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# CONSERVATIVENESS CRITERIA OF GIRSANOV TRANSFORMATION FOR NON-SYMMETRIC JUMP-DIFFUSION

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ABSTRACT. We develop the condition in our previous paper [The Conservativeness of Girsanov transformed for symetric jump-diffusion process (2018)] in the framework of nonsymmetric Markov process with jumps associated with regular Dirichlet form. We prove the conservativeness of it by relation in duality of Girsanov transformed process and recurrent criteria of Dirichlet form.

*Key words and phrases:* Conservativeness; Girsanov transformed process; Non-symmetric Markov process.

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## 1. INTRODUCTION

Let $E$ be a metrizable Lusin space, i.e., a space topologically isomorphic to a Borel subset of a complete separable metric space, $\mathcal{B}(E)$ the Borel $\sigma$-field of $E$, and $m$ a $\sigma$-finite measure on $(E, \mathcal{B}(E))$. Let $X = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ be a diffusion on $E$, which is assumed to be associated with strongly local non symmetric Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E; m)$ in the sense that $T_t f = P_t f \ m-a.e.$ for any $f \in B_b(E)\cap L^2(E; m)$ and $t \geq 0$, where $B_b(E)$ is the set of all bounded $\mathcal{B}(E)$-measurable functions on $E$, $(T_t)_{t\geq 0}$ is the $L^2$-semigroup corresponding to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, and $(P_t)_{t\geq 0}$ is the transition semigroup of $X$. It is wel known that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ must be a quasi-regular Dirichlet form on $L^2(E; m)$ (cf. [13]). Let $\hat{X} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\hat{X}_t)_{t\geq 0}, (\hat{P}_x)_{x\in E})$ be the dual process of $X$. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E})_e)$ be the extended Dirichlet space of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. For $u \in D(\mathcal{E})_e$, we have the Fukushima's decomposition

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^u + N_t^u, \ P_x\text{-a.s. for q.e. } x \in E$$

and

$$\tilde{u}(X_t) - \tilde{u}(X_0) = \hat{M}_t^u + \hat{N}_t^u, \ \hat{P}_x\text{-a.s. for q.e. } x \in E,$$

where $M_t^u$ and $\hat{M}_t^u$ are square integrable martingales (MAFs) with respect to $X$ and $\hat{X}$, respectively; $N_t^u$ and $\hat{N}_t^u$ are continuous additive functional (CAF) of zero energy with respect to $X$ and $\hat{X}$, respectively.

We define a pair of local MAFs by

$$L_t := \exp(M_t^u - \frac{1}{2}\langle M^u\rangle_t) \text{ and } \widehat{L}_t := \exp(\widehat{M}_t^u - \frac{1}{2}\langle \widehat{M}^u\rangle_t).$$

Denote by $\zeta$ the lifetime of $X$. Then, by [15](cf. also [4, [6]]),

$$\mathrm{d}Q_x := L_t \mathrm{d}P_x \text{ and } \mathrm{d}\widehat{Q}_x := \widehat{L}_t \mathrm{d}\widehat{P}_x \text{ on } \mathcal{F}_t \cap \{t < \zeta\}, \ x \in E$$

define unique families of probability measures $(Q_x)_{x\in E}$ and $(\widehat{Q}_x)_{x\in E}$ on $(\Omega, \mathcal{F}_\infty)$, respectively, where $\mathcal{F}_\infty := \sigma(\cup_{t\geq 0}\mathcal{F}_t)$. Note that $X$ and $\widehat{X}$ are still Markov processes with state space $E$ under these measures. We denote them by

$$Y = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (Y_t)_{t\geq 0}, (Q_x)_{x\in E}) \text{ and } \widehat{Y} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\widehat{Y}_t)_{t\geq 0}, (\widehat{Q}_x)_{x\in E})$$

and call them the Girsanov transformed processes of $X$ and $\widehat{X}$, respectively. For a fixed $\omega \in \Omega$, if $t < \zeta(\omega)$, we define the time reversal operator $r_t$ by

$$r_t\omega(u) = \begin{cases} \omega(t-u), & \text{if } 0 \leq u \leq t, \\ \omega(0), & \text{if } u > t. \end{cases}$$

The following theorem has been proved in [3, Theorem 2.2]

**Theorem 1.** *$Y$ and $\widehat{Y}$ are in duality with respect to $e^{2u}m$ if and only if*

$$N_t^u + \frac{1}{2}\langle M^u\rangle_t = \widehat{N}_t^u + \frac{1}{2}\langle \widehat{M}^u\rangle_t \text{ for } t < \zeta, P_m\text{-a.s.} \tag{1.1}$$

## 2. SEMI-DIRICHLET FORM ASSOCIATED WITH GIRSANOV TRANSFORMED PROCESS

Since $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a general (non-symmetric) Dirichlet form, the characterization of the (semi-) Dirichlet form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ associated with $Y$ becomes much more difficult, since $Y$ and $\widehat{Y}$ may not be in duality (cf. Theorem 1) and some powerful tools like Lyons-Zheng decomposition are mainly developed for symmetric Dirichlet forms. Based on the idea is that if $N^u$ is of bounded variation, then an $h$-transformation of $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ can be characterized by the perturbation of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Let $\mu = \mu^+ - \mu^-$ be a smooth signed measure, where $\mu^+, \mu^- \in S$ and $S$ denotes the set of all smooth measures on $(E, \mathcal{B}(E))$. Define the perturbation of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ with respect to $\mu$ by

$$\begin{aligned} &\mathcal{E}^\mu(f,g) = \mathcal{E}(f,g) + \langle f, g\rangle_\mu, \quad f, g \in \mathcal{D}(\mathcal{E}^\mu), \\ &\mathcal{D}(\mathcal{E}^\mu) = \mathcal{D}(\mathcal{E}) \cap L^2(E; |\mu|), \end{aligned}$$

where $\langle f, g\rangle_\mu := \int_E fg\mu(\mathrm{d}x)$. We use $A_t^{\mu^+}$ and $A_t^{\mu^-}$ to denote the positive CAFs (PCAFs) with the Revuz measures $\mu^+$ and $\mu^-$, respectively. Define $A_t^\mu = A_t^{\mu^+} - A_t^{\mu^-}$ and the corresponding generalized Feynman-Kac semigroup by

$$P_t^\mu f(x) := E_x[e^{-A_t^\mu} f(X_t); t < \zeta], t > 0,$$

provided the right-hand side makes sense.

**Definition 1.** A measure $\mu \in S$ is said to be of the Hardy class, denoted by $\mu \in S_H$, if there exist constants $\delta_\mu, \gamma_\mu \in (0, \infty)$ such that

$$\int_E \tilde{f}^2 \mathrm{d}\mu \le \delta_\mu \mathcal{E}(f,f) + \gamma_\mu (f,f)_m, \ \forall f \in \mathcal{D}(\mathcal{E}).$$

**Proposition 1.** (See [17, Theorem 5.2.7]) *Let $u \in \mathcal{D}_e(\mathcal{E})$. Then $N^u$ is of bounded variation if and only if there exist $\nu_1, \nu_2 \in S$ and an $\mathcal{E}$-nest $\{F_k\}_{k\ge1}$ such that*

$$\mathcal{E}(u,v) = \int_E \tilde{v}\mathrm{d}(\nu_1 - \nu_2), \ \forall v \in \bigcup_{k\ge1} \mathcal{D}(\mathcal{E})_{F_k}.$$

The following Theorem has been proved in [3, Corollary 3.5].

**Theorem 2.** *Let $u \in \mathcal{D}(\mathcal{E})_e$. Suppose that the Girsanov transformed processes $Y$ and $\widehat{Y}$ are in duality with respect to $e^{2u}m$ and $N^u$ is of bounded variation with*

$$N_t^u = N_t^{(1)} - N_t^{(2)} \text{ for } t < \zeta,$$

*where $N^{(1)}, N^{(2)}$ are the PCAFs with the respective Revuz measure $\nu_1$ and $\nu_2$, and $\nu_2 \in S_H$ with $\delta_{\nu_2} < 1$. Define $\mu = (\frac{1}{2}\mu_{\langle u\rangle} + \nu_1) - \nu_2$, where $\mu_{\langle u\rangle}$ is the Revuz measure of $\langle M^u\rangle$. Then $(Y, \widehat{Y})$ are associated with the Dirichlet form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ on $L^2(E; e^{2u}m)$ defined by*

$$\begin{aligned} &\mathcal{Q}(f,g) = \mathcal{E}(fe^u, ge^u) + \langle fe^u, ge^u\rangle_\mu, \qquad \forall f, g \in \mathcal{D}(\mathcal{Q}), \\ &\mathcal{D}(\mathcal{Q}) = \{f \in L^2(E; e^{2u}m) | fe^u \in \mathcal{D}(\mathcal{E}^\mu)\}. \end{aligned} \tag{2.1}$$

## 3. MAIN RESULT

**Theorem 3.** *Assume that $u$ is bounded and let $u \in \mathcal{D}_e(\mathcal{E})$. Then the Dirichlet form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ is conservative.*

*Proof.* The proof is based on an idea in [4]. We see from [7, Lemma 1.6.7] that there exists a strictly positive bounded function $h \in L^1(E; m)$ such that $u \in \mathcal{D}_e(\mathcal{E}^h)$ and then $u \in \mathcal{D}_e(\mathcal{E}^h)$, where $\mathcal{E}^h$ is perturbed form on $L^2(E; m)$ defined by

$$\mathcal{E}^h(f, g) = \mathcal{E}(f, g) + \langle f, g\rangle_{h.m}\ f, g \in \mathcal{D}(\mathcal{E}^h).$$

Then $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}))$ is a transient Dirichlet form and thus its extended Dirichlet space $\mathcal{D}_e(\mathcal{E}^h)$ is a Hilbert space with inner product $\mathcal{E}^h$ ([7, Theorem 1.6.2]). By Proposition 3.1, for $\nu_1, \nu_2 \in S$, we can take $\mathcal{E}$-nest $\{F_k\}_{k\geq 1}$ such that

$$\mathcal{E}(u, v) = \int_E \tilde{v} \mathrm{d}(\nu_1 - \nu_2)\ \forall v \in \bigcup_{k\geq 1} \mathcal{D}(\mathcal{E})_{F_k}.$$

Set $\rho = e^u$, and let $F_k^{(1)} = \{x \in F_k : \rho(x) \geq \frac{1}{k}\}$. Then $\{F_k^{(1)} : k \geq \frac{1}{k}\}$ is an $\mathcal{E}$-nest because $E \backslash \cup_{k\geq 1}\{\rho \geq \frac{1}{k}\}$ is an $\mathcal{E}$-exceptional. Since the norm $\sqrt{\mathcal{E}_1^h(\cdot,\cdot)}$ is equivalent to $\sqrt{\mathcal{E}_1(\cdot,\cdot)}$, $\{F_k^{(1)}\}$ is $\mathcal{E}^h$-nest as well. Set $\mathcal{D}_e(\mathcal{E}^h)_{F_k^{(1)}} := \{f \in \mathcal{D}_e(\mathcal{E}^h) : f = 0\ m-$a.e. on $E\backslash F_k^{(1)}\}$. Then $\mathcal{D}_e(\mathcal{E}^h)_{F_k^{(1)}}$ is a closed set of the Hilbert space $(\mathcal{D}_e(\mathcal{E}^h), \mathcal{E}^h)$ and $\bigcup_{k\geq 1} \mathcal{D}_e(\mathcal{E})_{F_k^{(1)}}$ is dense in $\mathcal{D}_e(\mathcal{E})$ by [5, Corollary 3.4.4].

Let $\rho_{F_k^{(1)}}$ be the $\mathcal{E}^h$-orthogonal projection of $\rho$ onto $\mathcal{D}_e(\mathcal{E}^h)F_k^{(1)}$. Then $\rho_{F_k^{(1)}}$ converges to $\rho$ in $(\mathcal{D}_e(\mathcal{E}^h), \mathcal{E}^h)$. Let $\rho_k = (0 \vee \rho_{F_k^{(1)}}) \wedge \rho$. Then $\rho_k \in \mathcal{D}(\mathcal{E})_{F_k^{(1)}}$ for each $k$ and $\rho_k \to \rho$ $m$-a.e. as $k \to \infty$. Since $\rho - \rho_k = (\rho - \rho_{F_k^{(1)}})^+$, we have

$$\begin{aligned}\mathcal{E}(\rho - \rho_k, \rho - \rho_k) &\leq \mathcal{E}^h(\rho - \rho_k, \rho - \rho_k)\\ &\leq \mathcal{E}^h(\rho - \rho_{F_k^{(1)}}, \rho - \rho_{F_k^{(1)}}).\end{aligned}$$

The last inequality tends to 0 as $k \to \infty$. By taking subsequence if necessary, we may assume that $\rho_k$ converges to $\rho$ $\mathcal{E}$-q.e. on $E$ ([5, Theorem 2.3.4]). For $k \geq 1$, define a function $w_k$ by

$$w_k := \begin{cases} \frac{\rho_k(x)}{\rho(x)}, & \text{if } \rho(x) > 0,\\ 0, & \text{if } \rho(x) = 0.\end{cases}$$

Then $0 \leq w_k \leq 1$ and $w_k \to 1$ $\mathcal{E}$-q.e. on $E$ as $k \to \infty$. Noting that $\rho_k \in \mathcal{D}(\mathcal{E})_{F_k^{(1)}}$ then $w_k$ is also in $\mathcal{D}(\mathcal{E})_{F_k^{(1)}}$. Since $\bigcup_{k\geq 1} \mathcal{D}(\mathcal{E})_{F_k}$ is dense in $\mathcal{D}(\mathcal{E})$ and $\mathcal{D}(\mathcal{E}) \subset \mathcal{D}(\mathcal{Q})$ then $\bigcup_{k\geq 1} \mathcal{D}(\mathcal{E})_{F_k} \subset \mathcal{D}(\mathcal{Q})$. Hence, since $\rho \in \mathcal{D}_e(\mathcal{E}) \bigcap (E_\partial)$ then $w_k$ belongs to $\mathcal{D}(\mathcal{Q})$.

For $w_k \in \mathcal{D}(\mathcal{Q})$, by Theorem 2

$$\begin{aligned}\mathcal{Q}(w_k, w_k) &= \mathcal{E}(w_k\rho, w_k\rho) + \langle w_k\rho, w_k\rho\rangle_\mu\\ &= \mathcal{E}(w_k\rho, w_k\rho) + \langle w_k\rho, w_k\rho\rangle_{\frac{1}{2}\mu_{\langle u\rangle}} + \langle w_k\rho, w_k\rho\rangle_{\nu_1-\nu_2}\\ &= \mathcal{E}(w_k\rho, w_k\rho) + \langle w_k\rho, w_k\rho\rangle_{\frac{1}{2}\mu_{\langle u\rangle}} - \mathcal{E}(u, w_k^2\rho^2). \qquad (3.1)\end{aligned}$$

Since $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a strongly local Dirichlet form,

$$\mathcal{E}(w_k\rho, w_k\rho) = \frac{1}{2}\int_E \mathrm{d}\mu_{\langle w_k\rho . w_k\rho\rangle}$$

$$= \frac{1}{2}\left(\int_E w_k^2 \mathrm{d}\mu^c_{\langle\rho\rangle} + \int_E \rho^2 \mathrm{d}\mu^c_{\langle w_k\rangle} + \int_E w_k\rho \mathrm{d}\mu^c_{\langle\rho, w_k\rangle} + \int_E \rho w_k \mathrm{d}\mu^c_{\langle w_k,\rho\rangle}\right). \tag{3.2}$$

and by the Schwarz inequality [12, Lemma 5.2]

$$\mathcal{E}(u, w_k^2\rho^2) = \frac{1}{2}\int_E \mathrm{d}\mu^c_{\langle u, w_k^2\rho^2\rangle}$$

$$= \frac{1}{2}\int_E w_k^2 \mathrm{d}\mu^c_{\langle u,\rho^2\rangle} + \frac{1}{2}\int_E \rho^2 \mathrm{d}\mu^c_{\langle u, w_k^2\rangle}$$

$$= \int_E w_k^2\rho \mathrm{d}\mu^c_{\langle u,\rho\rangle} + \int_E \rho^2 w_k \mathrm{d}\mu^c_{\langle u, w_k\rangle}$$

$$\leq \left(\int_E w_k^2\rho^2 \mathrm{d}\mu^c_{\langle u\rangle}\right)^{1/2}\left(\int_E w_k^2 \mathrm{d}\mu^c_{\langle\rho\rangle}\right)^{1/2} + \left(\int_E \rho^2 w_k^2 \mathrm{d}\mu^c_{\langle u\rangle}\right)^{1/2}\left(\int_E \rho^2 \mathrm{d}\mu^c_{\langle w_k\rangle}\right)^{1/2} \tag{3.3}$$

Since

$$\mathrm{d}\mu^c_{\langle\rho_k\rangle} = \mathrm{d}\mu^c_{\langle\rho w_k\rangle}$$

$$= 2w_k \mathrm{d}\mu^c_{\langle\rho w_k,\rho} + \rho^2 \mathrm{d}\mu^c_{\langle w_k\rangle} - w_k^2 \mathrm{d}\mu^c_{\langle\rho\rangle}$$

we have

$$\int_E \rho^2 \mathrm{d}\mu^c_{\langle w_k\rangle} = \frac{1}{2}\int_E \mathrm{d}\mu^c_{\langle\rho_k\rangle} + \frac{1}{2}\int_E w_k^2 \mathrm{d}^c_{\langle\rho\rangle} - \int_E w_k \mathrm{d}\mu^c_{\langle\rho_k,\rho\rangle}.$$

Since $w_k \to 1$ and $\rho_k \to \rho$ as $k \to \infty$, the first and second term in the right hand side of the equality above tend to $\frac{1}{2}\mu^c_{\langle\rho\rangle}(E)$. The third term tends to $\mu^c_{\langle\rho\rangle}(E)$ because

$$\left|\int_E w_k \mathrm{d}\mu^c_{\langle\rho_k,\rho\rangle} - \int_E \mathrm{d}\mu^c_{\langle\rho\rangle}\right| \leq \left(\int_E w_k^2 \mathrm{d}\mu^c_{\langle\rho\rangle}\right)^{1/2}\left(\int_E \mathrm{d}\mu^c_{\langle\rho_k-\rho\rangle}\right)^{1/2}.$$

Hence we have

$$\lim_{k\to\infty}\int_E \rho^2 \mathrm{d}\mu^c_{\langle w_k\rangle} = 0. \tag{3.4}$$

By the Schwarz inequality [12, Lemma 5.2] and (3.4), the third and the fourth terms tend to 0 and the last part of (3.3) also tends to 0 as $k \to \infty$ by (3.4) then we have $\lim_{k\to\infty} \mathcal{Q}(w_k, w_k) = 0$. Therefore $w_k \to 1$ q.e. and $\mathcal{Q}(w_k, w_k) \to 0$ as $k \to \infty$ which implies that $1 \in \mathcal{D}_e(\mathcal{Q})$ and $\mathcal{Q}(1, 1) = 0$. Hence, by [7, Theorem 1.6.3]$(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ is recurrent, in particular conservative. ∎

## 4. EXAMPLE

Let $E = \mathbb{R}^d$ and $m = \mathrm{d}x$ be the Lebesgue measure on $\mathbb{R}^d$. Suppose that $a_{ij} \in C^1(\mathbb{R}^d), 1 \leq i, j \leq d$, satisfying the following conditions:

$$\sum_{i,j=1}^d \bar{a}_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2_{\mathbb{R}^d}, \forall x, (\xi_1, \xi_2, ..., \xi_d) \in \mathbb{R}^d \text{ and } |\check{a}_{ij}(x)| \leq C, \ \forall x \in \mathbb{R}^d,$$

where $\bar{a}_{ij} := \frac{1}{2}(a_{ij} + a_{ji}), \check{a}_{ij} := \frac{1}{2}(a_{ij} - a_{ji}), 1 \leq i, j \leq d,\ \lambda, C \in (0, \infty)$. Define

$$\mathcal{E}(f,g) = \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\mathrm{d}x,\ \forall f,g \in C_0^\infty(\mathbb{R}^d).$$

Then $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ is closable and its closure $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a regular (non-symmetric) Dirichlet form on $L^2(\mathbb{R}^d; \mathrm{d}x)$. It is easy to see that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and $(Y, \hat{Y})$ be the Girsanov transformed processes of $(X, \hat{X})$ associated with the Dirichlet form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ on $L^2(\mathbb{R}^d; e^{2u}\mathrm{d}x)$ satisfying

$$\mathcal{Q}(f,g) = \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}e^{2u}\mathrm{d}x + \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} \check{a}_{ij}\frac{\partial f}{\partial x_i}\frac{\partial u}{\partial x_j}ge^{2u}\mathrm{d}x$$

for $f, g \in C_0^\infty(\mathbb{R}^d.)$ Suppose that $u \in C_0^2(\mathbb{R}^d)$ satisfying

$$\sum_{i,j=1}^{d}\frac{\partial \check{a}_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} = 0. \tag{4.1}$$

Then $Y$ and $\hat{Y}$ are in duality with respect to $e^{2u}\mathrm{d}x$ and $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ is conservative.

*Proof.*

$$N_t^u = \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \frac{\partial}{\partial x_i}\left[a_{ij}\frac{\partial u}{\partial x_j}\right](X_s)\mathrm{d}s. \tag{4.2}$$

$$\hat{N}_t^u = \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \frac{\partial}{\partial x_i}\left[a_{ji}\frac{\partial u}{\partial x_j}\right](X_s)\mathrm{d}s. \tag{4.3}$$

$$\langle M^u\rangle_t = \sum_{i,j=1}^{d}\int_0^t \left[a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j}\right](X_s)\mathrm{d}s. \tag{4.4}$$

$$\langle \hat{M}^u\rangle_t = \sum_{i,j=1}^{d}\int_0^t \left[a_{ji}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j}\right](X_s)\mathrm{d}s. \tag{4.5}$$

By (4.1) we get

$$\begin{aligned}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left[a_{ji}\frac{\partial u}{\partial x_j}\right] &= \sum_{i,j=1}^{d}\frac{\partial a_{ji}}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i,j=1}^{d}a_{ji}\frac{\partial^2 u}{\partial x_i \partial x_j}\\ &= \sum_{i,j=1}^{d}\frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i,j=1}^{d}a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\\ &= \sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left[a_{ij}\frac{\partial u}{\partial x_j}\right].\end{aligned}$$

Hence we obtain $N_t^u = \hat{N}_t^u$ and $\langle M^u\rangle_t = \langle \hat{M}^u\rangle_t$. Therefore $Y$ and $\hat{Y}$ are in duality with respect to $e^{2u}\mathrm{d}x$ by Theorem 1. Obviously, $N^u$ is of bounded variation. The Revuz measure of $\langle M^u\rangle$

and $N^u$ are respectively given by

$$\mu_{\langle u\rangle}(\mathrm{d}x) = \sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\mathrm{d}x \tag{4.6}$$

and

$$Lu\mathrm{d}x = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)\mathrm{d}x = \frac{1}{2}\left[\sum_{i,j=1}^{d}\frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i,j=1}^{d} a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}\right]\mathrm{d}x. \tag{4.7}$$

Since $a_{ij} \in C^1(\mathbb{R}^d),\ 1 \le i,j \le d$, and $u \in C_0^2(\mathbb{R}^d)$, $\left[\sum_{i,j=1}^{d}\frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i,j=1}^{d} a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}\right]$ is bounded on $\mathbb{R}^d$. Hence $\nu_2 \in S_H$ with $\delta_{\nu_2} < 1$. By Theorem 2, $Y$ is associated with Dirichlet form $(\mathcal{Q},\mathcal{D}(\mathcal{Q}))$:

$$\mathcal{Q}(f,g) = \mathcal{E}(fe^u, ge^u) + \langle fe^u, ge^u\rangle_{\frac{1}{2}\mu_{\langle u\rangle}} + \langle fe^u, ge^u\rangle_{Lu\mathrm{d}x},\ f,g \in \mathcal{D}(\mathcal{Q}). \tag{4.8}$$

Note that $\mathcal{D}(\mathcal{Q}) = \{f \in L^2(\mathbb{R}^d; e^{2u}\mathrm{d}x) | fe^u \in \mathcal{D}(\mathcal{E}^\mu)\}$ and assume that $\left(\sum_{i,j=1}^{d}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}a_{ij}\right)$ is bounded on $\mathbb{R}^d$. Then $C_0^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d; |\mu|)$ and thus $C_0^\infty(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{Q})$. By the definition of $(\mathcal{E},\mathcal{D}(\mathcal{E}))$,(4.6), (4.7), we get

$$\begin{aligned}
\mathcal{E}(fe^u, ge^u) &= \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}\frac{\partial fe^u}{\partial x_i}\frac{\partial ge^u}{\partial x_j}\mathrm{d}x \\
&= \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}e^{2u}\mathrm{d}x + \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial u}{\partial x_j}ge^{2u}\mathrm{d}x \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial g}{\partial x_j}fe^{2u}\mathrm{d}x + \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}fge^{2u}\mathrm{d}x.
\end{aligned}$$

and

$$\begin{aligned}
\langle fe^u, ge^u\rangle_{Lu\mathrm{d}x} &= -\mathcal{E}(u, fge^{2u}) \\
&= -\frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial f}{\partial x_j}ge^{2u}\mathrm{d}x - \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial g}{\partial x_j}fe^{2u}\mathrm{d}x \\
&\quad - \sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}fge^{2u}\mathrm{d}x
\end{aligned}$$

and the last part of the right hand side of (4.8)

$$\langle fe^u, ge^u\rangle_{\frac{1}{2}\mu_{\langle u\rangle}} = \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}fge^{2u}\mathrm{d}x$$

We thus have

$$\begin{aligned}\mathcal{Q}(f,g) &= \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}e^{2u}\mathrm{d}x + \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial u}{\partial x_j}ge^{2u}\mathrm{d}x \\ &\quad - \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial f}{\partial x_j}ge^{2u}\mathrm{d}x \\ &= \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}e^{2u}\mathrm{d}x + \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} \breve{a}_{ij}\frac{\partial f}{\partial x_i}\frac{\partial u}{\partial x_j}ge^{2u}\mathrm{d}x.\end{aligned}$$

Since $1 \in \mathcal{D}(\mathcal{Q})$ and $\mathcal{Q}(1,1) = 0$ then $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ is conservative. ∎

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