = \epsilon \|\mathbf{x}\| \, \|\mathbf{y}\| \cosh\theta$, where $\epsilon = +1$ or $\epsilon = -1$ according to $sign(x_2) = sign(y_2)$ or $sign(x_2) \neq sign(y_2)$, respectively. In this case θ is called the central angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

ii) Let $\mathbf{x}$ and $\mathbf{y}$ are timelike vectors.

Then, there is a unique real number $\theta \geq 0$ such that $\langle \mathbf{x}, \mathbf{y}\rangle = \epsilon \|\mathbf{x}\| \, \|\mathbf{y}\| \cosh\theta$, where $\epsilon = +1$ or $\epsilon = -1$ according to $\mathbf{x}$ and $\mathbf{y}$ have different time-orientation or the same time-orientation, respectively. θ is called the Lorentzian timelike angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

iii) Let $\mathbf{x}$ be a spacelike vector and $\mathbf{y}$ be timelike.

Then, there is a unique real number $\theta \geq 0$ such that $\langle \mathbf{x}, \mathbf{y}\rangle = \epsilon \|\mathbf{x}\| \, \|\mathbf{y}\| \sinh\theta$, where $\epsilon = +1$ or $\epsilon = -1$ according to $sign(x_2) = sign(y_1)$ or $sign(x_2) \neq sign(y_1)$. This number is called the Lorentzian timelike angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

Let $\alpha = \alpha(s)$ be a unit speed non-null curve in $\mathbb{E}_1^3$; $\kappa(s)$ and $\tau(s)$ denote the natural curvature and torsion of $\alpha = \alpha(s)$, respectively. Consider the Frenet frame $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ associated with curve $\alpha = \alpha(s)$ such that $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are the unit tangent, the principal normal and the binormal vector fields, respectively. Then, there are two cases for the Frenet formulae: (i) $\mathbf{T}$ and $\mathbf{N}$ (resp. $\mathbf{T}$ and $\mathbf{B}$) are spacelike vectors while $\mathbf{B}$ (resp. $\mathbf{N}$) is timelike vector (similar procedures will be applied):

$$\frac{d}{ds}\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}, \tag{2.1}$$

and

$$\mathbf{T} \times \mathbf{N} = -\mathbf{B}, \;\; \mathbf{B} \times \mathbf{T} = \mathbf{N}, \quad \mathbf{N} \times \mathbf{B} = \mathbf{T}. \tag{2.2}$$

(ii) $\mathbf{T}$ is a timelike vector while $\mathbf{N}$ and $\mathbf{B}$ are spacelike vectors

$$\frac{d}{ds}\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}. \tag{2.3}$$

Let $\mathbf{P} = \mathbf{P}(s,t)$ be a parametric timelike surface in $\mathbb{E}_1^3$ based on a given spacelike space curve $\alpha = \alpha(s)$ as follows:

$$\mathbf{P}(s,t) = \alpha(s) + a(s,t)\mathbf{T}(s) + b(s,t)\mathbf{N}(s) + c(s,t)\mathbf{B}(s); \;\; 0 \le t_0 \le T, \;\; 0 \le s \le L, \tag{2.4}$$

where $a(s,t)$, $b(s,t)$ and $c(s,t)$ are all C^1 functions. If the parameter t is seen as the time, the functions $a(s,t)$, $b(s,t)$ and $c(s,t)$ can then be viewed as directed marching distances of a point unit in the time t in the direction $\mathbf{T}$; $\mathbf{N}$; and $\mathbf{B}$, respectively, and the position vector $\alpha(s)$ is seen as the initial location of this point. The normal vector field is given by

$$\mathbf{n}(s,t) := \frac{\partial \mathbf{P}(s,t)}{\partial s} \times \frac{\partial \mathbf{P}(s,t)}{\partial t} = \zeta_1(s,t)\mathbf{T}(s) + \zeta_2(s,t)\mathbf{N}(s) + \eta_3(s,t)\mathbf{B}(s), \tag{2.5}$$

where

$$\left.\begin{aligned} \zeta_1(s,t) &= -\left(\tfrac{\partial c(s,t)}{\partial s} + b(s,t)\tau(s)\right)\tfrac{\partial b(s,t)}{\partial t} + \left(\tfrac{\partial b(s,t)}{\partial s} + a(s,t)\kappa(s) + c(s,t)\tau(s)\right)\tfrac{\partial c(s,t)}{\partial t}, \\ \zeta_2(s,t) &= -\left(1 + \tfrac{\partial a(s,t)}{\partial s} + b(s,t)\kappa(s)\right)\tfrac{\partial c(s,t)}{\partial t} + \left(\tfrac{\partial c(s,t)}{\partial s} + b(s,t)\tau(s)\right)\tfrac{\partial a(s,t)}{\partial t}, \\ \zeta_3(s,t) &= -(1 + \tfrac{\partial a(s,t)}{\partial s} + b(s,t)\kappa(s))\tfrac{\partial b(s,t)}{\partial t} + (\tfrac{\partial b(s,t)}{\partial s} + a(s,t)\kappa(s) + c(s,t)\tau(s))\tfrac{\partial a(s,t)}{\partial t}. \end{aligned}\right\}$$

$\mathbf{P}$ is called a timelike surface if the induced metric on $\mathbf{P}$ is a Lorentzian metric on each tangent plane [8, 9, 10]. This is equivalent to saying that the normal vector $\mathbf{n}$ is spacelike at each point of M. A non-null curve $\alpha = \alpha(s)$ is called isoparametric line of curvature of timelike surface $\mathbf{P}$ if it is both a line of curvature and an isoparametric curve on $\mathbf{P}$.

The same argument used to timelike surface based on a given spacelike curve can be repeated to timelike surface based on a given timelike curve; we omit the details here.

3. TIMELIKE SURFACES WITH A COMMON LINE OF CURVATURE

Our goal is to derive a necessary and sufficient conditions for which the given spacelike curve $\alpha(s)$ is an isoparametric line of curvature on the timelike surface $\mathbf{P}(s,t)$.
Firstly, since the directrix $\alpha(s)$ is an isoparametric curve on the surface there exists a parameter $t = t_0$ such that $\alpha(s) = \mathbf{P}(s, t_0)$, that is, we have:

$$a(s,t_0) = b(s,t_0) = c(s,t_0) = 0. \tag{3.1}$$

Thus, the normal vector field is

$$\mathbf{n}(s,t_0) := \frac{\partial \mathbf{P}(s,t_0)}{\partial s} \times \frac{\partial \mathbf{P}(s,t_0)}{\partial t} = \zeta_1(s,t_0)\mathbf{T}(s) + \zeta_2(s,t_0)\mathbf{N}(s) + \zeta_3(s,t_0)\mathbf{B}(s), \tag{3.2}$$

where

$$\left.\begin{aligned} \zeta_1(s,t_0) &= -\tfrac{\partial c(s,t_0)}{\partial s}\tfrac{\partial b(s,t_0)}{\partial t} + \tfrac{\partial c(s,t_0)}{\partial t}\tfrac{\partial b(s,t_0)}{\partial s}, \\ \zeta_2(s,t_0) &= -(1 + \tfrac{\partial a(s,t_0)}{\partial s})\tfrac{\partial c(s,t_0)}{\partial t} + \tfrac{\partial a(s,t_0)}{\partial t}\tfrac{\partial c(s,t_0)}{\partial s}, \\ \zeta_3(s,t_0) &= -(1 + \tfrac{\partial a(s,t_0)}{\partial s})\tfrac{\partial b(s,t_0)}{\partial t} + \tfrac{\partial a(s,t_0)}{\partial t}\tfrac{\partial b(s,t_0)}{\partial s}). \end{aligned}\right\} \tag{3.3}$$

Secondly, let us choose a spacelike unit vector

$$\mathbf{e}(s) = \cosh\theta \mathbf{N}(s) + \sinh\theta \mathbf{B}(s). \tag{3.4}$$

Hence, from Eqs. 3.2 and 3.4, we have that $\mathbf{e}(s) \| \mathbf{n}(s,t_0)$ if and only if there exists a function $\lambda(s)$ such that

$$\zeta_1(s,t_0) = 0,\ \ \zeta_2(s,t_0) = \lambda(s)\cosh\theta,\ \ \zeta_3(s,t_0) = \lambda(s)\sinh\theta. \tag{3.5}$$

Differentiating Eq. 3.4 and using the corresponding Frenet formulae 2.1, we find

$$\frac{d\mathbf{e}}{ds} = (\frac{d\theta}{ds} + \tau)\mathbf{e}^{\perp} + \kappa\cosh\theta\mathbf{T}.$$

However, according to the Rodrigues' formula, $\alpha = \alpha(s)$ is spacelike line of curvature on the timelike surface $\mathbf{P}(s,t)$ if and only if $\frac{d\theta}{ds} + \tau = 0$. This means that

$$\theta(s) = \theta_0 - \int_{s_0}^{s} \tau(s)ds, \tag{3.6}$$

where s_0 is the starting value of arc length and $\theta_0 = \theta(s_0)$. From the analysis above, we can draw a conclusion as follows:

Theorem 3.1. *The given spacelike curve $\alpha(s)$ is a line of curvature on the timelike surface $\mathbf{P}(s,t)$ if and only if*

$$\left.\begin{aligned} &a(s,t_0) = b(s,t_0) = c(s,t_0) = 0,\ \ 0 \le t_0 \le T,\ \ 0 \le s \le L,\ \ \lambda(s) \ne 0, \\ &\zeta_1(s,t_0) = 0,\ \ \zeta_2(s,t_0) = \lambda(s)\cosh\theta,\ \ \zeta_3(s,t_0) = \lambda(s)\sinh\theta, \end{aligned}\right\} \tag{3.7}$$

where the functions $\lambda(s)$ and $\theta(s)$ are called controlling functions.

We call the set of surfaces defined by 2.4 and 3.7 the family of timelike surfaces with common spacelike line of curvature. Any surface $\mathbf{P}(s,t)$ defined by 2.4 and satisfying 3.7 is a member of this family. Similar with [7], for the purposes of simplification and analysis, we also consider the case when the marching-scale functions $a(s,t)$, $b(s,t)$ and $w(s,t)$ can be written into two factors:

$$\begin{aligned} a(s,t) &= l(s)A(t), \\ b(s,t) &= m(s)B(t), \\ c(s,t) &= n(s)C(t), \end{aligned}$$

where $l(s), m(s), n(s)$, $A(t)$, $B(t)$ and $C(t)$ are C^1 functions and $l(s), m(s)$ and $n(s)$ are not identically zero. Thus, from the Theorem 3.1, we can get the following corollary:

Corollary 3.2. *A necessary and sufficient condition of the spacelike curve $\alpha(s)$ being a line of curvature on the timelike surface $\mathbf{P}(s,t)$ is*

$$\left.\begin{aligned} A(t_0) = B(t_0) = C(t_0) = 0, \\ -n(s)\tfrac{dC(t_0)}{dt} = \lambda(s)\cosh\theta,\ -m(s)\tfrac{dB(t_0)}{dt} = \lambda(s)\sinh\theta. \end{aligned}\right\} \tag{3.8}$$

However, we can assume that the marching-scale functions depend only on the parameter t; that is $l(s) = m(s) = n(s) = 1$. Then, we analyze the condition 3.8 according to the different expressions of $\theta(s)$:
(i) In the case of $\tau(s) \neq 0$, then $\theta(s)$ is a non-constant function of variable s and the condition 3.8 can be represented as

$$\left.\begin{array}{c} A(t_0) = B(t_0) = C(t_0) = 0, \\ -\frac{dC(t_0)}{dt} = \lambda(s)\cosh\theta,\ \frac{dB(t_0)}{dt} = -\lambda(s)\sinh\theta, \end{array}\right\} \tag{3.9}$$

(ii) In the case of $\tau(s) = 0$, that is the curve is a spacelike planar curve, then $\theta(s) = \theta_0$ is a constant and we have
(a) In the case of $\theta_0 \neq 0$, the condition 3.9 can be represented as

$$\left.\begin{array}{c} A(t_0) = B(t_0) = C(t_0) = 0, \\ -\frac{dC(t_0)}{dt} = \lambda(s)\cosh\theta_0,\ \frac{dB(t_0)}{dt} = -\lambda(s)\sinh\theta_0. \end{array}\right\} \tag{3.10}$$

(b) In the case of $\theta_0 = 0$, the condition 3.9 can be represented as

$$\left.\begin{array}{c} A(t_0) = B(t_0) = C(t_0) = 0, \\ -\frac{dC(t_0)}{dt} = \lambda(s),\ \frac{dB(t_0)}{dt} = o, \end{array}\right\} \tag{3.11}$$

and from Eq. 3.4 the normal $\mathbf{n}(s, t_0)$ (resp. $\mathbf{e}(s)$) is coincident with $\mathbf{N}$. In this case, the curve $\alpha = \alpha(s)$ is not only a spacelike line of curvature line but also a spacelike geodesic.

3.1. Examples of timelike surfaces with a common spacelike line of curvature.

In this paragraph, some representative examples are illustrated to verify the method.

Example 3.1. *In this example, we construct timelike surface pencil in which all the timelike surfaces share a spacelike helix as common spacelike line of curvature. Given the spacelike circle helix:*

$$\alpha(s) = (a_1 \sinh\frac{s}{a_3}, a_2\frac{s}{a_3}, a_1\cosh\frac{s}{a_3}),\ a_1 > 0,\ \ a_2 \neq 0,\ a_1^2 + a_2^2 = a_3^2,\ -4 \leq s \leq 4.$$

It is easy to show that

$$\left.\begin{array}{c} \mathbf{T}(s) = (\frac{a_1}{c}\cosh\frac{s}{a_3}, \frac{a_2}{a_3}, \frac{a_1}{a_3}\sinh\frac{s}{a_3}), \\ \mathbf{N}(s) = (\sinh\frac{s}{a_3}, 0, \cosh\frac{s}{a_3}), \\ \mathbf{B}(s) = (\frac{a_2}{a_3}\cosh\frac{s}{a_3}, -\frac{a_1}{a_3}, \frac{a_2}{a_3}\sinh\frac{s}{a_3}), \end{array}\right\}$$

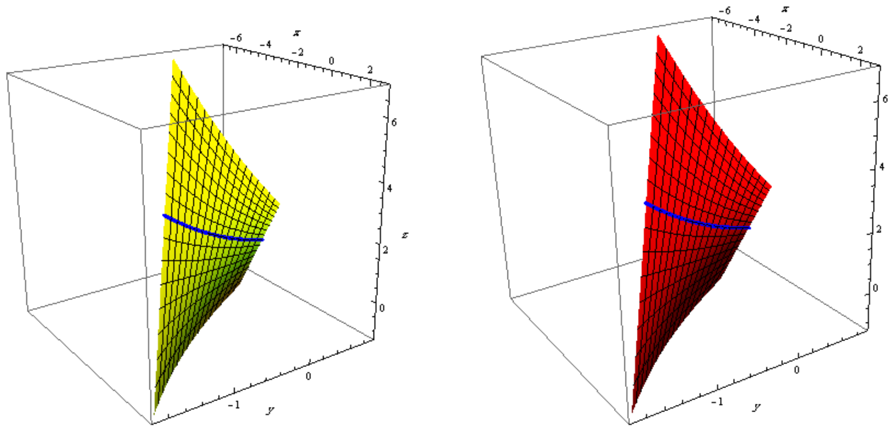
and $\tau = \frac{a_2}{a_3^2}$, then $\theta(s) = \frac{a_2}{a_3^2}s + \theta_0$. If $\theta_0 = 0$, we have $\theta(s) = \frac{a_2}{a_3^2}s$.
By choosing

$$\begin{aligned} l(s) &= m(s) = n(s) = 1, \\ A(t) &= \alpha t,\ -B(t) = t\lambda(s)\sinh\theta,\ \ -C(t) = t\lambda(s)\cosh\theta,\ \lambda \neq 0, \end{aligned}$$

and from formula 2.4, we obtain the following timelike surface pencil

$$\begin{aligned} \mathbf{P}(s, t; \alpha, \lambda) &= (a_1\sinh\frac{s}{a_3}, a_2\frac{s}{a_3}, a_1\cosh\frac{s}{a_3}) + t(\alpha, -\lambda\sinh\theta, -\lambda\cosh\theta) \\ &\quad \times \begin{pmatrix} \frac{a_1}{a_3}\cosh\frac{s}{a_3} & \frac{a_2}{a_3} & \frac{a_1}{a_3}\sinh\frac{s}{a_3} \\ \sinh\frac{s}{a_3} & 0 & \cosh\frac{s}{a_3} \\ \frac{a_2}{a_3}\cosh\frac{s}{a_3} & -\frac{a_1}{c} & \frac{a_2}{a_3}\sinh\frac{s}{a_3} \end{pmatrix}. \end{aligned}$$

So, if we choose $t \in [-2, 0]$, $a_1 = 2, a_2 = 1$, then for $\alpha = 1$, $\lambda = -1$ and $\alpha = -1, \lambda = 1$, the corresponding timelike surfaces are shown in Fig. 1A and Fig. 1B, respectively.



(A) (B)

Figure 1: (A) $P(s,t;1,-1)$ (B) $P(s,t;-\sqrt{5}/4,-\sqrt{5}/2)$.

Example 3.2. *Suppose we are given a parametric spacelike curve*

$$\alpha(s) = (\cos s, \sin s, 0), \quad -\pi \leq s \leq \pi.$$

After simple computation, we have

$$\mathbf{T}(s) = (-\sin s, \cos s, 0), \ \mathbf{N}(s) = (-\cos s, -\sin s, 0), \ \mathbf{B}(s) = (0,0,1),$$

and $\tau = 0$ which follows $\theta(s) = \theta_0$ is a constant. By choosing

$$\begin{aligned} l(s) &= m(s) = n(s) = 1, \\ A(t) &= \alpha t, \ -B(t) = t\lambda(s)\sinh\theta_0, \ \ -C(t) = t\lambda(s)\cosh\theta_0, \ \lambda \neq 0, \end{aligned}$$

and from formula 2.4, we obtain the following timelike surface pencil

$$\mathbf{P}(s,t;\alpha,\lambda) = (\cos s, \sin s, 0) + t(\alpha, -\lambda\sinh\theta_0, -\lambda\cosh\theta_0)\begin{pmatrix} -\sin s & \cos s & 0 \\ -\cos s & -\sin s & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

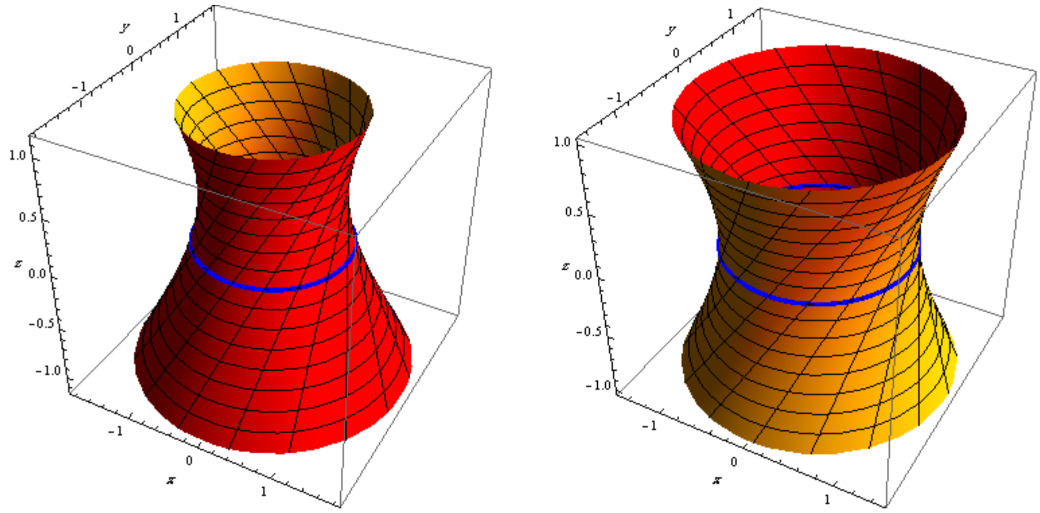
So, if we choose $t \in [-1,1]$ and $\theta_0 = 0.5$, then for $\alpha = 1, \ \lambda = -1$ and $\alpha = -1, \ \lambda = 1$, the corresponding timelike surfaces are shown in Fig. 2A, and Fig. 2B, respectively.

3.2. $\alpha = \alpha(s)$ **is a timelike line of curvature.** This time the directrix $\alpha(s)$ is an isoparametric timelike curve on the surface. As stated in the above case, we get the corresponding conditions and we omit the details here.

Theorem 3.3. *The given timelike curve $\alpha(s)$ is a line of curvature on the timelike surface $\mathbf{P}(s,t)$ if and only if*

$$\left.\begin{array}{l} a(s,t_0) = b(s,t_0) = c(s,t_0) = 0, \\ \frac{\partial c(s,t_0)}{\partial t}\frac{\partial b(s,t_0)}{\partial s} - \frac{\partial c(s,t_0)}{\partial s}\frac{\partial b(s,t_0)}{\partial t} = 0, \\ -(1 + \frac{\partial a(s,t_0)}{\partial s})\frac{\partial c(s,t_0)}{\partial t} + \frac{\partial a(s,t_0)}{\partial t}\frac{\partial c(s,t_0)}{\partial s} = \lambda(s)\cos\theta, \\ (1 + \frac{\partial a(s,t_0)}{\partial s})\frac{\partial b(s,t_0)}{\partial t} - \frac{\partial a(s,t_0)}{\partial t}\frac{\partial b(s,t_0)}{\partial s}) = \lambda(s)\sin\theta. \end{array}\right\} \tag{3.12}$$

where $\lambda(s) \neq 0$.



(A) (B)

Figure 2: (A) $P(s,t;0.3,-0.5)$ (B) $P(s,t;3,-1)$.

Corollary 3.4. *A necessary and sufficient condition of the timelike curve $\alpha(s)$ being a line of curvature on the timelike surface $\mathbf{P}(s,t)$ is*

$$\left.\begin{array}{c} A(t_0)=B(t_0)=C(t_0)=0, \\ -n(s)\frac{dC(t_0)}{dt}=\lambda(s)\cos\theta,\ m(s)\frac{dB(t_0)}{dt}=\lambda(s)\sin\theta, \end{array}\right\} \tag{3.13}$$

where $\lambda(s)\neq 0$.

By a similar procedure, we also have the following:
(i) In the case of $\tau(s)\neq 0$, then $\theta(s)$ is a non-constant function of variable s and the condition3.13 can be represented as

$$\left.\begin{array}{c} A(t_0)=B(t_0)=C(t_0)=0, \\ -\frac{dC(t_0)}{dt}=\lambda(s)\cos\theta,\ \frac{dB(t_0)}{dt}=\lambda(s)\sin\theta. \end{array}\right\} \tag{3.14}$$

(ii) In the case of $\tau(s)=0$, that is the curve is a spacelike planar curve, then $\theta(s)=\theta_0$ is a constant and we have two different cases:
(a) In the case of $\sin\theta_0\neq 0$, the condition 3.14 can be represented as

$$\left.\begin{array}{c} A(t_0)=B(t_0)=C(t_0)=0, \\ -\frac{dC(t_0)}{dt}=\lambda(s)\cos\theta_0,\ \frac{dB(t_0)}{dt}=\lambda(s)\sin\theta_0. \end{array}\right\} \tag{3.15}$$

(b) In the case of $\sin\theta_0=0$, the condition 3.14 can be represented as

$$\left.\begin{array}{c} A(t_0)=B(t_0)=C(t_0)=0, \\ -\frac{dC(t_0)}{dt}=\lambda(s),\ \frac{dB(t_0)}{dt}=0. \end{array}\right\} \tag{3.16}$$

In this case, the curve $\alpha=\alpha(s)$ is not only a timelike line of curvature but also a timelike geodesic of the timelike surface $\mathbf{P}(s,t)$.

3.3. **Examples of timelike surfaces with a common timelike line of curvature.** In this paragraph, by a similar arguments, some representative examples are illustrated to verify the method.

Example 3.3. *Suppose we are given a parametric timelike helix*

$$\alpha(s)=(a_1\cosh\frac{s}{c},b\frac{s}{c},a_1\sinh\frac{s}{c}),\ a_1>0,\ \ a_2\neq 0,\ a_1^2-a_2^2=a_3^2,\ -2\leq s\leq 2. \tag{3.17}$$

We will construct a pencil of timelike surfaces sharing the helix $\alpha(s)$ as the timelike curvature line. After simple computation, we have

$$\left.\begin{array}{r}\mathbf{T}(s)=(\frac{a_1}{a_3}\sinh\frac{s}{a_3},\frac{a_2}{a_3},\frac{a_1}{a_3}\cosh\frac{s}{a_3}),\\ \mathbf{N}(s)=(\cosh\frac{s}{a_3},0,\sinh\frac{s}{a_3}),\\ \mathbf{B}(s)=(\frac{a_2}{a_3}\sinh\frac{s}{a_3},\frac{a_1}{a_3},\frac{a_2}{a_3}\cosh\frac{s}{a_3}),\end{array}\right\} \tag{3.18}$$

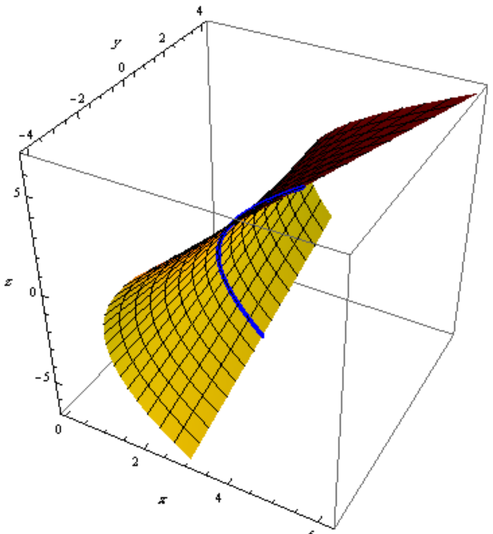
where $\tau=\frac{a_2}{a_3}$, then $\theta(s)=\frac{a_2}{a_3^2}s+\theta_0$ If $\theta_0=0$, we have $\theta(s)=\frac{a_2}{a_3^2}s$.
By choosing

$$\begin{aligned} l(s) &= m(s)=n(s)=1,\\ A(t) &= \alpha t,\ B(t)=t\lambda(s)\sin\theta,\ \ -C(t)=t\lambda(s)\cos\theta,\ \lambda\neq 0,\ \ 0\le t\le T. \end{aligned}$$

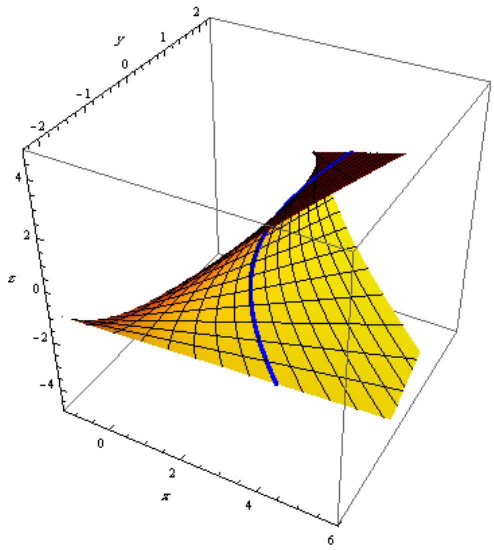
By putting these choices of functions A, B and C into Eq. 2.4, we obtain the following timelike surface pencil:

$$\mathbf{P}(s,t;\alpha,\lambda)=(a_1\cosh\frac{s}{a_3},a_2\frac{s}{a_3},a_1\sinh\frac{s}{a_3})+t(\alpha,\lambda\sin\theta,-\lambda\cos\theta)\begin{pmatrix}\frac{a_1}{a_3}\sinh\frac{s}{a_3} & \frac{a_2}{a_3} & \frac{a_1}{a_3}\cosh\frac{s}{a_3}\\ \cosh\frac{s}{a_3} & 0 & \sinh\frac{s}{a_3}\\ \frac{a_2}{a_3}\sinh\frac{s}{a_3} & \frac{a_1}{a_3} & \frac{a_2}{a_3}\cosh\frac{s}{a_3}\end{pmatrix}.$$

Here, we chose $t\in[-2,2]$, $a_1=2, a_2=1$. For $\alpha=1,\lambda=-1$ and $\alpha=-\frac{\sqrt{3}}{4},\lambda=-\frac{\sqrt{3}}{2}$, the corresponding timelike surfaces are shown in Fig. 3A and Fig. 3B, respectively.



(A)



(B)

Figure 3: (A) $P(s,t;1,-1)$ (B) $P(s,t;-\sqrt{5}/4,-\sqrt{5}/2)$.

Example 3.4. *Suppose we are given a parametric timelike curve*

$$\alpha(s)=(\cosh s,0,\sinh s),\quad -2\le s\le 2.$$

After simple computation, we have

$$\mathbf{T}(s)=(\sinh s,0,\cosh s),\ \mathbf{N}(s)=(\cosh s,0,\sinh s),\ \mathbf{B}(s)=(0,1,0),$$

and $\tau=0$ which follows θ is a nonzero constant. By choosing

$$\begin{aligned} l(s) &= m(s)=n(s)=1,\\ A(t) &= \alpha t,\ B(t)=t\lambda(s)\sin\theta,\ \ -C(t)=t\lambda(s)\cos\theta,\ \lambda\neq 0,\ \ 0\le t\le T, \end{aligned}$$

and from formula 2.4, we obtain the following timelike surface pencil

$$\mathbf{P}(s,t;\alpha,\lambda) = (\cosh s, 0, \sinh s) + t(\alpha, \lambda\sin\theta, -\lambda\cos\theta)\begin{pmatrix} \sinh s & 0 & \cosh s \\ \cosh s & 0 & \sinh s \\ 0 & 1 & 0 \end{pmatrix}.$$

We chose $\theta = 0$. For α=0.3, λ=-1, and $t \in$[-2,0], the corresponding timelike surface is shown in Fig. 4A. Fig. 4B shows the timelike surface with α=-0.3, λ=1, and $t \in$[0,2].

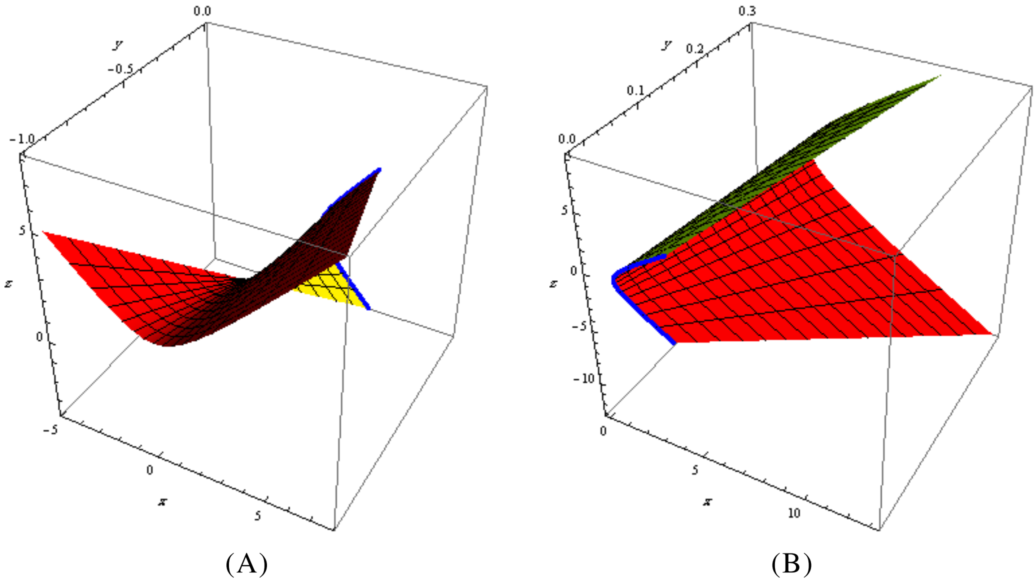


Figure 4: (*A*) $P(s,t;0.3,-0.5)$ (*B*) $P(s,t;3,-1)$.

3.4. **Timelike surfaces of revolution.** In this paragraph, for a given circle, we construct timelike surface of revolution with this circle as a line of curvature. By a circle in $\mathbb{E}_1^3$, as in Euclidean space, we mean a planar curve with constant curvature. In particular, given a surface of revolution there exists a uniparametric family of planes of $\mathbb{E}_1^3$ whose intersection with it is a circle. Since the circles are contained in a timelike surface, each circle of the foliation must be a spacelike curve. However, the planes containing the circles can be of any causal type. After an isometry of the ambient space $\mathbb{E}_1^3$, a spacelike circle parametrizes as follows:
If Γ is the horizontal plane $z = 0$, the circle is given by

$$\alpha(r) = (a_1\cos r, a_1\sin r, 0),\ a_1 > 0,\ \ 0 \le r \le 2\pi.$$

In this case, the curve is a Euclidean horizontal circle. It is easy to get that

$$\mathbf{T}(s) = (-\sin r, \cos r, 0),\ \mathbf{N}(s) = (-\cos r, -\sin r, 0),\ \mathbf{B}(s) = (0,0,1),$$

and $\tau = 0$, which follows $\theta(s) = \theta_0$ is a constant. Let

$$\begin{aligned} l(r) &= m(r) = n(r) = 1,\ \lambda(r) = \left\|\alpha^{'}(r)\right\|\mu \\ A(t) &= \alpha t,\ -B(t) = t\mu\sinh\theta_0,\ \ -CC(t) = t\lambda\mu\cosh\theta_0,\ \mu \neq 0, \end{aligned}$$

thus the timelike surface pencil can be expressed as

$$\mathbf{P}(r,t;\alpha,\mu) = (a_1\cos r, a_1\sin r, 0) + t(\alpha, -\mu\sinh\theta_0, -\mu\cosh\theta_0)\begin{pmatrix} -\sin r & \cos r & 0 \\ -\cos r & -\sin r & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

On the other hand, a surface of revolution (or rotational surface) of $\mathbb{E}_1^3$ with axis of rotation L is a surface which is invariant under the action of the group of motions in $\mathbb{E}_1^3$. If the axis is timelike (resp. spacelike), we may suppose that the axis is the $z-axis$ (resp. $y-axis$), since every timelike (resp. spacelike) unit vector is transformed to (0,0,1) (resp. (0,1,0)) by Lorentzian transformation. Then the surface is expressed as follows:

$$\bar{\mathbf{P}}(r,t) = (h_y(t)\cos r, h_y(t)\sin r, h_z(t)) \text{ if the axis-is timelike,}$$

where $\mathbf{h}(t)= (0, h_y(t), h_z(t))$ is a timelike curve in the oyz-plane and $h_y(t) \neq 0$. By combining the surfaces $\mathbf{P}(r,t)$ and $\mathbf{P}(r,t;\alpha,\mu)$ represent the same surface, we have

$$h_y(t) = a_1 + \mu t \sinh\theta_0, \;\; h_z(t) = -\mu t \cosh\theta_0.$$

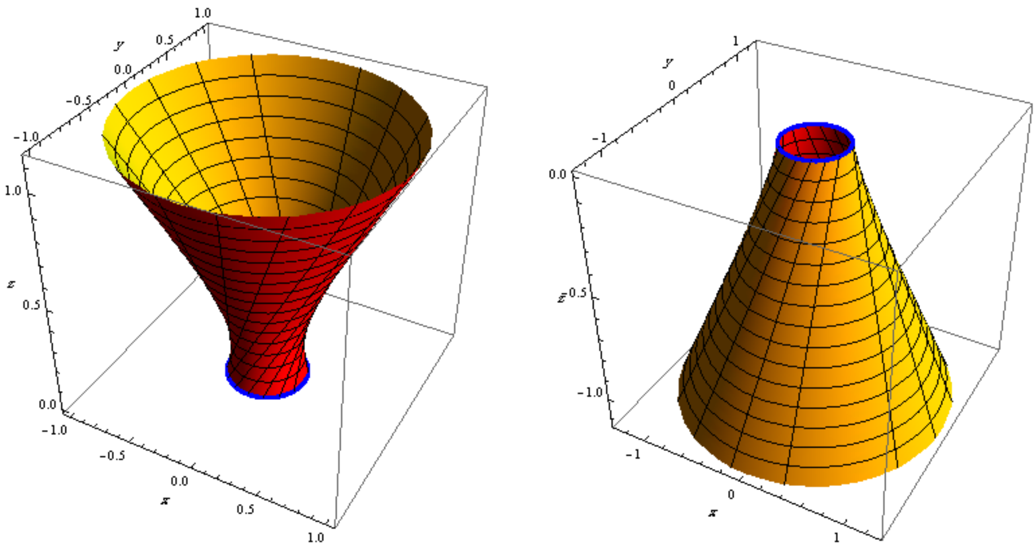
Hence, the timelike surface pencil of revolution can be written as

$$\mathbf{P}(r,t;a_1,\mu,\theta_0) = ((a_1 + \mu t\sinh\theta_0)\cos r, (a_1 + \mu t\sinh\theta_0)\sin r, -\mu t\cosh\theta_0))\,.$$

The parametric curve $\mathbf{h}(t)$ is presented by

$$\mathbf{h}(t)= (0, a_1 + \mu t\sinh\theta_0, -\mu t\cosh\theta_0).$$

This means that $\mathbf{P}$ is formed by a uniparametric family of horizontal circles. We chose $t \in [0,1]$, $\theta_0 = 0.5$, and $r \in [0, 2\pi]$. For $a_1 = \mu = 1$, and $a_1 = -\mu = 1$, the corresponding timelike surfaces are shown in Fig. 5A and Fig. 5B, respectively.



(A) (B)

Figure 5: (A) $P(r,t;0.3,-0.5,0.5)$ (B) $P(r,t;3,-1,1)$.

If Γ is the vertical plane $x = 0$, the circle is given by

$$\alpha(r) = (0, a_1\sinh r, a_1\cosh r), \;\; a_1 > 0.$$

The curve describes a hyperbola in a vertical plane. As stated in the above case, it is easy to get that

$$\mathbf{T}(s) = (0,\cosh r, \sinh r), \;\; \mathbf{N}(s) = (0, \sinh r, \cosh r), \;\; \mathbf{B}(s) = (1,0,0),$$

and $\tau = 0$, which follows $\theta(s) = \theta_0$ is a constant and we have

$$\begin{aligned} l(r) &= m(r) = n(r) = 1,\ \lambda(r) = \left\| \alpha^{'}(r) \right\| \mu \\ A(t) &= \alpha t,\ -B(t) = t\mu \sinh\theta_0,\ \ -C(t) = t\lambda, \mu\cosh\theta_0,\ \mu \neq 0. \end{aligned}$$

Thus, the timelike surface pencil can be expressed as

$$\mathbf{P}(r,t;\alpha,\mu) = (0, a_1\sinh r, a_1\cosh r) + t(\alpha, -\mu\sinh\theta_0, -\mu\cosh\theta_0)\begin{pmatrix} 0 & \cosh r & \sinh r \\ 0 & \sinh r & \cosh r \\ 1 & 0 & 0 \end{pmatrix}.$$

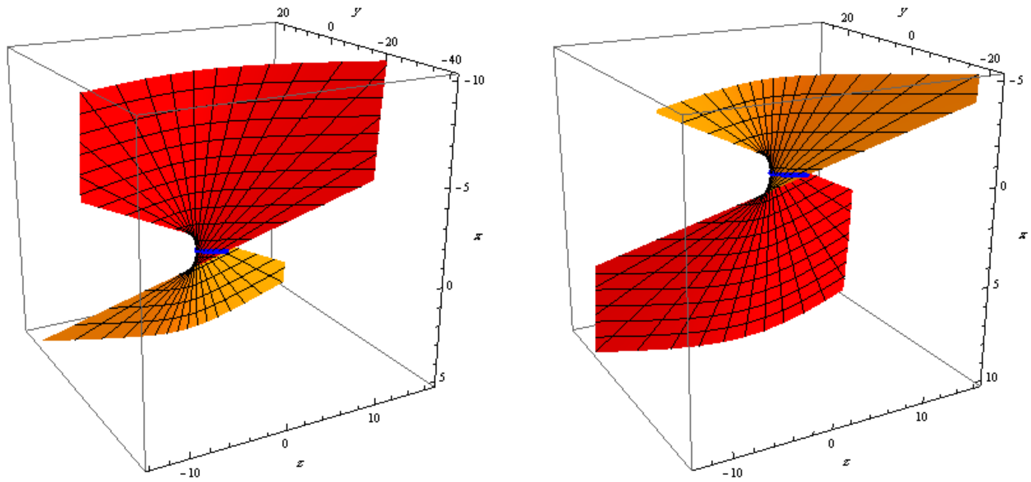
Analogously, the surface is expressed as follows:

$$\bar{\mathbf{P}}(r,t) = (h_x(t), h_z(t)\sinh r, h_z(t)\cosh r)\ \text{, if the axis-is spacelike,}$$

where $\mathbf{h}(t)$= $(h_x(t), 0, h_z(t))$ is a spacelike curve in the oxz-plane and $h_z(t) \neq 0$. Also, we can get

$$\begin{aligned} \mathbf{P}(r,t;a,\mu,\theta_0) &= (-\mu t\cosh\theta_0, (a_1 - \mu t\sinh\theta_0)\sinh r, (a_1 - \mu t\sinh\theta_0)\cosh r). \\ \mathbf{h}(t) &= (\,-\mu t\cosh\theta_0, 0, a_1 - \mu t\sinh\theta_0). \end{aligned}$$

This means that $\mathbf{P}$ is formed by a family of vertical hyperbolas. We chose $-2 \leq r \leq 2$. For $t \in [-5, 0]$, $a_1 = \mu = 1$, $\theta_0 = 0$ and $t \in [0, 5]$, $a_1 = \mu = 1$, $\theta_0 = 0.01$, the corresponding timelike surfaces are shown in Fig. 6A and Fig. 6B, respectively.



(A) (B)

Figure 6: $(A)\ P(r,t;0.3,-0.5,.5)\ (B)\ P(r,t;3,-1,1)$.

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