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TRACE INEQUALITIES FOR OPERATORS IN HILBERT SPACES: A SURVEY OF RECENT RESULTS

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ABSTRACT. In this paper we survey some recent trace inequalities for operators in Hilbert spaces that are connected to Schwarz's, Buzano's and Kato's inequalities and the reverses of Schwarz inequality known in the literature as Cassels' inequality and Shisha-Mond's inequality. Applications for some functionals that are naturally associated to some of these inequalities and for functions of operators defined by power series are given. Further, various trace inequalities for convex functions are presented including refinements of Jensen inequality and several reverses of Jensen's inequality. Hermite-Hadamard type inequalities and the trace version of Slater's inequality are given. Some Lipschitz type inequalities are also surveyed. Examples for fundamental functions such as the power, logarithmic, resolvent and exponential functions are provided as well.

Key words and phrases: Trace class operators, Hilbert-Schmidt operators, Trace, Schwarz inequality, Kato inequality, Cassels and Shisha-Mond inequalities, Jensen's inequality, Hermite-Hadamard inequality, Slater inequality, Trace inequalities for matrices, Power series of operators.

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Contents

1. INTRODUCTION

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i \in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i \in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in I} \|Af_j\|^2 = \sum_{j \in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator if and only if A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i \in I} \|Ae_i\|^2 \right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i \in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm*.

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt if and only if $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

THEOREM 1.1. *We have*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B \rangle_2 := \sum_{i \in I} \langle Ae_i, Be_i \rangle = \sum_{i \in I} \langle B^*Ae_i, e_i \rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$;

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and*

$$\|AT\|_2, \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

for any $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$;

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H);$$

(iv) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_2(H)$;

(v) $\mathcal{B}_2(H) \subseteq \mathcal{K}(H)$, *where* $\mathcal{K}(H)$ *denotes the algebra of compact operators on* H.

If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

PROPOSITION 1.2. *If $A\in\mathcal{B}(H)$, then the following are equivalent:*
(i) $A\in\mathcal{B}_1(H)$;
(ii) $|A|^{1/2}\in\mathcal{B}_2(H)$;
(ii) A *(or* $|A|$*) is the product of two elements of* $\mathcal{B}_2(H)$.

The following properties are also well known:

THEOREM 1.3. *With the above notations:*
(i) We have

$$\|A\|_1=\|A^*\|_1 \text{ and } \|A\|_2\le\|A\|_1 \tag{1.8}$$

for any $A\in\mathcal{B}_1(H)$;
(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H)\subseteq\mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H)=\mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1=\sup\left\{|\langle A,B\rangle_2|\mid B\in\mathcal{B}_2(H),\ \|B\|_2\le1\right\};$$

(v) $(\mathcal{B}_1(H),\|\cdot\|_1)$ *is a Banach space.*
(iv) We have the following isometric isomorphisms

$$\mathcal{B}_1(H)\cong K(H)^* \text{ and } \mathcal{B}_1(H)^*\cong\mathcal{B}(H),$$

where $K(H)^*$ *is the dual space of* $K(H)$ *and* $\mathcal{B}_1(H)^*$ *is the dual space of* $\mathcal{B}_1(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

THEOREM 1.4. *We have*
(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT)=\operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;
(iv) If $A,B\in\mathcal{B}_2(H)$ *then* $AB,BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;
(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

Utilising the trace notation we obviously have that

$$\langle A, B\rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*) \text{ and } \|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}\left(|A|^2\right)$$

for any $A, B \in \mathcal{B}_2(H)$.

Now, for the finite dimensional case, it is well known that the trace functional is *submultiplicative*, that is, for *positive semidefinite matrices* A and B in $M_n(\mathbb{C})$,

$$0 \le \operatorname{tr}(AB) \le \operatorname{tr}(A)\operatorname{tr}(B).$$

Therefore

$$0 \le \operatorname{tr}(A^k) \le [\operatorname{tr}(A)]^k,$$

where k is any positive integer.

In 2000, Yang [**135**] proved a matrix trace inequality

$$\operatorname{tr}\left[(AB)^k\right] \le (\operatorname{tr} A)^k (\operatorname{tr} B)^k, \tag{1.12}$$

where A and B are positive semidefinite matrices over $\mathbb{C}$ of the same order n and k is any positive integer. For related works the reader can refer to [**20**], [**22**], [**114**] and [**137**], which are continuations of the work of Bellman [**7**].

If $(H, \langle\cdot,\cdot\rangle)$ is a separable infinite-dimensional Hilbert space then the inequality (1.12) is also valid for any positive operators $A, B \in \mathcal{B}_1(H)$. This result was obtained by L. Liu in 2007, see [**102**].

In 2001, Yang et al. [**136**] improved (1.12) as follows:

$$\operatorname{tr}\left[(AB)^m\right] \le \left[\operatorname{tr}\left(A^{2m}\right)\operatorname{tr}\left(B^{2m}\right)\right]^{1/2}, \tag{1.13}$$

where A and B are positive semidefinite matrices over $\mathbb{C}$ of the same order and m is any positive integer.

In [**124**] the authors have proved many trace inequalities for sums and products of matrices. For instance, if A and B are positive semidefinite matrices in $M_n(\mathbb{C})$ then

$$\operatorname{tr}\left[(AB)^k\right] \le \min\left\{\|A\|^k \operatorname{tr}\left(B^k\right), \|B\|^k \operatorname{tr}\left(A^k\right)\right\} \tag{1.14}$$

for any positive integer k. Also, if $A, B \in M_n(\mathbb{C})$ then for $r \ge 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have the following *Young type inequality*

$$\operatorname{tr}\left(|AB^*|^r\right) \le \operatorname{tr}\left[\left(\frac{|A|^p}{p} + \frac{|B|^q}{q}\right)^r\right]. \tag{1.15}$$

Ando [**4**] proved a very strong form of Young's inequality - it was shown that if A and B are in $M_n(\mathbb{C})$, then there is a *unitary matrix* U such that

$$|AB^*| \le U\left(\frac{1}{p}|A|^p + \frac{1}{q}|B|^q\right)U^*,$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, which immediately gives the trace inequality

$$\operatorname{tr}\left(|AB^*|\right) \le \frac{1}{p}\operatorname{tr}\left(|A|^p\right) + \frac{1}{q}\operatorname{tr}\left(|B|^q\right). \tag{1.16}$$

This inequality can also be obtained from (1.15) by taking $r = 1$.

Another Hölder type inequality has been proved by Manjegani in [**109**] and can be stated as follows:

$$\operatorname{tr}(AB) \le \left[\operatorname{tr}(A^p)\right]^{1/p}\left[\operatorname{tr}(B^q)\right]^{1/q}, \tag{1.17}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and A and B are positive semidefinite matrices.

For the theory of trace functionals and their applications the reader is referred to [**126**].

For other trace inequalities see [**8**], [**20**], [**52**], [**49**], [**83**], [**98**], [**123**] and [**132**].

In this paper we survey some recent trace inequalities obtained by the author for operators in Hilbert spaces that are connected to Schwarz's, Buzano's and Kato's inequalities and the reverses of Schwarz inequality known in the literature as Cassels' inequality and Shisha-Mond's inequality. Applications for some functionals that are naturally associated to some of these inequalities and for functions of operators defined by power series are given, see also [**50**]. Further, various trace inequalities for convex functions are presented including refinements of Jensen inequality and several reverses of Jensen's inequality. Hermite-Hadamard type inequalities and the trace version of Slater's inequality are given. Some Lipschitz type inequalities are also surveyed. Examples for fundamental functions such as the power, logarithmic, resolvent and exponential functions are provided as well.

For Grüss' type inequalities for positive maps, see [**5**], [**106**] and [**115**]. For Cassels, Diaz-Metcalf and Shisha-Mond type inequalities, see [**113**]. For other inequalities for positive maps see [**9**], [**10**], [**19**], [**129**] and [**138**].

For trace inequalities for Hilbert space operators that appeared in information theory and quantum information theory we refer to [**23**], [**72**], [**108**] and [**134**].

2. SCHWARZ TYPE TRACE INEQUALITIES

2.1. Some Trace Inequalities Via Hermitian Forms. Let P a selfadjoint operator with $P \geq 0$. For $A \in \mathcal{B}_2(H)$ and $\{e_i\}_{i\in I}$ an orthonormal basis of H we have

$$\|A\|_{2,P}^2 := \operatorname{tr}(A^*PA) = \sum_{i\in I} \langle PAe_i, Ae_i\rangle \leq \|P\| \sum_{i\in I} \|Ae_i\|^2 = \|P\| \, \|A\|_2^2 ,$$

which shows that $\langle \cdot, \cdot \rangle_{2,P}$ defined by

$$\langle A, B\rangle_{2,P} := \operatorname{tr}(B^*PA) = \sum_{i\in I} \langle PAe_i, Be_i\rangle = \sum_{i\in I} \langle B^*PAe_i, e_i\rangle$$

is a *nonnegative Hermitian form* on $\mathcal{B}_2(H)$, i.e. $\langle \cdot, \cdot \rangle_{2,P}$ satisfies the properties:

(h) $\langle A, A\rangle_{2,P} \geq 0$ for any $A \in \mathcal{B}_2(H)$;

(hh) $\langle \cdot, \cdot \rangle_{2,P}$ is linear in the first variable;

(hhh) $\langle B, A\rangle_{2,P} = \overline{\langle A, B\rangle}_{2,P}$ for any $A, B \in \mathcal{B}_2(H)$.

Using the properties of the trace we also have the following representations

$$\|A\|_{2,P}^2 := \operatorname{tr}\left(P|A^*|^2\right) = \operatorname{tr}\left(|A^*|^2 P\right)$$

and

$$\langle A, B\rangle_{2,P} := \operatorname{tr}(PAB^*) = \operatorname{tr}(AB^*P) = \operatorname{tr}(B^*PA)$$

for any $A, B \in \mathcal{B}_2(H)$.

We start with the following result:

THEOREM 2.1 (Dragomir, 2014, [**58**]). *Let P a selfadjoint operator with $P \geq 0$, i.e. $\langle Px, x\rangle \geq 0$ for any $x \in H$.*

(i) For any $A, B \in \mathcal{B}_2(H)$

$$|\operatorname{tr}(PAB^*)|^2 \leq \operatorname{tr}\left(P|A^*|^2\right) \operatorname{tr}\left(P|B^*|^2\right) \tag{2.1}$$

and

$$\left[\operatorname{tr}\left(P|A^*|^2\right) + 2\operatorname{Re}\operatorname{tr}(PAB^*) + \operatorname{tr}\left(P|B^*|^2\right)\right]^{1/2} \leq \left[\operatorname{tr}\left(P|A^*|^2\right)\right]^{1/2} + \left[\operatorname{tr}\left(P|B^*|^2\right)\right]^{1/2}; \tag{2.2}$$

(ii) For any $A, B, C \in \mathcal{B}_2(H)$

$$\begin{aligned}(2.3)\qquad & \left|\operatorname{tr}(PAB^*)\operatorname{tr}\left(P|C^*|^2\right) - \operatorname{tr}(PAC^*)\operatorname{tr}(PCB^*)\right|^2 \\ & \leq \left[\operatorname{tr}\left(P|A^*|^2\right)\operatorname{tr}\left(P|C^*|^2\right) - |\operatorname{tr}(PAC^*)|^2\right] \\ & \times \left[\operatorname{tr}\left(P|B^*|^2\right)\operatorname{tr}\left(P|C^*|^2\right) - |\operatorname{tr}(PBC^*)|^2\right],\end{aligned}$$

$$\begin{aligned}(2.4)\qquad & |\operatorname{tr}(PAB^*)|\operatorname{tr}\left(P|C^*|^2\right) \\ & \leq \left|\operatorname{tr}(PAB^*)\operatorname{tr}\left(P|C^*|^2\right) - \operatorname{tr}(PAC^*)\operatorname{tr}(PCB^*)\right| \\ & + |\operatorname{tr}(PAC^*)\operatorname{tr}(PCB^*)| \\ & \leq \left[\operatorname{tr}\left(P|A^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P|B^*|^2\right)\right]^{1/2}\operatorname{tr}\left(P|C^*|^2\right)\end{aligned}$$

and

$$\begin{aligned}(2.5)\qquad & |\operatorname{tr}(PAC^*)\operatorname{tr}(PCB^*)| \\ & \leq \frac{1}{2}\left[\left[\operatorname{tr}\left(P|A^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P|B^*|^2\right)\right]^{1/2} + |\operatorname{tr}(PAB^*)|\right]\operatorname{tr}\left(P|C^*|^2\right).\end{aligned}$$

PROOF. (i) Making use of the Schwarz inequality for the nonnegative hermitian form $\langle\cdot,\cdot\rangle_{2,P}$ we have

$$\left|\langle A, B\rangle_{2,P}\right|^2 \leq \langle A, A\rangle_{2,P}\langle B, B\rangle_{2,P}$$

for any $A, B \in \mathcal{B}_2(H)$ and the inequality (2.1) is proved.

We observe that $\|\cdot\|_{2,P}$ is a seminorm on $\mathcal{B}_2(H)$ and by the triangle inequality we have

$$\|A + B\|_{2,P} \leq \|A\|_{2,P} + \|B\|_{2,P}$$

for any $A, B \in \mathcal{B}_2(H)$ and the inequality (2.2) is proved.

(ii) Let $C \in \mathcal{B}_2(H)$, $C \neq 0$. Define the mapping $[\cdot,\cdot]_{2,P,C} : \mathcal{B}_2(H) \times \mathcal{B}_2(H) \to \mathbb{C}$ by

$$[A, B]_{2,P,C} := \langle A, B\rangle_{2,P}\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\langle C, B\rangle_{2,P}.$$

Observe that $[\cdot,\cdot]_{2,P,C}$ is a nonnegative Hermitian form on $\mathcal{B}_2(H)$ and by Schwarz inequality we have

$$\begin{aligned}(2.6)\qquad & \left|\langle A, B\rangle_{2,P}\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\langle C, B\rangle_{2,P}\right|^2 \\ & \leq \left[\|A\|_{2,P}^2\|C\|_{2,P}^2 - \left|\langle A, C\rangle_{2,P}\right|^2\right]\left[\|B\|_{2,P}^2\|C\|_{2,P}^2 - \left|\langle B, C\rangle_{2,P}\right|^2\right]\end{aligned}$$

for any $A, B \in \mathcal{B}_2(H)$, which proves (2.3).

The case $C = 0$ is obvious.

Utilising the elementary inequality for real numbers m, n, p, q

$$\left(m^2 - n^2\right)\left(p^2 - q^2\right) \leq (mp - nq)^2,$$

we can easily see that

$$\begin{aligned}(2.7a)\qquad & \left[\|A\|_{2,P}^2\|C\|_{2,P}^2 - \left|\langle A, C\rangle_{2,P}\right|^2\right]\left[\|B\|_{2,P}^2\|C\|_{2,P}^2 - \left|\langle B, C\rangle_{2,P}\right|^2\right] \\ & \leq \left(\|A\|_{2,P}\|B\|_{2,P}\|C\|_{2,P}^2 - \left|\langle A, C\rangle_{2,P}\right|\left|\langle B, C\rangle_{2,P}\right|\right)^2\end{aligned}$$

for any $A, B, C \in \mathcal{B}_2(H)$.

Since, by Schwarz's inequality we have

$$\|A\|_{2,P}\|C\|_{2,P} \geq \left|\langle A, C\rangle_{2,P}\right|$$

and

$$\|B\|_{2,P}\|C\|_{2,P} \geq \left|\langle B, C\rangle_{2,P}\right|,$$

then by multiplying these inequalities we have

$$\|A\|_{2,P}\|B\|_{2,P}\|C\|_{2,P}^2 \geq \left|\langle A, C\rangle_{2,P}\right|\left|\langle B, C\rangle_{2,P}\right|$$

for any $A, B, C \in \mathcal{B}_2(H)$.

Utilizing the inequalities (2.6) and (2.7a) and taking the square root we get

$$\left|\langle A, B\rangle_{2,P}\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\langle C, B\rangle_{2,P}\right| \leq \|A\|_{2,P}\|B\|_{2,P}\|C\|_{2,P}^2 - \left|\langle A, C\rangle_{2,P}\right|\left|\langle B, C\rangle_{2,P}\right| \tag{2.8}$$

for any $A, B, C \in \mathcal{B}_2(H)$, which proves the second inequality in (2.4).

The first inequality is obvious by the modulus properties.

By the triangle inequality for modulus we also have

$$\left|\langle A, C\rangle_{2,P}\langle C, B\rangle_{2,P}\right| - \left|\langle A, B\rangle_{2,P}\right|\|C\|_{2,P}^2 \leq \left|\langle A, B\rangle_{2,P}\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\langle C, B\rangle_{2,P}\right| \tag{2.9}$$

for any $A, B, C \in \mathcal{B}_2(H)$.

On making use of (2.8) and (2.9) we have

$$\left|\langle A, C\rangle_{2,P}\langle C, B\rangle_{2,P}\right| - \left|\langle A, B\rangle_{2,P}\right|\|C\|_{2,P}^2 \leq \|A\|_{2,P}\|B\|_{2,P}\|C\|_{2,P}^2 - \left|\langle A, C\rangle_{2,P}\right|\left|\langle B, C\rangle_{2,P}\right|,$$

which is equivalent to the desired inequality (2.5). ∎

REMARK 2.1. By the triangle inequality for the hermitian form $[\cdot,\cdot]_{2,P,C} : \mathcal{B}_2(H)\times\mathcal{B}_2(H) \to \mathbb{C}$,

$$[A, B]_{2,P,C} := \langle A, B\rangle_{2,P}\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\langle C, B\rangle_{2,P}$$

we get

$$\left(\|A+B\|_{2,P}^2\|C\|_{2,P}^2 - \left|\langle A+B, C\rangle_{2,P}\right|^2\right)^{1/2} \leq \left(\|A\|_{2,P}^2\|C\|_{2,P}^2 - \left|\langle A, C\rangle_{2,P}\right|^2\right)^{1/2} + \left(\|B\|_{2,P}^2\|C\|_{2,P}^2 - \left|\langle B, C\rangle_{2,P}\right|^2\right)^{1/2},$$

which can be written as

$$\begin{aligned}&\left(\operatorname{tr}\left[P\left|(A+B)^*\right|^2\right]\operatorname{tr}\left(P\left|C^*\right|^2\right) - \left|\operatorname{tr}\left[P(A+B)C^*\right]\right|^2\right)^{1/2}\\ &\leq \left(\operatorname{tr}\left(P\left|A^*\right|^2\right)\operatorname{tr}\left(P\left|C^*\right|^2\right) - \left|\operatorname{tr}(PAC^*)\right|^2\right)^{1/2}\\ &+ \left(\operatorname{tr}\left(P\left|B^*\right|^2\right)\operatorname{tr}\left(P\left|C^*\right|^2\right) - \left|\operatorname{tr}(PBC^*)\right|^2\right)^{1/2}\end{aligned} \tag{2.10}$$

for any $A, B, C \in \mathcal{B}_2(H)$.

REMARK 2.2. If we take $B = \lambda C$ in (2.10), then we get

$$0 \le \operatorname{tr}\left[P\left|(A+\lambda C)^*\right|^2\right] \operatorname{tr}\left(P\left|C^*\right|^2\right) - \left|\operatorname{tr}\left[P\left(A+\lambda C\right)C^*\right]\right|^2 \le \operatorname{tr}\left(P\left|A^*\right|^2\right)\operatorname{tr}\left(P\left|C^*\right|^2\right) - \left|\operatorname{tr}\left(C^*PA\right)\right|^2 \tag{2.11}$$

for any $\lambda \in \mathbb{C}$ and $A, C \in \mathcal{B}_2(H)$.

Therefore, we have the bound

$$\sup_{\lambda\in\mathbb{C}}\left\{\operatorname{tr}\left[P\left|(A+\lambda C)^*\right|^2\right] \operatorname{tr}\left(P\left|C^*\right|^2\right) - \left|\operatorname{tr}\left[P\left(A+\lambda C\right)C^*\right]\right|^2\right\} = \operatorname{tr}\left(P\left|A^*\right|^2\right)\operatorname{tr}\left(P\left|C^*\right|^2\right) - \left|\operatorname{tr}\left(PAC^*\right)\right|^2. \tag{2.12}$$

We also have the inequalities

$$0 \le \operatorname{tr}\left[P\left|(A\pm C)^*\right|^2\right] \operatorname{tr}\left(P\left|C^*\right|^2\right) - \left|\operatorname{tr}\left[P\left(A\pm C\right)C^*\right]\right|^2 \le \operatorname{tr}\left(P\left|A^*\right|^2\right)\operatorname{tr}\left(P\left|C^*\right|^2\right) - \left|\operatorname{tr}\left(PAC^*\right)\right|^2 \tag{2.13}$$

for any $A, C \in \mathcal{B}_2(H)$.

REMARK 2.3. We observe that, by replacing A^* with A, B^* with B etc...above, we can get the dual inequalities, like, for instance

$$\left|\operatorname{tr}\left(PA^*C\right)\operatorname{tr}\left(PC^*B\right)\right| \le \frac{1}{2}\left[\left[\operatorname{tr}\left(P\left|A\right|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P\left|B\right|^2\right)\right]^{1/2} + \left|\operatorname{tr}\left(PA^*B\right)\right|\right]\operatorname{tr}\left(P\left|C\right|^2\right), \tag{2.14}$$

that holds for any $A, B, C \in \mathcal{B}_2(H)$.

This is an operator version of Buzano's inequality in inner product spaces, namely

$$\left|\langle x,e\rangle\langle e,y\rangle\right| \le \frac{1}{2}\left[\|x\|\|y\| + \left|\langle x,y\rangle\right|\right] \tag{2.15}$$

for $x, y, e \in H$ with $\|e\| = 1$.

Since

$$\left|\operatorname{tr}\left(PA^*C\right)\right| = \left|\overline{\operatorname{tr}\left(PA^*C\right)}\right| = \left|\operatorname{tr}\left[\left(PA^*C\right)^*\right]\right| = \left|\operatorname{tr}\left(C^*AP\right)\right| = \left|\operatorname{tr}\left(PC^*A\right)\right|,$$

$$\left|\operatorname{tr}\left(PC^*B\right)\right| = \left|\operatorname{tr}\left(PB^*C\right)\right|$$

and

$$\left|\operatorname{tr}\left(PA^*B\right)\right| = \left|\operatorname{tr}\left(PB^*A\right)\right|$$

then the inequality (2.14) can be also written as

$$\left|\operatorname{tr}\left(PC^*A\right)\operatorname{tr}\left(PB^*C\right)\right| \le \frac{1}{2}\left[\left[\operatorname{tr}\left(P\left|A\right|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P\left|B\right|^2\right)\right]^{1/2} + \left|\operatorname{tr}\left(PB^*A\right)\right|\right]\operatorname{tr}\left(P\left|C\right|^2\right), \tag{2.16}$$

that holds for any $A, B, C \in \mathcal{B}_2(H)$.

If we take in (2.16) $B = A^*$ then we get the following inequality

$$\left|\operatorname{tr}\left(PC^*A\right)\operatorname{tr}\left(PAC\right)\right| \le \frac{1}{2}\left[\left[\operatorname{tr}\left(P\left|A\right|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P\left|A^*\right|^2\right)\right]^{1/2} + \left|\operatorname{tr}\left(PA^2\right)\right|\right]\operatorname{tr}\left(P\left|C\right|^2\right), \tag{2.17}$$

for any $A, B, C \in \mathcal{B}_2(H)$.

If A is a normal operator, i.e. $|A|^2 = |A^*|^2$ then we have from (2.17) that

$$\left|\operatorname{tr}\left(PC^*A\right)\operatorname{tr}\left(PAC\right)\right| \le \frac{1}{2}\left[\operatorname{tr}\left(P\left|A\right|^2\right) + \left|\operatorname{tr}\left(PA^2\right)\right|\right]\operatorname{tr}\left(P\left|C\right|^2\right), \tag{2.18}$$

In particular, if C is selfadjoint and $C \in \mathcal{B}_2(H)$, then

$$|\text{tr}(PAC)|^2 \le \frac{1}{2}\left[\text{tr}\left(P|A|^2\right) + \left|\text{tr}\left(PA^2\right)\right|\right]\text{tr}\left(PC^2\right), \tag{2.19}$$

for any $A \in \mathcal{B}_2(H)$ a normal operator.

We notice that (2.19) is a trace operator version of *de Bruijn inequality* obtained in 1960 in [**11**], which gives the following refinement of the Cauchy-Bunyakovsky-Schwarz inequality:

$$\left|\sum_{i=1}^{n} a_i z_i\right|^2 \le \frac{1}{2}\sum_{i=1}^{n} a_i^2\left[\sum_{i=1}^{n}|z_i|^2 + \left|\sum_{i=1}^{n} z_i^2\right|\right], \tag{2.20}$$

provided that a_i are real numbers while z_i are complex for each $i \in \{1, ..., n\}$.

We notice that, if $P \in \mathcal{B}_1(H)$, $P \ge 0$ and $A, B \in \mathcal{B}(H)$, then

$$\langle A, B\rangle_{2,P} := \text{tr}(PAB^*) = \text{tr}(AB^*P) = \text{tr}(B^*PA)$$

is a *nonnegative Hermitian form* on $\mathcal{B}(H)$ and all the inequalities above will hold for A, B, $C \in \mathcal{B}(H)$. The details are left to the reader.

2.2. Some Functional Properties. We consider now the convex cone $\mathcal{B}_+(H)$ of nonnegative operators on the complex Hilbert space H and, for $A, B \in \mathcal{B}_2(H)$ define the functional $\sigma_{A,B} : \mathcal{B}_+(H) \to [0, \infty)$ by

$$\sigma_{A,B}(P) := \left[\text{tr}\left(P|A|^2\right)\right]^{1/2}\left[\text{tr}\left(P|B|^2\right)\right]^{1/2} - |\text{tr}(PA^*B)| \,(\ge 0). \tag{2.21}$$

The following theorem collects some fundamental properties of this functional.

THEOREM 2.2 (Dragomir, 2014, [**58**]). *Let* $A, B \in \mathcal{B}_2(H)$.

(i) For any $P, Q \in \mathcal{B}_+(H)$

$$\sigma_{A,B}(P + Q) \ge \sigma_{A,B}(P) + \sigma_{A,B}(Q) \,(\ge 0) \tag{2.22}$$

,namely, $\sigma_{A,B}$ *is a superadditive functional on* $\mathcal{B}_+(H)$*;*

(ii) For any $P, Q \in \mathcal{B}_+(H)$ *with* $P \ge Q$

$$\sigma_{A,B}(P) \ge \sigma_{A,B}(Q) \,(\ge 0), \tag{2.23}$$

namely, $\sigma_{A,B}$ *is a monotonic nondecreasing functional on* $\mathcal{B}_+(H)$*;*

(iii) If $P, Q \in \mathcal{B}_+(H)$ *and there exist the constants* $M > m > 0$ *such that* $MQ \ge P \ge mQ$ *then*

$$M\sigma_{A,B}(Q) \ge \sigma_{A,B}(P) \ge m\sigma_{A,B}(Q) \,(\ge 0). \tag{2.24}$$

PROOF. (i) Let $P, Q \in \mathcal{B}_+(H)$. On utilizing the elementary inequality

$$\left(a^2 + b^2\right)^{1/2}\left(c^2 + d^2\right)^{1/2} \ge ac + bd, \ a, b, c, d \ge 0$$

and the triangle inequality for the modulus, we have

$$\begin{aligned}
&\sigma_{A,B}(P+Q) \\
&= \left[\operatorname{tr}\left((P+Q)|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left((P+Q)|B|^2\right)\right]^{1/2} - |\operatorname{tr}((P+Q)A^*B)| \\
&= \left[\operatorname{tr}\left(P|A|^2+Q|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P|B|^2+Q|B|^2\right)\right]^{1/2} \\
&- |\operatorname{tr}(PA^*B+QA^*B)| \\
&= \left[\operatorname{tr}\left(P|A|^2\right)+\operatorname{tr}\left(Q|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P|B|^2\right)+\operatorname{tr}\left(Q|B|^2\right)\right]^{1/2} \\
&- |\operatorname{tr}(PA^*B)+\operatorname{tr}(QA^*B)| \\
&\geq \left[\operatorname{tr}\left(P|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2} + \left[\operatorname{tr}\left(Q|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(Q|B|^2\right)\right]^{1/2} \\
&- |\operatorname{tr}(PA^*B)| - |\operatorname{tr}(QA^*B)| \\
&= \sigma_{A,B}(P) + \sigma_{A,B}(Q)
\end{aligned}$$

and the inequality (2.22) is proved.

(ii) Let $P, Q \in \mathcal{B}_+(H)$ with $P \geq Q$. Utilising the superadditivity property we have

$$\sigma_{A,B}(P) = \sigma_{A,B}((P-Q)+Q) \geq \sigma_{A,B}(P-Q) + \sigma_{A,B}(Q) \geq \sigma_{A,B}(Q)$$

and the inequality (2.23) is obtained.

(iii) From the monotonicity property we have

$$\sigma_{A,B}(P) \geq \sigma_{A,B}(mQ) = m\sigma_{A,B}(Q)$$

and a similar inequality for M, which prove the desired result (2.24). ∎

COROLLARY 2.3. *Let $A, B \in \mathcal{B}_2(H)$ and $P \in \mathcal{B}(H)$ such that there exist the constants $M > m > 0$ with $M1_H \geq P \geq m1_H$. Then*

$$\begin{aligned}
&M\left(\left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} - |\operatorname{tr}(A^*B)|\right) \\
&\geq \left[\operatorname{tr}\left(P|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2} - |\operatorname{tr}(PA^*B)| \\
&\geq m\left(\left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} - |\operatorname{tr}(A^*B)|\right).
\end{aligned} \tag{2.25}$$

Let $P = |V|^2$ with $V \in \mathcal{B}(H)$. If $A, B \in \mathcal{B}_2(H)$ then

$$\begin{aligned}
\sigma_{A,B}\left(|V|^2\right) &= \left[\operatorname{tr}\left(|V|^2|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|V|^2|B|^2\right)\right]^{1/2} - \left|\operatorname{tr}\left(|V|^2A^*B\right)\right| \\
&= [\operatorname{tr}(V^*VA^*A)]^{1/2}[\operatorname{tr}(V^*VB^*B)]^{1/2} - |\operatorname{tr}(V^*VA^*B)| \\
&= [\operatorname{tr}(VA^*AV^*)]^{1/2}[\operatorname{tr}(VB^*BV^*)]^{1/2} - |\operatorname{tr}(VA^*BV^*)| \\
&= [\operatorname{tr}((AV^*)^*AV^*)]^{1/2}[\operatorname{tr}((BV^*)^*BV^*)]^{1/2} - |\operatorname{tr}((AV^*)^*BV^*)| \\
&= \left[\operatorname{tr}\left(|AV^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|BV^*|^2\right)\right]^{1/2} - |\operatorname{tr}((AV^*)^*BV^*)|.
\end{aligned}$$

On utilizing the property (2.22) for $P = |V|^2$, $Q = |U|^2$ with $V, U \in \mathcal{B}(H)$, then we have for any $A, B \in \mathcal{B}_2(H)$ the following trace inequality

$$\begin{aligned}
&\left[\operatorname{tr}\left(|AV^*|^2+|AU^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|BV^*|^2+|BU^*|^2\right)\right]^{1/2} \\
&- |\operatorname{tr}((AV^*)^*BV^*+(AU^*)^*BU^*)| \\
&\geq \left[\operatorname{tr}\left(|AV^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|BV^*|^2\right)\right]^{1/2} - |\operatorname{tr}((AV^*)^*BV^*)| \\
&+ \left[\operatorname{tr}\left(|AU^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|BU^*|^2\right)\right]^{1/2} - |\operatorname{tr}((AU^*)^*BU^*)| \;(\geq 0).
\end{aligned} \tag{2.26}$$

Also, if $|V|^2 \geq |U|^2$ with $V, U \in \mathcal{B}(H)$, then we have for any $A, B \in \mathcal{B}_2(H)$ that

$$\begin{aligned} (2.27) \quad & \left[\operatorname{tr}\left(|AV^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|BV^*|^2\right)\right]^{1/2} - \left|\operatorname{tr}\left((AV^*)^* BV^*\right)\right| \\ & \geq \left[\operatorname{tr}\left(|AU^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|BU^*|^2\right)\right]^{1/2} - \left|\operatorname{tr}\left((AU^*)^* BU^*\right)\right| (\geq 0). \end{aligned}$$

If $U \in \mathcal{B}(H)$ is invertible, then

$$\frac{1}{\|U^{-1}\|}\|x\| \leq \|Ux\| \leq \|U\|\|x\| \text{ for any } x \in H,$$

which implies that

$$\frac{1}{\|U^{-1}\|^2} 1_H \leq |U|^2 \leq \|U\|^2 1_H.$$

By making use of (2.25) we have the following trace inequality

$$\begin{aligned} (2.28) \quad & \|U\|^2 \left(\left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} - \left|\operatorname{tr}(A^*B)\right|\right) \\ & \geq \left[\operatorname{tr}\left(|AU^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|BU^*|^2\right)\right]^{1/2} - \left|\operatorname{tr}\left((AU^*)^* BU^*\right)\right| \\ & \geq \frac{1}{\|U^{-1}\|^2}\left(\left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} - \left|\operatorname{tr}(A^*B)\right|\right) \end{aligned}$$

for any $A, B \in \mathcal{B}_2(H)$.

Similar results may be stated for $P \in \mathcal{B}_1(H)$, $P \geq 0$ and $A, B \in \mathcal{B}(H)$. The details are omitted.

2.3. Inequalities for Sequences of Operators. For $n \geq 2$, define the Cartesian products $\mathcal{B}^{(n)}(H) := \mathcal{B}(H) \times ... \times \mathcal{B}(H)$, $\mathcal{B}_2^{(n)}(H) := \mathcal{B}_2(H) \times ... \times \mathcal{B}_2(H)$ and $\mathcal{B}_+^{(n)}(H) := \mathcal{B}_+(H) \times ... \times \mathcal{B}_+(H)$ where $\mathcal{B}_+(H)$ denotes the convex cone of nonnegative selfadjoint operators on H, i.e. $P \in \mathcal{B}_+(H)$ if $\langle Px, x\rangle \geq 0$ for any $x \in H$.

PROPOSITION 2.4 (Dragomir, 2014, [**58**]). *Let* $\mathbf{P} = (P_1, ..., P_n) \in \mathcal{B}_+^{(n)}(H)$ *and* $\mathbf{A} = (A_1, ..., A_n)$, $\mathbf{B} = (B_1, ..., B_n) \in \mathcal{B}_2^{(n)}(H)$ *and* $\mathbf{z} = (z_1, ..., z_n) \in \mathbb{C}^n$ *with* $n \geq 2$. *Then*

$$(2.29) \quad \left|\operatorname{tr}\left(\sum_{k=1}^n z_k P_k A_k^* B_k\right)\right|^2 \leq \operatorname{tr}\left(\sum_{k=1}^n |z_k| P_k |A_k|^2\right) \operatorname{tr}\left(\sum_{k=1}^n |z_k| P_k |B_k|^2\right).$$

PROOF. Using the properties of modulus and the inequality (2.1) we have

$$\begin{aligned} \left|\operatorname{tr}\left(\sum_{k=1}^n z_k P_k A_k^* B_k\right)\right| &= \left|\sum_{k=1}^n z_k \operatorname{tr}(P_k A_k^* B_k)\right| \\ &\leq \sum_{k=1}^n |z_k| \left|\operatorname{tr}(P_k A_k^* B_k)\right| \\ &\leq \sum_{k=1}^n |z_k| \left[\operatorname{tr}\left(P_k |A_k|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P_k |B_k|^2\right)\right]^{1/2}. \end{aligned}$$

Utilizing the weighted discrete Cauchy-Bunyakovsky-Schwarz inequality we also have

$$\begin{aligned}
&\sum_{k=1}^{n}|z_k|\left[\operatorname{tr}\left(P_k|A_k|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P_k|B_k|^2\right)\right]^{1/2}\\
&\le\left(\sum_{k=1}^{n}|z_k|\left(\left[\operatorname{tr}\left(P_k|A_k|^2\right)\right]^{1/2}\right)^2\right)^{1/2}\left(\sum_{k=1}^{n}|z_k|\left(\left[\operatorname{tr}\left(P_k|B_k|^2\right)\right]^{1/2}\right)^2\right)^{1/2}\\
&=\left(\sum_{k=1}^{n}|z_k|\operatorname{tr}\left(P_k|A_k|^2\right)\right)^{1/2}\left(\sum_{k=1}^{n}|z_k|\operatorname{tr}\left(P_k|B_k|^2\right)\right)^{1/2}\\
&=\left(\operatorname{tr}\left(\sum_{k=1}^{n}|z_k|P_k|A_k|^2\right)\right)^{1/2}\left(\operatorname{tr}\left(\sum_{k=1}^{n}|z_k|P_k|B_k|^2\right)\right)^{1/2},
\end{aligned}$$

which is equivalent to the desired result (2.29). ∎

We consider the functional for n-tuples of nonnegative operators as follows:

$$\begin{aligned}
\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{P}):=&\left[\operatorname{tr}\left(\sum_{k=1}^{n}P_k|A_k|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{k=1}^{n}P_k|B_k|^2\right)\right]^{1/2}\\
&-\left|\operatorname{tr}\left(\sum_{k=1}^{n}P_kA_k^*B_k\right)\right|.
\end{aligned}\tag{2.30}$$

Utilising a similar argument to the one in Theorem 2.2 we can state:

PROPOSITION 2.5. *Let* $\mathbf{A}=(A_1,...,A_n)$, $\mathbf{B}=(B_1,...,B_n)\in\mathcal{B}_2^{(n)}(H)$.

(i) For any $\mathbf{P},\mathbf{Q}\in\mathcal{B}_+^{(n)}(H)$

$$\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{P}+\mathbf{Q})\ge\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{P})+\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{Q})\,(\ge 0),\tag{2.31}$$

namely, $\sigma_{\mathbf{A},\mathbf{B}}$ *is a superadditive functional on* $\mathcal{B}_+^{(n)}(H)$;

(ii) For any $\mathbf{P},\mathbf{Q}\in\mathcal{B}_+^{(n)}(H)$ *with* $\mathbf{P}\ge\mathbf{Q}$, *namely* $P_k\ge Q_k$ *for all* $k\in\{1,...,n\}$

$$\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{P})\ge\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{Q})\,(\ge 0),\tag{2.32}$$

namely, $\sigma_{\mathbf{A},\mathbf{B}}$ *is a monotonic nondecreasing functional on* $\mathcal{B}_+^{(n)}(H)$;

(iii) If $\mathbf{P},\mathbf{Q}\in\mathcal{B}_+^{(n)}(H)$ *and there exist the constants* $M>m>0$ *such that* $M\mathbf{Q}\ge\mathbf{P}\ge m\mathbf{Q}$ *then*

$$M\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{Q})\ge\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{P})\ge m\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{Q})\,(\ge 0).\tag{2.33}$$

If $\mathbf{P}=(p_1 1_H,...,p_n 1_H)$ with $p_k\ge 0$, $k\in\{1,...,n\}$ then the functional of nonnegative weights $\mathbf{p}=(p_1,...,p_n)$ defined by

$$\begin{aligned}
\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{p}):=&\left[\operatorname{tr}\left(\sum_{k=1}^{n}p_k|A_k|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{k=1}^{n}p_k|B_k|^2\right)\right]^{1/2}\\
&-\left|\operatorname{tr}\left(\sum_{k=1}^{n}p_kA_k^*B_k\right)\right|.
\end{aligned}\tag{2.34}$$

has the same properties as in (2.31)-(2.33).

Moreover, we have the simple bounds:

(2.35)
$$\max_{k\in\{1,\dots,n\}}\{p_k\}\left\{\left[\operatorname{tr}\left(\sum_{k=1}^{n}|A_k|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{k=1}^{n}|B_k|^2\right)\right]^{1/2}\right.$$
$$\left.-\left|\operatorname{tr}\left(\sum_{k=1}^{n}A_k^*B_k\right)\right|\right\}$$
$$\geq\left[\operatorname{tr}\left(\sum_{k=1}^{n}p_k|A_k|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{k=1}^{n}p_k|B_k|^2\right)\right]^{1/2}-\left|\operatorname{tr}\left(\sum_{k=1}^{n}p_kA_k^*B_k\right)\right|$$
$$\geq\min_{k\in\{1,\dots,n\}}\{p_k\}\left\{\left[\operatorname{tr}\left(\sum_{k=1}^{n}|A_k|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{k=1}^{n}|B_k|^2\right)\right]^{1/2}\right.$$
$$\left.-\left|\operatorname{tr}\left(\sum_{k=1}^{n}A_k^*B_k\right)\right|\right\}.$$

2.4. Inequalities for Power Series of Operators. Denote by:

$$D(0,R)=\begin{cases}\{z\in\mathbb{C}:|z|<R\}, & \text{if } R<\infty\\ \mathbb{C}, & \text{if } R=\infty,\end{cases}$$

and consider the functions:

$$\lambda\mapsto f(\lambda):D(0,R)\to\mathbb{C},\ f(\lambda):=\sum_{n=0}^{\infty}\alpha_n\lambda^n$$

and

$$\lambda\mapsto f_a(\lambda):D(0,R)\to\mathbb{C},\ f_a(\lambda):=\sum_{n=0}^{\infty}|\alpha_n|\lambda^n.$$

As some natural examples that are useful for applications, we can point out that, if

(2.36)
$$f(\lambda)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \lambda\in D(0,1);$$
$$g(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \lambda\in\mathbb{C};$$
$$h(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \lambda\in\mathbb{C};$$
$$l(\lambda)=\sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \lambda\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(\lambda)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n=\ln\frac{1}{1-\lambda},\ \lambda\in D(0,1);\tag{2.37}$$
$$g_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}=\cosh\lambda,\ \lambda\in\mathbb{C};$$
$$h_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}=\sinh\lambda,\ \lambda\in\mathbb{C};$$
$$l_a(\lambda)=\sum_{n=0}^{\infty}\lambda^n=\frac{1}{1-\lambda},\ \lambda\in D(0,1).$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\exp(\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^n\qquad\lambda\in\mathbb{C},\tag{2.38}$$
$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad\lambda\in D(0,1);$$
$$\sin^{-1}(\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1},\qquad\lambda\in D(0,1);$$
$$\tanh^{-1}(\lambda)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad\lambda\in D(0,1)$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n,\alpha,\beta,\gamma>0,$$
$$\lambda\in D(0,1);$$

where Γ is *Gamma function*.

PROPOSITION 2.6 (Dragomir, 2014, [**58**]). *Let* $f(\lambda):=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ *be a power series with complex coefficients and convergent on the open disk* $D(0,R)$, $R>0$. *If* $(H,\langle\cdot,\cdot\rangle)$ *is a separable infinite-dimensional Hilbert space and* A, $B\in\mathcal{B}_1(H)$ *are positive operators with* $\operatorname{tr}(A),\operatorname{tr}(B)<R^{1/2}$*, then*

$$|\operatorname{tr}(f(AB))|^2\le f_a^2(\operatorname{tr}A\operatorname{tr}B)\le f_a\left((\operatorname{tr}A)^2\right)f_a\left((\operatorname{tr}B)^2\right).\tag{2.39}$$

PROOF. By the inequality (1.12) for the positive operators A, $B\in\mathcal{B}_1(H)$ we have

$$\left|\operatorname{tr}\left[\sum_{k=0}^{n}\alpha_k(AB)^k\right]\right|=\left|\sum_{k=0}^{n}\alpha_k\operatorname{tr}\left[(AB)^k\right]\right|\tag{2.40}$$
$$\le\sum_{k=0}^{n}|\alpha_k|\left|\operatorname{tr}\left[(AB)^k\right]\right|=\sum_{k=0}^{n}|\alpha_k|\operatorname{tr}\left[(AB)^k\right]$$
$$\le\sum_{k=0}^{n}|\alpha_k|(\operatorname{tr}A)^k(\operatorname{tr}B)^k=\sum_{k=0}^{n}|\alpha_k|(\operatorname{tr}A\operatorname{tr}B)^k.$$

Utilising the weighted Cauchy-Bunyakovsky-Schwarz inequality for sums we have

$$\sum_{k=0}^{n} |\alpha_k| (\operatorname{tr} A)^k (\operatorname{tr} B)^k \le \left(\sum_{k=0}^{n} |\alpha_k| (\operatorname{tr} A)^{2k}\right)^{1/2} \left(\sum_{k=0}^{n} |\alpha_k| (\operatorname{tr} B)^{2k}\right)^{1/2}. \tag{2.41}$$

Then by (2.40) and (2.41) we have

$$\begin{aligned} \left|\operatorname{tr}\left[\sum_{k=0}^{n} \alpha_k (AB)^k\right]\right|^2 &\le \left[\sum_{k=0}^{n} |\alpha_k| (\operatorname{tr} A \operatorname{tr} B)^k\right]^2 \\ &\le \sum_{k=0}^{n} |\alpha_k| \left[(\operatorname{tr} A)^2\right]^k \sum_{k=0}^{n} |\alpha_k| \left[(\operatorname{tr} B)^2\right]^k \end{aligned} \tag{2.42}$$

for $n \ge 1$.

Since $0 \le \operatorname{tr}(A), \operatorname{tr}(B) < R^{1/2}$, the numerical series

$$\sum_{k=0}^{\infty} |\alpha_k| (\operatorname{tr} A \operatorname{tr} B)^k, \ \sum_{k=0}^{\infty} |\alpha_k| \left[(\operatorname{tr} A)^2\right]^k \text{ and } \sum_{k=0}^{\infty} |\alpha_k| \left[(\operatorname{tr} B)^2\right]^k$$

are convergent.

Also, since $0 \le \operatorname{tr}(AB) \le \operatorname{tr}(A) \operatorname{tr}(B) < R$, the operator series $\sum_{k=0}^{\infty} \alpha_k (AB)^k$ is convergent in $\mathcal{B}_1(H)$.

Letting $n \to \infty$ in (2.42) and utilizing the continuity property of $\operatorname{tr}(\cdot)$ on $\mathcal{B}_1(H)$ we get the desired result (2.39). ∎

EXAMPLE 2.1. *a) If we take in (2.39)* $f(\lambda) = (1 \pm \lambda)^{-1}$, $|\lambda| < 1$ *then we get the inequality*

$$\left|\operatorname{tr}\left((1_H \pm AB)^{-1}\right)\right|^2 \le \left(1 - (\operatorname{tr} A)^2\right)^{-1} \left(1 - (\operatorname{tr} B)^2\right)^{-1} \tag{2.43}$$

for any $A, B \in \mathcal{B}_1(H)$ *positive operators with* $\operatorname{tr}(A), \operatorname{tr}(B) < 1$.

b) If we take in (2.39) $f(\lambda) = \ln(1 \pm \lambda)^{-1}$, $|\lambda| < 1$, *then we get the inequality*

$$\left|\operatorname{tr}\left(\ln(1_H \pm AB)^{-1}\right)\right|^2 \le \ln\left(1 - (\operatorname{tr} A)^2\right)^{-1} \ln\left(1 - (\operatorname{tr} B)^2\right)^{-1} \tag{2.44}$$

for any $A, B \in \mathcal{B}_1(H)$ *positive operators with* $\operatorname{tr}(A), \operatorname{tr}(B) < 1$.

We have the following result as well:

THEOREM 2.7 (Dragomir, 2014, [**58**]). *Let* $f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n$ *be a power series with complex coefficients and convergent on the open disk* $D(0, R)$, $R > 0$. *If* $A, B \in \mathcal{B}_2(H)$ *are normal operators with* $A^*B = BA^*$ *and* $\operatorname{tr}\left(|A|^2\right), \operatorname{tr}\left(|B|^2\right) < R$ *then the inequality*

$$|\operatorname{tr}(f(A^*B))|^2 \le \operatorname{tr}\left(f_a\left(|A|^2\right)\right) \operatorname{tr}\left(f_a\left(|B|^2\right)\right). \tag{2.45}$$

PROOF. From the inequality (2.29) we have

$$\left|\operatorname{tr}\left(\sum_{k=0}^{n} \alpha_k (A^*)^k B^k\right)\right|^2 \le \operatorname{tr}\left(\sum_{k=0}^{n} |\alpha_k| \left|A^k\right|^2\right) \operatorname{tr}\left(\sum_{k=0}^{n} |\alpha_k| \left|B^k\right|^2\right). \tag{2.46}$$

Since A, B are normal operators, then we have $\left|A^k\right|^2 = |A|^{2k}$ and $\left|B^k\right|^2 = |B|^{2k}$ for any $k \ge 0$. Also, since $A^*B = BA^*$ then we also have $(A^*)^k B^k = (A^*B)^k$ for any $k \ge 0$.

Due to the fact that $A, B \in \mathcal{B}_2(H)$ and $\operatorname{tr}\left(|A|^2\right), \operatorname{tr}\left(|B|^2\right) < R$, it follows that $\operatorname{tr}(A^*B) \le R$ and the operator series

$$\sum_{k=0}^{\infty} \alpha_k (A^*B)^k, \ \sum_{k=0}^{\infty} |\alpha_k| |A|^{2k} \text{ and } \sum_{k=0}^{\infty} |\alpha_k| |B|^{2k}$$

are convergent in the Banach space $\mathcal{B}_1(H)$.

Taking the limit over $n \to \infty$ in (2.46) and using the continuity of the $\operatorname{tr}(\cdot)$ on $\mathcal{B}_1(H)$ we deduce the desired result (2.45). ∎

EXAMPLE 2.2. *a) If we take in (2.45) $f(\lambda) = (1 \pm \lambda)^{-1}$, $|\lambda| < 1$ then we get the inequality*

$$\left|\operatorname{tr}\left((1_H \pm A^*B)^{-1}\right)\right|^2 \le \operatorname{tr}\left(\left(1 - |A|^2\right)^{-1}\right) \operatorname{tr}\left(\left(1 - |B|^2\right)^{-1}\right) \tag{2.47}$$

for any $A, B \in \mathcal{B}_2(H)$ normal operators with $A^*B = BA^*$ and $\operatorname{tr}\left(|A|^2\right), \operatorname{tr}\left(|B|^2\right) < 1$.

b) If we take in (2.45) $f(\lambda) = \exp(\lambda)$, $\lambda \in \mathbb{C}$ then we get the inequality

$$|\operatorname{tr}(\exp(A^*B))|^2 \le \operatorname{tr}\left(\exp\left(|A|^2\right)\right) \operatorname{tr}\left(\exp\left(|B|^2\right)\right) \tag{2.48}$$

for any $A, B \in \mathcal{B}_2(H)$ normal operators with $A^*B = BA^*$.

THEOREM 2.8 (Dragomir, 2014, [**58**]). *Let $f(z) := \sum_{j=0}^{\infty} p_j z^j$ and $g(z) := \sum_{j=0}^{\infty} q_j z^j$ be two power series with nonnegative coefficients and convergent on the open disk $D(0, R)$, $R > 0$. If T and V are two normal and commuting operators from $\mathcal{B}_2(H)$ with $\operatorname{tr}\left(|T|^2\right)$, $\operatorname{tr}\left(|V|^2\right) < R$, then*

$$\begin{aligned} &\left[\operatorname{tr}\left(f\left(|T|^2\right) + g\left(|T|^2\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(f\left(|V|^2\right) + g\left(|V|^2\right)\right)\right]^{1/2} \\ &- |\operatorname{tr}(f(T^*V) + g(T^*V))| \\ &\ge \left[\operatorname{tr}\left(f\left(|T|^2\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(f\left(|V|^2\right)\right)\right]^{1/2} - |\operatorname{tr}(f(T^*V))| \\ &+ \left[\operatorname{tr}\left(g\left(|T|^2\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(g\left(|V|^2\right)\right)\right]^{1/2} - |\operatorname{tr}(g(T^*V))| \, (\ge 0). \end{aligned} \tag{2.49}$$

Moreover, if $p_j \ge q_j$ for any $j \in \mathbb{N}$, then, with the above assumptions on T and V,

$$\begin{aligned} &\left[\operatorname{tr}\left(f\left(|T|^2\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(f\left(|V|^2\right)\right)\right]^{1/2} - |\operatorname{tr}(f(T^*V))| \\ &\ge \left[\operatorname{tr}\left(g\left(|T|^2\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(g\left(|V|^2\right)\right)\right]^{1/2} - |\operatorname{tr}(g(T^*V))| \, (\ge 0). \end{aligned} \tag{2.50}$$

PROOF. Utilising the superadditivity property of the functional $\sigma_{\mathbf{A},\mathbf{B}}(\cdot)$ above as a function of weights $\mathbf{p}$ and the fact that T and V are two normal and commuting operators we can state that

$$\begin{aligned} &\left[\operatorname{tr}\left(\sum_{k=0}^{n} (p_k + q_k) |T|^{2k}\right)\right]^{1/2} \left[\operatorname{tr}\left(\sum_{k=0}^{n} (p_k + q_k) |V|^{2k}\right)\right]^{1/2} \\ &- \left|\operatorname{tr}\left(\sum_{k=0}^{n} (p_k + q_k) (T^*V)^k\right)\right| \\ &\ge \left[\operatorname{tr}\left(\sum_{k=0}^{n} p_k |T|^{2k}\right)\right]^{1/2} \left[\operatorname{tr}\left(\sum_{k=0}^{n} p_k |V|^{2k}\right)\right]^{1/2} - \left|\operatorname{tr}\left(\sum_{k=0}^{n} p_k (T^*V)^k\right)\right| \\ &+ \left[\operatorname{tr}\left(\sum_{k=0}^{n} q_k |T|^{2k}\right)\right]^{1/2} \left[\operatorname{tr}\left(\sum_{k=0}^{n} q_k |V|^{2k}\right)\right]^{1/2} - \left|\operatorname{tr}\left(\sum_{k=0}^{n} q_k (T^*V)^k\right)\right| \end{aligned} \tag{2.51}$$

for any $n \ge 1$.

Since all the series whose partial sums are involved in (2.51) are convergent in $\mathcal{B}_1(H)$, by letting $n \to \infty$ in (2.51) we get (2.49).

The inequality (2.50) follows by the monotonicity property of $\sigma_{\mathbf{A},\mathbf{B}}(\cdot)$ and the details are omitted. ∎

EXAMPLE 2.3. *Now, observe that if we take*

$$f(\lambda) = \sinh \lambda = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1}$$

and

$$g(\lambda) = \cosh \lambda = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n}$$

then

$$f(\lambda) + g(\lambda) = \exp \lambda = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^{n}$$

for any $\lambda \in \mathbb{C}$.

If T *and* V *are two normal and commuting operators from* $\mathcal{B}_2(H)$*, then by (2.11)*

$$\begin{aligned} &\left[\operatorname{tr}\left(\exp\left(|T|^2\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(\exp\left(|V|^2\right)\right)\right]^{1/2} - |\operatorname{tr}(\exp(T^*V))| \\ &\geq \left[\operatorname{tr}\left(\sinh\left(|T|^2\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(\sinh\left(|V|^2\right)\right)\right]^{1/2} - |\operatorname{tr}(\sinh(T^*V))| \\ &+ \left[\operatorname{tr}\left(\cosh\left(|T|^2\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(\cosh\left(|V|^2\right)\right)\right]^{1/2} - |\operatorname{tr}(\cosh(T^*V))| \ (\geq 0). \end{aligned} \tag{2.52}$$

Now, consider the series $\frac{1}{1-\lambda} = \sum_{n=0}^{\infty} \lambda^n$, $\lambda \in D(0,1)$ *and* $\ln \frac{1}{1-\lambda} = \sum_{n=1}^{\infty} \frac{1}{n}\lambda^n$, $\lambda \in D(0,1)$ *and define* $p_n = 1$, $n \geq 0$, $q_0 = 0$, $q_n = \frac{1}{n}$, $n \geq 1$, *then we observe that for any* $n \geq 0$, $p_n \geq q_n$.

If T *and* V *are two normal and commuting operators from* $\mathcal{B}_2(H)$ *with* $\operatorname{tr}\left(|T|^2\right), \operatorname{tr}\left(|V|^2\right) < 1$, *then by (2.12)*

$$\begin{aligned} &\left[\operatorname{tr}\left(\left(1_H - |T|^2\right)^{-1}\right)\right]^{1/2}\left[\operatorname{tr}\left(\left(1_H - |V|^2\right)^{-1}\right)\right]^{1/2} \\ &- \left|\operatorname{tr}\left((1_H - T^*V)^{-1}\right)\right| \\ &\geq \left[\operatorname{tr}\left(\ln\left(1_H - |T|^2\right)^{-1}\right)\right]^{1/2}\left[\operatorname{tr}\left(\ln\left(1_H - |V|^2\right)^{-1}\right)\right]^{1/2} \\ &- \left|\operatorname{tr}\left(\ln(1_H - T^*V)^{-1}\right)\right| \ (\geq 0). \end{aligned} \tag{2.53}$$

2.5. Inequalities for Matrices.

We have the following result for matrices.

PROPOSITION 2.9 (Dragomir, 2014, [**58**]). *Let* $f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n$ *be a power series with complex coefficients and convergent on the open disk* $D(0,R)$, $R > 0$. *If* A *and* B *are positive semidefinite matrices in* $M_n(\mathbb{C})$ *with* $\operatorname{tr}(A^2), \operatorname{tr}(B^2) < R$, *then the inequality*

$$|\operatorname{tr}[f(AB)]|^2 \leq \operatorname{tr}\left[f_a\left(A^2\right)\right]\operatorname{tr}\left[f_a\left(B^2\right)\right]. \tag{2.54}$$

If $\operatorname{tr}(A), \operatorname{tr}(B) < \sqrt{R}$, *then also*

$$|\operatorname{tr}[f(AB)]| \leq \min\left\{\operatorname{tr}(f_a(\|A\|B)), \operatorname{tr}(f_a(\|B\|A))\right\}. \tag{2.55}$$

PROOF. We observe that (1.13) holds for $m = 0$ with equality.

By utilizing the generalized triangle inequality for the modulus and the inequality (1.13) we have

$$\begin{aligned}(2.56)\qquad & \left|\operatorname{tr}\left[\sum_{n=0}^{m}\alpha_n(AB)^n\right]\right| \\ &= \left|\sum_{n=0}^{m}\alpha_n \operatorname{tr}\left[(AB)^n\right]\right| \le \sum_{n=0}^{m}|\alpha_n|\left|\operatorname{tr}\left[(AB)^n\right]\right| \\ &= \sum_{n=0}^{m}|\alpha_n| \operatorname{tr}\left[(AB)^n\right] \le \sum_{n=0}^{m}|\alpha_n|\left[\operatorname{tr}\left(A^{2n}\right)\right]^{1/2}\left[\operatorname{tr}\left(B^{2n}\right)\right]^{1/2},\end{aligned}$$

for any $m \ge 1$.

Applying the weighted Cauchy-Bunyakowsky-Schwarz discrete inequality we also have

$$\begin{aligned}(2.57)\qquad & \sum_{n=0}^{m}|\alpha_n|\left[\operatorname{tr}\left(A^{2n}\right)\right]^{1/2}\left[\operatorname{tr}\left(B^{2n}\right)\right]^{1/2} \\ &\le \left(\sum_{n=0}^{m}|\alpha_n|\left(\left[\operatorname{tr}\left(A^{2n}\right)\right]^{1/2}\right)^2\right)^{1/2}\left(\sum_{n=0}^{m}|\alpha_n|\left(\left[\operatorname{tr}\left(B^{2n}\right)\right]^{1/2}\right)^2\right)^{1/2} \\ &= \left(\sum_{n=0}^{m}|\alpha_n|\left[\operatorname{tr}\left(A^{2n}\right)\right]\right)^{1/2}\left(\sum_{n=0}^{m}|\alpha_n|\left[\operatorname{tr}\left(B^{2n}\right)\right]\right)^{1/2} \\ &= \left[\operatorname{tr}\left(\sum_{n=0}^{m}|\alpha_n| A^{2n}\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{n=0}^{m}|\alpha_n| B^{2n}\right)\right]^{1/2}\end{aligned}$$

for any $m \ge 1$.

Therefore, by (2.56) and (2.57) we get

$$(2.58)\qquad \left|\operatorname{tr}\left[\sum_{n=0}^{m}\alpha_n(AB)^n\right]\right|^2 \le \operatorname{tr}\left(\sum_{n=0}^{m}|\alpha_n| A^{2n}\right)\operatorname{tr}\left(\sum_{n=0}^{m}|\alpha_n| B^{2n}\right)$$

for any $m \ge 1$.

Since $\operatorname{tr}(A^2), \operatorname{tr}(B^2) < R$, then $\operatorname{tr}(AB) \le \sqrt{\operatorname{tr}(A^2)\operatorname{tr}(B^2)} < R$ and the series

$$\sum_{n=0}^{\infty}\alpha_n(AB)^n,\ \sum_{n=0}^{\infty}|\alpha_n| A^{2n} \text{ and } \sum_{n=0}^{\infty}|\alpha_n| B^{2n}$$

are convergent in $M_n(\mathbb{C})$.

Taking the limit over $m \to \infty$ in (2.58) and utilizing the continuity property of $\operatorname{tr}(\cdot)$ on $M_n(\mathbb{C})$ we get (2.54).

The inequality (2.55) follows from (1.14) in a similar way and the details are omitted. ∎

EXAMPLE 2.4. *a) If we take $f(\lambda) = (1 \pm \lambda)^{-1}$, $|\lambda| < 1$ then we get the inequality*

$$(2.59)\qquad \left|\operatorname{tr}\left[(I_n \pm AB)^{-1}\right]\right|^2 \le \operatorname{tr}\left[\left(I_n - A^2\right)^{-1}\right]\operatorname{tr}\left[\left(I_n - B^2\right)^{-1}\right]$$

for any A and B positive semidefinite matrices in $M_n(\mathbb{C})$ with $\operatorname{tr}(A^2), \operatorname{tr}(B^2) < 1$. Here I_n is the identity matrix in $M_n(\mathbb{C})$.

We also have the inequality

$$(2.60)\qquad \left|\operatorname{tr}\left[(I_n \pm AB)^{-1}\right]\right| \le \min\left\{\operatorname{tr}\left((I_n - \|A\| B)^{-1}\right), \operatorname{tr}\left((I_n - \|B\| A)^{-1}\right)\right\}$$

for any A and B positive semidefinite matrices in $M_n(\mathbb{C})$ with $\operatorname{tr}(A), \operatorname{tr}(B) < 1$.

b) If we take $f(\lambda) = \exp\lambda$, then

$$(\operatorname{tr}[\exp(AB)])^2 \le \operatorname{tr}\left[\exp\left(A^2\right)\right]\operatorname{tr}\left[\exp\left(B^2\right)\right] \tag{2.61}$$

and

$$\operatorname{tr}[\exp(AB)] \le \min\{\operatorname{tr}(\exp(\|A\| B)), \operatorname{tr}(\exp(\|B\| A))\} \tag{2.62}$$

for any A and B positive semidefinite matrices in $M_n(\mathbb{C})$.

PROPOSITION 2.10 (Dragomir, 2014, [**58**]). *Let $f(\lambda) := \sum_{n=0}^{\infty}\alpha_n\lambda^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R > 0$. If A and B are matrices in $M_n(\mathbb{C})$ with $\operatorname{tr}(|A|^p), \operatorname{tr}(|B|^q) < R$, where $p, q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, then*

$$\begin{aligned} |\operatorname{tr}(f(|AB^*|))| &\le \operatorname{tr}\left[f_a\left(\frac{|A|^p}{p}+\frac{|B|^q}{q}\right)\right] \\ &\le \operatorname{tr}\left[\frac{1}{p}f_a(|A|^p)+\frac{1}{q}f_a(|B|^q)\right]. \end{aligned} \tag{2.63}$$

PROOF. The inequality (1.15) holds with equality for $r = 0$.

By utilizing the generalized triangle inequality for the modulus and the inequality (1.15) we have

$$\begin{aligned} \left|\operatorname{tr}\left(\sum_{n=0}^{m}\alpha_n |AB^*|^n\right)\right| &= \left|\sum_{n=0}^{m}\alpha_n \operatorname{tr}(|AB^*|^n)\right| \\ &\le \sum_{n=0}^{m}|\alpha_n|\,|\operatorname{tr}(|AB^*|^n)| = \sum_{n=0}^{m}|\alpha_n|\operatorname{tr}(|AB^*|^n) \\ &\le \sum_{n=0}^{m}|\alpha_n|\operatorname{tr}\left[\left(\frac{|A|^p}{p}+\frac{|B|^q}{q}\right)^n\right] \\ &= \operatorname{tr}\left[\sum_{n=0}^{m}|\alpha_n|\left(\frac{|A|^p}{p}+\frac{|B|^q}{q}\right)^n\right] \end{aligned} \tag{2.64}$$

for any $m \ge 1$ and $p, q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$.

It is know that if $f : [0,\infty) \to \mathbb{R}$ is a convex function, then $\operatorname{tr} f(\cdot)$ is convex on the cone $M_n^+(\mathbb{C})$ of positive semidefinite matrices in $M_n(\mathbb{C})$. Therefore, for $n \ge 1$ we have

$$\operatorname{tr}\left[\left(\frac{|A|^p}{p}+\frac{|B|^q}{q}\right)^n\right] \le \frac{1}{p}\operatorname{tr}(|A|^{pn}) + \frac{1}{q}\operatorname{tr}(|B|^{qn}) \tag{2.65}$$

where $p, q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$.

The inequality reduces to equality if $n = 0$.

Then we have

$$\begin{aligned} \sum_{n=0}^{m}|\alpha_n|\operatorname{tr}\left[\left(\frac{|A|^p}{p}+\frac{|B|^q}{q}\right)^n\right] &\le \sum_{n=0}^{m}|\alpha_n|\left[\frac{1}{p}\operatorname{tr}(|A|^{pn}) + \frac{1}{q}\operatorname{tr}(|B|^{qn})\right] \\ &= \operatorname{tr}\left[\frac{1}{p}\sum_{n=0}^{m}|\alpha_n|\,|A|^{pn} + \frac{1}{q}\sum_{n=0}^{m}|\alpha_n|\,|B|^{qn}\right] \end{aligned} \tag{2.66}$$

for any $m \ge 1$ and $p, q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$.

From (2.64) and (2.66) we get

$$\left|\operatorname{tr}\left(\sum_{n=0}^{m}\alpha_n |AB^*|^r\right)\right| \le \operatorname{tr}\left[\sum_{n=0}^{m}|\alpha_n|\left(\frac{|A|^p}{p}+\frac{|B|^q}{q}\right)^n\right] \le \operatorname{tr}\left[\frac{1}{p}\sum_{n=0}^{m}|\alpha_n||A|^{pn}+\frac{1}{q}\sum_{n=0}^{m}|\alpha_n||B|^{qn}\right] \tag{2.67}$$

for any $m \ge 1$ and $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Since $\operatorname{tr}(|A|^p), \operatorname{tr}(|B|^q) < R$, then all the series whose partial sums are involved in (2.67) are convergent, then by letting $m \to \infty$ in (2.67) we deduce the desired inequality (2.63). ∎

EXAMPLE 2.5. *a) If we take $f(\lambda) = (1 \pm \lambda)^{-1}$, $|\lambda| < 1$ then we get the inequalities*

$$\left|\operatorname{tr}\left((I_n \pm |AB^*|)^{-1}\right)\right| \le \operatorname{tr}\left(\left[I_n - \left(\frac{|A|^p}{p}+\frac{|B|^q}{q}\right)\right]^{-1}\right) \le \operatorname{tr}\left[\frac{1}{p}(I_n - |A|^p)^{-1} + \frac{1}{q}(I_n - |B|^q)^{-1}\right], \tag{2.68}$$

where A and B are matrices in $M_n(\mathbb{C})$ with $\operatorname{tr}(|A|^p), \operatorname{tr}(|B|^q) < 1$ and $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

b) If we take $f(\lambda) = \exp\lambda$, then

$$\operatorname{tr}(\exp(|AB^*|)) \le \operatorname{tr}\left[\exp\left(\frac{|A|^p}{p}+\frac{|B|^q}{q}\right)\right] \le \operatorname{tr}\left[\frac{1}{p}\exp(|A|^p) + \frac{1}{q}\exp(|B|^q)\right], \tag{2.69}$$

where A and B are matrices in $M_n(\mathbb{C})$ and $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Finally, we can state the following result:

PROPOSITION 2.11 (Dragomir, 2014, [**58**]). *Let $f(\lambda) := \sum_{n=0}^{\infty}\alpha_n\lambda^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R > 0$. If A and B are commuting positive semidefinite matrices in $M_n(\mathbb{C})$ with $\operatorname{tr}(A^p), \operatorname{tr}(B^q) < R$, where $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$|\operatorname{tr}(f(AB))| \le [\operatorname{tr}(f_a(A^p))]^{1/p}[\operatorname{tr}(f_a(B^q))]^{1/q}. \tag{2.70}$$

PROOF. Since A and B are commuting positive semidefinite matrices in $M_n(\mathbb{C})$, then by (1.17) we have for any natural number n including $n = 0$ that

$$\operatorname{tr}((AB)^n) = \operatorname{tr}(A^nB^n) \le [\operatorname{tr}(A^{np})]^{1/p}[\operatorname{tr}(B^{nq})]^{1/q}, \tag{2.71}$$

where $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

By (2.71) and the weighted Hölder discrete inequality we have

$$\begin{aligned}\left|\operatorname{tr}\left(\sum_{n=0}^{m}\alpha_n(AB)^n\right)\right| &= \left|\sum_{n=0}^{m}\alpha_n \operatorname{tr}(A^nB^n)\right| \le \sum_{n=0}^{m}|\alpha_n|\,|\operatorname{tr}(A^nB^n)| \\ &\le \sum_{n=0}^{m}|\alpha_n|\left[\operatorname{tr}(A^{np})\right]^{1/p}\left[\operatorname{tr}(B^{nq})\right]^{1/q} \\ &\le \left(\sum_{n=0}^{m}|\alpha_n|\left(\left[\operatorname{tr}(A^{np})\right]^{1/p}\right)^p\right)^{1/p} \\ &\times \left(\sum_{n=0}^{m}|\alpha_n|\left(\left[\operatorname{tr}(B^{nq})\right]^{1/q}\right)^q\right)^{1/q} \\ &= \left(\sum_{n=0}^{m}|\alpha_n|\operatorname{tr}(A^{np})\right)^{1/p}\left(\sum_{n=0}^{m}|\alpha_n|\operatorname{tr}(B^{nq})\right)^{1/q} \\ &= \left(\operatorname{tr}(\sum_{n=0}^{m}|\alpha_n|\,A^{np})\right)^{1/p}\left(\operatorname{tr}(\sum_{n=0}^{m}|\alpha_n|\,B^{nq})\right)^{1/q}\end{aligned}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

The proof follows now in a similar way with the ones from above and the details are omitted. ∎

EXAMPLE 2.6. *a) If we take* $f(\lambda) = (1 \pm \lambda)^{-1}$, $|\lambda| < 1$ *then we get the inequality*

$$\left|\operatorname{tr}\left((I_n \pm AB)^{-1}\right)\right| \le \left[\operatorname{tr}((I_n - A^p)^{-1})\right]^{1/p}\left[\operatorname{tr}((I_n - B^q)^{-1})\right]^{1/q}, \tag{2.72}$$

for any A *and* B *commuting positive semidefinite matrices in* $M_n(\mathbb{C})$ *with* $\operatorname{tr}(A^p), \operatorname{tr}(B^q) < 1$, *where* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

b) If we take $f(\lambda) = \exp\lambda$, *then*

$$\operatorname{tr}(\exp(AB)) \le \left[\operatorname{tr}(\exp(A^p))\right]^{1/p}\left[\operatorname{tr}(\exp(B^q))\right]^{1/q}, \tag{2.73}$$

for any A *and* B *commuting positive semidefinite matrices in* $M_n(\mathbb{C})$ *and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

3. KATO'S TYPE TRACE INEQUALITIES

3.1. Kato's Inequality. We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle\cdot,\cdot\rangle)$.

If P is a positive selfadjoint operator on H, i.e. $\langle Px, x\rangle \ge 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px, y\rangle|^2 \le \langle Px, x\rangle\langle Py, y\rangle, \tag{3.1}$$

for any $x, y \in H$.

The following inequality is of interest as well, see [**91**, p. 221].

Let P be a positive selfadjoint operator on H. Then

$$\|Px\|^2 \le \|P\|\langle Px, x\rangle \tag{3.2}$$

for any $x \in H$.

The *"square root"* of a positive bounded selfadjoint operator on H can be defined as follows, see for instance [**91**, p. 240]: *If the operator* $A \in B(H)$ *is selfadjoint and positive, then there* *exists a unique positive selfadjoint operator* $B := \sqrt{A} \in B(H)$ *such that* $B^2 = A$. *If* A *is invertible, then so is* B.

If $A \in \mathcal{B}(H)$, then the operator A^*A is selfadjoint and positive. Define the *"absolute value"* operator by $|A| := \sqrt{A^*A}$.

In 1952, Kato [**92**] proved the following celebrated *generalization of Schwarz inequality* for any bounded linear operator T on H:

$$|\langle Tx, y\rangle|^2 \le \langle (T^*T)^\alpha x, x\rangle \langle (TT^*)^{1-\alpha} y, y\rangle, \tag{3.3}$$

for any $x, y \in H$, $\alpha \in [0,1]$. Utilizing the modulus notation introduced before, we can write (3.3) as follows

$$|\langle Tx, y\rangle|^2 \le \left\langle |T|^{2\alpha} x, x\right\rangle \left\langle |T^*|^{2(1-\alpha)} y, y\right\rangle \tag{3.4}$$

for any $x, y \in H$, $\alpha \in [0,1]$.

It is useful to observe that, if $T = N$, a normal operator, i.e., we recall that $NN^* = N^*N$, then the inequality (3.4) can be written as

$$|\langle Nx, y\rangle|^2 \le \left\langle |N|^{2\alpha} x, x\right\rangle \left\langle |N|^{2(1-\alpha)} y, y\right\rangle, \tag{3.5}$$

and in particular, for selfadjoint operators A we can state it as

$$|\langle Ax, y\rangle| \le \||A|^\alpha x\| \left\||A|^{1-\alpha} y\right\| \tag{3.6}$$

for any $x, y \in H$, $\alpha \in [0,1]$.

If $T = U$, a unitary operator, i.e., we recall that $UU^* = U^*U = 1_H$, then the inequality (3.4) becomes

$$|\langle Ux, y\rangle| \le \|x\| \|y\|$$

for any $x, y \in H$, which provides a natural generalization for the Schwarz inequality in H.

The symmetric powers in the inequalities above are natural to be considered, so if we choose in (3.4), (3.5) and in (3.6) $\alpha = 1/2$ then we get for any $x, y \in H$

$$|\langle Tx, y\rangle|^2 \le \langle |T| x, x\rangle \langle |T^*| y, y\rangle, \tag{3.7}$$

$$|\langle Nx, y\rangle|^2 \le \langle |N| x, x\rangle \langle |N| y, y\rangle, \tag{3.8}$$

and

$$|\langle Ax, y\rangle| \le \left\||A|^{1/2} x\right\| \left\||A|^{1/2} y\right\| \tag{3.9}$$

respectively.

It is also worthwhile to observe that, if we take the supremum over $y \in H$, $\|y\| = 1$ in (3.4) then we get

$$\|Tx\|^2 \le \|T\|^{2(1-\alpha)} \left\langle |T|^{2\alpha} x, x\right\rangle \tag{3.10}$$

for any $x \in H$, or in an equivalent form

$$\|Tx\| \le \||T|^\alpha x\| \|T\|^{1-\alpha} \tag{3.11}$$

for any $x \in H$.

If we take $\alpha = 1/2$ in (3.10), then we get

$$\|Tx\|^2 \le \|T\| \langle |T| x, x\rangle \tag{3.12}$$

for any $x \in H$, which in the particular case of $T = P$, a positive operator, provides the result from (3.2).

For various interesting generalizations, extension and Kato related results, see the papers [**75**]-[**86**], [**102**]-[**109**] and [**131**].

3.2. Trace Inequalities Via Kato's Result. We start with the following result:

THEOREM 3.1 (Dragomir, 2014, [**65**]). *Let* $T \in \mathcal{B}(H)$.

(i) If for some $\alpha \in (0,1)$, $|T|^{2\alpha}, |T^*|^{2(1-\alpha)} \in \mathcal{B}_1(H)$, *then* $T \in \mathcal{B}_1(H)$ *and*

$$|\mathrm{tr}(T)|^2 \le \mathrm{tr}\left(|T|^{2\alpha}\right) \mathrm{tr}\left(|T^*|^{2(1-\alpha)}\right); \tag{3.13}$$

(ii) If for some $\alpha \in [0,1]$ *and an orthonormal basis* $\{e_i\}_{i\in I}$ *the sum*

$$\sum_{i\in I} \|Te_i\|^{\alpha} \|T^* e_i\|^{1-\alpha}$$

is finite, then $T \in \mathcal{B}_1(H)$ *and*

$$|\mathrm{tr}(T)| \le \sum_{i\in I} \|Te_i\|^{\alpha} \|T^* e_i\|^{1-\alpha}. \tag{3.14}$$

Moreover, if the sums $\sum_{i\in I} \|Te_i\|$ *and* $\sum_{i\in I} \|T^*e_i\|$ *are finite for an orthonormal basis* $\{e_i\}_{i\in I}$, *then* $T \in \mathcal{B}_1(H)$ *and*

$$|\mathrm{tr}(T)| \le \inf_{\alpha\in[0,1]} \left\{ \sum_{i\in I} \|Te_i\|^{\alpha} \|T^* e_i\|^{1-\alpha} \right\} \le \min\left\{ \sum_{i\in F} \|Te_i\|, \sum_{i\in F} \|T^* e_i\| \right\}. \tag{3.15}$$

PROOF. (i) Assume that $\alpha \in (0,1)$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis in H and F a finite part of I. Then by Kato's inequality (3.4) we have

$$\left| \sum_{i\in F} \langle Te_i, e_i \rangle \right| \le \sum_{i\in F} |\langle Te_i, e_i\rangle| \le \sum_{i\in F} \left\langle |T|^{2\alpha} e_i, e_i \right\rangle^{1/2} \left\langle |T^*|^{2(1-\alpha)} e_i, e_i \right\rangle^{1/2}. \tag{3.16}$$

By Cauchy-Buniakovski-Schwarz inequality for finite sums we have

$$\begin{aligned} &\sum_{i\in F} \left\langle |T|^{2\alpha} e_i, e_i \right\rangle^{1/2} \left\langle |T^*|^{2(1-\alpha)} e_i, e_i \right\rangle^{1/2} \\ &\le \left(\sum_{i\in F} \left[\left\langle |T|^{2\alpha} e_i, e_i \right\rangle^{1/2} \right]^2 \right)^{1/2} \left(\sum_{i\in F} \left[\left\langle |T^*|^{2(1-\alpha)} e_i, e_i \right\rangle^{1/2} \right]^2 \right)^{1/2} \\ &= \left(\sum_{i\in F} \left\langle |T|^{2\alpha} e_i, e_i \right\rangle \right)^{1/2} \left(\sum_{i\in F} \left\langle |T^*|^{2(1-\alpha)} e_i, e_i \right\rangle \right)^{1/2}. \end{aligned} \tag{3.17}$$

Therefore, by (3.16) and (3.17) we have

$$\left| \sum_{i\in F} \langle Te_i, e_i \rangle \right| \le \left(\sum_{i\in F} \left\langle |T|^{2\alpha} e_i, e_i \right\rangle \right)^{1/2} \left(\sum_{i\in F} \left\langle |T^*|^{2(1-\alpha)} e_i, e_i \right\rangle \right)^{1/2} \tag{3.18}$$

for any finite part F of I.

If for some $\alpha \in (0,1)$ we have $|T|^{2\alpha}, |T^*|^{2(1-\alpha)} \in \mathcal{B}_1(H)$, then the sums $\sum_{i\in I} \left\langle |T|^{2\alpha} e_i, e_i \right\rangle$ and $\sum_{i\in I} \left\langle |T^*|^{2(1-\alpha)} e_i, e_i \right\rangle$ are finite and by (3.18) we have that $\sum_{i\in I} \langle Te_i, e_i\rangle$ is also finite and we have the inequality (3.13).

(ii) Assume that $\alpha \in [0,1]$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis in H and F a finite part of I. Utilising McCarthy's inequality for the positive operator P, namely

$$\left\langle P^{\beta} x, x \right\rangle \le \langle Px, x\rangle^{\beta},$$

that holds for $\beta \in [0,1]$ and $x \in H$, $\|x\| = 1$, we have

$$\left\langle |T|^{2\alpha} e_i, e_i \right\rangle \leq \left\langle |T|^2 e_i, e_i \right\rangle^{\alpha}$$

and

$$\left\langle |T^*|^{2(1-\alpha)} e_i, e_i \right\rangle \leq \left\langle |T^*|^2 e_i, e_i \right\rangle^{1-\alpha}$$

for any $i \in I$.

Making use of (3.16) we have

$$\begin{aligned} \left| \sum_{i\in F} \langle Te_i, e_i \rangle \right| &\leq \sum_{i\in F} |\langle Te_i, e_i \rangle| \leq \sum_{i\in F} \left\langle |T|^{2\alpha} e_i, e_i \right\rangle^{1/2} \left\langle |T^*|^{2(1-\alpha)} e_i, e_i \right\rangle^{1/2} \\ &\leq \sum_{i\in F} \left\langle |T|^2 e_i, e_i \right\rangle^{\alpha/2} \left\langle |T^*|^2 e_i, e_i \right\rangle^{(1-\alpha)/2} \\ &= \sum_{i\in F} \langle T^*Te_i, e_i \rangle^{\alpha/2} \langle TT^*e_i, e_i \rangle^{(1-\alpha)/2} \\ &= \sum_{i\in F} \|Te_i\|^{\alpha} \|T^*e_i\|^{1-\alpha}. \end{aligned} \tag{3.19}$$

Utilizing Hölder's inequality for finite sums and $p = \frac{1}{\alpha}$, $q = \frac{1}{1-\alpha}$ we also have

$$\begin{aligned} &\sum_{i\in F} \|Te_i\|^{\alpha} \|T^*e_i\|^{1-\alpha} \\ &\leq \left[\sum_{i\in F} (\|Te_i\|^{\alpha})^{1/\alpha} \right]^{\alpha} \left[\sum_{i\in F} \left(\|T^*e_i\|^{1-\alpha}\right)^{1/(1-\alpha)} \right]^{1-\alpha} \\ &= \left[\sum_{i\in F} \|Te_i\| \right]^{\alpha} \left[\sum_{i\in F} \|T^*e_i\| \right]^{1-\alpha}. \end{aligned} \tag{3.20}$$

Since all the series involved in (3.19) and (3.20) are convergent, then we get

$$\begin{aligned} \left| \sum_{i\in I} \langle Te_i, e_i \rangle \right| &\leq \sum_{i\in I} \|Te_i\|^{\alpha} \|T^*e_i\|^{1-\alpha} \\ &\leq \left[\sum_{i\in I} \|Te_i\| \right]^{\alpha} \left[\sum_{i\in I} \|T^*e_i\| \right]^{1-\alpha} \end{aligned} \tag{3.21}$$

for any $\alpha \in [0,1]$.

Taking the infimum over $\alpha \in [0,1]$ in (3.21) produces

$$\begin{aligned} \left| \sum_{i\in I} \langle Te_i, e_i \rangle \right| &\leq \inf_{\alpha\in[0,1]} \left\{ \sum_{i\in F} \|Te_i\|^{\alpha} \|T^*e_i\|^{1-\alpha} \right\} \\ &\leq \inf_{\alpha\in[0,1]} \left[\sum_{i\in F} \|Te_i\| \right]^{\alpha} \left[\sum_{i\in F} \|T^*e_i\| \right]^{1-\alpha} \\ &= \min \left\{ \sum_{i\in F} \|Te_i\|, \sum_{i\in F} \|T^*e_i\| \right\}. \end{aligned} \tag{3.22}$$

∎

COROLLARY 3.2. *Let* $T \in \mathcal{B}(H)$.
(i) If $|T|, |T^*| \in \mathcal{B}_1(H)$, *then* $T \in \mathcal{B}_1(H)$ *and*

$$|\operatorname{tr}(T)|^2 \le \operatorname{tr}(|T|)\operatorname{tr}(|T^*|); \tag{3.23}$$

(ii) If for an orthonormal basis $\{e_i\}_{i \in I}$ *the sum* $\sum_{i\in I}\sqrt{\|Te_i\|\|T^*e_i\|}$ *is finite, then* $T \in \mathcal{B}_1(H)$ *and*

$$|\operatorname{tr}(T)| \le \sum_{i\in I}\sqrt{\|Te_i\|\|T^*e_i\|}. \tag{3.24}$$

COROLLARY 3.3. *Let* $N \in \mathcal{B}(H)$ *be a normal operator. If for some* $\alpha \in (0,1)$, $|N|^{2\alpha}$, $|N|^{2(1-\alpha)} \in \mathcal{B}_1(H)$, *then* $N \in \mathcal{B}_1(H)$ *and*

$$|\operatorname{tr}(N)|^2 \le \operatorname{tr}\left(|N|^{2\alpha}\right)\operatorname{tr}\left(|N|^{2(1-\alpha)}\right). \tag{3.25}$$

In particular, if $|N| \in \mathcal{B}_1(H)$, *then* $N \in \mathcal{B}_1(H)$ *and*

$$|\operatorname{tr}(N)| \le \operatorname{tr}(|N|). \tag{3.26}$$

The following result also holds.

THEOREM 3.4 (Dragomir, 2014, [**65**]). *Let* $T \in \mathcal{B}(H)$ *and* $A, B \in \mathcal{B}_2(H)$.
(i) For any $\alpha \in [0,1]$, $|A^*|^2|T|^{2\alpha}$, $|B^*|^2|T^*|^{2(1-\alpha)}$ *and* $B^*TA \in \mathcal{B}_1(H)$ *and*

$$|\operatorname{tr}(AB^*T)|^2 \le \operatorname{tr}\left(|A^*|^2|T|^{2\alpha}\right)\operatorname{tr}\left(|B^*|^2|T^*|^{2(1-\alpha)}\right); \tag{3.27}$$

(ii) We also have

$$\begin{aligned} &|\operatorname{tr}(AB^*T)|^2 \\ &\le \min\left\{\operatorname{tr}\left(|B|^2\right)\operatorname{tr}\left(|A^*|^2|T|^2\right), \operatorname{tr}\left(|A|^2\right)\operatorname{tr}\left(|B^*|^2|T^*|^2\right)\right\}. \end{aligned} \tag{3.28}$$

PROOF. (i) Let $\{e_i\}_{i\in I}$ be an orthonormal basis in H and F a finite part of I. Then by Kato's inequality (3.4) we have

$$|\langle TAe_i, Be_i\rangle|^2 \le \left\langle |T|^{2\alpha}Ae_i, Ae_i\right\rangle\left\langle |T^*|^{2(1-\alpha)}Be_i, Be_i\right\rangle \tag{3.29}$$

for any $i \in I$. This is equivalent to

$$|\langle B^*TAe_i, e_i\rangle| \le \left\langle A^*|T|^{2\alpha}Ae_i, e_i\right\rangle^{1/2}\left\langle B^*|T^*|^{2(1-\alpha)}Be_i, e_i\right\rangle^{1/2} \tag{3.30}$$

for any $i \in I$.

Using the generalized triangle inequality for the modulus and the Cauchy-Bunyakowsky-Schwarz inequality for finite sums we have from (3.30) that

$$\begin{aligned}(3.31)\qquad &\left|\sum_{i\in F}\langle B^*TAe_i,e_i\rangle\right| \\ &\le \sum_{i\in F}|\langle B^*TAe_i,e_i\rangle| \\ &\le \sum_{i\in F}\left\langle A^*|T|^{2\alpha}Ae_i,e_i\right\rangle^{1/2}\left\langle B^*|T^*|^{2(1-\alpha)}Be_i,e_i\right\rangle^{1/2} \\ &\le \left[\sum_{i\in F}\left(\left\langle A^*|T|^{2\alpha}Ae_i,e_i\right\rangle^{1/2}\right)^2\right]^{1/2} \\ &\times\left[\sum_{i\in F}\left(\left\langle B^*|T^*|^{2(1-\alpha)}Be_i,e_i\right\rangle^{1/2}\right)^2\right]^{1/2} \\ &= \left[\sum_{i\in F}\left\langle A^*|T|^{2\alpha}Ae_i,e_i\right\rangle\right]^{1/2}\left[\sum_{i\in F}\left\langle B^*|T^*|^{2(1-\alpha)}Be_i,e_i\right\rangle\right]^{1/2}\end{aligned}$$

for any F a finite part of I.

Let $\alpha\in[0,1]$. Since $A,\ B\in\mathcal{B}_2(H)$, then $A^*|T|^{2\alpha}A$, $B^*|T^*|^{2(1-\alpha)}B$ and $B^*TA\in\mathcal{B}_1(H)$ and by (3.31) we have

$$(3.32)\qquad |\operatorname{tr}(B^*TA)|\le\left[\operatorname{tr}\left(A^*|T|^{2\alpha}A\right)\right]^{1/2}\left[\operatorname{tr}\left(B^*|T^*|^{2(1-\alpha)}B\right)\right]^{1/2}.$$

Since, by the properties of trace we have

$$\operatorname{tr}(B^*TA)=\operatorname{tr}(AB^*T),$$

$$\operatorname{tr}\left(A^*|T|^{2\alpha}A\right)=\operatorname{tr}\left(AA^*|T|^{2\alpha}\right)=\operatorname{tr}\left(|A^*|^2|T|^{2\alpha}\right)$$

and

$$\operatorname{tr}\left(B^*|T^*|^{2(1-\alpha)}B\right)=\operatorname{tr}\left(|B^*|^2|T^*|^{2(1-\alpha)}\right),$$

then by (3.32) we get (3.27).

(ii) Utilising McCarthy's inequality [**109**] for the positive operator P

$$\left\langle P^\beta x,x\right\rangle\le\langle Px,x\rangle^\beta$$

that holds for $\beta\in(0,1)$ and $x\in H$, $\|x\|=1$, we have

$$(3.33)\qquad \left\langle P^\beta y,y\right\rangle\le\|y\|^{2(1-\beta)}\langle Py,y\rangle^\beta$$

for any $y\in H$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis in H and F a finite part of I. From (3.33) we have

$$\left\langle|T|^{2\alpha}Ae_i,Ae_i\right\rangle\le\|Ae_i\|^{2(1-\alpha)}\left\langle|T|^2Ae_i,Ae_i\right\rangle^\alpha$$

and

$$\left\langle|T^*|^{2(1-\alpha)}Be_i,Be_i\right\rangle\le\|Be_i\|^{2\alpha}\left\langle|T^*|^2Be_i,Be_i\right\rangle^{1-\alpha}$$

for any $i\in I$.

Making use of the inequality (3.29) we get

$$\begin{aligned}|\langle TAe_i, Be_i\rangle|^2 &\le \|Ae_i\|^{2(1-\alpha)} \left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\alpha} \|Be_i\|^{2\alpha} \left\langle |T^*|^2 Be_i, Be_i\right\rangle^{1-\alpha} \\ &= \|Be_i\|^{2\alpha} \left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\alpha} \|Ae_i\|^{2(1-\alpha)} \left\langle |T^*|^2 Be_i, Be_i\right\rangle^{1-\alpha}\end{aligned}$$

and taking the square root we get

$$|\langle TAe_i, Be_i\rangle| \le \|Be_i\|^{\alpha} \left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\frac{\alpha}{2}} \|Ae_i\|^{1-\alpha} \left\langle |T^*|^2 Be_i, Be_i\right\rangle^{\frac{1-\alpha}{2}} \tag{3.34}$$

for any $i \in I$.

Using the generalized triangle inequality for the modulus and the Hölder's inequality for finite sums and $p = \frac{1}{\alpha}$, $q = \frac{1}{1-\alpha}$ we get from (3.34) that

$$\begin{aligned}&\left|\sum_{i\in F} \langle B^*TAe_i, e_i\rangle\right| \\ &\le \sum_{i\in F} |\langle B^*TAe_i, e_i\rangle| \\ &\le \sum_{i\in F} \|Be_i\|^{\alpha} \left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\frac{\alpha}{2}} \|Ae_i\|^{1-\alpha} \left\langle |T^*|^2 Be_i, Be_i\right\rangle^{\frac{1-\alpha}{2}} \\ &\le \left(\sum_{i\in F} \left[\|Be_i\|^{\alpha} \left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\frac{\alpha}{2}}\right]^{1/\alpha}\right)^{\alpha} \\ &\times \left(\sum_{i\in F} \left[\|Ae_i\|^{1-\alpha} \left\langle |T^*|^2 Be_i, Be_i\right\rangle^{\frac{1-\alpha}{2}}\right]^{1/(1-\alpha)}\right)^{1-\alpha} \\ &= \left(\sum_{i\in F} \|Be_i\| \left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\frac{1}{2}}\right)^{\alpha} \left(\sum_{i\in F} \|Ae_i\| \left\langle |T^*|^2 Be_i, Be_i\right\rangle^{\frac{1}{2}}\right)^{1-\alpha}.\end{aligned} \tag{3.35}$$

By Cauchy-Bunyakowsky-Schwarz inequality for finite sums we also have

$$\begin{aligned}\sum_{i\in F} \|Be_i\| \left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\frac{1}{2}} &\le \left(\sum_{i\in F} \|Be_i\|^2\right)^{1/2} \left(\sum_{i\in F} \left\langle |T|^2 Ae_i, Ae_i\right\rangle\right)^{1/2} \\ &= \left(\sum_{i\in F} \left\langle |B|^2 e_i, e_i\right\rangle\right)^{1/2} \left(\sum_{i\in F} \left\langle A^* |T|^2 Ae_i, e_i\right\rangle\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\sum_{i\in F} \|Ae_i\| \left\langle |T^*|^2 Be_i, Be_i\right\rangle^{\frac{1}{2}} &\le \left(\sum_{i\in F} \|Ae_i\|^2\right)^{1/2} \left(\sum_{i\in F} \left\langle |T^*|^2 Be_i, Be_i\right\rangle\right)^{1/2} \\ &= \left(\sum_{i\in F} \left\langle |A|^2 e_i, e_i\right\rangle\right)^{1/2} \left(\sum_{i\in F} \left\langle B^* |T^*|^2 Be_i, e_i\right\rangle\right)^{1/2}\end{aligned}$$

and by (3.35) we obtain

$$\begin{aligned}(3.36)\qquad &\left|\sum_{i\in F}\left\langle B^{*}TAe_{i},e_{i}\right\rangle\right| \\ &\leq\left(\sum_{i\in F}\left\langle\left|B\right|^{2}e_{i},e_{i}\right\rangle\right)^{\alpha/2}\left(\sum_{i\in F}\left\langle A^{*}\left|T\right|^{2}Ae_{i},e_{i}\right\rangle\right)^{\alpha/2} \\ &\times\left(\sum_{i\in F}\left\langle\left|A\right|^{2}e_{i},e_{i}\right\rangle\right)^{(1-\alpha)/2}\left(\sum_{i\in F}\left\langle B^{*}\left|T^{*}\right|^{2}Be_{i},e_{i}\right\rangle\right)^{(1-\alpha)/2}\end{aligned}$$

for any F a finite part of I.

Let $\alpha\in[0,1]$. Since $A,B\in\mathcal{B}_{2}(H)$, then $A^{*}|T|^{2}A$ and $B^{*}|T^{*}|^{2}B\in\mathcal{B}_{1}(H)$ and by (3.36) we get

$$\begin{aligned}(3.37)\qquad &\left|\operatorname{tr}\left(AB^{*}T\right)\right|^{2} \\ &\leq\left[\operatorname{tr}\left(|B|^{2}\right)\operatorname{tr}\left(A^{*}|T|^{2}A\right)\right]^{\alpha}\left[\operatorname{tr}\left(|A|^{2}\right)\operatorname{tr}\left(B^{*}|T^{*}|^{2}B\right)\right]^{1-\alpha} \\ &=\left[\operatorname{tr}\left(|B|^{2}\right)\operatorname{tr}\left(|A^{*}|^{2}|T|^{2}\right)\right]^{\alpha}\left[\operatorname{tr}\left(|A|^{2}\right)\operatorname{tr}\left(|B^{*}|^{2}|T^{*}|^{2}\right)\right]^{1-\alpha}.\end{aligned}$$

Taking the infimum over $\alpha\in[0,1]$ we get (3.28). ∎

COROLLARY 3.5. *Let $T\in\mathcal{B}(H)$ and $A,B\in\mathcal{B}_{2}(H)$. We have $|A^{*}|^{2}|T|$, $|B^{*}|^{2}|T^{*}|$ and $B^{*}TA\in\mathcal{B}_{1}(H)$ and*

$$(3.38)\qquad \left|\operatorname{tr}\left(AB^{*}T\right)\right|^{2}\leq\operatorname{tr}\left(|A^{*}|^{2}|T|\right)\operatorname{tr}\left(|B^{*}|^{2}|T^{*}|\right).$$

COROLLARY 3.6. *Let $N\in\mathcal{B}(H)$ be a normal operator and $A,B\in\mathcal{B}_{2}(H)$.*

(i) For any $\alpha\in[0,1]$, $|A^{*}|^{2}|N|^{2\alpha}$, $|B^{*}|^{2}|N|^{2(1-\alpha)}$ and $B^{*}NA\in\mathcal{B}_{1}(H)$ and

$$(3.39)\qquad \left|\operatorname{tr}\left(AB^{*}N\right)\right|^{2}\leq\operatorname{tr}\left(|A^{*}|^{2}|N|^{2\alpha}\right)\operatorname{tr}\left(|B^{*}|^{2}|N|^{2(1-\alpha)}\right).$$

In particular, $|A^{*}|^{2}|N|$, $|B^{*}|^{2}|N|$ and $B^{*}NA\in\mathcal{B}_{1}(H)$ and

$$(3.40)\qquad \left|\operatorname{tr}\left(AB^{*}N\right)\right|^{2}\leq\operatorname{tr}\left(|A^{*}|^{2}|N|\right)\operatorname{tr}\left(|B^{*}|^{2}|N|\right).$$

(ii) We also have

$$\begin{aligned}(3.41)\qquad &\left|\operatorname{tr}\left(AB^{*}N\right)\right|^{2} \\ &\leq\min\left\{\operatorname{tr}\left(|B|^{2}\right)\operatorname{tr}\left(|A^{*}|^{2}|N|^{2}\right),\operatorname{tr}\left(|A|^{2}\right)\operatorname{tr}\left(|B^{*}|^{2}|N|^{2}\right)\right\}.\end{aligned}$$

REMARK 3.1. Let $\alpha\in[0,1]$. By replacing A with A^{*} and B with B^{*} in (3.27) we get

$$(3.42)\qquad \left|\operatorname{tr}\left(A^{*}BT\right)\right|^{2}\leq\operatorname{tr}\left(|A|^{2}|T|^{2\alpha}\right)\operatorname{tr}\left(|B|^{2}|T^{*}|^{2(1-\alpha)}\right)$$

for any $T\in\mathcal{B}(H)$ and $A,B\in\mathcal{B}_{2}(H)$.

If in this inequality we take $A=B$, then we get

$$(3.43)\qquad \left|\operatorname{tr}\left(|B|^{2}T\right)\right|^{2}\leq\operatorname{tr}\left(|B|^{2}|T|^{2\alpha}\right)\operatorname{tr}\left(|B|^{2}|T^{*}|^{2(1-\alpha)}\right)$$

for any $T\in\mathcal{B}(H)$ and $B\in\mathcal{B}_{2}(H)$.

If in (3.42) we take $A=B^{*}$, then we get

$$(3.44)\qquad \left|\operatorname{tr}\left(B^{2}T\right)\right|^{2}\leq\operatorname{tr}\left(|B^{*}|^{2}|T|^{2\alpha}\right)\operatorname{tr}\left(|B|^{2}|T^{*}|^{2(1-\alpha)}\right)$$

for any $T\in\mathcal{B}(H)$ and $B\in\mathcal{B}_{2}(H)$.

Also, if $T = N$, a normal operator, then (3.43) and (3.44) become

$$\left|\operatorname{tr}\left(|B|^2 N\right)\right|^2 \leq \operatorname{tr}\left(|B|^2 |N|^{2\alpha}\right) \operatorname{tr}\left(|B|^2 |N|^{2(1-\alpha)}\right) \tag{3.45}$$

and

$$\left|\operatorname{tr}\left(B^2 N\right)\right|^2 \leq \operatorname{tr}\left(|B^*|^2 |N|^{2\alpha}\right) \operatorname{tr}\left(|B|^2 |N|^{2(1-\alpha)}\right), \tag{3.46}$$

for any $B \in \mathcal{B}_2(H)$.

3.3. Some Functional Properties. Let $A \in \mathcal{B}_2(H)$ and $P \in \mathcal{B}(H)$ with $P \geq 0$. Then $Q := A^*PA \in \mathcal{B}_1(H)$ with $Q \geq 0$ and writing the inequality (3.43) for $B = (A^*PA)^{1/2} \in \mathcal{B}_2(H)$ we get

$$|\operatorname{tr}(A^*PAT)|^2 \leq \operatorname{tr}\left(A^*PA|T|^{2\alpha}\right) \operatorname{tr}\left(A^*PA|T^*|^{2(1-\alpha)}\right),$$

which, by the properties of trace, is equivalent to

$$|\operatorname{tr}(PATA^*)|^2 \leq \operatorname{tr}\left(PA|T|^{2\alpha} A^*\right) \operatorname{tr}\left(PA|T^*|^{2(1-\alpha)} A^*\right), \tag{3.47}$$

where $T \in \mathcal{B}(H)$ and $\alpha \in [0,1]$.

For a given $A \in \mathcal{B}_2(H)$, $T \in \mathcal{B}(H)$ and $\alpha \in [0,1]$, we consider the functional $\sigma_{A,T,\alpha}$ defined on the cone $\mathcal{B}_+(H)$ of nonnegative operators on $\mathcal{B}(H)$ by

$$\begin{aligned}\sigma_{A,T,\alpha}(P) := {} & \left[\operatorname{tr}\left(PA|T|^{2\alpha} A^*\right)\right]^{1/2} \left[\operatorname{tr}\left(PA|T^*|^{2(1-\alpha)} A^*\right)\right]^{1/2} \\ & - |\operatorname{tr}(PATA^*)|.\end{aligned}$$

The following theorem collects some fundamental properties of this functional.

THEOREM 3.7 (Dragomir, 2014, [**65**]). *Let* $A \in \mathcal{B}_2(H)$, $T \in \mathcal{B}(H)$ *and* $\alpha \in [0,1]$.

(i) For any $P, Q \in \mathcal{B}_+(H)$

$$\sigma_{A,T,\alpha}(P+Q) \geq \sigma_{A,T,\alpha}(P) + \sigma_{A,T,\alpha}(Q) (\geq 0), \tag{3.48}$$

namely, $\sigma_{A,T,\alpha}$ *is a superadditive functional on* $\mathcal{B}_+(H)$;

(ii) For any $P, Q \in \mathcal{B}_+(H)$ *with* $P \geq Q$

$$\sigma_{A,T,\alpha}(P) \geq \sigma_{A,T,\alpha}(Q) (\geq 0), \tag{3.49}$$

namely, $\sigma_{A,T,\alpha}$ *is a monotonic nondecreasing functional on* $\mathcal{B}_+(H)$;

(iii) If $P, Q \in \mathcal{B}_+(H)$ *and there exist the constants* $M > m > 0$ *such that* $MQ \geq P \geq mQ$ *then*

$$M\sigma_{A,T,\alpha}(Q) \geq \sigma_{A,T,\alpha}(P) \geq m\sigma_{A,T,\alpha}(Q) (\geq 0). \tag{3.50}$$

PROOF. (i) Let $P, Q \in \mathcal{B}_+(H)$. On utilizing the elementary inequality

$$\left(a^2+b^2\right)^{1/2} \left(c^2+d^2\right)^{1/2} \geq ac+bd, \ a,b,c,d \geq 0$$

and the triangle inequality for the modulus, we have

$$\begin{aligned}
&\sigma_{A,T,\alpha}(P+Q) \\
&= \left[\operatorname{tr}\left((P+Q)A|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left((P+Q)A|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} \\
&- |\operatorname{tr}((P+Q)ATA^*)| \\
&= \left[\operatorname{tr}\left(PA|T|^{2\alpha}A^* + QA|T|^{2\alpha}A^*\right)\right]^{1/2} \\
&\times \left[\operatorname{tr}\left(PA|T^*|^{2(1-\alpha)}A^* + QA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} \\
&- |\operatorname{tr}(PATA^* + QATA^*)| \\
&= \left[\operatorname{tr}\left(PA|T|^{2\alpha}A^*\right) + \operatorname{tr}\left(QA|T|^{2\alpha}A^*\right)\right]^{1/2} \\
&\times \left[\operatorname{tr}\left(PA|T^*|^{2(1-\alpha)}A^*\right) + \operatorname{tr}\left(QA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} \\
&- |\operatorname{tr}(PATA^*) + \operatorname{tr}(QATA^*)| \\
&\geq \left[\operatorname{tr}\left(PA|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(PA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} \\
&+ \left[\operatorname{tr}\left(QA|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(QA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} \\
&- |\operatorname{tr}(PATA^*)| - |\operatorname{tr}(QATA^*)| \\
&= \sigma_{A,T,\alpha}(P) + \sigma_{A,T,\alpha}(Q)
\end{aligned}$$

and the inequality (3.48) is proved.

(ii) Let $P, Q \in \mathcal{B}_+(H)$ with $P \geq Q$. Utilising the superadditivity property we have

$$\begin{aligned}
\sigma_{A,T,\alpha}(P) &= \sigma_{A,T,\alpha}((P-Q)+Q) \geq \sigma_{A,T,\alpha}(P-Q) + \sigma_{A,T,\alpha}(Q) \\
&\geq \sigma_{A,T,\alpha}(Q)
\end{aligned}$$

and the inequality (3.49) is obtained.

(iii) From the monotonicity property we have

$$\sigma_{A,T,\alpha}(P) \geq \sigma_{A,T,\alpha}(mQ) = m\sigma_{A,T,\alpha}(Q)$$

and a similar inequality for M, which prove the desired result (3.50). ∎

COROLLARY 3.8. *Let $A \in \mathcal{B}_2(H)$, $T \in \mathcal{B}(H)$ and $\alpha \in [0,1]$. If $P \in \mathcal{B}(H)$ is such that there exist the constants $M > m > 0$ with $M1_H \geq P \geq m1_H$, then*

$$\begin{aligned}
&M\left(\left[\operatorname{tr}\left(A|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(A|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} - |\operatorname{tr}(ATA^*)|\right) \\
&\geq \left[\operatorname{tr}\left(PA|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(PA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} - |\operatorname{tr}(PATA^*)| \\
&\geq m\left(\left[\operatorname{tr}\left(A|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(A|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} - |\operatorname{tr}(ATA^*)|\right).
\end{aligned} \tag{3.51}$$

For a given $A\in\mathcal{B}_2(H)$, $T\in\mathcal{B}(H)$ and $\alpha\in[0,1]$, if we take $P=|V|^2$ with $V\in\mathcal{B}(H)$, we have

$$\begin{aligned}\sigma_{A,T,\alpha}\left(|V|^2\right) &= \left[\operatorname{tr}\left(|V|^2 A|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(|V|^2 A|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} \\ &- \left|\operatorname{tr}\left(|V|^2 ATA^*\right)\right| \\ &= \left[\operatorname{tr}\left(V^*VA|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(V^*VA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} \\ &- \left|\operatorname{tr}\left(V^*VATA^*\right)\right| \\ &= \left[\operatorname{tr}\left(A^*V^*VA|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(A^*V^*VA|T^*|^{2(1-\alpha)}\right)\right]^{1/2} \\ &- \left|\operatorname{tr}\left(A^*V^*VAT\right)\right| \\ &= \left[\operatorname{tr}\left((VA)^*VA|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left((VA)^*VA|T^*|^{2(1-\alpha)}\right)\right]^{1/2} \\ &- \left|\operatorname{tr}\left((VA)^*VAT\right)\right| \\ &= \left[\operatorname{tr}\left(|VA|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|VA|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(|VA|^2T\right)\right|.\end{aligned}$$

Assume that $A\in\mathcal{B}_2(H)$, $T\in\mathcal{B}(H)$ and $\alpha\in[0,1]$.

If we use the superadditivity property of the functional $\sigma_{A,T,\alpha}$ we have for any $V,U\in\mathcal{B}(H)$ that

$$\begin{aligned}&\left[\operatorname{tr}\left(\left(|VA|^2+|UA|^2\right)|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(\left(|VA|^2+|UA|^2\right)|T^*|^{2(1-\alpha)}\right)\right]^{1/2} \\ &- \left|\operatorname{tr}\left(\left(|VA|^2+|UA|^2\right)T\right)\right| \\ &\geq \left[\operatorname{tr}\left(|VA|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|VA|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(|VA|^2T\right)\right| \\ &+ \left[\operatorname{tr}\left(|UA|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|UA|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(|UA|^2T\right)\right| \\ &(\geq 0).\end{aligned} \tag{3.52}$$

Also, if $|V|^2\geq|U|^2$ with $V,U\in\mathcal{B}(H)$, then

$$\begin{aligned}&\left[\operatorname{tr}\left(|VA|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|VA|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(|VA|^2T\right)\right| \\ &\geq \left[\operatorname{tr}\left(|UA|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|UA|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(|UA|^2T\right)\right| \\ &(\geq 0).\end{aligned} \tag{3.53}$$

If $U\in\mathcal{B}(H)$ is invertible, then

$$\frac{1}{\|U^{-1}\|}\|x\|\leq\|Ux\|\leq\|U\|\|x\| \text{ for any } x\in H,$$

which implies that

$$\frac{1}{\|U^{-1}\|^2}1_H\leq|U|^2\leq\|U\|^2 1_H.$$

Utilising (3.51) we get

$$\begin{aligned} (3.54)\qquad & \|U\|^2 \left(\left[\operatorname{tr}\left(|A|^2 |T|^{2\alpha}\right)\right]^{1/2} \left[\operatorname{tr}\left(|A|^2 |T^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(|A|^2 T\right)\right| \right) \\ & \geq \left[\operatorname{tr}\left(|UA|^2 |T|^{2\alpha}\right)\right]^{1/2} \left[\operatorname{tr}\left(|UA|^2 |T^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(|UA|^2 T\right)\right| \\ & \geq \frac{1}{\|U^{-1}\|^2} \\ & \times \left(\left[\operatorname{tr}\left(|A|^2 |T|^{2\alpha}\right)\right]^{1/2} \left[\operatorname{tr}\left(|A|^2 |T^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(|A|^2 T\right)\right| \right). \end{aligned}$$

3.4. Inequalities for n-Tuples of Operators. We have:

PROPOSITION 3.9 (Dragomir, 2014, [**65**]). *Let* $\mathbf{P} = (P_1, ..., P_n) \in \mathcal{B}_+^{(n)}(H)$, $\mathbf{T} = (T_1, ..., T_n) \in \mathcal{B}^{(n)}(H)$, $\mathbf{A} = (A_1, ..., A_n) \in \mathcal{B}_2^{(n)}(H)$ *and* $\mathbf{z} = (z_1, ..., z_n) \in \mathbb{C}^n$ *with* $n \geq 2$. *Then*

$$\begin{aligned} (3.55)\qquad & \left|\operatorname{tr}\left(\sum_{k=1}^n z_k P_k A_k T_k A_k^*\right)\right|^2 \\ & \leq \operatorname{tr}\left(\sum_{k=1}^n |z_k| P_k A_k |T_k|^{2\alpha} A_k^*\right) \operatorname{tr}\left(\sum_{k=1}^n |z_k| P_k A_k |T_k^*|^{2(1-\alpha)} A_k^*\right) \end{aligned}$$

for any $\alpha \in [0, 1]$.

PROOF. Using the properties of modulus and the inequality (3.47) we have

$$\begin{aligned} & \left|\operatorname{tr}\left(\sum_{k=1}^n z_k P_k A_k T_k A_k^*\right)\right| \\ & = \left|\sum_{k=1}^n z_k \operatorname{tr}\left(P_k A_k T_k A_k^*\right)\right| \leq \sum_{k=1}^n |z_k| \left|\operatorname{tr}\left(P_k A_k T_k A_k^*\right)\right| \\ & \leq \sum_{k=1}^n |z_k| \left[\operatorname{tr}\left(P_k A_k |T_k|^{2\alpha} A_k^*\right)\right]^{1/2} \left[\operatorname{tr}\left(P_k A_k |T_k^*|^{2(1-\alpha)} A_k^*\right)\right]^{1/2}. \end{aligned}$$

Utilizing the weighted discrete Cauchy-Bunyakovsky-Schwarz inequality we also have

$$\begin{aligned} & \sum_{k=1}^n |z_k| \left[\operatorname{tr}\left(P_k A_k |T_k|^{2\alpha} A_k^*\right)\right]^{1/2} \left[\operatorname{tr}\left(P_k A_k |T_k^*|^{2(1-\alpha)} A_k^*\right)\right]^{1/2} \\ & \leq \left(\sum_{k=1}^n |z_k| \left(\left[\operatorname{tr}\left(P_k A_k |T_k|^{2\alpha} A_k^*\right)\right]^{1/2}\right)^2\right)^{1/2} \\ & \times \left(\sum_{k=1}^n |z_k| \left(\left[\operatorname{tr}\left(P_k A_k |T_k^*|^{2(1-\alpha)} A_k^*\right)\right]^{1/2}\right)^2\right)^{1/2} \\ & = \left(\sum_{k=1}^n |z_k| \operatorname{tr}\left(P_k A_k |T_k|^{2\alpha} A_k^*\right)\right)^{1/2} \left(\sum_{k=1}^n |z_k| \operatorname{tr}\left(P_k A_k |T_k^*|^{2(1-\alpha)} A_k^*\right)\right)^{1/2}, \end{aligned}$$

which imply the desired result (3.55). ∎

REMARK 3.2. If we take $P_k = 1_H$ for any $k \in \{1, ..., n\}$ in (3.55), then

$$\left|\operatorname{tr}\left(\sum_{k=1}^{n} z_k |A_k|^2 T_k\right)\right|^2 \leq \operatorname{tr}\left(\sum_{k=1}^{n} |z_k| |A_k|^2 |T_k|^{2\alpha}\right) \operatorname{tr}\left(\sum_{k=1}^{n} |z_k| |A_k|^2 |T_k^*|^{2(1-\alpha)}\right) \tag{3.56}$$

provided that $\mathbf{T} = (T_1, ..., T_n) \in \mathcal{B}^{(n)}(H)$, $\mathbf{A} = (A_1, ..., A_n) \in \mathcal{B}_2^{(n)}(H)$, $\alpha \in [0,1]$ and $\mathbf{z} = (z_1, ..., z_n) \in \mathbb{C}^n$.

We consider the functional for n-tuples of nonnegative operators $\mathbf{P} = (P_1, ..., P_n) \in \mathcal{B}_+^{(n)}(H)$ as follows:

$$\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{P}) := \left[\operatorname{tr}\left(\sum_{k=1}^{n} P_k A_k |T_k|^{2\alpha} A_k^*\right)\right]^{1/2} \times \left[\operatorname{tr}\left(\sum_{k=1}^{n} P_k A_k |T_k^*|^{2(1-\alpha)} A_k^*\right)\right]^{1/2} - \left|\operatorname{tr}\left(\sum_{k=1}^{n} P_k A_k T_k A_k^*\right)\right|, \tag{3.57}$$

where $\mathbf{T} = (T_1, ..., T_n) \in \mathcal{B}^{(n)}(H)$, $\mathbf{A} = (A_1, ..., A_n) \in \mathcal{B}_2^{(n)}(H)$ and $\alpha \in [0,1]$.

Utilising a similar argument to the one in Theorem 3.7 we can state:

PROPOSITION 3.10 (Dragomir, 2014, [**65**]). *Let* $\mathbf{T} = (T_1, ..., T_n) \in \mathcal{B}^{(n)}(H)$, $\mathbf{A} = (A_1, ..., A_n) \in \mathcal{B}_2^{(n)}(H)$ *and* $\alpha \in [0,1]$.

(i) For any $\mathbf{P}, \mathbf{Q} \in \mathcal{B}_+^{(n)}(H)$

$$\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{P}+\mathbf{Q}) \geq \sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{P}) + \sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{Q}) \,(\geq 0), \tag{3.58}$$

namely, $\sigma_{\mathbf{A},\mathbf{T},\alpha}$ *is a superadditive functional on* $\mathcal{B}_+^{(n)}(H)$;

(ii) For any $\mathbf{P}, \mathbf{Q} \in \mathcal{B}_+^{(n)}(H)$ *with* $\mathbf{P} \geq \mathbf{Q}$, *namely* $P_k \geq Q_k$ *for all* $k \in \{1, ..., n\}$

$$\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{P}) \geq \sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{Q}) \,(\geq 0), \tag{3.59}$$

namely, $\sigma_{\mathbf{A},\mathbf{B}}$ *is a monotonic nondecreasing functional on* $\mathcal{B}_+^{(n)}(H)$;

(iii) If $\mathbf{P}, \mathbf{Q} \in \mathcal{B}_+^{(n)}(H)$ *and there exist the constants* $M > m > 0$ *such that* $M\mathbf{Q} \geq \mathbf{P} \geq m\mathbf{Q}$ *then*

$$M\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{Q}) \geq \sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{P}) \geq m\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{Q}) \,(\geq 0). \tag{3.60}$$

If $\mathbf{P} = (p_1 1_H, ..., p_n 1_H)$ with $p_k \geq 0$, $k \in \{1, ..., n\}$ then the functional of real nonnegative weights $\mathbf{p} = (p_1, ..., p_n)$ defined by

$$\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{p}) := \left[\operatorname{tr}\left(\sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2\alpha}\right)\right]^{1/2} \times \left[\operatorname{tr}\left(\sum_{k=1}^{n} p_k |A_k|^2 |T_k^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(\sum_{k=1}^{n} p_k |A_k|^2 T_k\right)\right| \tag{3.61}$$

has the same properties as in Theorem 3.7.

Moreover, we have the simple bounds

$$\max_{k\in\{1,\dots,n\}}\{p_k\}\left(\left[\operatorname{tr}\left(\sum_{k=1}^{n}|A_k|^2|T_k|^{2\alpha}\right)\right]^{1/2}\right.$$
$$\times\left[\operatorname{tr}\left(\sum_{k=1}^{n}|A_k|^2|T_k^*|^{2(1-\alpha)}\right)\right]^{1/2}-\left.\left|\operatorname{tr}\left(\sum_{k=1}^{n}p_k|A_k|^2T_k\right)\right|\right)$$
$$\geq\left[\operatorname{tr}\left(\sum_{k=1}^{n}p_k|A_k|^2|T_k|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{k=1}^{n}p_k|A_k|^2|T_k^*|^{2(1-\alpha)}\right)\right]^{1/2}$$
$$-\left|\operatorname{tr}\left(\sum_{k=1}^{n}p_k|A_k|^2T_k\right)\right|$$
$$\geq\min_{k\in\{1,\dots,n\}}\{p_k\}\left(\left[\operatorname{tr}\left(\sum_{k=1}^{n}|A_k|^2|T_k|^{2\alpha}\right)\right]^{1/2}\right.$$
$$\times\left[\operatorname{tr}\left(\sum_{k=1}^{n}|A_k|^2|T_k^*|^{2(1-\alpha)}\right)\right]^{1/2}-\left.\left|\operatorname{tr}\left(\sum_{k=1}^{n}p_k|A_k|^2T_k\right)\right|\right). \tag{3.62}$$

3.5. Further Inequalities for Power Series. We have the following version of Kato's inequality for functions defined by power series:

THEOREM 3.11 (Dragomir, 2014, [**65**]). *Let $f(\lambda):=\sum_{n=1}^{\infty}\alpha_n\lambda^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R>0$. Let $N\in\mathcal{B}(H)$ be a normal operator. If for some $\alpha\in(0,1)$, $|N|^{2\alpha}$, $|N|^{2(1-\alpha)}\in\mathcal{B}_1(H)$ with $\operatorname{tr}\left(|N|^{2\alpha}\right)$, $\operatorname{tr}\left(|N|^{2(1-\alpha)}\right)<R$, then*

$$|\operatorname{tr}(f(N))|^2\leq\operatorname{tr}\left(f_a\left(|N|^{2\alpha}\right)\right)\operatorname{tr}\left(f_a\left(|N|^{2(1-\alpha)}\right)\right). \tag{3.63}$$

PROOF. Since N is a normal operator, then for any natural number $k\geq1$ we have $\left|N^k\right|^{2\alpha}=|N|^{2\alpha k}$ and $\left|N^k\right|^{2(1-\alpha)}=|N|^{2(1-\alpha)k}$.

By the generalized triangle inequality for the modulus we have for $n\geq2$

$$\left|\operatorname{tr}\left(\sum_{k=1}^{n}\alpha_kN^k\right)\right|=\left|\sum_{k=1}^{n}\alpha_k\operatorname{tr}\left(N^k\right)\right|\leq\sum_{k=1}^{n}|\alpha_k|\left|\operatorname{tr}\left(N^k\right)\right|. \tag{3.64}$$

If for some $\alpha\in(0,1)$ we have $|N|^{2\alpha}$, $|N|^{2(1-\alpha)}\in\mathcal{B}_1(H)$, then by Corollary 3.3 we have $N\in\mathcal{B}_1(H)$. Now, since N, $|N|^{2\alpha}$, $|N|^{2(1-\alpha)}\in\mathcal{B}_1(H)$ then any natural power of these operators belong to $\mathcal{B}_1(H)$ and by (3.25) we have

$$\left|\operatorname{tr}\left(N^k\right)\right|^2\leq\operatorname{tr}\left(|N|^{2\alpha k}\right)\operatorname{tr}\left(|N|^{2(1-\alpha)k}\right), \tag{3.65}$$

for any natural number $k\geq1$.

Making use of (3.65) we have

$$\sum_{k=1}^{n}|\alpha_k|\left|\operatorname{tr}\left(N^k\right)\right|\leq\sum_{k=1}^{n}|\alpha_k|\left(\operatorname{tr}\left(|N|^{2\alpha k}\right)\right)^{1/2}\left(\operatorname{tr}\left(|N|^{2(1-\alpha)k}\right)\right)^{1/2}. \tag{3.66}$$

Utilising the weighted Cauchy-Bunyakovsky-Schwarz inequality for sums we also have

$$\begin{aligned}(3.67)\qquad &\sum_{k=1}^{n}|\alpha_k|\left(\operatorname{tr}\left(|N|^{2\alpha k}\right)\right)^{1/2}\left(\operatorname{tr}\left(|N|^{2(1-\alpha)k}\right)\right)^{1/2}\\ &\leq\left[\sum_{k=1}^{n}|\alpha_k|\left(\left(\operatorname{tr}\left(|N|^{2\alpha k}\right)\right)^{1/2}\right)^2\right]^{1/2}\\ &\times\left[\sum_{k=1}^{n}|\alpha_k|\left(\left(\operatorname{tr}\left(|N|^{2(1-\alpha)k}\right)\right)^{1/2}\right)^2\right]^{1/2}\\ &=\left[\sum_{k=1}^{n}|\alpha_k|\operatorname{tr}\left(|N|^{2\alpha k}\right)\right]^{1/2}\left[\sum_{k=1}^{n}|\alpha_k|\operatorname{tr}\left(|N|^{2(1-\alpha)k}\right)\right]^{1/2}.\end{aligned}$$

Making use of (3.64), (3.66) and (3.67) we get the inequality

$$(3.68)\qquad \left|\operatorname{tr}\left(\sum_{k=1}^{n}\alpha_k N^k\right)\right|^2\leq\operatorname{tr}\left(\sum_{k=1}^{n}|\alpha_k|\,|N|^{2\alpha k}\right)\operatorname{tr}\left(\sum_{k=1}^{n}|\alpha_k|\,|N|^{2(1-\alpha)k}\right)$$

for any $n\geq 2$.

Due to the fact that $\operatorname{tr}\left(|N|^{2\alpha}\right),\operatorname{tr}\left(|N|^{2(1-\alpha)}\right)<R$ it follows by (3.25) that $\operatorname{tr}(|N|)<R$ and the operator series

$$\sum_{k=1}^{\infty}\alpha_k N^k,\ \sum_{k=1}^{\infty}|\alpha_k|\,|N|^{2\alpha k}\text{ and }\sum_{k=1}^{\infty}|\alpha_k|\,|N|^{2(1-\alpha)k}$$

are convergent in the Banach space $\mathcal{B}_1(H)$.

Taking the limit over $n\to\infty$ in (3.68) and using the continuity of the $\operatorname{tr}(\cdot)$ on $\mathcal{B}_1(H)$ we deduce the desired result (3.63). ∎

EXAMPLE 3.1. *a) If we take in* $f(\lambda)=(1\pm\lambda)^{-1}-1=\mp\lambda\left((1\pm\lambda)^{-1}\right)$*,* $|\lambda|<1$ *then we get from (3.63) the inequality*

$$\begin{aligned}(3.69)\qquad &\left|\operatorname{tr}\left(N\left((1_H\pm N)^{-1}\right)\right)\right|^2\\ &\leq\operatorname{tr}\left(|N|^{2\alpha}\left(1_H-|N|^{2\alpha}\right)^{-1}\right)\operatorname{tr}\left(|N|^{2(1-\alpha)}\left(1_H-|N|^{2(1-\alpha)}\right)^{-1}\right),\end{aligned}$$

provided that $N\in\mathcal{B}(H)$ *is a normal operator and for* $\alpha\in(0,1)$, $|N|^{2\alpha},|N|^{2(1-\alpha)}\in\mathcal{B}_1(H)$ *with* $\operatorname{tr}\left(|N|^{2\alpha}\right),\operatorname{tr}\left(|N|^{2(1-\alpha)}\right)<1$.

b) If we take in (3.63) $f(\lambda)=\exp(\lambda)-1$*,* $\lambda\in\mathbb{C}$ *then we get the inequality*

$$(3.70)\qquad |\operatorname{tr}(\exp(N)-1_H)|^2\leq\operatorname{tr}\left(\exp\left(|N|^{2\alpha}\right)-1_H\right)\operatorname{tr}\left(\exp\left(|N|^{2(1-\alpha)}\right)-1_H\right),$$

provided that $N\in\mathcal{B}(H)$ *is a normal operator and for* $\alpha\in(0,1)$*,* $|N|^{2\alpha},|N|^{2(1-\alpha)}\in\mathcal{B}_1(H)$.

The following result also holds:

THEOREM 3.12 (Dragomir, 2014, [**65**]). *Let* $f(\lambda):=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ *be a power series with complex coefficients and convergent on the open disk* $D(0,R)$, $R>0$. *If* $T\in\mathcal{B}(H)$, $A\in$

$\mathcal{B}_2(H)$ *are normal operators that double commute, i.e.* $TA = AT$ *and* $TA^* = A^*T$ *and* $\operatorname{tr}\left(|A|^2|T|^{2\alpha}\right), \operatorname{tr}\left(|A|^2|T|^{2(1-\alpha)}\right) < R$ *for some* $\alpha \in [0,1]$, *then*

$$\left|\operatorname{tr}\left(f\left(|A|^2 T\right)\right)\right|^2 \leq \operatorname{tr}\left(f_a\left(|A|^2|T|^{2\alpha}\right)\right)\operatorname{tr}\left(f_a\left(|A|^2|T|^{2(1-\alpha)}\right)\right). \tag{3.71}$$

PROOF. From the inequality (3.56) we have

$$\left|\operatorname{tr}\left(\sum_{k=0}^{n}\alpha_k\left|A^k\right|^2T^k\right)\right|^2 \leq \operatorname{tr}\left(\sum_{k=0}^{n}|\alpha_k|\left|A^k\right|^2\left|T^k\right|^{2\alpha}\right)\operatorname{tr}\left(\sum_{k=0}^{n}|\alpha_k|\left|A^k\right|^2\left|T^k\right|^{2(1-\alpha)}\right). \tag{3.72}$$

Since A and T are normal operators, then $\left|A^k\right|^2 = |A|^{2k}$, $\left|T^k\right|^{2\alpha} = |T|^{2\alpha k}$ and $\left|T^k\right|^{2(1-\alpha)} = |T|^{2(1-\alpha)k}$ for any natural number $k \geq 0$ and $\alpha \in [0,1]$.

Since T and A double commute, then is easy to see that

$$|A|^{2k}T^k = \left(|A|^2T\right)^k,\ |A|^{2k}|T|^{2\alpha k} = \left(|A|^2|T|^{2\alpha}\right)^k$$

and

$$|A|^{2k}|T|^{2(1-\alpha)k} = \left(|A|^2|T|^{2(1-\alpha)}\right)^k$$

for any natural number $k \geq 0$ and $\alpha \in [0,1]$.

Therefore (3.72) is equivalent to

$$\left|\operatorname{tr}\left(\sum_{k=0}^{n}\alpha_k\left(|A|^2T\right)^k\right)\right|^2 \leq \operatorname{tr}\left(\sum_{k=0}^{n}|\alpha_k|\left(|A|^2|T|^{2\alpha}\right)^k\right)\operatorname{tr}\left(\sum_{k=0}^{n}|\alpha_k|\left(|A|^2|T|^{2(1-\alpha)}\right)^k\right), \tag{3.73}$$

for any natural number $n \geq 1$and $\alpha \in [0,1]$.

Due to the fact that $\operatorname{tr}\left(|A|^2|T|^{2\alpha}\right), \operatorname{tr}\left(|A|^2|T|^{2(1-\alpha)}\right) < R$ it follows by (3.56) for $n = 1$ that $\operatorname{tr}\left(|A|^2T\right) < R$ and the operator series

$$\sum_{k=1}^{\infty}\alpha_kN^k,\ \sum_{k=1}^{\infty}|\alpha_k||N|^{2\alpha k} \text{ and } \sum_{k=1}^{\infty}|\alpha_k||N|^{2(1-\alpha)k}$$

are convergent in the Banach space $\mathcal{B}_1(H)$.

Taking the limit over $n \to \infty$ in (3.73) and using the continuity of the $\operatorname{tr}(\cdot)$ on $\mathcal{B}_1(H)$ we deduce the desired result (3.71). ∎

EXAMPLE 3.2. *a) If we take* $f(\lambda) = (1 \pm \lambda)^{-1}$, $|\lambda| < 1$ *then we get from (3.71) the inequality*

$$\left|\operatorname{tr}\left(\left(1_H \pm |A|^2T\right)^{-1}\right)\right|^2 \leq \operatorname{tr}\left(\left(1_H - |A|^2|T|^{2\alpha}\right)^{-1}\right)\operatorname{tr}\left(\left(1_H - |A|^2|T|^{2(1-\alpha)}\right)^{-1}\right), \tag{3.74}$$

provided that $T \in \mathcal{B}(H)$, $A \in \mathcal{B}_2(H)$ *are normal operators that double commute and* $\operatorname{tr}\left(|A|^2|T|^{2\alpha}\right), \operatorname{tr}\left(|A|^2|T|^{2(1-\alpha)}\right) < 1$ *for* $\alpha \in [0,1]$.

b) If we take in (3.71) $f(\lambda) = \exp(\lambda)$, $\lambda \in \mathbb{C}$ *then we get the inequality*

$$\left|\operatorname{tr}\left(\exp\left(|A|^2 T\right)\right)\right|^2 \leq \operatorname{tr}\left(\exp\left(|A|^2 |T|^{2\alpha}\right)\right) \operatorname{tr}\left(\exp\left(|A|^2 |T|^{2(1-\alpha)}\right)\right), \tag{3.75}$$

provided that $T \in \mathcal{B}(H)$ *and* $A \in \mathcal{B}_2(H)$ *are normal operators that double commute and* $\alpha \in [0,1]$.

THEOREM 3.13 (Dragomir, 2014, [**65**]). *Let* $f(z) := \sum_{j=0}^{\infty} p_j z^j$ *and* $g(z) := \sum_{j=0}^{\infty} q_j z^j$ *be two power series with nonnegative coefficients and convergent on the open disk* $D(0,R)$, $R > 0$. *If* $T \in \mathcal{B}(H)$, $A \in \mathcal{B}_2(H)$ *are normal operators that double commute and* $\operatorname{tr}\left(|A|^2 |T|^{2\alpha}\right)$, $\operatorname{tr}\left(|A|^2 |T|^{2(1-\alpha)}\right) < R$ *for* $\alpha \in [0,1]$, *then*

$$\begin{aligned} &\left[\operatorname{tr}\left(f\left(|A|^2 |T|^{2\alpha}\right) + g\left(|A|^2 |T|^{2\alpha}\right)\right)\right]^{1/2} \\ &\times \left[\operatorname{tr}\left(f\left(|A|^2 |T|^{2(1-\alpha)}\right) + g\left(|A|^2 |T|^{2(1-\alpha)}\right)\right)\right]^{1/2} \\ &- \left|\operatorname{tr}\left(f\left(|A|^2 T\right) + g\left(|A|^2 T\right)\right)\right| \\ &\geq \left[\operatorname{tr}\left(f\left(|A|^2 |T|^{2\alpha}\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(f\left(|A|^2 |T|^{2(1-\alpha)}\right)\right)\right]^{1/2} \\ &- \left|\operatorname{tr}\left(f\left(|A|^2 T\right)\right)\right| \\ &+ \left[\operatorname{tr}\left(g\left(|A|^2 |T|^{2\alpha}\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(g\left(|A|^2 |T|^{2(1-\alpha)}\right)\right)\right]^{1/2} \\ &- \left|\operatorname{tr}\left(g\left(|A|^2 T\right)\right)\right| (\geq 0). \end{aligned} \tag{3.76}$$

Moreover, if $p_j \geq q_j$ *for any* $j \in \mathbb{N}$, *then, with the above assumptions on* T *and* A,

$$\begin{aligned} &\left[\operatorname{tr}\left(f\left(|A|^2 |T|^{2\alpha}\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(f\left(|A|^2 |T|^{2(1-\alpha)}\right)\right)\right]^{1/2} \\ &- \left|\operatorname{tr}\left(f\left(|A|^2 T\right)\right)\right| \\ &\geq \left[\operatorname{tr}\left(g\left(|A|^2 |T|^{2\alpha}\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(g\left(|A|^2 |T|^{2(1-\alpha)}\right)\right)\right]^{1/2} \\ &- \left|\operatorname{tr}\left(g\left(|A|^2 T\right)\right)\right| (\geq 0). \end{aligned} \tag{3.77}$$

The proof follows in a similar way to the proof of Theorem 3.12 by making use of the superadditivity and monotonicity properties of the functional $\sigma_{\mathbf{A},\mathbf{T},\alpha}(\cdot)$. We omit the details.

EXAMPLE 3.3. *Now, observe that if we take*

$$f(\lambda) = \sinh \lambda = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1}$$

and

$$g(\lambda) = \cosh \lambda = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n}$$

then

$$f(\lambda) + g(\lambda) = \exp \lambda = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^{n}$$

for any $\lambda \in \mathbb{C}$.

If $T\in\mathcal{B}(H)$, $A\in\mathcal{B}_2(H)$ are normal operators that double commute and $\alpha\in[0,1]$, then by (3.76)

$$\begin{aligned}
(3.78)\qquad & \left[\operatorname{tr}\left(\exp\left(|A|^2|T|^{2\alpha}\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(\exp\left(|A|^2|T|^{2(1-\alpha)}\right)\right)\right]^{1/2}\\
&-\left|\operatorname{tr}\left(\exp\left(|A|^2T\right)\right)\right|\\
&\geq\left[\operatorname{tr}\left(\sinh\left(|A|^2|T|^{2\alpha}\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(\sinh\left(|A|^2|T|^{2(1-\alpha)}\right)\right)\right]^{1/2}\\
&-\left|\operatorname{tr}\left(\sinh\left(|A|^2T\right)\right)\right|\\
&+\left[\operatorname{tr}\left(\cosh\left(|A|^2|T|^{2\alpha}\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(\cosh\left(|A|^2|T|^{2(1-\alpha)}\right)\right)\right]^{1/2}\\
&-\left|\operatorname{tr}\left(\cosh\left(|A|^2T\right)\right)\right|(\geq 0).
\end{aligned}$$

Now, consider the series $\frac{1}{1-\lambda}=\sum_{n=0}^{\infty}\lambda^n$, $\lambda\in D(0,1)$ and $\ln\frac{1}{1-\lambda}=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n$, $\lambda\in D(0,1)$ and define $p_n=1$, $n\geq 0$, $q_0=0$, $q_n=\frac{1}{n}$, $n\geq 1$, then we observe that for any $n\geq 0$, $p_n\geq q_n$.

If $T\in\mathcal{B}(H)$, $A\in\mathcal{B}_2(H)$ are normal operators that double commute, $\alpha\in[0,1]$ and $\operatorname{tr}\left(|A|^2|T|^{2\alpha}\right)$, $\operatorname{tr}\left(|A|^2|T|^{2(1-\alpha)}\right)<1$, then by (3.77)

$$\begin{aligned}
(3.79)\qquad & \left[\operatorname{tr}\left(\left(1_H-|A|^2|T|^{2\alpha}\right)^{-1}\right)\right]^{1/2}\left[\operatorname{tr}\left(\left(1_H-|A|^2|T|^{2(1-\alpha)}\right)^{-1}\right)\right]^{1/2}\\
&-\left|\operatorname{tr}\left(\left(1_H-|A|^2T\right)^{-1}\right)\right|\\
&\geq\left[\operatorname{tr}\left(\ln\left(1_H-|A|^2|T|^{2\alpha}\right)^{-1}\right)\right]^{1/2}\\
&\times\left[\operatorname{tr}\left(\ln\left(1_H-|A|^2|T|^{2(1-\alpha)}\right)^{-1}\right)\right]^{1/2}\\
&-\left|\operatorname{tr}\left(\ln\left(1_H-|A|^2T\right)^{-1}\right)\right|(\geq 0).
\end{aligned}$$

4. REVERSES OF SCHWARZ INEQUALITY

4.1. Some Classical Facts. Let $\bar{\mathbf{a}}=(a_1,\dots,a_n)$ and $\bar{\mathbf{b}}=(b_1,\dots,b_n)$ be two positive n*-tuples* with

$$0<m_1\leq a_i\leq M_1<\infty \text{ and } 0<m_2\leq b_i\leq M_2<\infty; \tag{4.1}$$

for each $i\in\{1,\dots,n\}$, and some constants m_1, m_2, M_1, M_2.

The following reverses of the Cauchy-Bunyakovsky-Schwarz inequality for positive sequences of real numbers are well known:

a) *Pólya-Szegö's inequality* [**120**]:

$$\frac{\sum_{k=1}^n a_k^2\sum_{k=1}^n b_k^2}{\left(\sum_{k=1}^n a_kb_k\right)^2}\leq\frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}}+\sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2.$$

b) *Shisha-Mond's inequality* [**125**]:

$$\frac{\sum_{k=1}^n a_k^2}{\sum_{k=1}^n a_kb_k}-\frac{\sum_{k=1}^n a_kb_k}{\sum_{k=1}^n b_k^2}\leq\left[\left(\frac{M_1}{m_2}\right)^{\frac{1}{2}}-\left(\frac{m_1}{M_2}\right)^{\frac{1}{2}}\right]^2.$$

If $\overline{\mathbf{w}} = (w_1, \ldots, w_n)$ is a positive sequence, then the following weighted inequalities also hold:

c) *Cassels' inequality* [**133**]. If the positive real sequences $\overline{\mathbf{a}} = (a_1, \ldots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \ldots, b_n)$ satisfy the condition

$$0 < m \leq \frac{a_k}{b_k} \leq M < \infty \text{ for each } k \in \{1, \ldots, n\}, \tag{4.2}$$

then

$$\frac{\left(\sum_{k=1}^n w_k a_k^2\right)\left(\sum_{k=1}^n w_k b_k^2\right)}{\left(\sum_{k=1}^n w_k a_k b_k\right)^2} \leq \frac{(M+m)^2}{4mM}.$$

For other recent results providing discrete reverse inequalities, see the monograph online [**33**].

The following reverse of Schwarz's inequality in inner product spaces holds [**34**].

THEOREM 4.1 (Dragomir, 2003, [**34**]). *Let $A, a \in \mathbb{C}$ and $x, y \in H$, a complex inner product space with the inner product $\langle \cdot, \cdot \rangle$. If*

$$\operatorname{Re} \langle Ay - x, x - ay \rangle \geq 0, \tag{4.3}$$

or equivalently,

$$\left\| x - \frac{a+A}{2} \cdot y \right\| \leq \frac{1}{2} |A - a| \|y\|, \tag{4.4}$$

holds, then

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{4} |A - a|^2 \|y\|^4. \tag{4.5}$$

The constant $\frac{1}{4}$ is sharp in (4.5).

In 1935, G. Grüss [**90**] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integrals means as follows:

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx \right| \leq \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma), \tag{4.6}$$

where $f, g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$ and satisfy the condition

$$\phi \leq f(x) \leq \Phi, \gamma \leq g(x) \leq \Gamma \tag{4.7}$$

for each $x \in [a, b]$, where $\phi, \Phi, \gamma, \Gamma$ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In [**25**], in order to generalize the Grüss integral inequality in abstract structures the author has proved the following inequality in inner product spaces.

THEOREM 4.2 (Dragomir, 1999, [**25**]). *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and $e \in H$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0 \text{ and } \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0 \tag{4.8}$$

hold, then

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \varphi| |\Gamma - \gamma|. \tag{4.9}$$

The constant $\frac{1}{4}$ is best possible in the sense that it can not be replaced by a smaller constant.

For other results of this type, see the recent monograph [**36**] and the references therein.

For other Grüss type results for integral and sums see the papers [**1**]-[**3**], [**13**]-[**15**], [**16**]-[**18**], [**24**]-[**69**], [**73**], [**117**], [**139**] and the references therein.

4.2. Additive Reverses of Schwarz Trace Inequality.

We denote by

$$\mathcal{B}_1^+(H) := \{P : P \in \mathcal{B}_1(H),\ P \text{ is selfadjoint and } P \geq 0\}.$$

We obtained recently the following result [**51**]:

THEOREM 4.3 (Dragomir, 2014, [**51**]). *For any* $A, C \in \mathcal{B}(H)$ *and* $P \in \mathcal{B}_1^+(H) \setminus \{0\}$

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\
&\leq \inf_{\lambda\in\mathbb{C}} \|A - \lambda \cdot 1_H\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\
&\leq \inf_{\lambda\in\mathbb{C}} \|A - \lambda \cdot 1_H\| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2},
\end{aligned} \tag{4.10}$$

where $\|\cdot\|$ *is the operator norm.*

PROOF. We observe that, for any $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}
&\frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left[P(A - \lambda 1_H)\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right] \\
&= \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left[PA\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right] \\
&- \frac{\lambda}{\operatorname{tr}(P)} \operatorname{tr}\left[P\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right] \\
&= \frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}.
\end{aligned} \tag{4.11}$$

Taking the modulus in (4.11) and utilizing the properties of the trace, we have

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\
&= \frac{1}{\operatorname{tr}(P)}\left|\operatorname{tr}\left[P(A - \lambda 1_H)\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right]\right| \\
&= \frac{1}{\operatorname{tr}(P)}\left|\operatorname{tr}\left[(A - \lambda 1_H)\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right]\right| \\
&\leq \|A - \lambda 1_H\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)
\end{aligned}$$

for any $\lambda \in \mathbb{C}$.

Utilising Schwarz's inequality we also have

$$\begin{aligned}
(4.12)\qquad & \operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\
&=\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}P^{1/2}\right|\right)\\
&\le\left[\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right|^2\right)\right]^{1/2}[\operatorname{tr}(P)]^{1/2}.
\end{aligned}$$

Observe that

$$\begin{aligned}
(4.13)\qquad & \operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right|^2\right)\\
&=\operatorname{tr}\left(\left(\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right)^*\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right)\\
&=\operatorname{tr}\left(P^{1/2}\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)^*\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right)\\
&=\operatorname{tr}\left(\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)^*\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right)\\
&=\operatorname{tr}\left(\left(C^*-\frac{\overline{\operatorname{tr}(PC)}}{\operatorname{tr}(P)}1_H\right)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right)\\
&=\operatorname{tr}\left[\left(|C|^2-\frac{\overline{\operatorname{tr}(PC)}}{\operatorname{tr}(P)}C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}C^*+\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2 1_H\right)P\right]\\
&=\left(\frac{\operatorname{tr}\left(|C|^2P\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right)\operatorname{tr}(P).
\end{aligned}$$

By (4.12) and (4.13) we get

$$\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\le\left(\frac{\operatorname{tr}\left(|C|^2P\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right)^{1/2}\operatorname{tr}(P)$$

and by (4.24) we have

$$\begin{aligned}
(4.14)\qquad & \left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\\
&\le\|A-\lambda\cdot 1_H\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\
&\le\|A-\lambda\cdot 1_H\|\left(\frac{\operatorname{tr}\left(|C|^2P\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right)^{1/2}
\end{aligned}$$

for any $\lambda\in\mathbb{C}$.

Taking the infimum over $\lambda\in\mathbb{C}$ in (4.14) we get the desired result (4.23). ∎

We also have [**51**]:

COROLLARY 4.4. *Let* $\alpha, \beta \in \mathbb{C}$ *and* $A \in B(H)$ *such that*

$$\left\| A - \frac{\alpha+\beta}{2} \cdot 1_H \right\| \le \frac{1}{2} |\beta - \alpha| .$$

For any $C \in \mathcal{B}(H)$ *and* $P \in \mathcal{B}_1^+(H) \setminus \{0\}$

$$\begin{aligned} &\left| \frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \\ &\le \frac{1}{2} |\beta - \alpha| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right) \\ &\le \frac{1}{2} |\beta - \alpha| \left[\frac{\operatorname{tr}\left(P |C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \right]^{1/2} . \end{aligned} \tag{4.15}$$

In particular, if $C \in \mathcal{B}(H)$ *is such that*

$$\left\| C - \frac{\alpha+\beta}{2} \cdot 1_H \right\| \le \frac{1}{2} |\beta - \alpha| ,$$

then

$$\begin{aligned} 0 &\le \frac{\operatorname{tr}\left(P |C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \\ &\le \frac{1}{2} |\beta - \alpha| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right) \\ &\le \frac{1}{2} |\beta - \alpha| \left[\frac{\operatorname{tr}\left(P |C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \right]^{1/2} \le \frac{1}{4} |\beta - \alpha|^2 . \end{aligned} \tag{4.16}$$

Also

$$\begin{aligned} &\left| \frac{\operatorname{tr}(PC^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right)^2 \right| \\ &\le \frac{1}{2} |\beta - \alpha| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right) \\ &\le \frac{1}{2} |\beta - \alpha| \left[\frac{\operatorname{tr}\left(P |C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \right]^{1/2} \le \frac{1}{4} |\beta - \alpha|^2 . \end{aligned} \tag{4.17}$$

For other related results see [**51**].
In order to simplify writing, we use the following notation

$$\mathcal{B}_+(H) := \{ P \in \mathcal{B}(H) ,\ P \text{ is selfadjoint and } P \ge 0 \} .$$

The following result holds:

THEOREM 4.5 (Dragomir, 2014, [**54**]). *Let, either* $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ *and* $\gamma, \Gamma \in \mathbb{C}$.

(i) We have

$$\begin{aligned}
0 &\leq \operatorname{tr}\left(P\left|A\right|^{2}\right)\operatorname{tr}\left(P\left|B\right|^{2}\right)-\left|\operatorname{tr}\left(PB^{*}A\right)\right|^{2} \\
&= \operatorname{Re}\left[\left(\Gamma\operatorname{tr}\left(P\left|B\right|^{2}\right)-\operatorname{tr}\left(PB^{*}A\right)\right)\left(\operatorname{tr}\left(PA^{*}B\right)-\overline{\gamma}\operatorname{tr}\left(P\left|B\right|^{2}\right)\right)\right] \\
&- \operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^{*}-\overline{\gamma}B^{*}\right)\left(\Gamma B-A\right)\right]\right) \\
&\leq \frac{1}{4}\left|\Gamma-\gamma\right|^{2}\left[\operatorname{tr}\left(P\left|B\right|^{2}\right)\right]^{2} \\
&- \operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^{*}-\overline{\gamma}B^{*}\right)\left(\Gamma B-A\right)\right]\right).
\end{aligned} \tag{4.18}$$

(ii) If

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^{*}-\overline{\gamma}B^{*}\right)\left(\Gamma B-A\right)\right]\right)\geq 0 \tag{4.19}$$

or, equivalently

$$\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}B\right|^{2}\right)\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}\operatorname{tr}\left(P\left|B\right|^{2}\right), \tag{4.20}$$

then

$$\begin{aligned}
0 &\leq \operatorname{tr}\left(P\left|A\right|^{2}\right)\operatorname{tr}\left(P\left|B\right|^{2}\right)-\left|\operatorname{tr}\left(PB^{*}A\right)\right|^{2} \\
&\leq \operatorname{Re}\left[\left(\Gamma\operatorname{tr}\left(P\left|B\right|^{2}\right)-\operatorname{tr}\left(PB^{*}A\right)\right)\left(\operatorname{tr}\left(PA^{*}B\right)-\overline{\gamma}\operatorname{tr}\left(P\left|B\right|^{2}\right)\right)\right] \\
&\leq \frac{1}{4}\left|\Gamma-\gamma\right|^{2}\left[\operatorname{tr}\left(P\left|B\right|^{2}\right)\right]^{2}
\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}
0 &\leq \operatorname{tr}\left(P\left|A\right|^{2}\right)\operatorname{tr}\left(P\left|B\right|^{2}\right)-\left|\operatorname{tr}\left(PB^{*}A\right)\right|^{2} \\
&\leq \frac{1}{4}\left|\Gamma-\gamma\right|^{2}\left[\operatorname{tr}\left(P\left|B\right|^{2}\right)\right]^{2} \\
&- \operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^{*}-\overline{\gamma}B^{*}\right)\left(\Gamma B-A\right)\right]\right) \\
&\leq \frac{1}{4}\left|\Gamma-\gamma\right|^{2}\left[\operatorname{tr}\left(P\left|B\right|^{2}\right)\right]^{2}.
\end{aligned} \tag{4.22}$$

PROOF. Observe that, by the trace properties, we have

$$\begin{aligned}
I_{1} &:= \operatorname{Re}\left[\left(\Gamma\operatorname{tr}\left(P\left|B\right|^{2}\right)-\operatorname{tr}\left(PB^{*}A\right)\right)\left(\operatorname{tr}\left(PA^{*}B\right)-\overline{\gamma}\operatorname{tr}\left(P\left|B\right|^{2}\right)\right)\right] \\
&= \operatorname{Re}\left[\left(\Gamma\operatorname{tr}\left(P\left|B\right|^{2}\right)-\operatorname{tr}\left(PB^{*}A\right)\right)\left(\overline{\operatorname{tr}\left(PB^{*}A\right)}-\overline{\gamma}\operatorname{tr}\left(P\left|B\right|^{2}\right)\right)\right] \\
&= \operatorname{Re}\left[\Gamma\operatorname{tr}\left(P\left|B\right|^{2}\right)\overline{\operatorname{tr}\left(PB^{*}A\right)}+\overline{\gamma}\operatorname{tr}\left(PB^{*}A\right)\operatorname{tr}\left(P\left|B\right|^{2}\right)\right. \\
&- \left.\left|\operatorname{tr}\left(PB^{*}A\right)\right|^{2}-\Gamma\overline{\gamma}\left[\operatorname{tr}\left(P\left|B\right|^{2}\right)\right]^{2}\right] \\
&= \operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{Re}\left[\Gamma\overline{\operatorname{tr}\left(PB^{*}A\right)}+\overline{\gamma}\operatorname{tr}\left(PB^{*}A\right)\right] \\
&- \left|\operatorname{tr}\left(PB^{*}A\right)\right|^{2}-\left[\operatorname{tr}\left(P\left|B\right|^{2}\right)\right]^{2}\operatorname{Re}\left(\Gamma\overline{\gamma}\right)
\end{aligned} \tag{4.23}$$

and

$$
\begin{aligned}
I_2 &:= \operatorname{tr}\left(P|B|^2\right)\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}B^*\right)(\Gamma B-A)\right]\right)\\
&= \operatorname{tr}\left(P|B|^2\right)\operatorname{Re}\left[\operatorname{tr}\left(\Gamma PA^*B+\overline{\gamma}PB^*A-\overline{\gamma}\Gamma PB^*B-PA^*A\right)\right]\\
&= \operatorname{tr}\left(P|B|^2\right)\operatorname{Re}\left[\Gamma\operatorname{tr}\left(PA^*B\right)+\overline{\gamma}\operatorname{tr}\left(PB^*A\right)\right]\\
&-\overline{\gamma}\Gamma\operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}\left(P|A|^2\right)\Big]\\
&= \operatorname{tr}\left(P|B|^2\right)\operatorname{Re}\left[\Gamma\overline{\operatorname{tr}\left(PB^*A\right)}+\overline{\gamma}\operatorname{tr}\left(PB^*A\right)\right]\\
&-\left[\operatorname{tr}\left(P|B|^2\right)\right]^2\operatorname{Re}\left(\overline{\gamma}\Gamma\right)-\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right),
\end{aligned}
$$

for P a selfadjoint operator with $P\geq 0$, A, $B\in\mathcal{B}_2(H)$ and $\gamma,\Gamma\in\mathbb{C}$.

Then we have

$$
I_1-I_2=\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right)-\left|\operatorname{tr}\left(PB^*A\right)\right|^2,
$$

which proves the equality in (4.18).

Utilising the elementary inequality for complex numbers

$$
\operatorname{Re}\left(u\overline{v}\right)\leq\frac{1}{4}|u+v|^2,\ u,v\in\mathbb{C},
$$

we have

$$
\begin{aligned}
&\operatorname{Re}\left[\left(\Gamma\operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}\left(PB^*A\right)\right)\left(\operatorname{tr}\left(PA^*B\right)-\overline{\gamma}\operatorname{tr}\left(P|B|^2\right)\right)\right]\\
&=\operatorname{Re}\left[\left(\Gamma\operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}\left(PB^*A\right)\right)\left(\overline{\operatorname{tr}\left(PB^*A\right)-\gamma\operatorname{tr}\left(P|B|^2\right)}\right)\right]\\
&\leq\frac{1}{4}\left[\Gamma\operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}\left(PB^*A\right)+\operatorname{tr}\left(PB^*A\right)-\gamma\operatorname{tr}\left(P|B|^2\right)\right]^2\\
&=\frac{1}{4}|\Gamma-\gamma|^2\left[\operatorname{tr}\left(P|B|^2\right)\right]^2,
\end{aligned}
$$

which proves the last inequality in (4.18).

We have the equalities

$$
\begin{aligned}
&\frac{1}{4}|\Gamma-\gamma|^2P|B|^2-P\left|A-\frac{\gamma+\Gamma}{2}B\right|^2 \qquad (4.24)\\
&=P\left[\frac{1}{4}|\Gamma-\gamma|^2|B|^2-\left|A-\frac{\gamma+\Gamma}{2}B\right|^2\right]\\
&=P\left[\frac{1}{4}|\Gamma-\gamma|^2|B|^2-\left(A-\frac{\gamma+\Gamma}{2}B\right)^*\left(A-\frac{\gamma+\Gamma}{2}B\right)\right]\\
&=P\left[\frac{1}{4}|\Gamma-\gamma|^2|B|^2\right.\\
&\left.-|A|^2+\frac{\overline{\gamma+\Gamma}}{2}B^*A+\frac{\gamma+\Gamma}{2}A^*B-\left|\frac{\gamma+\Gamma}{2}\right|^2|B|^2\right]
\end{aligned}
$$

$$= P\left[-|A|^2 + \frac{\overline{\gamma+\Gamma}}{2}B^*A + \frac{\gamma+\Gamma}{2}A^*B + \left(\frac{1}{4}|\Gamma-\gamma|^2 - \left|\frac{\gamma+\Gamma}{2}\right|^2\right)|B|^2\right]$$
$$= P\left[-|A|^2 + \frac{\overline{\gamma+\Gamma}}{2}B^*A + \frac{\gamma+\Gamma}{2}A^*B - \operatorname{Re}(\Gamma\overline{\gamma})|B|^2\right]$$

for any bounded operators A, B, P and the complex numbers $\gamma, \Gamma \in \mathbb{C}$.

Let P be a selfadjoint operator with $P \geq 0$, A, $B \in \mathcal{B}_2(H)$ and $\gamma, \Gamma \in \mathbb{C}$. Taking the trace in (4.24) we get

$$\frac{1}{4}|\Gamma-\gamma|^2 \operatorname{tr}\left(P|B|^2\right) - \operatorname{tr}\left(P\left|A - \frac{\gamma+\Gamma}{2}B\right|^2\right) \tag{4.25}$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}(PB^*A) + \frac{\gamma+\Gamma}{2}\operatorname{tr}(PA^*B)$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}(PB^*A) + \frac{\gamma+\Gamma}{2}\overline{\operatorname{tr}(PB^*A)}$$

$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}(PB^*A) + \overline{\frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}(PB^*A)}$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + 2\operatorname{Re}\left[\frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}(PB^*A)\right]$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right] + \operatorname{Re}\left[\overline{\Gamma}\operatorname{tr}(PB^*A)\right]$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right] + \operatorname{Re}\left[\overline{\overline{\Gamma}\operatorname{tr}(PB^*A)}\right]$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right] + \operatorname{Re}\left[\Gamma\overline{\operatorname{tr}(PB^*A)}\right].$$

Utilising the equality for I_2 above, we conclude that (4.19) holds if and only if (4.20) holds, and the inequalities (4.21) and (4.22) thus follow from (4.18).

The case $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ goes likewise and the details are omitted. ∎

For two given operators T, $U \in B(H)$ and two given scalars α, $\beta \in \mathbb{C}$ consider the transform

$$\mathcal{C}_{\alpha,\beta}(T,U) = (T^* - \bar{\alpha}U^*)(\beta U - T).$$

This transform generalizes the transform

$$\mathcal{C}_{\alpha,\beta}(T) := (T^* - \bar{\alpha}1_H)(\beta 1_H - T) = \mathcal{C}_{\alpha,\beta}(T, 1_H),$$

where 1_H is the identity operator, which has been introduced in [**38**] in order to provide some generalizations of the well known Kantorovich inequality for operators in Hilbert spaces.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty, y\rangle \geq 0$ for any $y \in H$.

Utilizing the following identity

$$\begin{aligned}(4.26)\qquad \operatorname{Re}\langle \mathcal{C}_{\alpha,\beta}(T,U)x,x\rangle &= \operatorname{Re}\langle \mathcal{C}_{\beta,\alpha}(T,U)x,x\rangle \\ &= \frac{1}{4}|\beta-\alpha|^2\|Ux\|^2 - \left\|Tx-\frac{\alpha+\beta}{2}\cdot Ux\right\|^2 \\ &= \frac{1}{4}|\beta-\alpha|^2\left\langle |U|^2 x,x\right\rangle - \left\langle \left|T-\frac{\alpha+\beta}{2}\cdot U\right|^2 x,x\right\rangle\end{aligned}$$

that holds for any scalars α, β and any vector $x \in H$, we can give a simple characterization result that is useful in the following:

LEMMA 4.6. *For $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$ the following statements are equivalent:*

(i) *The transform $\mathcal{C}_{\alpha,\beta}(T,U)$ (or, equivalently, $\mathcal{C}_{\beta,\alpha}(T,U)$) is accretive;*
(ii) *We have the norm inequality*

$$(4.27)\qquad \left\|Tx-\frac{\alpha+\beta}{2}\cdot Ux\right\| \le \frac{1}{2}|\beta-\alpha|\,\|Ux\|$$

for any $x \in H$;
(iii) *We have the following inequality in the operator order*

$$\left|T-\frac{\alpha+\beta}{2}\cdot U\right|^2 \le \frac{1}{4}|\beta-\alpha|^2|U|^2.$$

As a consequence of the above lemma we can state:

COROLLARY 4.7. *Let $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$. If $\mathcal{C}_{\alpha,\beta}(T,U)$ is accretive, then*

$$(4.28)\qquad \left\|T-\frac{\alpha+\beta}{2}\cdot U\right\| \le \frac{1}{2}|\beta-\alpha|\,\|U\|.$$

REMARK 4.1. In order to give examples of linear operators $T, U \in B(H)$ and numbers α, $\beta \in \mathbb{C}$ such that the transform $\mathcal{C}_{\alpha,\beta}(T,U)$ is accretive, it suffices to select two bounded linear operator S and V and the complex numbers z, w $(w \neq 0)$ with the property that $\|Sx - zVx\| \le |w|\,\|Vx\|$ for any $x \in H$, and, by choosing $T = S$, $U = V$, $\alpha = \frac{1}{2}(z+w)$ and $\beta = \frac{1}{2}(z-w)$ we observe that T and U satisfy (4.27), i.e., $\mathcal{C}_{\alpha,\beta}(T,U)$ is accretive.

COROLLARY 4.8. *Let, either $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ and $\gamma, \Gamma \in \mathbb{C}$. If the transform $\mathcal{C}_{\gamma,\Gamma}(A,B)$ is accretive, then we have the inequalities (4.21) and (4.22).*

The case of selfadjoint operators is as follows.

COROLLARY 4.9. *Let P, A, B be selfadjoint operators with either $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ and $m, M \in \mathbb{R}$ with $M > m$. If $(A - mB)(MB - A) \ge 0$, then*

$$\begin{aligned}(4.29)\qquad 0 &\le \operatorname{tr}(PA^2)\operatorname{tr}(PB^2) - [\operatorname{tr}(PBA)]^2 \\ &\le \left[\left(M\operatorname{tr}(PB^2) - \operatorname{tr}(PBA)\right)\left(\operatorname{tr}(PAB) - m\operatorname{tr}(PB^2)\right)\right] \\ &\le \frac{1}{4}(M-m)^2\left[\operatorname{tr}(PB^2)\right]^2\end{aligned}$$

and

$$0 \le \operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right) - \left[\operatorname{tr}\left(PBA\right)\right]^2 \le \frac{1}{4}\left(M-m\right)^2\left[\operatorname{tr}\left(PB^2\right)\right]^2 - \operatorname{tr}\left(PB^2\right)\operatorname{tr}\left[P\left(A-mB\right)\left(MB-A\right)\right] \le \frac{1}{4}\left(M-m\right)^2\left[\operatorname{tr}\left(PB^2\right)\right]^2. \tag{4.30}$$

We also have the following result:

THEOREM 4.10 (Dragomir, 2014, [**54**]). *Let, either* $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ *and* $\lambda \in \mathbb{C}$.

(i) We have

$$0 \le \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right) - \left|\operatorname{tr}\left(PB^*A\right)\right|^2 = \operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A - \lambda B\right|^2\right) - \left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\lambda - \operatorname{tr}\left(PB^*A\right)\right|^2. \tag{4.31}$$

(ii) If there is $r > 0$ *such that*

$$\operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A - \lambda B\right|^2\right) \le r^2\left[\operatorname{tr}\left(P|B|^2\right)\right],$$

then we have the reverse of Schwarz inequality

$$0 \le \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right) - \left|\operatorname{tr}\left(PB^*A\right)\right|^2 \le r^2\left[\operatorname{tr}\left(P|B|^2\right)\right] - \left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\lambda - \operatorname{tr}\left(PB^*A\right)\right|^2 \le r^2\left[\operatorname{tr}\left(P|B|^2\right)\right]. \tag{4.32}$$

PROOF. Using the properties of trace, we have for $P \ge 0$, $A, B \in \mathcal{B}_2(H)$ and $\lambda \in \mathbb{C}$ that

$$\begin{aligned}
J_1 &:= \operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A - \lambda B\right|^2\right) \\
&= \operatorname{tr}\left(P\left(\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A - \lambda B\right)^*\left(\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A - \lambda B\right)\right) \\
&= \operatorname{tr}\left(P\left[\operatorname{tr}\left(P|B|^2\right)|A|^2 + |\lambda|^2|B|^2\right.\right. \\
&\quad \left.\left. -\overline{\lambda}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}B^*A - \lambda\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A^*B\right]\right) \\
&= \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right) + |\lambda|^2\operatorname{tr}\left(P|B|^2\right) \\
&\quad - \overline{\lambda}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\operatorname{tr}\left(PB^*A\right) - \lambda\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\operatorname{tr}\left(PA^*B\right) \\
&= \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right) + |\lambda|^2\operatorname{tr}\left(P|B|^2\right) \\
&\quad - \overline{\lambda}\operatorname{tr}\left(PB^*A\right)\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2} - \overline{\overline{\lambda}\operatorname{tr}\left(PB^*A\right)}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2} \\
&= \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right) + |\lambda|^2\operatorname{tr}\left(P|B|^2\right) \\
&\quad - 2\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\operatorname{Re}\left(\overline{\lambda}\operatorname{tr}\left(PB^*A\right)\right)
\end{aligned}$$

and

$$\begin{aligned} J_2 \quad : \quad &= \left| \left[\operatorname{tr} \left(P |B|^2 \right) \right]^{1/2} \lambda - \operatorname{tr} \left(PB^* A \right) \right|^2 \\ &= \left(\left[\operatorname{tr} \left(P |B|^2 \right) \right]^{1/2} \lambda - \operatorname{tr} \left(PB^* A \right) \right) \overline{\left(\left[\operatorname{tr} \left(P |B|^2 \right) \right]^{1/2} \lambda - \operatorname{tr} \left(PB^* A \right) \right)} \\ &= \operatorname{tr} \left(P |B|^2 \right) |\lambda|^2 - 2 \left[\operatorname{tr} \left(P |B|^2 \right) \right]^{1/2} \operatorname{Re} \left(\overline{\lambda} \operatorname{tr} \left(PB^* A \right) \right) + |\operatorname{tr} \left(PB^* A \right)|^2 . \end{aligned}$$

Therefore

$$\begin{aligned} & J_1 - J_2 \\ & = \operatorname{tr} \left(P \left| \left[\operatorname{tr} \left(P |B|^2 \right) \right]^{1/2} A - \lambda B \right|^2 \right) - \left| \left[\operatorname{tr} \left(P |B|^2 \right) \right]^{1/2} \lambda - \operatorname{tr} \left(PB^* A \right) \right|^2 \end{aligned}$$

and the equality (4.31) is proved.

The inequality (4.32) follows from (4.31).

The other case is similar. ∎

COROLLARY 4.11. *Let, either* $P \in \mathcal{B}_+ (H)$, $C, D \in \mathcal{B}_2 (H)$ *or* $P \in \mathcal{B}_1^+ (H)$, $C, D \in \mathcal{B} (H)$ *and* $\delta, \Delta \in \mathbb{C}$.

If

$$\operatorname{Re} \left(\operatorname{tr} \left[P \left(C^* - \overline{\delta} D^* \right) \left(\Delta D - C \right) \right] \right) \geq 0 \tag{4.33}$$

or, equivalently

$$\operatorname{tr} \left(P \left| C - \frac{\delta + \Delta}{2} D \right|^2 \right) \leq \frac{1}{4} |\Delta - \delta|^2 \operatorname{tr} \left(P |D|^2 \right), \tag{4.34}$$

then

$$\begin{aligned} 0 &\leq \operatorname{tr} \left(P |C|^2 \right) \operatorname{tr} \left(P |D|^2 \right) - |\operatorname{tr} \left(PD^* C \right)|^2 \\ &\leq \frac{1}{4} |\Delta - \delta|^2 \left[\operatorname{tr} \left(P |D|^2 \right) \right]^2 - \left| \frac{\delta + \Delta}{2} \operatorname{tr} \left(P |D|^2 \right) - \operatorname{tr} \left(PD^* C \right) \right|^2 \\ &\leq \frac{1}{4} |\Delta - \delta|^2 \left[\operatorname{tr} \left(P |D|^2 \right) \right]^2 . \end{aligned} \tag{4.35}$$

PROOF. The equivalence of the inequalities (4.33) and (4.34) follows from Theorem 4.5 (ii).

If we write the inequality (4.34) for $C = A$ and $D = B$, we have

$$\operatorname{tr} \left(P \left| A - \frac{\delta + \Delta}{2} B \right|^2 \right) \leq \frac{1}{4} |\Delta - \delta|^2 \operatorname{tr} \left(P |B|^2 \right).$$

If we multiply this inequality by $\operatorname{tr} \left(P |B|^2 \right) \geq 0$ we get

$$\begin{aligned} & \operatorname{tr} \left(P \left| \left[\operatorname{tr} \left(P |B|^2 \right) \right]^{1/2} A - \frac{\delta + \Delta}{2} \left[\operatorname{tr} \left(P |B|^2 \right) \right]^{1/2} B \right|^2 \right) \\ & \leq \frac{1}{4} |\Delta - \delta|^2 \operatorname{tr} \left(P |B|^2 \right) \operatorname{tr} \left(P |B|^2 \right) . \end{aligned} \tag{4.36}$$

Let

$$\lambda = \frac{\delta + \Delta}{2} \left[\operatorname{tr} \left(P |B|^2 \right) \right]^{1/2} \text{ and } r = \frac{1}{2} |\Delta - \delta| \left[\operatorname{tr} \left(P |B|^2 \right) \right]^{1/2} .$$

Then by (4.36) we have

$$\operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A-\lambda B\right|^2\right)\leq r^2\operatorname{tr}\left(P|B|^2\right),$$

and by (4.32) we get

$$\begin{aligned}0&\leq\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right)-\left|\operatorname{tr}\left(PB^*A\right)\right|^2\\&\leq\frac{1}{4}|\Delta-\delta|^2\left[\operatorname{tr}\left(P|B|^2\right)\right]^2-\left|\frac{\delta+\Delta}{2}\operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}\left(PB^*A\right)\right|^2\\&\leq\frac{1}{4}|\Delta-\delta|^2\left[\operatorname{tr}\left(P|B|^2\right)\right]^2,\end{aligned}$$

and the inequality (4.35) is proved. ∎

COROLLARY 4.12. *Let, either $P\in\mathcal{B}_+(H)$, C, $D\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^+(H)$, C, $D\in\mathcal{B}(H)$ and δ, $\Delta\in\mathbb{C}$. If the transform $\mathcal{C}_{\delta,\Delta}(C,D)$ is accretive, then we have the inequalities (4.35).*

The case of selfadjoint operators is as follows.

COROLLARY 4.13. *Let P, C, D be selfadjoint operators with either $P\in\mathcal{B}_+(H)$, C, $D\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^+(H)$, C, $D\in\mathcal{B}(H)$ and n, $N\in\mathbb{R}$ with $N>n$. If $(C-nD)(ND-C)\geq 0$, then*

$$\begin{aligned}0&\leq\operatorname{tr}\left(PC^2\right)\operatorname{tr}\left(PD^2\right)-\left[\operatorname{tr}\left(PDC\right)\right]^2\\&\leq\frac{1}{4}(N-n)^2\left[\operatorname{tr}\left(PD^2\right)\right]^2-\left(\frac{n+N}{2}\operatorname{tr}\left(PD^2\right)-\operatorname{tr}\left(PDC\right)\right)^2\\&\leq\frac{1}{4}(N-n)^2\left[\operatorname{tr}\left(PD^2\right)\right]^2.\end{aligned}\tag{4.37}$$

4.3. Trace Inequalities of Grüss Type. Let P be a selfadjoint operator with $P\geq 0$. The functional $\langle\cdot,\cdot\rangle_{2,P}$ defined by

$$\langle A,B\rangle_{2,P}:=\operatorname{tr}\left(PB^*A\right)=\operatorname{tr}\left(APB^*\right)=\operatorname{tr}\left(B^*AP\right)$$

is a *nonnegative Hermitian form* on $\mathcal{B}_2(H)$.

For a pair of complex numbers (α,β) and $P\in\mathcal{B}_+(H)$, in order to simplify the notations, we say that the pair of operators $(U,V)\in\mathcal{B}_2(H)\times\mathcal{B}_2(H)$ has the *trace P-(α,β)-property* if

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(U^*-\overline{\alpha}V^*\right)\left(\beta V-U\right)\right]\right)\geq 0$$

or, equivalently

$$\operatorname{tr}\left(P\left|U-\frac{\alpha+\beta}{2}V\right|^2\right)\leq\frac{1}{4}|\beta-\alpha|^2\operatorname{tr}\left(P|V|^2\right).$$

The above definitions can be also considered in the case when $P\in\mathcal{B}_1^+(H)$ and A, $B\in\mathcal{B}(H)$.

THEOREM 4.14 (Dragomir, 2014, [**54**]). *Let, either $P\in\mathcal{B}_+(H)$, A, B, $C\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^+(H)$, A, B, $C\in\mathcal{B}(H)$ and λ, Γ, δ, $\Delta\in\mathbb{C}$. If (A,C) has the trace P-(λ,Γ)-property* *and* (B,C) *has the trace* P-(δ,Δ)*-property, then*

$$
\begin{aligned}
(4.38)\qquad & \left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right| \\
&\le \operatorname{tr}\left(P|C|^2\right)\left[\frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\operatorname{tr}\left(P|C|^2\right)\right. \\
&-\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}C^*\right)(\Gamma C-A)\right]\right)\right]^{1/2} \\
&\left.\times\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C-B)\right]\right)\right]^{1/2}\right] \\
&\le \frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\left[\operatorname{tr}\left(P|C|^2\right)\right]^2.
\end{aligned}
$$

PROOF. We prove in the case that $P\in\mathcal{B}_+(H)$ and $A,\,B,\,C\in\mathcal{B}_2(H)$.

Making use of the Schwarz inequality for the nonnegative hermitian form $\langle\cdot,\cdot\rangle_{2,P}$ we have

$$\left|\langle A,B\rangle_{2,P}\right|^2\le\langle A,A\rangle_{2,P}\langle B,B\rangle_{2,P}$$

for any $A,\,B\in\mathcal{B}_2(H)$.

Let $C\in\mathcal{B}_2(H),\ C\ne 0$. Define the mapping $[\cdot,\cdot]_{2,P,C}:\mathcal{B}_2(H)\times\mathcal{B}_2(H)\to\mathbb{C}$ by

$$[A,B]_{2,P,C}:=\langle A,B\rangle_{2,P}\|C\|_{2,P}^2-\langle A,C\rangle_{2,P}\langle C,B\rangle_{2,P}.$$

Observe that $[\cdot,\cdot]_{2,P,C}$ is a nonnegative Hermitian form on $\mathcal{B}_2(H)$ and by Schwarz inequality we also have

$$
\begin{aligned}
&\left|\langle A,B\rangle_{2,P}\|C\|_{2,P}^2-\langle A,C\rangle_{2,P}\langle C,B\rangle_{2,P}\right|^2 \\
&\le\left[\|A\|_{2,P}^2\|C\|_{2,P}^2-\left|\langle A,C\rangle_{2,P}\right|^2\right]\left[\|B\|_{2,P}^2\|C\|_{2,P}^2-\left|\langle B,C\rangle_{2,P}\right|^2\right]
\end{aligned}
$$

for any $A,\,B\in\mathcal{B}_2(H)$, namely

$$
\begin{aligned}
(4.39)\qquad & \left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \\
&\le\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)-\left|\operatorname{tr}(PC^*A)\right|^2\right] \\
&\times\left[\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)-\left|\operatorname{tr}(PC^*B)\right|^2\right],
\end{aligned}
$$

where for the last term we used the equality $\left|\langle B,C\rangle_{2,P}\right|^2=\left|\langle C,B\rangle_{2,P}\right|^2$.

Since (A,C) has the trace P-(λ,Γ)-property and (B,C) has the trace P-(δ,Δ) -property, then by (4.22) we have

$$
\begin{aligned}
(4.40)\qquad & 0\le\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)-\left|\operatorname{tr}(PC^*A)\right|^2 \\
&\le\operatorname{tr}\left(P|C|^2\right) \\
&\times\left[\frac{1}{4}|\Gamma-\gamma|^2\left[\operatorname{tr}\left(P|C|^2\right)\right]-\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}C^*\right)(\Gamma C-A)\right]\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(4.41)\qquad & 0\le\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)-\left|\operatorname{tr}(PC^*B)\right|^2 \\
&\le\operatorname{tr}\left(P|C|^2\right) \\
&\times\left[\frac{1}{4}|\Delta-\delta|^2\left[\operatorname{tr}\left(P|C|^2\right)\right]-\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C-B)\right]\right)\right].
\end{aligned}
$$

If we multiply (4.40) with (4.41) and use (4.39), then we get

$$
\begin{aligned}
&\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \\
&\le \left[\operatorname{tr}\left(P|C|^2\right)\right]^2 \\
&\times\left[\frac{1}{4}|\Gamma-\gamma|^2\left[\operatorname{tr}\left(P|C|^2\right)\right]-\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}C^*\right)(\Gamma C-A)\right]\right)\right] \\
&\times\left[\frac{1}{4}|\Delta-\delta|^2\left[\operatorname{tr}\left(P|C|^2\right)\right]-\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C-B)\right]\right)\right].
\end{aligned} \tag{4.42}
$$

Utilising the elementary inequality for positive numbers $m,\ n,\ p,\ q$

$$
\left(m^2-n^2\right)\left(p^2-q^2\right)\le(mp-nq)^2,
$$

we can state that

$$
\begin{aligned}
&\left[\frac{1}{4}|\Gamma-\gamma|^2\left[\operatorname{tr}\left(P|C|^2\right)\right]-\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}C^*\right)(\Gamma C-A)\right]\right)\right] \\
&\times\left[\frac{1}{4}|\Delta-\delta|^2\left[\operatorname{tr}\left(P|C|^2\right)\right]-\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C-B)\right]\right)\right] \\
&\le\left(\frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\left[\operatorname{tr}\left(P|C|^2\right)\right]\right. \\
&-\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}C^*\right)(\Gamma C-A)\right]\right)\right]^{1/2} \\
&\left.\times\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C-B)\right]\right)\right]^{1/2}\right)^2
\end{aligned} \tag{4.43}
$$

with the term in the right hand side in the brackets being nonnegative.

Making use of (4.42) and (4.43) we then get

$$
\begin{aligned}
&\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \\
&\le\left[\operatorname{tr}\left(P|C|^2\right)\right]^2\left(\frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\left[\operatorname{tr}\left(P|C|^2\right)\right]\right. \\
&-\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}C^*\right)(\Gamma C-A)\right]\right)\right]^{1/2} \\
&\left.\times\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C-B)\right]\right)\right]^{1/2}\right)^2.
\end{aligned} \tag{4.44}
$$

Taking the square root in (4.44) we obtain the desired result (4.38). ∎

COROLLARY 4.15. *Let, either* $P\in\mathcal{B}_+(H),\ A,\ B,\ C\in\mathcal{B}_2(H)$ *or* $P\in\mathcal{B}_1^+(H),\ A,\ B,\ C\in\mathcal{B}(H)$ *and* $\lambda,\ \Gamma,\ \delta,\ \Delta\in\mathbb{C}$. *If the transforms* $\mathcal{C}_{\lambda,\Gamma}(A,C)$ *and* $\mathcal{C}_{\delta,\Delta}(B,C)$ *are accretive, then the inequality (4.38) is valid.*

We have:

COROLLARY 4.16. *Let* $P,\ A,\ B,\ C$ *be selfadjoint operators with either* $P\in\mathcal{B}_+(H),\ A,\ B,\ C\in\mathcal{B}_2(H)$ *or* $P\in\mathcal{B}_1^+(H),\ A,\ B,\ C\in\mathcal{B}(H)$ *and* $m,\ M,\ n,\ N\in\mathbb{R}$ *with* $M>m$ *and*

$N > n$. *If* $(A - mC)(MC - A) \geq 0$ *and* $(B - nC)(NC - B) \geq 0$ *then*

$$\begin{aligned}(4.45)\qquad &\left|\operatorname{tr}(PBA)\operatorname{tr}\left(PC^2\right) - \operatorname{tr}(PCA)\operatorname{tr}(PBC)\right| \\ &\leq \operatorname{tr}\left(PC^2\right)\left[\frac{1}{4}(M-m)(N-n)\operatorname{tr}\left(PC^2\right)\right. \\ &- [\operatorname{Re}(\operatorname{tr}(A - mC)(MC - A))]^{1/2} \\ &\left.\times [\operatorname{Re}(\operatorname{tr}[P(B - nC)(NC - B)])]^{1/2}\right] \\ &\leq \frac{1}{4}(M-m)(N-n)\left[\operatorname{tr}\left(PC^2\right)\right]^2.\end{aligned}$$

Finally, we have:

THEOREM 4.17 (Dragomir, 2014, [**54**]). *With the assumptions of Theorem 4.14*

$$\begin{aligned}(4.46)\qquad &\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right| \\ &\leq \operatorname{tr}\left(P|C|^2\right)\left[\frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\operatorname{tr}\left(P|C|^2\right)\right. \\ &- \left|\frac{\Gamma+\gamma}{2}\operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A)\right| \\ &\left.\times\left|\frac{\delta+\Delta}{2}\operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*B)\right|\right] \\ &\leq \frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\left[\operatorname{tr}\left(P|C|^2\right)\right]^2.\end{aligned}$$

If the transforms $\mathcal{C}_{\lambda,\Gamma}(A, C)$ *and* $\mathcal{C}_{\delta,\Delta}(B, C)$ *are accretive, then the inequality (4.46) also holds.*

The proof is similar to the one for Theorem 4.14 via the Corollary 4.11 and the details are omitted.

COROLLARY 4.18. *With the assumptions of Corollary 4.16*

$$\begin{aligned}(4.47)\qquad &\left|\operatorname{tr}(PBA)\operatorname{tr}\left(PC^2\right) - \operatorname{tr}(PCA)\operatorname{tr}(PBC)\right| \\ &\leq \operatorname{tr}\left(PC^2\right)\left[\frac{1}{4}(M-m)(N-n)\operatorname{tr}\left(PC^2\right)\right. \\ &- \left|\frac{M+m}{2}\operatorname{tr}\left(PC^2\right) - \operatorname{tr}(PCA)\right| \\ &\left.\times\left|\frac{n+N}{2}\operatorname{tr}\left(PC^2\right) - \operatorname{tr}(PCB)\right|\right] \\ &\leq \frac{1}{4}(M-m)(N-n)\left[\operatorname{tr}\left(PC^2\right)\right]^2.\end{aligned}$$

4.4. Some Examples in the Case of $P \in \mathcal{B}_1(H)$. Utilising the above results in the case when $P \in \mathcal{B}_1^+(H)$, $A \in \mathcal{B}(H)$ and $B = 1_H$ we can also state the following inequalities that complement the earlier results obtained in [**51**]:

PROPOSITION 4.19 (Dragomir, 2014, [**54**]). *Let* $P \in \mathcal{B}_1^+(H)$, $A \in \mathcal{B}(H)$ *and* $\gamma, \Gamma \in \mathbb{C}$.

(i) We have

$$0 \leq \frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \tag{4.48}$$
$$= \operatorname{Re}\left[\left(\Gamma - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA^*)}{\operatorname{tr}(P)} - \overline{\gamma}\right)\right]$$
$$- \frac{1}{\operatorname{tr}(P)} \operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}1_H\right)\left(\Gamma 1_H - A\right)\right]\right)$$
$$\leq \frac{1}{4}|\Gamma - \gamma|^2 - \frac{1}{\operatorname{tr}(P)} \operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}1_H\right)\left(\Gamma 1_H - A\right)\right]\right).$$

(ii) If

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}1_H\right)\left(\Gamma 1_H - A\right)\right]\right) \geq 0 \tag{4.49}$$

or, equivalently

$$\frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(P\left|A - \frac{\gamma + \Gamma}{2}1_H\right|^2\right) \leq \frac{1}{4}|\Gamma - \gamma|^2, \tag{4.50}$$

and we say for simplicity that A has the trace P-(λ, Γ)-property, then

$$0 \leq \frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \tag{4.51}$$
$$\leq \operatorname{Re}\left[\left(\Gamma - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA^*)}{\operatorname{tr}(P)} - \overline{\gamma}\right)\right] \leq \frac{1}{4}|\Gamma - \gamma|^2$$

and

$$0 \leq \frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \tag{4.52}$$
$$\leq \frac{1}{4}|\Gamma - \gamma|^2 - \frac{1}{\operatorname{tr}(P)} \operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}1_H\right)\left(\Gamma 1_H - A\right)\right]\right) \leq \frac{1}{4}|\Gamma - \gamma|^2.$$

(iii) If the transform $\mathcal{C}_{\lambda,\Gamma}(A) := \left(A^* - \overline{\gamma}1_H\right)\left(\Gamma 1_H - A\right)$ is accretive, then the inequalities (4.51) and (4.52) also hold.

COROLLARY 4.20. *Let $P \in \mathcal{B}_1^+(H)$, A be a selfadjoint operator and $m, M \in \mathbb{R}$ with $M > m$.*

(i) If $\left(A - m1_H\right)\left(M1_H - A\right) \geq 0$, then

$$0 \leq \frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right]^2 \tag{4.53}$$
$$\leq \left[\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)\right] \leq \frac{1}{4}(M - m)^2$$

and

$$0 \leq \frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right]^2 \tag{4.54}$$
$$\leq \frac{1}{4}(M - m)^2 - \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left[P(A - mB)(MB - A)\right] \leq \frac{1}{4}(M - m)^2.$$

(ii) If $m1_H \leq A \leq M1_H$, then (4.53) and (4.54) also hold.

We have the following reverse of Schwarz inequality as well:

PROPOSITION 4.21 (Dragomir, 2014, [**54**]). *Let* $P \in \mathcal{B}_1^+(H)$, $A \in \mathcal{B}(H)$ *and* $\gamma, \Gamma \in \mathbb{C}$.

(i) If A *has the trace* P-(λ, Γ)*-property, then*

$$0 \le \frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \le \frac{1}{4}|\Gamma-\gamma|^2 - \left|\frac{\Gamma+\gamma}{2} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \le \frac{1}{4}|\Gamma-\gamma|^2. \tag{4.55}$$

(ii) If the transform $\mathcal{C}_{\lambda,\Gamma}(A) := (A^* - \overline{\gamma}1_H)(\Gamma 1_H - A)$ *is accretive, then the inequality (4.55) also holds.*

COROLLARY 4.22. *Let* $P \in \mathcal{B}_1^+(H)$, A *be a selfadjoint operator and* $m, M \in \mathbb{R}$ *with* $M > m$.

(i) If $(A - m1_H)(M1_H - A) \ge 0$, *then*

$$0 \le \frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right]^2 \le \frac{1}{4}(M-m)^2 - \left|\frac{m+M}{2} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \le \frac{1}{4}(M-m)^2. \tag{4.56}$$

(ii) If $m1_H \le A \le M1_H$, *then (4.56) also holds.*

Finally, we have the following Grüss type inequality as well:

PROPOSITION 4.23 (Dragomir, 2014, [**54**]). *Let* $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ *and* $\lambda, \Gamma, \delta, \Delta \in \mathbb{C}$.

(i) If A *has the trace* P-(λ, Γ)*-property and* B *has the trace* P-(δ, Δ)*-property, then*

$$\begin{aligned} &\left|\frac{\operatorname{tr}(PB^*A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB^*)}{\operatorname{tr}(P)}\right| \\ &\le \left[\frac{1}{4}|\Gamma-\gamma||\Delta-\delta| \right. \\ &- \frac{1}{\operatorname{tr}(P)}\left[\operatorname{Re}\left(\operatorname{tr}\left[P(A^*-\overline{\gamma}1_H)(\Gamma 1_H - A)\right]\right)\right]^{1/2} \\ &\left.\times \frac{1}{\operatorname{tr}(P)}\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}1_H\right)(\Delta 1_H - B)\right]\right)\right]^{1/2}\right] \le \frac{1}{4}|\Gamma-\gamma||\Delta-\delta| \end{aligned} \tag{4.57}$$

and

$$\begin{aligned} &\left|\frac{\operatorname{tr}(PB^*A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB^*)}{\operatorname{tr}(P)}\right| \\ &\le \frac{1}{4}|\Gamma-\gamma||\Delta-\delta| - \left|\frac{\Gamma+\gamma}{2} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|\left|\frac{\delta+\Delta}{2} - \frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right| \\ &\le \frac{1}{4}|\Gamma-\gamma||\Delta-\delta|. \end{aligned} \tag{4.58}$$

(ii) If the transforms $\mathcal{C}_{\lambda,\Gamma}(A)$ *and* $\mathcal{C}_{\delta,\Delta}(B)$ *are accretive then (4.57) and (4.58) also hold.*

The case of selfadjoint operators is as follows:

COROLLARY 4.24. *Let* P, A, B *be selfadjoint operators with* $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ *and* $m, M, n, N \in \mathbb{R}$ *with* $M > m$ *and* $N > n$.

(i) If $(A - m1_H)(M1_H - A) \geq 0$ *and* $(B - n1_H)(N1_H - B) \geq 0$ *then*

$$\begin{aligned}(4.59)\quad & \left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right| \\ & \leq \left[\frac{1}{4}(M-m)(N-n)\right. \\ & - \frac{1}{\operatorname{tr}(P)}\left[\operatorname{Re}\left(\operatorname{tr}(A-m1_H)(M1_H-A)\right)\right]^{1/2} \\ & \left.\times\frac{1}{\operatorname{tr}(P)}\left[\operatorname{Re}\left(\operatorname{tr}\left[P(B-n1_H)(N1_H-B)\right]\right)\right]^{1/2}\right] \\ & \leq \frac{1}{4}(M-m)(N-n)\end{aligned}$$

and

$$\begin{aligned}(4.60)\quad & \left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right| \\ & \leq \frac{1}{4}(M-m)(N-n) - \left|\frac{m+M}{2} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|\left|\frac{n+N}{2} - \frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right| \\ & \leq \frac{1}{4}(M-m)(N-n).\end{aligned}$$

(ii) If $m1_H \leq A \leq M1_H$ *and* $n1_H \leq B \leq N1_H$ *then (4.59) and (4.60) also hold.*

5. CASSELS TYPE INEQUALITIES

5.1. General Inequalities. We have the following result:

THEOREM 5.1 (Dragomir, 2014, [**59**]). *Let, either* $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ *and* $\gamma, \Gamma \in \mathbb{C}$ *with* $\operatorname{Re}(\Gamma\overline{\gamma}) = \operatorname{Re}(\Gamma)\operatorname{Re}(\gamma) + \operatorname{Im}(\Gamma)\operatorname{Im}(\gamma) > 0$.

(i) If (A, B) *satisfies the* P-(γ, Γ)*-trace property, then*

$$\begin{aligned}(5.1)\quad & \operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right) \\ & \leq \frac{1}{4}\cdot\frac{\left[\operatorname{Re}(\gamma+\Gamma)\operatorname{Re}\operatorname{tr}(PB^*A) + \operatorname{Im}(\gamma+\Gamma)\operatorname{Im}\operatorname{tr}(PB^*A)\right]^2}{\operatorname{Re}(\Gamma)\operatorname{Re}(\gamma) + \operatorname{Im}(\Gamma)\operatorname{Im}(\gamma)} \\ & \leq \frac{1}{4}\cdot\frac{|\gamma+\Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}\left|\operatorname{tr}(PB^*A)\right|^2.\end{aligned}$$

(ii) If the transform $\mathcal{C}_{\gamma,\Gamma}(A, B)$ *is accretive, then the inequality (5.1) also holds.*

PROOF. (i) If (A,B) satisfies the P-(γ,Γ)-trace property, then, on utilizing the calculations above, we have

$$\begin{aligned}
0 &\le \frac{1}{4}\left|\Gamma-\gamma\right|^2 \operatorname{tr}\left(P\left|B\right|^2\right) - \operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}B\right|^2\right) \\
&= -\operatorname{tr}\left(P\left|A\right|^2\right) - \operatorname{Re}\left(\Gamma\overline{\gamma}\right)\operatorname{tr}\left(P\left|B\right|^2\right) \\
&+ \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}\left(PB^*A\right)\right] + \operatorname{Re}\left[\Gamma\overline{\operatorname{tr}\left(PB^*A\right)}\right] \\
&= -\operatorname{tr}\left(P\left|A\right|^2\right) - \operatorname{Re}\left(\Gamma\overline{\gamma}\right)\operatorname{tr}\left(P\left|B\right|^2\right) \\
&+ \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}\left(PB^*A\right)\right] + \operatorname{Re}\left[\overline{\Gamma\overline{\operatorname{tr}\left(PB^*A\right)}}\right] \\
&= -\operatorname{tr}\left(P\left|A\right|^2\right) - \operatorname{Re}\left(\Gamma\overline{\gamma}\right)\operatorname{tr}\left(P\left|B\right|^2\right) \\
&+ \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}\left(PB^*A\right)\right] + \operatorname{Re}\left[\overline{\Gamma}\operatorname{tr}\left(PB^*A\right)\right] \\
&= -\operatorname{tr}\left(P\left|A\right|^2\right) - \operatorname{Re}\left(\Gamma\overline{\gamma}\right)\operatorname{tr}\left(P\left|B\right|^2\right) + \operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^*A\right)\right],
\end{aligned}$$

which implies that

$$\begin{aligned}
&\operatorname{tr}\left(P\left|A\right|^2\right) + \operatorname{Re}\left(\Gamma\overline{\gamma}\right)\operatorname{tr}\left(P\left|B\right|^2\right) \\
&\le \operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^*A\right)\right] \\
&= \operatorname{Re}\left(\gamma+\Gamma\right)\operatorname{Re}\operatorname{tr}\left(PB^*A\right) + \operatorname{Im}\left(\gamma+\Gamma\right)\operatorname{Im}\operatorname{tr}\left(PB^*A\right).
\end{aligned} \tag{5.2}$$

Making use of the elementary inequality

$$2\sqrt{pq} \le p+q,\ p,q\ge 0,$$

we also have

$$2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)\operatorname{tr}\left(P\left|A\right|^2\right)\operatorname{tr}\left(P\left|B\right|^2\right)} \le \operatorname{tr}\left(P\left|A\right|^2\right) + \operatorname{Re}\left(\Gamma\overline{\gamma}\right)\operatorname{tr}\left(P\left|B\right|^2\right). \tag{5.3}$$

Utilising (5.2) and (5.3) we get

$$\begin{aligned}
&\sqrt{\operatorname{tr}\left(P\left|A\right|^2\right)\operatorname{tr}\left(P\left|B\right|^2\right)} \\
&\le \frac{\operatorname{Re}\left(\gamma+\Gamma\right)\operatorname{Re}\operatorname{tr}\left(PB^*A\right) + \operatorname{Im}\left(\gamma+\Gamma\right)\operatorname{Im}\operatorname{tr}\left(PB^*A\right)}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}
\end{aligned} \tag{5.4}$$

that is equivalent with the first inequality in (5.1).

The second inequality in (5.1) is obvious by Schwarz inequality

$$\left(ab+cd\right)^2 \le \left(a^2+c^2\right)\left(b^2+d^2\right),\ a,b,c,d\in\mathbb{R}.$$

The (ii) is obvious from (i). ∎

REMARK 5.1. We observe that the inequality between the first and last term in (5.1) is equivalent to

$$0 \le \operatorname{tr}\left(P\left|A\right|^2\right)\operatorname{tr}\left(P\left|B\right|^2\right) - \left|\operatorname{tr}\left(PB^*A\right)\right|^2 \le \frac{1}{4}\cdot\frac{\left|\gamma-\Gamma\right|^2}{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}\left|\operatorname{tr}\left(PB^*A\right)\right|^2. \tag{5.5}$$

COROLLARY 5.2. *Let, either* $P\in\mathcal{B}_+(H)$, $A\in\mathcal{B}_2(H)$ *or* $P\in\mathcal{B}_1^+(H)$, $A\in\mathcal{B}(H)$ *and* $\gamma,\Gamma\in\mathbb{C}$ *with* $\operatorname{Re}\left(\Gamma\overline{\gamma}\right) = \operatorname{Re}\left(\Gamma\right)\operatorname{Re}\left(\gamma\right) + \operatorname{Im}\left(\Gamma\right)\operatorname{Im}\left(\gamma\right) > 0$.

(i) If A *satisfies the* P-(γ,Γ)*-trace property, namely*

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}1_H\right)\left(\Gamma 1_H - A\right)\right]\right) \ge 0 \tag{5.6}$$

or, equivalently

$$\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}1_H\right|^2\right)\leq\frac{1}{4}\left|\Gamma-\gamma\right|^2\operatorname{tr}(P),\tag{5.7}$$

then

$$\begin{aligned}\frac{\operatorname{tr}\left(P\left|A\right|^2\right)}{\operatorname{tr}(P)}&\\ &\leq\frac{1}{4}\cdot\frac{\left[\operatorname{Re}(\gamma+\Gamma)\frac{\operatorname{Re}\operatorname{tr}(PA)}{\operatorname{tr}(P)}+\operatorname{Im}(\gamma+\Gamma)\frac{\operatorname{Im}\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right]^2}{\operatorname{Re}(\Gamma)\operatorname{Re}(\gamma)+\operatorname{Im}(\Gamma)\operatorname{Im}(\gamma)}\\ &\leq\frac{1}{4}\cdot\frac{\left|\gamma+\Gamma\right|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}\left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2.\end{aligned}\tag{5.8}$$

(ii) If the transform $\mathcal{C}_{\gamma,\Gamma}(A)$ is accretive, then the inequality (5.1) also holds.

(iii) We have

$$0\leq\frac{\operatorname{tr}\left(P\left|A\right|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2\leq\frac{1}{4}\cdot\frac{\left|\gamma-\Gamma\right|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}\left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2.\tag{5.9}$$

REMARK 5.2. The case of selfadjoint operators is as follows.

Let A,B be selfadjoint operators and either $P\in\mathcal{B}_+(H)$, $A,B\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^+(H)$, $A,B\in\mathcal{B}(H)$ and $m,M\in\mathbb{R}$ with $mM>0.$

(i) If (A,B) satisfies the P-(m,M)-trace property, then

$$\operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right)\leq\frac{(m+M)^2}{4mM}\left[\operatorname{tr}(PBA)\right]^2\tag{5.10}$$

or, equivalently

$$0\leq\operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right)-\left[\operatorname{tr}(PBA)\right]^2\leq\frac{(m-M)^2}{4mM}\left[\operatorname{tr}(PBA)\right]^2.\tag{5.11}$$

(ii) If the transform $\mathcal{C}_{m,M}(A,B)$ is accretive, then the inequality (5.10) also holds.

(iii) If $(A-mB)(MB-A)\geq0,$ then (5.10) is valid.

5.2. Trace Inequalities of Grüss Type.

We have the following Grüss type inequality:

THEOREM 5.3 (Dragomir, 2014, [**59**]). *Let, either $P\in\mathcal{B}_+(H)$, $A,B,C\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^+(H)$, $A,B,C\in\mathcal{B}(H)$ with $P|A|^2$, $P|B|^2$, $P|C|^2\neq0$ and $\lambda,\Gamma,\delta,\Delta\in\mathbb{C}$ with $\operatorname{Re}(\Gamma\overline{\gamma})$, $\operatorname{Re}(\Delta\overline{\delta})>0.$ If (A,C) has the trace P-(λ,Γ)-property and (B,C) has the trace P-(δ,Δ)-property, then*

$$\left|\frac{\operatorname{tr}(PB^*A)\operatorname{tr}\left(P\left|C\right|^2\right)}{\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)}-1\right|\leq\frac{1}{4}\cdot\frac{\left|\gamma-\Gamma\right|\left|\delta-\Delta\right|}{\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{Re}\left(\Delta\overline{\delta}\right)}}.\tag{5.12}$$

PROOF. We prove in the case that $P\in\mathcal{B}_+(H)$ and $A,B,C\in\mathcal{B}_2(H)$.

Making use of the Schwarz inequality for the nonnegative hermitian form $\langle\cdot,\cdot\rangle_{2,P}$ we have

$$\left|\langle A,B\rangle_{2,P}\right|^2\leq\langle A,A\rangle_{2,P}\langle B,B\rangle_{2,P}$$

for any $A,B\in\mathcal{B}_2(H)$.

Let $C\in\mathcal{B}_2(H)$, $C\neq0$. Define the mapping $[\cdot,\cdot]_{2,P,C}:\mathcal{B}_2(H)\times\mathcal{B}_2(H)\to\mathbb{C}$ by

$$[A,B]_{2,P,C}:=\langle A,B\rangle_{2,P}\left\|C\right\|_{2,P}^2-\langle A,C\rangle_{2,P}\langle C,B\rangle_{2,P}.$$

Observe that $[\cdot,\cdot]_{2,P,C}$ is a nonnegative Hermitian form on $\mathcal{B}_2(H)$ and by Schwarz inequality we also have

$$\left|\langle A,B\rangle_{2,P}\|C\|_{2,P}^2-\langle A,C\rangle_{2,P}\langle C,B\rangle_{2,P}\right|^2 \le \left[\|A\|_{2,P}^2\|C\|_{2,P}^2-\left|\langle A,C\rangle_{2,P}\right|^2\right]\left[\|B\|_{2,P}^2\|C\|_{2,P}^2-\left|\langle B,C\rangle_{2,P}\right|^2\right]$$

for any $A,B\in\mathcal{B}_2(H)$, namely

$$\begin{aligned}&\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2\\ &\le\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)-|\operatorname{tr}(PC^*A)|^2\right]\\ &\times\left[\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)-|\operatorname{tr}(PB^*C)|^2\right],\end{aligned}\tag{5.13}$$

where for the last term we used the equality $\left|\langle B,C\rangle_{2,P}\right|^2=\left|\langle C,B\rangle_{2,P}\right|^2$.

Since (A,C) has the trace P-(λ,Γ)-property and (B,C) has the trace P-(δ,Δ) -property, then by (5.5) we have

$$0\le\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)-|\operatorname{tr}(PC^*A)|^2\le\frac{1}{4}\cdot\frac{|\gamma-\Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}|\operatorname{tr}(PC^*A)|^2\tag{5.14}$$

and

$$0\le\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)-|\operatorname{tr}(PB^*C)|^2\le\frac{1}{4}\cdot\frac{|\delta-\Delta|^2}{\operatorname{Re}\left(\Delta\overline{\delta}\right)}|\operatorname{tr}(PB^*C)|^2.\tag{5.15}$$

If we multiply the inequalities (5.14) and (5.15) we get

$$\begin{aligned}&\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)-|\operatorname{tr}(PC^*A)|^2\right]\\ &\times\left[\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)-|\operatorname{tr}(PB^*C)|^2\right]\\ &\le\frac{1}{16}\cdot\frac{|\gamma-\Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}\frac{|\delta-\Delta|^2}{\operatorname{Re}\left(\Delta\overline{\delta}\right)}|\operatorname{tr}(PC^*A)|^2|\operatorname{tr}(PB^*C)|^2.\end{aligned}\tag{5.16}$$

If we use (5.13) and (5.16) we get

$$\begin{aligned}&\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2\\ &\le\frac{1}{16}\cdot\frac{|\gamma-\Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}\frac{|\delta-\Delta|^2}{\operatorname{Re}\left(\Delta\overline{\delta}\right)}|\operatorname{tr}(PC^*A)|^2|\operatorname{tr}(PB^*C)|^2.\end{aligned}\tag{5.17}$$

Since $P,A,B,C\neq0$ then by (5.14) and (5.15) we get $\operatorname{tr}(PC^*A)\neq0$ and $\operatorname{tr}(PB^*C)\neq0$. Now, if we take the square root in (5.17) and divide by $|\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)|$ we obtain the desired result (5.12). ∎

COROLLARY 5.4. *Let, either $P\in\mathcal{B}_+(H)$, $A,B\in\mathcal{B}_2$ or $P\in\mathcal{B}_1^+(H)$, $A,B\in\mathcal{B}(H)$ with $P|A|^2$, $P|B|^2\neq0$ and $\lambda,\Gamma,\delta,\Delta\in\mathbb{C}$ with $\operatorname{Re}(\Gamma\overline{\gamma})$, $\operatorname{Re}\left(\Delta\overline{\delta}\right)>0$. If A has the trace P-(λ,Γ)-property and B has the trace P-(δ,Δ)-property, then*

$$\left|\frac{\operatorname{tr}(PB^*A)\operatorname{tr}(P)}{\operatorname{tr}(PA)\operatorname{tr}(PB^*)}-1\right|\le\frac{1}{4}\cdot\frac{|\gamma-\Gamma||\delta-\Delta|}{\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{Re}\left(\Delta\overline{\delta}\right)}}.\tag{5.18}$$

The case of selfadjoint operators is useful for applications.

REMARK 5.3. Assume that A, B, C are selfadjoint operators. If, either $P \in \mathcal{B}_+(H)$, A, B, $C \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, B, $C \in \mathcal{B}(H)$ with PA^2, PB^2, $PC^2 \neq 0$ and m, M, n, $N \in \mathbb{R}$ with mM, $nN > 0$. If (A, C) has the trace P-(m, M)-*property and* (B, C) has the trace P-(n, N)-*property, then*

$$\left|\frac{\operatorname{tr}(PBA)\operatorname{tr}(PC^2)}{\operatorname{tr}(PCA)\operatorname{tr}(PBC)} - 1\right| \leq \frac{1}{4} \cdot \frac{(M-m)(N-n)}{\sqrt{mnMN}}. \tag{5.19}$$

If A has the trace P-(k, K)-*property and* B has the trace P-(l, L)-*property, then*

$$\left|\frac{\operatorname{tr}(PBA)\operatorname{tr}(P)}{\operatorname{tr}(PA)\operatorname{tr}(PB)} - 1\right| \leq \frac{1}{4} \cdot \frac{(K-k)(L-l)}{\sqrt{klKL}}, \tag{5.20}$$

where kK, $lL > 0$.

We observe that, if $0 < k1_H \leq A \leq K1_H$ and $0 < l1_H \leq B \leq L1_H$, then by (5.21)

$$|\operatorname{tr}(PBA)\operatorname{tr}(P) - \operatorname{tr}(PA)\operatorname{tr}(PB)| \leq \frac{1}{4} \cdot \frac{(K-k)(L-l)}{\sqrt{klKL}} \operatorname{tr}(PA)\operatorname{tr}(PB) \tag{5.21}$$

or, equivalently

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right| \leq \frac{1}{4} \cdot \frac{(K-k)(L-l)}{\sqrt{klKL}}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}. \tag{5.22}$$

5.3. Applications for Convex Functions. In the paper **[53]** we obtained amongst other the following reverse of the Jensen trace inequality:

Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$, then we have

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\ &\leq \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \\ &\leq \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \frac{\operatorname{tr}\left(P\left|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \\ \\ \frac{1}{2}(M-m) \frac{\operatorname{tr}\left(P\left|f'(A) - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \end{cases} \\ &\leq \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m) \left[\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\ &\leq \frac{1}{4}[f'(M) - f'(m)](M-m). \end{aligned} \tag{5.23}$$

Let $\mathcal{M}_n(\mathbb{C})$ be the space of all square matrices of order n with complex elements and $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$, then by taking $P = I_n$

in (5.23) we get

$$0 \le \frac{\operatorname{tr}(f(A))}{n} - f\left(\frac{\operatorname{tr}(A)}{n}\right) \tag{5.24}$$
$$\le \frac{\operatorname{tr}(f'(A)A)}{n} - \frac{\operatorname{tr}(A)}{n} \cdot \frac{\operatorname{tr}(f'(A))}{n}$$
$$\le \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \frac{\operatorname{tr}\left(\left|A - \frac{\operatorname{tr}(A)}{n} 1_H\right|\right)}{n} \\ \\ \frac{1}{2}(M-m) \frac{\operatorname{tr}\left(\left|f'(A) - \frac{\operatorname{tr}(f'(A))}{n} 1_H\right|\right)}{n} \end{cases}$$
$$\le \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \left[\frac{\operatorname{tr}(A^2)}{n} - \left(\frac{\operatorname{tr}(A)}{n}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m) \left[\frac{\operatorname{tr}([f'(A)]^2)}{n} - \left(\frac{\operatorname{tr}(f'(A))}{n}\right)^2\right]^{1/2} \end{cases}$$
$$\le \frac{1}{4}[f'(M) - f'(m)](M-m).$$

The following reverse inequality also holds:

PROPOSITION 5.5 (Dragomir, 2014, [**59**]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If f is a continuously differentiable convex function on $[m, M]$ with $f'(m) > 0$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \ge 0$, then*

$$0 \le \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \tag{5.25}$$
$$\le \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}$$
$$\le \frac{1}{4} \cdot \frac{(M-m)[f'(M) - f'(m)]}{\sqrt{mMf'(m)f'(M)}} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}.$$

The proof follows by the inequality (5.22) and the details are omitted.

Let $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$ with $f'(m) > 0$ then by taking $P = I_n$ in (5.25) we get

$$0 \le \frac{\operatorname{tr}(f(A))}{n} - f\left(\frac{\operatorname{tr}(A)}{n}\right) \tag{5.26}$$
$$\le \frac{\operatorname{tr}(f'(A)A)}{n} - \frac{\operatorname{tr}(A)}{n} \cdot \frac{\operatorname{tr}(f'(A))}{n}$$
$$\le \frac{1}{4} \cdot \frac{(M-m)[f'(M) - f'(m)]}{\sqrt{mMf'(m)f'(M)}} \frac{\operatorname{tr}(A)}{n} \frac{\operatorname{tr}(f'(A))}{n}.$$

We consider the power function $f : (0, \infty) \to (0, \infty)$, $f(t) = t^r$ with $t \in \mathbb{R} \setminus \{0\}$. For $r \in (-\infty, 0) \cup [1, \infty)$, f is convex while for $r \in (0, 1)$, f is concave.

Let $r \geq 1$ and A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$0 \leq \frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^r \leq r\left[\frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(PA^{r-1})}{\operatorname{tr}(P)}\right] \leq \frac{1}{4} r \frac{(M-m)(M^{r-1}-m^{r-1})}{m^{r/2}M^{r/2}} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PA^{r-1})}{\operatorname{tr}(P)}. \tag{5.27}$$

If we take the first and last term in (5.27) we get the inequality:

$$0 \leq \frac{\operatorname{tr}(P)\operatorname{tr}(PA^r)}{\operatorname{tr}(PA)\operatorname{tr}(PA^{r-1})} - \frac{\operatorname{tr}(P)[\operatorname{tr}(PA)]^{r-1}}{\operatorname{tr}(PA^{p-1})[\operatorname{tr}(P)]^{r-1}} \leq \frac{1}{4} r \frac{(M-m)(M^{r-1}-m^{r-1})}{m^{r/2}M^{r/2}}. \tag{5.28}$$

Consider the convex function $f : \mathbb{R} \to (0, \infty)$, $f(t) = \exp t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then using (5.25) we have

$$0 \leq \frac{\operatorname{tr}(P \exp A)}{\operatorname{tr}(P)} - \exp\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \leq \frac{\operatorname{tr}(PA \exp A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(P \exp A)}{\operatorname{tr}(P)} \leq \frac{1}{4} \cdot \frac{(M-m)(\exp M - \exp m)}{\sqrt{mM \exp(m+M)}} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(P \exp A)}{\operatorname{tr}(P)}. \tag{5.29}$$

If we take the first and last term in (5.29) we get the inequality:

$$0 \leq \frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)} - \frac{[\operatorname{tr}(P)]^2 \exp\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)}{\operatorname{tr}(PA)\operatorname{tr}(P \exp A)} \leq \frac{1}{4} \cdot \frac{(M-m)(\exp M - \exp m)}{\sqrt{mM \exp(m+M)}}. \tag{5.30}$$

6. SHISHA-MOND TYPE TRACE INEQUALITIES

6.1. General Results.

We have the following result:

THEOREM 6.1 (Dragomir, 2014, [**60**]). *Let, either* $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ *and* $\gamma, \Gamma \in \mathbb{C}$ *with* $\Gamma + \gamma \neq 0$.

(i) If (A, B) *satisfies the* P-(γ, Γ)*-trace property, then*

$$\sqrt{\operatorname{tr}(P|A|^2)\operatorname{tr}(P|B|^2)} \leq \frac{\operatorname{Re}(\gamma+\Gamma)\operatorname{Re}\operatorname{tr}(PB^*A) + \operatorname{Im}(\gamma+\Gamma)\operatorname{Im}\operatorname{tr}(PB^*A)}{|\Gamma+\gamma|} + \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}(P|B|^2) \leq |\operatorname{tr}(PB^*A)| + \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}(P|B|^2). \tag{6.1}$$

(ii) If the transform $\mathcal{C}_{\gamma,\Gamma}(A,B)$ is accretive, then the inequality (6.1) also holds.

PROOF. (i) If (A,B) satisfies the P-(γ,Γ)-trace property, then

$$\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}B\right|^2\right)\leq\frac{1}{4}\left|\Gamma-\gamma\right|^2\operatorname{tr}\left(P\left|B\right|^2\right)$$

that is equivalent to

$$\operatorname{tr}\left(P\left|A\right|^2\right)-\operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^*A\right)\right]+\frac{1}{4}\left|\Gamma+\gamma\right|^2\operatorname{tr}\left(P\left|B\right|^2\right)\leq\frac{1}{4}\left|\Gamma-\gamma\right|^2\operatorname{tr}\left(P\left|B\right|^2\right),$$

which implies that

$$\begin{aligned}\operatorname{tr}\left(P\left|A\right|^2\right)+\frac{1}{4}\left|\Gamma+\gamma\right|^2\operatorname{tr}\left(P\left|B\right|^2\right)\\ \leq\operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^*A\right)\right]+\frac{1}{4}\left|\Gamma-\gamma\right|^2\operatorname{tr}\left(P\left|B\right|^2\right).\end{aligned}\tag{6.2}$$

Making use of the elementary inequality

$$2\sqrt{pq}\leq p+q,\ p,q\geq 0,$$

we also have

$$\left|\Gamma+\gamma\right|\left[\operatorname{tr}\left(P\left|A\right|^2\right)\operatorname{tr}\left(P\left|B\right|^2\right)\right]^{1/2}\leq\operatorname{tr}\left(P\left|A\right|^2\right)+\frac{1}{4}\left|\Gamma+\gamma\right|^2\operatorname{tr}\left(P\left|B\right|^2\right).\tag{6.3}$$

Utilising (6.2) and (6.3) we get

$$\begin{aligned}\left|\Gamma+\gamma\right|\left[\operatorname{tr}\left(P\left|A\right|^2\right)\operatorname{tr}\left(P\left|B\right|^2\right)\right]^{1/2}\\ \leq\operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^*A\right)\right]+\frac{1}{4}\left|\Gamma-\gamma\right|^2\operatorname{tr}\left(P\left|B\right|^2\right).\end{aligned}\tag{6.4}$$

Dividing by $|\Gamma+\gamma|>0$ and observing that

$$\operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^*A\right)\right]=\operatorname{Re}\left(\gamma+\Gamma\right)\operatorname{Re}\operatorname{tr}\left(PB^*A\right)+\operatorname{Im}\left(\gamma+\Gamma\right)\operatorname{Im}\operatorname{tr}\left(PB^*A\right)$$

we get the first inequality in (6.1).

The second inequality in (6.1) is obvious by Schwarz inequality

$$\left(ab+cd\right)^2\leq\left(a^2+c^2\right)\left(b^2+d^2\right),\ a,b,c,d\in\mathbb{R}.$$

The (ii) is obvious from (i). ∎

REMARK 6.1. We observe that the inequality between the first and last term in (6.1) is equivalent to

$$0\leq\sqrt{\operatorname{tr}\left(P\left|A\right|^2\right)\operatorname{tr}\left(P\left|B\right|^2\right)}-\left|\operatorname{tr}\left(PB^*A\right)\right|\leq\frac{1}{4}\frac{\left|\Gamma-\gamma\right|^2}{\left|\Gamma+\gamma\right|}\operatorname{tr}\left(P\left|B\right|^2\right).\tag{6.5}$$

COROLLARY 6.2. *Let, either $P\in\mathcal{B}_+(H)$, $A\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^+(H)$, $A\in\mathcal{B}(H)$ and $\gamma,\Gamma\in\mathbb{C}$ with $\gamma+\Gamma\neq 0$.*

(i) If A satisfies the P-(γ,Γ)-trace property, namely

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}1_H\right)\left(\Gamma 1_H-A\right)\right]\right)\geq 0\tag{6.6}$$

or, equivalently

$$\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}1_H\right|^2\right)\leq\frac{1}{4}\left|\Gamma-\gamma\right|^2\operatorname{tr}\left(P\right),\tag{6.7}$$

then

$$\sqrt{\frac{\operatorname{tr}\left(P\left|A\right|^{2}\right)}{\operatorname{tr}\left(P\right)}} \tag{6.8}$$
$$\leq \frac{\operatorname{Re}\left(\gamma+\Gamma\right)\frac{\operatorname{Re}\operatorname{tr}(PA)}{\operatorname{tr}(P)}+\operatorname{Im}\left(\gamma+\Gamma\right)\frac{\operatorname{Im}\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{\left|\Gamma+\gamma\right|}+\frac{1}{4}\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}$$
$$\leq \left|\frac{\operatorname{tr}\left(PA\right)}{\operatorname{tr}\left(P\right)}\right|+\frac{1}{4}\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}.$$

(ii) If the transform $\mathcal{C}_{\gamma,\Gamma}\left(A\right)$ is accretive, then the inequality (6.1) also holds.

(iii) We have

$$0\leq\sqrt{\frac{\operatorname{tr}\left(P\left|A\right|^{2}\right)}{\operatorname{tr}\left(P\right)}}-\left|\frac{\operatorname{tr}\left(PA\right)}{\operatorname{tr}\left(P\right)}\right|\leq\frac{1}{4}\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}. \tag{6.9}$$

REMARK 6.2. The case of selfadjoint operators is as follows.

Let A,B be selfadjoint operators and either $P\in\mathcal{B}_{+}\left(H\right),A,B\in\mathcal{B}_{2}\left(H\right)$ or $P\in\mathcal{B}_{1}^{+}\left(H\right)$, $A,B\in\mathcal{B}\left(H\right)$ and $m,M\in\mathbb{R}$ with $m+M\neq0$.

(i) If (A,B) satisfies the P-(m,M)-trace property, then

$$\sqrt{\operatorname{tr}\left(PA^{2}\right)\operatorname{tr}\left(PB^{2}\right)}\leq\operatorname{Re}\operatorname{tr}\left(PBA\right)+\frac{\left(M-m\right)^{2}}{4\left|M+m\right|}\operatorname{tr}\left(PB^{2}\right) \tag{6.10}$$
$$\leq\left|\operatorname{tr}\left(PBA\right)\right|+\frac{\left(M-m\right)^{2}}{4\left|M+m\right|}\operatorname{tr}\left(PB^{2}\right)$$

and

$$0\leq\sqrt{\operatorname{tr}\left(PA^{2}\right)\operatorname{tr}\left(PB^{2}\right)}-\operatorname{Re}\operatorname{tr}\left(PBA\right)\leq\frac{\left(M-m\right)^{2}}{4\left|M+m\right|}\operatorname{tr}\left(PB^{2}\right).$$

(ii) If the transform $\mathcal{C}_{m,M}\left(A,B\right)$ is accretive, then the inequality (6.10) also holds.

(iii) If $\left(A-mB\right)\left(MB-A\right)\geq0$, then (6.10) is valid.

COROLLARY 6.3. *Let A,B be selfadjoint operators and either $P\in\mathcal{B}_{+}\left(H\right),A,B\in\mathcal{B}_{2}\left(H\right)$ or $P\in\mathcal{B}_{1}^{+}\left(H\right),A,B\in\mathcal{B}\left(H\right)$ and $m,M\in\mathbb{R}$ with $m+M\neq0$.*

(i) If (A,B) satisfies the P-(m,M)-trace property, then

$$\left(\sqrt{\operatorname{tr}\left(PA^{2}\right)}+\sqrt{\operatorname{tr}\left(PB^{2}\right)}\right)^{2}-\operatorname{tr}\left(P\left(A+B\right)^{2}\right)\leq\frac{\left(M-m\right)^{2}}{4\left|M+m\right|}\operatorname{tr}\left(PB^{2}\right) \tag{6.11}$$

and

$$\sqrt{\operatorname{tr}\left(PA^{2}\right)}+\sqrt{\operatorname{tr}\left(PB^{2}\right)}-\sqrt{\operatorname{tr}\left(P\left(A+B\right)^{2}\right)}\leq\frac{\sqrt{2}}{2}\frac{M-m}{\sqrt{\left|M+m\right|}}\sqrt{\operatorname{tr}\left(PB^{2}\right)}. \tag{6.12}$$

PROOF. Observe that

$$\left(\sqrt{\operatorname{tr}\left(PA^{2}\right)}+\sqrt{\operatorname{tr}\left(PB^{2}\right)}\right)^{2}-\operatorname{tr}\left(P\left(A+B\right)^{2}\right)$$
$$=2\left(\sqrt{\operatorname{tr}\left(PA^{2}\right)\operatorname{tr}\left(PB^{2}\right)}-\operatorname{Re}\operatorname{tr}\left(PBA\right)\right).$$

Utilising (6.10) we deduce (6.11).

The inequality (6.12) follows from (6.11). ∎

6.2. Trace Inequalities of Grüss Type. We have the following Grüss type inequality:

THEOREM 6.4 (Dragomir, 2014, [**60**]). *Let, either* $P \in \mathcal{B}_+(H)$, $A, B, C \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A, B, C \in \mathcal{B}(H)$ *with* $P|A|^2, P|B|^2, P|C|^2 \neq 0$ *and* $\lambda, \Gamma, \delta, \Delta \in \mathbb{C}$ *with* $\gamma + \Gamma \neq 0$, $\delta + \Delta \neq 0$. *If* (A, C) *has the trace* P-(λ, Γ)*-property and* (B, C) *has the trace* P-(δ, Δ)*-property, then*

$$\left|\frac{\operatorname{tr}(PB^*A)}{\operatorname{tr}\left(P|C|^2\right)} - \frac{\operatorname{tr}(PC^*A)}{\operatorname{tr}\left(P|C|^2\right)}\frac{\operatorname{tr}(PB^*C)}{\operatorname{tr}\left(P|C|^2\right)}\right|^2 \leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\sqrt{\frac{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)}{\left[\operatorname{tr}\left(P|C|^2\right)\right]^2}}. \tag{6.13}$$

PROOF. We prove in the case that $P \in \mathcal{B}_+(H)$ and $A, B, C \in \mathcal{B}_2(H)$.

Making use of the Schwarz inequality for the nonnegative hermitian form $\langle\cdot,\cdot\rangle_{2,P}$ we have

$$\left|\langle A, B\rangle_{2,P}\right|^2 \leq \langle A, A\rangle_{2,P}\langle B, B\rangle_{2,P}$$

for any $A, B \in \mathcal{B}_2(H)$.

Let $C \in \mathcal{B}_2(H)$, $C \neq 0$. Define the mapping $[\cdot,\cdot]_{2,P,C} : \mathcal{B}_2(H) \times \mathcal{B}_2(H) \to \mathbb{C}$ by

$$[A, B]_{2,P,C} := \langle A, B\rangle_{2,P}\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\langle C, B\rangle_{2,P}.$$

Observe that $[\cdot,\cdot]_{2,P,C}$ is a nonnegative Hermitian form on $\mathcal{B}_2(H)$ and by Schwarz inequality we also have

$$\begin{aligned}&\left|\langle A, B\rangle_{2,P}\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\langle C, B\rangle_{2,P}\right|^2\\ &\leq \left[\|A\|_{2,P}^2\|C\|_{2,P}^2 - \left|\langle A, C\rangle_{2,P}\right|^2\right]\left[\|B\|_{2,P}^2\|C\|_{2,P}^2 - \left|\langle B, C\rangle_{2,P}\right|^2\right]\end{aligned}$$

for any $A, B \in \mathcal{B}_2(H)$, namely

$$\begin{aligned}&\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2\\ &\leq \left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2\right]\\ &\times\left[\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PB^*C)|^2\right],\end{aligned} \tag{6.14}$$

where for the last term we used the equality $\left|\langle B, C\rangle_{2,P}\right|^2 = \left|\langle C, B\rangle_{2,P}\right|^2$.

Since (A, C) has the trace P-(λ, Γ)-property and (B, C) has the trace P-(δ, Δ) -property, then by (6.5) we have

$$0 \leq \sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)} - |\operatorname{tr}(PC^*A)| \leq \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}\left(P|C|^2\right)$$

and

$$0 \leq \sqrt{\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)} - |\operatorname{tr}(PC^*B)| \leq \frac{1}{4}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\operatorname{tr}\left(P|C|^2\right),$$

which imply

$$0 \le \operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2 \le \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}\left(P|C|^2\right)\left(\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)} + |\operatorname{tr}(PC^*A)|\right) \le \frac{1}{2}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)} \tag{6.15}$$

and

$$0 \le \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PB^*C)|^2 \le \frac{1}{4}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\operatorname{tr}\left(P|C|^2\right)\left(\sqrt{\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)} + |\operatorname{tr}(PC^*B)|\right) \le \frac{1}{2}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)}. \tag{6.16}$$

If we multiply the inequalities (6.15) and (6.16) we get

$$\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2\right] \times \left[\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PB^*C)|^2\right] \le \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)} \times \operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)}. \tag{6.17}$$

If we use (6.14) and (6.17) we get

$$\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \le \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)} \times \operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)}. \tag{6.18}$$

Since $P|C|^2 \neq 0$ then by (6.18) we get the desired result (6.13). ∎

COROLLARY 6.5. *Let, either $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2$ or $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ with $P|A|^2$, $P|B|^2 \neq 0$ and λ, Γ, δ, $\Delta \in \mathbb{C}$ with $\gamma+\Gamma \neq 0$, $\delta+\Delta \neq 0$. If A has the trace P-(λ,Γ)-property and B has the trace P-(δ,Δ)-property, then*

$$\left|\frac{\operatorname{tr}(PB^*A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB^*)}{\operatorname{tr}(P)}\right|^2 \le \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\sqrt{\frac{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)}{[\operatorname{tr}(P)]^2}}. \tag{6.19}$$

The case of selfadjoint operators is useful for applications.

REMARK 6.3. Assume that A, B, C are selfadjoint operators. If, either $P \in \mathcal{B}_+(H)$, A, B, $C \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, B, $C \in \mathcal{B}(H)$ with PA^2, PB^2, $PC^2 \neq 0$ and m, M, n, $N \in \mathbb{R}$ with $m+M$, $n+N \neq 0$. If (A,C) has the trace P-(m,M)*-property and* (B,C) has the trace P-(n,N)-*property, then*

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(PC^2)}-\frac{\operatorname{tr}(PCA)}{\operatorname{tr}(PC^2)}\frac{\operatorname{tr}(PBC)}{\operatorname{tr}(PC^2)}\right|^2 \leq \frac{1}{4}\cdot\frac{(M-m)^2}{|M+m|}\frac{(N-n)^2}{|N+n|}\sqrt{\frac{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)}{[\operatorname{tr}(PC^2)]^2}}. \tag{6.20}$$

If A has the trace P-(k,K)-*property and* B has the trace P-(l,L)-*property, then*

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right|^2 \leq \frac{1}{4}\cdot\frac{(K-k)^2}{|K+k|}\frac{(L-l)^2}{|L+l|}\sqrt{\frac{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)}{[\operatorname{tr}(P)]^2}}, \tag{6.21}$$

where $k+K, l+L\neq 0$.

6.3. Applications for Convex Functions. In the paper **[53]** we obtained amongst other the following reverse of the Jensen trace inequality:

Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m<M$. If f is a continuously differentiable convex function on $[m,M]$ and $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\geq 0$, then we have

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)}-f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\ &\leq \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \\ &\leq \begin{cases} \frac{1}{2}[f'(M)-f'(m)]\frac{\operatorname{tr}\left(P\left|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \\ \\ \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(P\left|f'(A)-\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \end{cases} \\ &\leq \begin{cases} \frac{1}{2}[f'(M)-f'(m)]\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\ &\leq \frac{1}{4}[f'(M)-f'(m)](M-m). \end{aligned} \tag{6.22}$$

Let $\mathcal{M}_n(\mathbb{C})$ be the space of all square matrices of order n with complex elements and $A\in\mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m<M$. If f is a continuously differentiable convex function on $[m,M]$, then by taking $P=I_n$

in (6.22) we get

$$
\begin{aligned}
(6.23)\qquad 0 &\leq \frac{\operatorname{tr}(f(A))}{n}-f\left(\frac{\operatorname{tr}(A)}{n}\right) \\
&\leq \frac{\operatorname{tr}(f'(A)A)}{n}-\frac{\operatorname{tr}(A)}{n}\cdot\frac{\operatorname{tr}(f'(A))}{n} \\
&\leq \begin{cases} \frac{1}{2}\left[f'(M)-f'(m)\right]\frac{\operatorname{tr}\left(\left|A-\frac{\operatorname{tr}(A)}{n}1_H\right|\right)}{n} \\ \\ \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(\left|f'(A)-\frac{\operatorname{tr}(f'(A))}{n}1_H\right|\right)}{n} \end{cases} \\
&\leq \begin{cases} \frac{1}{2}\left[f'(M)-f'(m)\right]\left[\frac{\operatorname{tr}\left(A^2\right)}{n}-\left(\frac{\operatorname{tr}(A)}{n}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left([f'(A)]^2\right)}{n}-\left(\frac{\operatorname{tr}(f'(A))}{n}\right)^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4}\left[f'(M)-f'(m)\right](M-m).
\end{aligned}
$$

The following reverse inequality also holds:

PROPOSITION 6.6 (Dragomir, 2014, [**60**]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m+M\neq 0$. If f is a continuously differentiable convex function on $[m,M]$ with $f'(m)+f'(M)\neq 0$ and $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\geq 0$, then*

$$
\begin{aligned}
(6.24)\qquad 0 &\leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)}-f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\
&\leq \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \\
&\leq \frac{1}{2}\cdot\frac{|M-m|\,|f'(M)-f'(m)|}{\sqrt{|m+M|}\sqrt{|f'(m)+f'(M)|}}\sqrt[4]{\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)}}.
\end{aligned}
$$

The proof follows by the inequality (6.21) and the details are omitted.

Let $A\in\mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m+M\neq 0$. If f is a continuously differentiable convex function on $[m,M]$ with $f'(m)+f'(M)\neq 0$ then by taking $P=I_n$ in (6.24) we get

$$
\begin{aligned}
(6.25)\qquad 0 &\leq \frac{\operatorname{tr}(f(A))}{n}-f\left(\frac{\operatorname{tr}(A)}{n}\right) \\
&\leq \frac{\operatorname{tr}(f'(A)A)}{n}-\frac{\operatorname{tr}(A)}{n}\cdot\frac{\operatorname{tr}(f'(A))}{n} \\
&\leq \frac{1}{2}\cdot\frac{|M-m|\,|f'(M)-f'(m)|}{\sqrt{|m+M|}\sqrt{|f'(m)+f'(M)|}}\sqrt[4]{\frac{\operatorname{tr}(A^2)}{n}\frac{\operatorname{tr}\left([f'(A)]^2\right)}{n}}.
\end{aligned}
$$

We consider the power function $f:(0,\infty)\to(0,\infty)$, $f(t)=t^r$ with $t\in\mathbb{R}\setminus\{0\}$. For $r\in(-\infty,0)\cup[1,\infty)$, f is convex while for $r\in(0,1)$, f is concave.

Let $r \geq 1$ and A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$0 \leq \frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^r \leq r\left[\frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(PA^{r-1})}{\operatorname{tr}(P)}\right] \leq \frac{1}{2} r \frac{(M-m)(M^{r-1}-m^{r-1})}{(m+M)^{1/2}(m^{r-1}+M^{r-1})^{1/2}} \sqrt[4]{\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PA^{2(p-1)})}{\operatorname{tr}(P)}}. \tag{6.26}$$

Consider the convex function $f : \mathbb{R} \to (0, \infty)$, $f(t) = \exp t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then using (6.24) we have

$$0 \leq \frac{\operatorname{tr}(P \exp A)}{\operatorname{tr}(P)} - \exp\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \leq \frac{\operatorname{tr}(PA \exp A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(P \exp A)}{\operatorname{tr}(P)} \leq \frac{1}{2} \frac{|M-m|(\exp(M) - \exp(m))}{\sqrt{|m+M|}\sqrt{\exp m + \exp M}} \sqrt[4]{\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(P \exp(2A))}{\operatorname{tr}(P)}}. \tag{6.27}$$

7. SOME HÖLDER TYPE TRACE INEQUALITIES

7.1. Some Preliminary Facts. Assume that A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notation

$$A \sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2}\right)^\nu A^{1/2},$$

for the *weighted geometric mean.* When $\nu = \frac{1}{2}$, we write $A \sharp B$ for brevity.

We have the following Hölder type trace inequality:

THEOREM 7.1 (Dragomir, 2015, [**68**]). *If A, B are positive invertible operators, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $A^p, B^q \in \mathcal{B}_1(H)$, then $B^q \sharp_{1/p} A^p \in \mathcal{B}_1(H)$ and*

$$\operatorname{tr}\left(B^q \sharp_{1/p} A^p\right) \leq [\operatorname{tr}(A^p)]^{1/p} [\operatorname{tr}(B^q)]^{1/q}. \tag{7.1}$$

In particular, if $A^2, B^2 \in \mathcal{B}_1(H)$, then $B^2 \sharp A^2 \in \mathcal{B}_1(H)$ and

$$\left[\operatorname{tr}\left(B^2 \sharp A^2\right)\right]^2 \leq \operatorname{tr}\left(A^2\right) \operatorname{tr}\left(B^2\right). \tag{7.2}$$

PROOF. In [**74**], the authors obtained the following Hölder's type inequality for the weighted geometric mean:

$$\left\langle B^q \sharp_{1/p} A^p x, x \right\rangle \leq \langle A^p x, x \rangle^{1/p} \langle B^q x, x \rangle^{1/q} \tag{7.3}$$

for any $x \in H$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. Then by (7.3) and Hölder's inequality we have

$$\begin{aligned}
\operatorname{tr}\left(B^q\sharp_{1/p}A^p\right) &= \sum_{i\in I}\left\langle B^q\sharp_{1/p}A^p e_i, e_i\right\rangle \\
&\le \sum_{i\in I}\left\langle A^p e_i, e_i\right\rangle^{1/p}\left\langle B^q e_i, e_i\right\rangle^{1/q} \\
&\le \left(\sum_{i\in I}\left[\left\langle A^p e_i, e_i\right\rangle^{1/p}\right]^p\right)^{1/p}\left(\sum_{i\in I}\left[\left\langle B^q e_i, e_i\right\rangle^{1/q}\right]^q\right)^{1/q} \\
&= \left(\sum_{i\in I}\left\langle A^p e_i, e_i\right\rangle\right)^{1/p}\left(\sum_{i\in I}\left\langle B^q e_i, e_i\right\rangle\right)^{1/q} = \left[\operatorname{tr}\left(A^p\right)\right]^{1/p}\left[\operatorname{tr}\left(B^q\right)\right]^{1/q},
\end{aligned}$$

which proves the desired inequality (7.1). ∎

COROLLARY 7.2. *If A_k, B_k are positive invertible operators, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and A_k^p, $B_k^q \in \mathcal{B}_1(H)$ for $k\in\{1,...,n\}$, then $B_k^q\sharp_{1/p}A_k^p \in \mathcal{B}_1(H)$ for $k\in\{1,...,n\}$ and for any $p_k\ge 0$, $k\in\{1,...,n\}$ we have*

$$\operatorname{tr}\left(\sum_{k=1}^n p_k B_k^q\sharp_{1/p}A_k^p\right) \le \left(\operatorname{tr}\left(\sum_{k=1}^n p_k A_k^p\right)\right)^{1/p}\left(\operatorname{tr}\left(\sum_{k=1}^n p_k B_k^q\right)\right)^{1/q}. \tag{7.4}$$

In particular, if A_k^2, $B_k^2\in\mathcal{B}_1(H)$ for $k\in\{1,...,n\}$ then $B_k^2\sharp A_k^2\in\mathcal{B}_1(H)$ for $k\in\{1,...,n\}$ and for any $p_k\ge 0$, $k\in\{1,...,n\}$ we have

$$\left[\operatorname{tr}\left(\sum_{k=1}^n p_k B_k^2\sharp A_k^2\right)\right]^2 \le \operatorname{tr}\left(\sum_{k=1}^n p_k A_k^2\right)\operatorname{tr}\left(\sum_{k=1}^n p_k B_k^2\right). \tag{7.5}$$

PROOF. Using Hölder's weighted discrete inequality we have

$$\begin{aligned}
\operatorname{tr}\left(\sum_{k=1}^n p_k B_k^q\sharp_{1/p}A_k^p\right) &= \sum_{k=1}^n p_k \operatorname{tr}\left(B_k^q\sharp_{1/p}A_k^p\right) \le \sum_{k=1}^n p_k\left[\operatorname{tr}\left(A_k^p\right)\right]^{1/p}\left[\operatorname{tr}\left(B_k^q\right)\right]^{1/q} \\
&\le \left(\sum_{k=1}^n p_k\left(\left[\operatorname{tr}\left(A_k^p\right)\right]^{1/p}\right)^p\right)^{1/p}\left(\sum_{k=1}^n p_k\left(\left[\operatorname{tr}\left(B_k^q\right)\right]^{1/q}\right)^q\right)^{1/q} \\
&= \left(\sum_{k=1}^n p_k\operatorname{tr}\left(A_k^p\right)\right)^{1/p}\left(\sum_{k=1}^n p_k\operatorname{tr}\left(B_k^q\right)\right)^{1/q} \\
&= \left(\operatorname{tr}\left(\sum_{k=1}^n p_k A_k^p\right)\right)^{1/p}\left(\operatorname{tr}\left(\sum_{k=1}^n p_k B_k^q\right)\right)^{1/q}
\end{aligned}$$

and the inequality (7.4) is proved. ∎

THEOREM 7.3 (Dragomir, 2015, [**68**]). *If A, B are positive invertible operators, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $C\in\mathcal{B}_1(H)$, $C\ge 0$ then CA^p, CB^q, $C\left(B^q\sharp_{1/p}A^p\right)\in\mathcal{B}_1(H)$ and*

$$\operatorname{tr}\left(C\left(B^q\sharp_{1/p}A^p\right)\right) \le \left[\operatorname{tr}\left(CA^p\right)\right]^{1/p}\left[\operatorname{tr}\left(CB^q\right)\right]^{1/q}. \tag{7.6}$$

In particular, if $C\in\mathcal{B}_1(H)$, then CA^2, CB^2, $C\left(B^2\sharp A^2\right)\in\mathcal{B}_1(H)$ and

$$\left[\operatorname{tr}\left(C\left(B^2\sharp A^2\right)\right)\right]^2 \le \operatorname{tr}\left(CA^2\right)\operatorname{tr}\left(CB^2\right). \tag{7.7}$$

PROOF. From the inequality (7.3) we have

$$\left\langle B^{q}\sharp_{1/p}A^{p}C^{1/2}x,C^{1/2}x\right\rangle \leq \left\langle A^{p}C^{1/2}x,C^{1/2}x\right\rangle^{1/p}\left\langle B^{q}C^{1/2}x,C^{1/2}x\right\rangle^{1/q}$$

for any $x\in H$, which is equivalent to

$$\left\langle C^{1/2}B^{q}\sharp_{1/p}A^{p}C^{1/2}x,x\right\rangle \leq \left\langle C^{1/2}A^{p}C^{1/2}x,x\right\rangle^{1/p}\left\langle C^{1/2}B^{q}C^{1/2}x,x\right\rangle^{1/q} \tag{7.8}$$

for any $x\in H$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. Then by (7.8) and Hölder's inequality we have

$$\begin{aligned}
&\operatorname{tr}\left(C\left(B^{q}\sharp_{1/p}A^{p}\right)\right)\\
&=\operatorname{tr}\left(C^{1/2}\left(B^{q}\sharp_{1/p}A^{p}\right)C^{1/2}\right)=\sum_{i\in I}\left\langle C^{1/2}\left(B^{q}\sharp_{1/p}A^{p}\right)C^{1/2}e_i,e_i\right\rangle\\
&\leq\sum_{i\in I}\left\langle C^{1/2}A^{p}C^{1/2}e_i,e_i\right\rangle^{1/p}\left\langle C^{1/2}B^{q}C^{1/2}e_i,e_i\right\rangle^{1/q}\\
&\leq\left(\sum_{i\in I}\left[\left\langle C^{1/2}A^{p}C^{1/2}e_i,e_i\right\rangle^{1/p}\right]^{p}\right)^{1/p}\left(\sum_{i\in I}\left[\left\langle C^{1/2}B^{q}C^{1/2}e_i,e_i\right\rangle^{1/q}\right]^{q}\right)^{1/q}\\
&=\left(\sum_{i\in I}\left\langle C^{1/2}A^{p}C^{1/2}e_i,e_i\right\rangle\right)^{1/p}\left(\sum_{i\in I}\left\langle C^{1/2}B^{q}C^{1/2}e_i,e_i\right\rangle\right)^{1/q}\\
&=\left[\operatorname{tr}\left(C^{1/2}A^{p}C^{1/2}\right)\right]^{1/p}\left[\operatorname{tr}\left(C^{1/2}B^{q}C^{1/2}\right)\right]^{1/q}=\left[\operatorname{tr}\left(CA^{p}\right)\right]^{1/p}\left[\operatorname{tr}\left(CB^{q}\right)\right]^{1/q},
\end{aligned}$$

which proves the desired result (7.6). ∎

COROLLARY 7.4. *If A_k, B_k are positive invertible operators, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $C_k\in\mathcal{B}_1(H)$, $C_k\geq 0$ for $k\in\{1,...,n\}$ then $C_kA_k^p$, $C_kB_k^q$, $C_k\left(B_k^q\sharp_{1/p}A_k^p\right)\in\mathcal{B}_1(H)$ for $k\in\{1,...,n\}$ and we have*

$$\operatorname{tr}\left(\sum_{k=1}^{n}C_k\left(B_k^q\sharp_{1/p}A_k^p\right)\right)\leq\left(\operatorname{tr}\left(\sum_{k=1}^{n}C_kA_k^p\right)\right)^{1/p}\left(\operatorname{tr}\left(\sum_{k=1}^{n}C_kB_k^q\right)\right)^{1/q}. \tag{7.9}$$

In particular, $C_kA_k^2$, $C_kB_k^2$, $C_k\left(B_k^2\sharp A_k^2\right)\in\mathcal{B}_1(H)$ for $k\in\{1,...,n\}$ and we have

$$\left[\operatorname{tr}\left(\sum_{k=1}^{n}C_k\left(B_k^2\sharp A_k^2\right)\right)\right]^2\leq\operatorname{tr}\left(\sum_{k=1}^{n}C_kA_k^2\right)\operatorname{tr}\left(\sum_{k=1}^{n}C_kB_k^2\right). \tag{7.10}$$

The proof follows by (7.6) on making use of a similar argument to the one in the proof of Corollary 7.2.

7.1.1. *Some Reverse Vector Inequalities.* We have the following reverse of Hölder's vector inequality for operators:

THEOREM 7.5 (Dragomir, 2015, [**68**]). *Let A and B be two positive invertible operators, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and m, $M>0$ such that*

$$m^pB^q\leq A^p\leq M^pB^q. \tag{7.11}$$

Then

$$\left\langle A^px,x\right\rangle^{1/p}\left\langle B^qx,x\right\rangle^{1/q}\leq\exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p-1\right)^2\right]\left\langle B^q\sharp_{1/p}A^px,x\right\rangle \tag{7.12}$$

for any $x\in H$.

PROOF. In [**57**] we proved the following double inequality that provides a refinement and a reverse of the *arithmetic mean - geometric mean* inequality:

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right]\le\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\le\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right] \tag{7.13}$$

for any $a,b>0$ and $\nu\in[0,1]$.

If $a,b\in[t,T]\subset(0,\infty)$ and since

$$0<\frac{\max\{a,b\}}{\min\{a,b\}}-1\le\frac{T}{t}-1,$$

hence

$$\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\le\left(\frac{T}{t}-1\right)^2.$$

Therefore, by (7.13) we get

$$(1-\nu)a+\nu b\le a^{1-\nu}b^{\nu}\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{T}{t}-1\right)^2\right], \tag{7.14}$$

for any $a,b\in[t,T]$ and $\nu\in(0,1)$.

Now, if C is an operator with $tI\le C\le TI$ then for $p>1$ we have $t^pI\le C^p\le T^pI$. Using the functional calculus we get from (7.14) for $\nu=\frac{1}{p}$ that

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}C^p\le\exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]d^{1-\frac{1}{p}}C,$$

namely, the vector inequality,

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}\langle C^py,y\rangle\le\exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]d^{1-\frac{1}{p}}\langle Cy,y\rangle, \tag{7.15}$$

for any $y\in H$, $\|y\|=1$ and $d\in[t^p,T^p]$.

Since $d=\langle C^py,y\rangle\in[t^p,T^p]$ for any $y\in H$, $\|y\|=1$, hence by (7.15) we have

$$\begin{aligned}&\left(1-\frac{1}{p}\right)\langle C^py,y\rangle+\frac{1}{p}\langle C^py,y\rangle\\&\le\exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]\langle C^py,y\rangle^{1-\frac{1}{p}}\langle Cy,y\rangle,\end{aligned}$$

that is equivalent to

$$\langle C^py,y\rangle\le\exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]\langle C^py,y\rangle^{1-\frac{1}{p}}\langle Cy,y\rangle,$$

and by division with $\langle C^py,y\rangle^{1-\frac{1}{p}}>0$, $y\in H$, $\|y\|=1$, to

$$\langle C^py,y\rangle^{1/p}\le\exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]\langle Cy,y\rangle. \tag{7.16}$$

If $z \in H$ with $z \neq 0$, then by taking $y = \frac{z}{\|z\|}$ in (7.16) we get

$$\langle C^p z, z\rangle^{1/p} \langle z, z\rangle^{1/q} \le \exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p - 1\right)^2\right] \langle Cz, z\rangle, \tag{7.17}$$

for any $z \in H$.

Now, from (7.11) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^p I \le B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \le M^p I$ and by taking the power $\frac{1}{p}$ we get $mI \le \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} \le MI$.

By writing the inequality (7.17) for $C = \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}}$, $t = m$, $T = M$ and $z = B^{\frac{q}{2}} x$, with $x \in H$, we have

$$\begin{aligned}&\left\langle B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} B^{\frac{q}{2}} x, B^{\frac{q}{2}} x\right\rangle^{1/p} \left\langle B^{\frac{q}{2}} x, B^{\frac{q}{2}} x\right\rangle^{1/q} \\ &\le \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p - 1\right)^2\right] \left\langle \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} B^{\frac{q}{2}} x, B^{\frac{q}{2}} x\right\rangle,\end{aligned}$$

namely

$$\begin{aligned}&\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} \\ &\le \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p - 1\right)^2\right] \left\langle B^{\frac{q}{2}} \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} B^{\frac{q}{2}} x, x\right\rangle,\end{aligned}$$

for any $x \in H$, and the inequality (7.12) is proved. ∎

REMARK 7.1. We observe, for A and B two positive invertible operators, that the condition (7.11) is equivalent to following condition

$$mI \le \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} \le MI. \tag{7.18}$$

If we assume that $rB^q \le A^p \le RB^q$, then by (7.12) we have the inequality

$$\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} \le \exp\left[\frac{1}{2pq}\left(\frac{R}{r} - 1\right)^2\right] \left\langle B^q \sharp_{1/p} A^p x, x\right\rangle \tag{7.19}$$

for any $x \in H$.

The following particular case is related to Schwarz's trace inequality:

COROLLARY 7.6. *Let A and B be two positive invertible operators and $m, M > 0$ such that*

$$mI \le \left(B^{-1} A^2 B^{-1}\right)^{\frac{1}{2}} \le MI, \tag{7.20}$$

then we have

$$\left\langle A^2 x, x\right\rangle^{1/2} \left\langle B^2 x, x\right\rangle^{1/2} \le \exp\left[\frac{1}{8}\left(\left(\frac{M}{m}\right)^2 - 1\right)^2\right] \left\langle A^2 \sharp B^2 x, x\right\rangle \tag{7.21}$$

for any $x \in H$.

Under more suitable conditions for the operators involved, we have:

COROLLARY 7.7. *Assume that A and B satisfy the conditions*

$$m_1 I \leq A \leq M_1 I,\ m_2 I \leq B \leq M_2 I \tag{7.22}$$

for some $0 < m_1 < M_1$ and $0 < m_2 < M_2$. Then we have

$$\begin{aligned} &\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} \\ &\leq \exp\left[\frac{1}{2pq}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q - 1\right)^2\right] \left\langle B^q \sharp_{1/p} A^p x, x\right\rangle, \end{aligned} \tag{7.23}$$

for any $x \in H$.

In particular, we have

$$\left\langle A^2 x, x\right\rangle^{1/2} \left\langle B^2 x, x\right\rangle^{1/2} \leq \exp\left[\frac{1}{8}\left(\left(\frac{M_1 M_2}{m_1 m_2}\right)^2 - 1\right)^2\right] \left\langle A^2 \sharp B^2 x, x\right\rangle, \tag{7.24}$$

for any $x \in H$.

8. REFINEMENTS AND REVERSES OF YOUNG INEQUALITY

8.1. Trace Inequalities Via Kittaneh-Manasrah Results. Kittaneh and Manasrah [**95**], [**96**] provided a refinement and a reverse for *Young's inequality* as follows:

$$r\left(\sqrt{a} - \sqrt{b}\right)^2 \leq (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \leq R\left(\sqrt{a} - \sqrt{b}\right)^2, \tag{8.1}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. The case $\nu = \frac{1}{2}$ reduces (8.1) to an identity.

We can give a simple direct proof for (8.1) as follows. Recall the following result obtained by the author in 2006 [**37**] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} &n \min_{j\in\{1,2,\ldots,n\}} \{p_j\} \left[\frac{1}{n}\sum_{j=1}^n \Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^n x_j\right)\right] \\ &\leq \frac{1}{P_n}\sum_{j=1}^n p_j \Phi(x_j) - \Phi\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right) \\ &\leq n \max_{j\in\{1,2,\ldots,n\}} \{p_j\} \left[\frac{1}{n}\sum_{j=1}^n \Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^n x_j\right)\right], \end{aligned} \tag{8.2}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\ldots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\ldots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^n p_j > 0$. For $n = 2$, we deduce from (8.2) that

$$\begin{aligned} &2\min\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \\ &\leq \nu\Phi(x) + (1-\nu)\Phi(y) - \Phi[\nu x + (1-\nu) y] \\ &\leq 2\max\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{8.3}$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0,1]$. If we take $\Phi(x) = \exp(x)$, then we get from (8.3)

$$2\min\{\nu, 1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2} - \exp\left(\frac{x+y}{2}\right)\right] \\ \leq \nu\exp(x) + (1-\nu)\exp(y) - \exp[\nu x + (1-\nu)y] \\ \leq 2\max\{\nu, 1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2} - \exp\left(\frac{x+y}{2}\right)\right] \tag{8.4}$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0,1]$. Further, denote $\exp(x) = a$, $\exp(y) = b$ with $a, b > 0$, then from (8.4) we obtain the inequality (8.1).

We have:

THEOREM 8.1 (Dragomir, 2015, [**66**]). *Let A, B be two positive operators and $P, Q \in \mathcal{B}_1(H)$ with $P, Q > 0$. Then for any $\nu \in [0,1]$ we have*

$$r\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - 2\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}\right) \\ \leq (1-\nu)\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \nu\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} - \frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)} \\ \leq R\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - 2\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}\right), \tag{8.5}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

PROOF. Fix $b > 0$, and by using the functional calculus for the operator A, we have from (8.1) that

$$r\left(\langle Ax, x\rangle - 2\sqrt{b}\left\langle A^{1/2}x, x\right\rangle + b\langle x, x\rangle\right) \\ \leq (1-\nu)\langle Ax, x\rangle + \nu b\langle x, x\rangle - b^{\nu}\left\langle A^{1-\nu}x, x\right\rangle \\ \leq R\left(\langle Ax, x\rangle - 2\sqrt{b}\left\langle A^{1/2}x, x\right\rangle + b\langle x, x\rangle\right) \tag{8.6}$$

for any $x \in H$.

Now, fix $x \in H \setminus \{0\}$. Then by using the functional calculus for the operator B, we have by (8.6) that

$$r\left(\langle Ax, x\rangle\|y\|^2 - 2\left\langle A^{1/2}x, x\right\rangle\left\langle B^{1/2}y, y\right\rangle + \|x\|^2\langle By, y\rangle\right) \\ \leq (1-\nu)\langle Ax, x\rangle\|y\|^2 + \nu\|x\|^2\langle By, y\rangle - \langle B^{\nu}y, y\rangle\left\langle A^{1-\nu}x, x\right\rangle \\ \leq R\left(\langle Ax, x\rangle\|y\|^2 - 2\left\langle A^{1/2}x, x\right\rangle\left\langle B^{1/2}y, y\right\rangle + \|x\|^2\langle By, y\rangle\right) \tag{8.7}$$

for any $x, y \in H$ and $\nu \in [0,1]$.

This inequality is of interest in itself as well.

Now, let $x = P^{1/2}e$, $y = Q^{1/2}f$ where e, $f \in H$. Then by (8.7) we get

$$\begin{aligned} &r\left(\left\langle P^{1/2}AP^{1/2}e,e\right\rangle \langle Qf,f\rangle \right. \\ &-2\left\langle P^{1/2}A^{1/2}P^{1/2}e,e\right\rangle \left\langle Q^{1/2}B^{1/2}Q^{1/2}f,f\right\rangle \\ &\left.+\langle Pe,e\rangle \left\langle Q^{1/2}BQ^{1/2}f,f\right\rangle\right) \\ &\leq (1-\nu)\left\langle P^{1/2}AP^{1/2}e,e\right\rangle \langle Qf,f\rangle + \nu \langle Pe,e\rangle \left\langle Q^{1/2}BQ^{1/2}f,f\right\rangle \\ &-\left\langle P^{1/2}A^{1-\nu}P^{1/2}e,e\right\rangle \left\langle Q^{1/2}B^{\nu}Q^{1/2}f,f\right\rangle \\ &\leq R\left(\left\langle P^{1/2}AP^{1/2}e,e\right\rangle \langle Qf,f\rangle \right. \\ &-2\left\langle P^{1/2}A^{1/2}P^{1/2}e,e\right\rangle \left\langle Q^{1/2}B^{1/2}Q^{1/2}f,f\right\rangle \\ &\left.+\langle Pe,e\rangle \left\langle Q^{1/2}BQ^{1/2}f,f\right\rangle\right) \end{aligned} \tag{8.8}$$

for any e, $f \in H$.

Let $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ be two orthonormal bases of H. If we take in (8.8) $e = e_i$, $i \in I$ and $f = f_j$, $j \in J$ and summing over $i \in I$ and $j \in J$, then we get

$$\begin{aligned} &r\left(\sum_{i\in I}\left\langle P^{1/2}AP^{1/2}e_i,e_i\right\rangle \sum_{j\in J}\langle Qf_j,f_j\rangle \right. \\ &-2\sum_{i\in I}\left\langle P^{1/2}A^{1/2}P^{1/2}e_i,e_i\right\rangle \sum_{j\in J}\left\langle Q^{1/2}B^{1/2}Q^{1/2}f_j,f_j\right\rangle \\ &\left.+\sum_{i\in I}\langle Pe_i,e_i\rangle \sum_{j\in J}\left\langle Q^{1/2}BQ^{1/2}f_j,f_j\right\rangle\right) \\ &\leq (1-\nu)\sum_{i\in I}\left\langle P^{1/2}AP^{1/2}e_i,e_i\right\rangle \sum_{j\in J}\langle Qf_j,f_j\rangle \\ &+\nu\sum_{i\in I}\langle Pe_i,e_i\rangle \sum_{j\in J}\left\langle Q^{1/2}BQ^{1/2}f_j,f_j\right\rangle \\ &-\sum_{i\in I}\left\langle P^{1/2}A^{1-\nu}P^{1/2}e_i,e_i\right\rangle \sum_{j\in J}\left\langle Q^{1/2}B^{\nu}Q^{1/2}f_j,f_j\right\rangle \\ &\leq R\left(\sum_{i\in I}\left\langle P^{1/2}AP^{1/2}e_i,e_i\right\rangle \sum_{j\in J}\langle Qf_j,f_j\rangle \right. \\ &-2\sum_{i\in I}\left\langle P^{1/2}A^{1/2}P^{1/2}e_i,e_i\right\rangle \sum_{j\in J}\left\langle Q^{1/2}B^{1/2}Q^{1/2}f_j,f_j\right\rangle \\ &\left.+\sum_{i\in I}\langle Pe_i,e_i\rangle \sum_{j\in J}\left\langle Q^{1/2}BQ^{1/2}f_j,f_j\right\rangle\right). \end{aligned} \tag{8.9}$$

Using the properties of the trace we get

$$\begin{aligned} &r\left(\operatorname{tr}(PA)\operatorname{tr}(Q) - 2\operatorname{tr}\left(PA^{1/2}\right)\operatorname{tr}\left(QB^{1/2}\right) + \operatorname{tr}(P)\operatorname{tr}(QB)\right) \\ &\leq (1-\nu)\operatorname{tr}(PA)\operatorname{tr}(Q) + \nu\operatorname{tr}(P)\operatorname{tr}(QB) - \operatorname{tr}\left(PA^{1-\nu}\right)\operatorname{tr}(QB^{\nu}) \\ &\leq R\left(\operatorname{tr}(PA)\operatorname{tr}(Q) - 2\operatorname{tr}\left(PA^{1/2}\right)\operatorname{tr}\left(QB^{1/2}\right) + \operatorname{tr}(P)\operatorname{tr}(QB)\right) \end{aligned}$$

and the inequality (8.5) is proved. ∎

COROLLARY 8.2. *Let A be a positive operator and $P \in \mathcal{B}_1(H)$ with $P > 0$. Then for any $\nu \in [0,1]$ we have*

$$2r\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\right)^2\right) \le \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PA^{\nu})}{\operatorname{tr}(P)} \le 2R\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\right)^2\right), \tag{8.10}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

REMARK 8.1. If P, Q are positive invertible operators with $P, Q \in \mathcal{B}_1(H)$, then by (8.10) for $A = P^{-1/2}QP^{-1/2}$ we get

$$2r\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)^2\right) \le \frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(P\sharp_{1-\nu}Q)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\sharp_{\nu}Q)}{\operatorname{tr}(P)} \le 2R\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)^2\right), \tag{8.11}$$

where the operator weighted geometric mean is defined as

$$A\sharp_{\nu}B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2}. \tag{8.12}$$

When $\nu = \frac{1}{2}$, we write $A\sharp B$ for brevity.

COROLLARY 8.3. *Let A, B two positive operators and $P, Q \in \mathcal{B}_1(H)$ with $P, Q > 0$. If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have*

$$\begin{aligned} & t\left(\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - 2\frac{\operatorname{tr}\left(PA^{p/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{q/2}\right)}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}(QB^q)}{\operatorname{tr}(Q)}\right) \\ & \le \frac{1}{p}\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} + \frac{1}{q}\frac{\operatorname{tr}(QB^q)}{\operatorname{tr}(Q)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} \\ & \le T\left(\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - 2\frac{\operatorname{tr}\left(PA^{p/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{q/2}\right)}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}(QB^q)}{\operatorname{tr}(Q)}\right), \end{aligned} \tag{8.13}$$

where $t = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

The proof follows by (8.5) on replacing A with A^p, B with B^q and $\nu = \frac{1}{q}$.

REMARK 8.2. If P, Q, S, V are positive invertible operators with $P, Q, S, V \in \mathcal{B}_1(H)$, then by (8.13) we get for $A = P^{-1/2}SP^{-1/2}$ and $B = Q^{-1/2}VQ^{-1/2}$ that

$$\begin{aligned} & t\left(\frac{\operatorname{tr}(P\sharp_p S)}{\operatorname{tr}(P)} - 2\frac{\operatorname{tr}\left(P\sharp_{p/2}S\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\sharp_{q/2}V\right)}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}(Q\sharp_q V)}{\operatorname{tr}(Q)}\right) \\ & \le \frac{1}{p}\frac{\operatorname{tr}(P\sharp_p S)}{\operatorname{tr}(P)} + \frac{1}{q}\frac{\operatorname{tr}(Q\sharp_q V)}{\operatorname{tr}(Q)} - \frac{\operatorname{tr}(S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(V)}{\operatorname{tr}(Q)} \\ & \le T\left(\frac{\operatorname{tr}(P\sharp_p S)}{\operatorname{tr}(P)} - 2\frac{\operatorname{tr}\left(P\sharp_{p/2}S\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\sharp_{q/2}V\right)}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}(Q\sharp_q V)}{\operatorname{tr}(Q)}\right), \end{aligned} \tag{8.14}$$

where $t = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

In particular, if we take in (8.14) $S = Q$ and $V = P$, then we get

$$t\left(\frac{\operatorname{tr}(P\sharp_p Q)}{\operatorname{tr}(P)} - 2\frac{\operatorname{tr}\left(P\sharp_{p/2} Q\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\sharp_{q/2} P\right)}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}(Q\sharp_q P)}{\operatorname{tr}(Q)}\right)$$
$$\le \frac{1}{p}\frac{\operatorname{tr}(P\sharp_p Q)}{\operatorname{tr}(P)} + \frac{1}{q}\frac{\operatorname{tr}(Q\sharp_q P)}{\operatorname{tr}(Q)} - 1$$
$$\le T\left(\frac{\operatorname{tr}(P\sharp_p Q)}{\operatorname{tr}(P)} - 2\frac{\operatorname{tr}\left(P\sharp_{p/2} Q\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\sharp_{q/2} P\right)}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}(Q\sharp_q P)}{\operatorname{tr}(Q)}\right), \tag{8.15}$$

where $t = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

8.2. Trace Inequalities Via Liao-Wu-Zhao and Zuo-Shi-Fujii Results. We consider the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{8.16}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^r\left(\frac{a}{b}\right)a^{1-\nu}b^\nu \le (1-\nu)a + \nu b \le K^R\left(\frac{a}{b}\right)a^{1-\nu}b^\nu, \tag{8.17}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (8.17) was obtained by Zuo et al. in [**140**] while the second by Liao et al. [**99**].

We can give a simple direct proof for (8.17) as follows.

Indeed, if we write the inequality (8.3) for the convex function $\Phi(x) = -\ln x$, and for the positive numbers a and b we get

$$2\min\{\nu, 1-\nu\}\left[\ln\left(\frac{a+b}{2}\right) - \frac{\ln a + \ln b}{2}\right]$$
$$\le \ln[\nu b + (1-\nu)a] - (1-\nu)\ln a - \nu\ln b$$
$$\le 2\max\{\nu, 1-\nu\}\left[\ln\left(\frac{a+b}{2}\right) - \frac{\ln a + \ln b}{2}\right]$$

that is equivalent to

$$\min\{\nu, 1-\nu\}\ln\left(\frac{a+b}{2\sqrt{ab}}\right)^2 \le \ln\left[\frac{\nu b + (1-\nu)a}{a^{1-\nu}b^\nu}\right]$$
$$\le \max\{\nu, 1-\nu\}\ln\left(\frac{a+b}{2\sqrt{ab}}\right)^2$$

and to (8.17), as stated.

If $a \in [m_1, M_1]$ and $b \in [m_2, M_2]$ with $0 < m_1 < M_1$, $0 < m_2 < M_2$ then

$$\frac{m_1}{M_2} \le \frac{a}{b} \le \frac{M_1}{m_2}.$$

Denote

$$m =: \min_{(a,b)\in[m_1,M_1]\times[m_2,M_2]} K\left(\frac{a}{b}\right) \text{ and } M =: \max_{(a,b)\in[m_1,M_1]\times[m_2,M_2]} K\left(\frac{a}{b}\right).$$

Taking into account the properties of Kantorovich's ratio we have

$$(8.18)\qquad m := \begin{cases} K\left(\frac{M_1}{m_2}\right) > 1 \text{ if } \frac{M_1}{m_2} < 1, \\ 1 \text{ if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}, \\ K\left(\frac{m_1}{M_2}\right) > 1 \text{ if } 1 < \frac{m_1}{M_2}, \end{cases} = \begin{cases} K\left(\frac{m_2}{M_1}\right) > 1 \text{ if } \frac{M_1}{m_2} < 1, \\ 1 \text{ if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}, \\ K\left(\frac{M_2}{m_1}\right) > 1 \text{ if } 1 < \frac{m_1}{M_2} \end{cases}$$

and

$$(8.19)\qquad M := \begin{cases} K\left(\frac{m_1}{M_2}\right) > 1 \text{ if } \frac{M_1}{m_2} < 1, \\ \max\left\{K\left(\frac{m_1}{M_2}\right), K\left(\frac{M_1}{m_2}\right)\right\} > 1 \text{ if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}, \\ K\left(\frac{M_1}{m_2}\right) > 1 \text{ if } 1 < \frac{m_1}{M_2}, \end{cases}$$

$$= \begin{cases} K\left(\frac{M_2}{m_1}\right) > 1 \text{ if } \frac{M_1}{m_2} < 1, \\ \max\left\{K\left(\frac{M_2}{m_1}\right), K\left(\frac{M_1}{m_2}\right)\right\} > 1 \text{ if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}, \\ K\left(\frac{M_1}{m_2}\right) > 1 \text{ if } 1 < \frac{m_1}{M_2}. \end{cases}$$

We have the following result:

THEOREM 8.4 (Dragomir, 2015, [**66**]). *Let A, B be two operators such that*

$$(8.20)\qquad 0 < m_1 I \le A < M_1 I,\ 0 < m_2 I \le B \le M_2 I$$

and $P, Q \in \mathcal{B}_1(H)$ with $P, Q > 0$. Then for any $\nu \in [0,1]$, we have for m, M as defined by (8.18) and (8.19) that

$$(8.21)\qquad m^r \frac{\operatorname{tr}(PA^{1-\nu})}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)} \le (1-\nu)\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \nu \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} \le M^R \frac{\operatorname{tr}(PA^{1-\nu})}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)},$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

In particular, we have

$$(8.22)\qquad m^{1/2} \frac{\operatorname{tr}(PA^{1/2})}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB^{1/2})}{\operatorname{tr}(Q)} \le \frac{1}{2}\left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}\right] \le M^{1/2} \frac{\operatorname{tr}(PA^{1/2})}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB^{1/2})}{\operatorname{tr}(Q)}.$$

PROOF. From (8.17) we have

$$(8.23)\qquad m^r a^{1-\nu} b^{\nu} \le (1-\nu) a + \nu b \le M^R a^{1-\nu} b^{\nu},$$

where $a \in [m_1, M_1]$, $b \in [m_2, M_2]$ and $\nu \in [0,1]$.

Using the functional calculus for the operator A, we have

$$m^r b^\nu \left\langle A^{1-\nu}x, x\right\rangle \le (1-\nu)\langle Ax, x\rangle + \nu b\|x\|^2 \le M^R b^\nu \left\langle A^{1-\nu}x, x\right\rangle, \tag{8.24}$$

for any $x \in H$, $b \in [m_2, M_2]$ and $\nu \in [0,1]$.

Using the functional calculus for B we get from (8.24) that

$$\begin{aligned} m^r \left\langle A^{1-\nu}x, x\right\rangle \langle B^\nu y, y\rangle &\le (1-\nu)\langle Ax, x\rangle \|y\|^2 + \nu\|x\|^2\langle By, y\rangle \\ &\le M^R \left\langle A^{1-\nu}x, x\right\rangle \langle B^\nu y, y\rangle, \end{aligned} \tag{8.25}$$

for any $x, y \in H$ and $\nu \in [0,1]$.

This is an inequality of interest in itself as well.

Further, let $x = P^{1/2}e$, $y = Q^{1/2}f$ where $e, f \in H$. Then by (8.25) we have

$$\begin{aligned} & m^r \left\langle P^{1/2}A^{1-\nu}P^{1/2}e, e\right\rangle \left\langle Q^{1/2}B^\nu Q^{1/2}f, f\right\rangle \\ & \le (1-\nu)\left\langle P^{1/2}AP^{1/2}e, e\right\rangle \langle Qf, f\rangle + \nu\langle Pe, e\rangle \left\langle Q^{1/2}BQ^{1/2}f, f\right\rangle \\ & \le M^R \left\langle P^{1/2}A^{1-\nu}P^{1/2}e, e\right\rangle \left\langle Q^{1/2}B^\nu Q^{1/2}f, f\right\rangle, \end{aligned} \tag{8.26}$$

for any $e, f \in H$ and $\nu \in [0,1]$.

Now, on making use of a similar argument as in the proof of Theorem 8.1, we get the desired result (8.21). ∎

REMARK 8.3. Let A, B be two operators such that the condition (8.20) is valid and $P \in \mathcal{B}_1(H)$ with $P > 0$. Then for any $\nu \in [0,1]$, we have for m, M as defined by (8.18) and (8.19) that

$$\begin{aligned} m^r \frac{\operatorname{tr}(PA^{1-\nu})}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PB^\nu)}{\operatorname{tr}(P)} &\le \frac{\operatorname{tr}(P[(1-\nu)A + \nu B])}{\operatorname{tr}(P)} \\ &\le M^R \frac{\operatorname{tr}(PA^{1-\nu})}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PB^\nu)}{\operatorname{tr}(P)}, \end{aligned} \tag{8.27}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

In particular, we have

$$\begin{aligned} m^{1/2} \frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}\left(PB^{1/2}\right)}{\operatorname{tr}(P)} &\le \frac{\operatorname{tr}\left(P\left(\frac{A+B}{2}\right)\right)}{\operatorname{tr}(P)} \\ &\le M^{1/2} \frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}\left(PB^{1/2}\right)}{\operatorname{tr}(P)}. \end{aligned} \tag{8.28}$$

For $0 < m_1 < M_1$, $0 < m_2 < M_2$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we define

$$m_{p,q} := \begin{cases} K\left(\frac{M_1^p}{m_2^q}\right) > 1 \text{ if } \frac{M_1^p}{m_2^q} < 1, \\ \\ 1 \text{ if } \frac{m_1^p}{M_2^q} \le 1 \le \frac{M_1^p}{m_2^q}, \\ \\ K\left(\frac{M_2^q}{m_1^p}\right) > 1 \text{ if } 1 < \frac{m_1^p}{M_2^q} \end{cases} \tag{8.29}$$

and

$$M_{p,q} := \begin{cases} K\left(\frac{M_2^q}{m_1^p}\right) > 1 \text{ if } \frac{M_1^p}{m_2^q} < 1, \\ \max\left\{K\left(\frac{M_2^q}{m_1^p}\right), K\left(\frac{M_1^p}{m_2^q}\right)\right\} > 1 \text{ if } \frac{m_1^p}{M_2^q} \le 1 \le \frac{M_1^p}{m_2^q}, \\ K\left(\frac{M_1^p}{m_2^q}\right) > 1 \text{ if } 1 < \frac{m_1^p}{M_2^q}. \end{cases} \tag{8.30}$$

COROLLARY 8.5. *Let A, B be two operators such that (8.20) is valid and $P, Q \in \mathcal{B}_1(H)$ with $P, Q > 0$. Then for any $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have for $m_{p,q}$, $M_{p,q}$ as defined by (8.29) and (8.30) that*

$$m_{p,q}^t \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} \le \frac{1}{p} \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} + \frac{1}{q} \frac{\operatorname{tr}(QB^q)}{\operatorname{tr}(Q)} \le M_{p,q}^T \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}, \tag{8.31}$$

where $t = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

PROOF. From (8.20) we have

$$0 < m_1^p I \le A^p < M_1^p I, \; 0 < m_2^q I \le B^q \le M_2^q I.$$

By replacing A by A^p, B by B^q and $\nu = \frac{1}{q}$ in (8.21) then we get the desired result (8.31). ∎

REMARK 8.4. If we take $Q = P$ in (8.31), then we get

$$m_{p,q}^t \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)} \le \frac{\operatorname{tr}\left[P\left(\frac{1}{p}A^p + \frac{1}{q}B^q\right)\right]}{\operatorname{tr}(P)} \le M_{p,q}^T \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}. \tag{8.32}$$

For $p = q = 2$ we consider

$$\tilde{m}_2 := \begin{cases} K\left[\left(\frac{M_1}{m_2}\right)^2\right] > 1 \text{ if } \frac{M_1}{m_2} < 1, \\ 1 \text{ if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}, \\ K\left[\left(\frac{M_2}{m_1}\right)^2\right] > 1 \text{ if } 1 < \frac{m_1}{M_2} \end{cases} \tag{8.33}$$

and

$$\tilde{M}_2 := \begin{cases} K\left[\left(\frac{M_2}{m_1}\right)^2\right] > 1 \text{ if } \frac{M_1}{m_2} < 1, \\ \max\left\{K\left[\left(\frac{M_2}{m_1}\right)^2\right], K\left[\left(\frac{M_1}{m_2}\right)^2\right]\right\} > 1 \text{ if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}, \\ K\left[\left(\frac{M_1}{m_2}\right)^2\right] > 1 \text{ if } 1 < \frac{m_1}{M_2}. \end{cases} \tag{8.34}$$

COROLLARY 8.6. *Let A, B be two operators such that (8.20) is valid and $P, Q \in \mathcal{B}_1(H)$ with $P, Q > 0$. Then for $\tilde{m}_2$, $\tilde{M}_2$ as defined by (8.33) and (8.34) we have that*

$$\tilde{m}_2^{1/2} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} \leq \frac{1}{p} \frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} + \frac{1}{q} \frac{\operatorname{tr}(QB^2)}{\operatorname{tr}(Q)} \leq \tilde{M}_2^{1/2} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}. \tag{8.35}$$

In particular,

$$\tilde{m}_2^{1/2} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)} \leq \frac{\operatorname{tr}\left[P\left(\frac{A^2+B^2}{2}\right)\right]}{\operatorname{tr}(P)} \leq \tilde{M}_2^{1/2} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}. \tag{8.36}$$

COROLLARY 8.7. *If P, Q, S, V are positive invertible operators with $P, Q, S, V \in \mathcal{B}_1(H)$ and for $0 < m_1 < M_1$, $0 < m_2 < M_2$,*

$$0 < m_1 P \leq S \leq M_1 P,\ 0 < m_2 Q \leq V \leq M_2 Q. \tag{8.37}$$

Then for any $\nu \in [0,1]$, we have for m, M as defined by (8.18) and (8.19) that

$$m^r \frac{\operatorname{tr}(P\sharp_{1-\nu} S)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(Q\sharp_\nu V)}{\operatorname{tr}(Q)} \leq (1-\nu) \frac{\operatorname{tr}(S)}{\operatorname{tr}(P)} + \nu \frac{\operatorname{tr}(V)}{\operatorname{tr}(Q)} \leq M^R \frac{\operatorname{tr}(P\sharp_{1-\nu} S)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(Q\sharp_\nu V)}{\operatorname{tr}(Q)}, \tag{8.38}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

In particular, we have

$$m^{1/2} \frac{\operatorname{tr}(P\sharp S)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(Q\sharp V)}{\operatorname{tr}(Q)} \leq \frac{1}{2}\left[\frac{\operatorname{tr}(S)}{\operatorname{tr}(P)} + \frac{\operatorname{tr}(V)}{\operatorname{tr}(Q)}\right] \leq M^{1/2} \frac{\operatorname{tr}(P\sharp S)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(Q\sharp V)}{\operatorname{tr}(Q)}. \tag{8.39}$$

PROOF. From (8.37) we have

$$0 < m_1 \leq P^{-1/2} S P^{-1/2} \leq M_1,\ 0 < m_2 \leq Q^{-1/2} V Q^{-1/2} \leq M_2.$$

If we use the inequality (8.21) for $A = P^{-1/2} S P^{-1/2}$ and $B = Q^{-1/2} V Q^{-1/2}$ then

$$\begin{aligned} & m^r \frac{\operatorname{tr}\left(P\left(P^{-1/2} S P^{-1/2}\right)^{1-\nu}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}\left(Q\left(Q^{-1/2} V Q^{-1/2}\right)^{\nu}\right)}{\operatorname{tr}(Q)} \\ & \leq (1-\nu) \frac{\operatorname{tr}\left(P P^{-1/2} S P^{-1/2}\right)}{\operatorname{tr}(P)} + \nu \frac{\operatorname{tr}\left(Q Q^{-1/2} V Q^{-1/2}\right)}{\operatorname{tr}(Q)} \\ & \leq M^R \frac{\operatorname{tr}\left(P\left(P^{-1/2} S P^{-1/2}\right)^{1-\nu}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}\left(Q\left(Q^{-1/2} V Q^{-1/2}\right)^{\nu}\right)}{\operatorname{tr}(Q)}, \end{aligned}$$

which, by the properties of trace, is equivalent to (8.38). ∎

REMARK 8.5. If P, S, V are positive invertible operators with $P, S, V \in \mathcal{B}_1(H)$ and for $0 < m_1 < M_1$, $0 < m_2 < M_2$,

$$0 < m_1 P \leq S \leq M_1 P,\ 0 < m_2 P \leq V \leq M_2 P, \tag{8.40}$$

then for any $\nu \in [0,1]$, we have for m, M as defined by (8.18) and (8.19) that

$$m^{r}\frac{\operatorname{tr}(P\sharp_{1-\nu}S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\sharp_{\nu}V)}{\operatorname{tr}(P)} \leq \frac{\operatorname{tr}((1-\nu)S+\nu V)}{\operatorname{tr}(P)} \leq M^{R}\frac{\operatorname{tr}(P\sharp_{1-\nu}S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\sharp_{\nu}V)}{\operatorname{tr}(P)}, \tag{8.41}$$

where $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

In particular, we have

$$m^{1/2}\frac{\operatorname{tr}(P\sharp S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\sharp V)}{\operatorname{tr}(P)} \leq \frac{\operatorname{tr}\left(\frac{S+V}{2}\right)}{\operatorname{tr}(P)} \leq M^{1/2}\frac{\operatorname{tr}(P\sharp S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\sharp V)}{\operatorname{tr}(P)}. \tag{8.42}$$

8.3. Trace Inequalities Via Tominaga and Furuichi Results. We recall that *Specht's ratio* is defined by [**128**]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h=1. \end{cases} \tag{8.43}$$

It is well known that $\lim_{h\to1} S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu} \leq (1-\nu)a+\nu b \leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{8.44}$$

where $a, b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$.

The second inequality in (8.44) is due to Tominaga [**130**] while the first one is due to Furuichi [**82**].

If $a\in[m_1,M_1]$ and $b\in[m_2,M_2]$ with $0<m_1<M_1$, $0<m_2<M_2$ then

$$\frac{m_1}{M_2}\leq\frac{a}{b}\leq\frac{M_1}{m_2}.$$

Denote, for $r\in(0,1)$

$$\breve{m}_r =: \min_{(a,b)\in[m_1,M_1]\times[m_2,M_2]} S\left(\left(\frac{a}{b}\right)^{r}\right) \text{ and } \breve{M} =: \max_{(a,b)\in[m_1,M_1]\times[m_2,M_2]} S\left(\frac{a}{b}\right).$$

Taking into account the properties of Specht's ratio we have

$$\breve{m}_r := \begin{cases} S\left(\left(\frac{M_1}{m_2}\right)^{r}\right)>1 \text{ if } \frac{M_1}{m_2}<1, \\ \\ 1 \text{ if } \frac{m_1}{M_2}\leq1\leq\frac{M_1}{m_2}, \\ \\ S\left(\left(\frac{M_2}{m_1}\right)^{r}\right)>1 \text{ if } 1<\frac{m_1}{M_2}, \end{cases} \tag{8.45}$$

and

$$\breve{M} := \begin{cases} S\left(\frac{M_2}{m_1}\right) > 1 \text{ if } \frac{M_1}{m_2} < 1, \\ \\ \max\left\{S\left(\frac{M_2}{m_1}\right), S\left(\frac{M_1}{m_2}\right)\right\} > 1 \text{ if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}, \\ \\ S\left(\frac{M_1}{m_2}\right) > 1 \text{ if } 1 < \frac{m_1}{M_2}. \end{cases} \tag{8.46}$$

We have the following result:

THEOREM 8.8 (Dragomir, 2015, [**66**]). *Let A, B be two operators such that*

$$0 < m_1 I \le A < M_1 I,\ 0 < m_2 I \le B \le M_2 I \tag{8.47}$$

and $P, Q \in \mathcal{B}_1(H)$ with $P, Q > 0$. Then for any $\nu \in [0,1]$, we have for $\breve{m}_r$, $\breve{M}$ as defined by (8.45) and (8.46) that

$$\breve{m}_r \frac{\operatorname{tr}(PA^{1-\nu})}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)} \le (1-\nu)\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \nu \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} \le \breve{M} \frac{\operatorname{tr}(PA^{1-\nu})}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)}, \tag{8.48}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

In particular, we have

$$\breve{m}_{1/2} \frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)} \le \frac{1}{2}\left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}\right] \le \breve{M} \frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)}. \tag{8.49}$$

PROOF. From (8.17) we have

$$\breve{m}_r a^{1-\nu} b^{\nu} \le (1-\nu) a + \nu b \le \breve{M} a^{1-\nu} b^{\nu},$$

where $a \in [m_1, M_1]$, $b \in [m_2, M_2]$ and $\nu \in [0,1]$.

Now, on making use of a similar argument as in the proof of Theorem 8.4, we get the desired result (8.48). ∎

For $0 < m_1 < M_1$, $0 < m_2 < M_2$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we define for $r \in (0,1)$

$$\breve{m}_{r,p,q} := \begin{cases} S\left(\left(\frac{M_1^p}{m_2^q}\right)^r\right) > 1 \text{ if } \frac{M_1^p}{m_2^q} < 1, \\ \\ 1 \text{ if } \frac{m_1^p}{M_2^q} \le 1 \le \frac{M_1^p}{m_2^q}, \\ \\ S\left(\left(\frac{M_2^q}{m_1^p}\right)^r\right) > 1 \text{ if } 1 < \frac{m_1^p}{M_2^q} \end{cases} \tag{8.50}$$

and

$$\breve{M}_{p,q} := \begin{cases} S\left(\frac{M_2^q}{m_1^p}\right) > 1 \text{ if } \frac{M_1^p}{m_2^q} < 1, \\ \\ \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\} > 1 \text{ if } \frac{m_1^p}{M_2^q} \le 1 \le \frac{M_1^p}{m_2^q}, \\ \\ S\left(\frac{M_1^p}{m_2^q}\right) > 1 \text{ if } 1 < \frac{m_1^p}{M_2^q}. \end{cases} \tag{8.51}$$

COROLLARY 8.9. *Let* A, B *be two operators such that (8.20) is valid and* $P, Q \in \mathcal{B}_1(H)$ *with* $P, Q > 0$. *Then for any* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *we have for* $\breve{m}_{t,p,q}$, $\breve{M}_{p,q}$ *as defined by (8.50) and (8.51) that*

$$\breve{m}_{t,p,q}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} \le \frac{1}{p}\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} + \frac{1}{q}\frac{\operatorname{tr}(QB^q)}{\operatorname{tr}(Q)} \le \breve{M}_{p,q}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}, \tag{8.52}$$

where $t = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

9. FURTHER REFINEMENTS AND REVERSES OF YOUNG INEQUALITY

9.1. Some Results Via Kittaneh-Manasrah Inequality. Kittaneh and Manasrah [**95**], [**96**] provided a refinement and a reverse for *Young's inequality* as follows:

$$r\left(\sqrt{a} - \sqrt{b}\right)^2 \le (1-\nu)a + \nu b - a^{1-\nu}b^\nu \le R\left(\sqrt{a} - \sqrt{b}\right)^2, \tag{9.1}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. The case $\nu = \frac{1}{2}$ reduces (9.1) to an identity.

We have:

THEOREM 9.1 (Dragomir, 2015, [**64**]). *Let* C *be a positive operator and* $P \in \mathcal{B}_1(H)$, $P > 0$. *Then for any* $\nu \in [0,1]$ *we have*

$$\begin{aligned} 2r&\left(\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1/2} - \frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{\nu-\frac{1}{2}} \\ &\le \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{\nu} - \frac{\operatorname{tr}(PC^\nu)}{\operatorname{tr}(P)} \\ &\le 2R\left(\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1/2} - \frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{\nu-\frac{1}{2}} \end{aligned} \tag{9.2}$$

and

$$\begin{aligned} r&\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)^2\right) \\ &\le \nu\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} + (1-\nu)\left(\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)^2 - \frac{\operatorname{tr}(PC^\nu)}{\operatorname{tr}(P)}\left(\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)^{2(1-\nu)} \\ &\le R\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)^2\right) \end{aligned} \tag{9.3}$$

where $r = \min\{1-\nu, \nu\}$ *and* $R = \max\{1-\nu, \nu\}$.

PROOF. Fix $b \geq 0$, and by using the functional calculus for the operator C, we have from (9.1) that

$$r\left(\langle Cx,x\rangle - 2\sqrt{b}\left\langle C^{1/2}x,x\right\rangle + b\langle x,x\rangle\right) \leq (1-\nu)\langle Cx,x\rangle + \nu b\langle x,x\rangle - b^{\nu}\left\langle C^{1-\nu}x,x\right\rangle \leq R\left(\langle Cx,x\rangle - 2\sqrt{b}\left\langle C^{1/2}x,x\right\rangle + b\langle x,x\rangle\right) \tag{9.4}$$

for any $x \in H$.

Now, let $x = P^{1/2}e$ where $e \in H$. Then by (9.4) we get

$$r\left(\left\langle P^{1/2}CP^{1/2}e,e\right\rangle - 2\sqrt{b}\left\langle P^{1/2}C^{1/2}P^{1/2}e,e\right\rangle + b\langle Pe,e\rangle\right) \leq (1-\nu)\left\langle P^{1/2}CP^{1/2}e,e\right\rangle + \nu b\langle Pe,e\rangle - b^{\nu}\left\langle P^{1/2}C^{1-\nu}P^{1/2}e,e\right\rangle \leq R\left(\left\langle P^{1/2}CP^{1/2}e,e\right\rangle - 2\sqrt{b}\left\langle P^{1/2}C^{1/2}P^{1/2}e,e\right\rangle + b\langle Pe,e\rangle\right) \tag{9.5}$$

for any $e \in H$ and $b \geq 0$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. If we take in (9.5) $e = e_i$, $i \in I$ and summing over $i \in I$, then we get

$$r\left(\sum_{i\in I}\left\langle P^{1/2}CP^{1/2}e_i,e_i\right\rangle - 2\sqrt{b}\sum_{i\in I}\left\langle P^{1/2}C^{1/2}P^{1/2}e_i,e_i\right\rangle + b\sum_{i\in I}\langle Pe_i,e_i\rangle\right) \leq (1-\nu)\sum_{i\in I}\left\langle P^{1/2}CP^{1/2}e_i,e_i\right\rangle + \nu b\sum_{i\in I}\langle Pe_i,e_i\rangle - b^{\nu}\sum_{i\in I}\left\langle P^{1/2}C^{1-\nu}P^{1/2}e_i,e_i\right\rangle \leq R\left(\sum_{i\in I}\left\langle P^{1/2}CP^{1/2}e_i,e_i\right\rangle - 2\sqrt{b}\sum_{i\in I}\left\langle P^{1/2}C^{1/2}P^{1/2}e_i,e_i\right\rangle + b\sum_{i\in I}\langle Pe_i,e_i\rangle\right)$$

for any $b \geq 0$ and by using the properties of the trace, we obtain

$$r\left(\operatorname{tr}(PC) - 2\operatorname{tr}\left(PC^{1/2}\right)\sqrt{b} + \operatorname{tr}(P)\,b\right) \leq (1-\nu)\operatorname{tr}(PC) + \nu\operatorname{tr}(P)\,b - \operatorname{tr}\left(PC^{1-\nu}\right)b^{\nu} \leq R\left(\operatorname{tr}(PC) - 2\operatorname{tr}\left(PC^{1/2}\right)\sqrt{b} + \operatorname{tr}(P)\,b\right)$$

for any $b \geq 0$.

Dividing by $\operatorname{tr}(P) > 0$ we get

$$r\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - 2\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\sqrt{b} + b\right) \leq (1-\nu)\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} + \nu b - \frac{\operatorname{tr}\left(PC^{1-\nu}\right)}{\operatorname{tr}(P)}b^{\nu} \leq R\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - 2\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\sqrt{b} + b\right) \tag{9.6}$$

for any $b \geq 0$.

This inequality is of interest in itself as well.

Now, if we take in (9.6) $b = \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}$, then we get for $\nu \in [0,1]$ that

$$\begin{aligned} & 2r\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\sqrt{\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}}\right) \\ & \leq \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PC^{1-\nu})}{\operatorname{tr}(P)}\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{\nu} \\ & \leq 2R\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\sqrt{\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}}\right) \end{aligned}$$

which is equivalent to

$$\begin{aligned} & 2r\left(\sqrt{\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}} - \frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)\sqrt{\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}} \\ & \leq \left(\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1-\nu} - \frac{\operatorname{tr}(PC^{1-\nu})}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{\nu} \\ & \leq 2R\left(\sqrt{\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}} - \frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)\sqrt{\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}}, \end{aligned} \tag{9.7}$$

Now if we replace ν by $1-\nu$ in (9.7) we deduce (9.2).

Also, if we take in (9.6) $b = \left(\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)^2$ and replace ν by $1-\nu$ then we get the inequality (9.3). ∎

COROLLARY 9.2. *If P, Q are positive invertible operators with $P, Q \in \mathcal{B}_1(H)$, then for any $\nu \in [0,1]$ we have*

$$\begin{aligned} & 2r\left(\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}\right)^{1/2} - \frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}\right)^{\nu-\frac{1}{2}} \\ & \leq \left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}\right)^{\nu} - \frac{\operatorname{tr}(P\sharp_\nu Q)}{\operatorname{tr}(P)} \\ & \leq 2R\left(\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}\right)^{1/2} - \frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}\right)^{\nu-\frac{1}{2}}, \end{aligned} \tag{9.8}$$

$$\begin{aligned} & r\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)^2\right) \\ & \leq \nu\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)} + (1-\nu)\left(\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)^2 - \frac{\operatorname{tr}(P\sharp_\nu Q)}{\operatorname{tr}(P)}\left(\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)^{2(1-\nu)} \\ & \leq R\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)^2\right) \end{aligned} \tag{9.9}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

PROOF. The proof follows by (9.2) on choosing $C = P^{-1/2}QP^{-1/2}$. ∎

COROLLARY 9.3. *Let A and B be two positive invertible operators, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and such that A^p, $B^q\in\mathcal{B}_1(H)$. Then*

$$2t\left(\left(\frac{\operatorname{tr}(A^p)}{\operatorname{tr}(B^q)}\right)^{1/2}-\frac{\operatorname{tr}(B^q\sharp A^p)}{\operatorname{tr}(B^q)}\right)\left(\frac{\operatorname{tr}(B^q\sharp A^p)}{\operatorname{tr}(B^q)}\right)^{\frac{1}{p}-\frac{1}{2}} \tag{9.10}$$
$$\leq\left(\frac{\operatorname{tr}(A^p)}{\operatorname{tr}(B^q)}\right)^{\frac{1}{p}}-\frac{\operatorname{tr}\left(B^q\sharp_{1/p}A^p\right)}{\operatorname{tr}(B^q)}$$
$$\leq 2T\left(\left(\frac{\operatorname{tr}(A^p)}{\operatorname{tr}(B^q)}\right)^{1/2}-\frac{\operatorname{tr}(B^q\sharp A^p)}{\operatorname{tr}(B^q)}\right)\left(\frac{\operatorname{tr}(B^q\sharp A^p)}{\operatorname{tr}(B^q)}\right)^{\frac{1}{p}-\frac{1}{2}}$$

and

$$t\left(\frac{\operatorname{tr}(A^p)}{\operatorname{tr}(B^q)}-\left(\frac{\operatorname{tr}(B^q\sharp A^p)}{\operatorname{tr}(B^q)}\right)^2\right) \tag{9.11}$$
$$\leq\frac{1}{p}\frac{\operatorname{tr}(A^p)}{\operatorname{tr}(B^q)}+\frac{1}{q}\left(\frac{\operatorname{tr}(B^q\sharp A^p)}{\operatorname{tr}(B^q)}\right)^2-\frac{\operatorname{tr}\left(B^q\sharp_{1/p}A^p\right)}{\operatorname{tr}(B^q)}\left(\frac{\operatorname{tr}(B^q\sharp A^p)}{\operatorname{tr}(B^q)}\right)^{2/q}$$
$$\leq T\left(\frac{\operatorname{tr}(A^p)}{\operatorname{tr}(B^q)}-\left(\frac{\operatorname{tr}(B^q\sharp A^p)}{\operatorname{tr}(B^q)}\right)^2\right),$$

where $t=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ and $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

The proof follows by Corollary 9.2 for $P=A^p$, $Q=B^q$ and $\nu=\frac{1}{p}$.

9.2. Some Results Via Tominaga Inequality. We recall that *Specht's ratio* is defined by [**128**]

$$S(h):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty),\\ \\ 1 \text{ if } h=1.\end{cases} \tag{9.12}$$

It is well known that $\lim_{h\to1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a multiplicative reverse for Young's inequality

$$\left(a^{1-\nu}b^{\nu}\leq\right)(1-\nu)a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{9.13}$$

where $a,b>0$, $\nu\in[0,1]$. This inequality is due to Tominaga [**130**].

THEOREM 9.4 (Dragomir, 2015, [**64**]). *Let C be an operator with the property that*

$$mI\leq C\leq MI \tag{9.14}$$

for some constants m, M with $M>m>0$ and $P\in\mathcal{B}_1(H)$, $P\geq0$ with $\operatorname{tr}(P)>0$. Then for any $p>1$ we have

$$\left(\frac{\operatorname{tr}(PC^p)}{\operatorname{tr}(P)}\right)^{1/p}\leq S\left(\left(\frac{M}{m}\right)^p\right)\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}. \tag{9.15}$$

In particular, we have

$$\operatorname{tr}\left(PC^2\right)\operatorname{tr}(P) \le S^2\left(\left(\frac{M}{m}\right)^2\right)\left[\operatorname{tr}(PC)\right]^2. \tag{9.16}$$

PROOF. Assume that $\nu \in (0,1)$. Let $a, b \in [m, M] \subset (0, \infty)$, then $\frac{m}{M} \le \frac{a}{b} \le \frac{M}{m}$ with $\frac{m}{M} < 1 < \frac{M}{m}$. If $\frac{a}{b} \in \left[\frac{m}{M}, 1\right)$ then $S\left(\frac{a}{b}\right) \le S\left(\frac{m}{M}\right) = S\left(\frac{M}{m}\right)$. If $\frac{a}{b} \in \left(1, \frac{M}{m}\right]$ then also $S\left(\frac{a}{b}\right) \le S\left(\frac{M}{m}\right)$. Therefore for any $a, b \in [m, M]$ we have by Tominaga's inequality (9.13) that

$$(1-\nu)a + \nu b \le S\left(\frac{M}{m}\right)a^{1-\nu}b^{\nu}. \tag{9.17}$$

Now, if C is an operator with $mI \le C \le MI$ then for $p > 1$ we have $m^pI \le C^p \le M^pI$. Using the functional calculus we get from (9.17) for $\nu = \frac{1}{p}$ that

$$\left(1-\frac{1}{p}\right)d + \frac{1}{p}C^p \le S\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}C,$$

namely, the vector inequality,

$$\left(1-\frac{1}{p}\right)d\langle y,y\rangle + \frac{1}{p}\langle C^py,y\rangle \le S\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}\langle Cy,y\rangle, \tag{9.18}$$

for any $y \in H$ and $d \in [m^p, M^p]$.

Now, let $y = P^{1/2}e$ where $e \in H$. Then by (9.18) we get

$$\begin{aligned}\left(1-\frac{1}{p}\right)d\langle Pe,e\rangle + \frac{1}{p}\left\langle P^{1/2}C^pP^{1/2}e,e\right\rangle \\ \le S\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}\left\langle P^{1/2}CP^{1/2}e,e\right\rangle,\end{aligned} \tag{9.19}$$

for any $e \in H$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. If we take in (9.19) $e = e_i$, $i \in I$ and summing over $i \in I$, then we get

$$\begin{aligned}\left(1-\frac{1}{p}\right)d\sum_{i\in I}\langle Pe_i,e_i\rangle + \frac{1}{p}\sum_{i\in I}\left\langle P^{1/2}C^pP^{1/2}e_i,e_i\right\rangle \\ \le S\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}\sum_{i\in I}\left\langle P^{1/2}CP^{1/2}e_i,e_i\right\rangle,\end{aligned}$$

and by the properties of trace

$$\left(1-\frac{1}{p}\right)d\operatorname{tr}(P) + \frac{1}{p}\operatorname{tr}(PC^p) \le S\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}\operatorname{tr}(PC),$$

for any $d \in [m^p, M^p]$.

This inequality can be written as

$$\left(1-\frac{1}{p}\right)d + \frac{1}{p}\frac{\operatorname{tr}(PC^p)}{\operatorname{tr}(P)} \le S\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}, \tag{9.20}$$

for any $d \in [m^p, M^p]$, that is of interest in itself.

Now, if we take in (9.20) $d = \frac{\operatorname{tr}(PC^p)}{\operatorname{tr}(P)} \in [m^p, M^p]$, then we get

$$\frac{\operatorname{tr}(PC^p)}{\operatorname{tr}(P)} \le S\left(\left(\frac{M}{m}\right)^p\right)\left(\frac{\operatorname{tr}(PC^p)}{\operatorname{tr}(P)}\right)^{1-\frac{1}{p}}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)},$$

which is equivalent to (9.15). ∎

COROLLARY 9.5. *Let* A *and* B *be two positive invertible operators,* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $m,M>0$ *such that* $B^{q}\in\mathcal{B}_{1}(H)$ *and*

$$m^{p}B^{q}\leq A^{p}\leq M^{p}B^{q}. \tag{9.21}$$

Then

$$\left[\operatorname{tr}(A^{p})\right]^{1/p}\left[\operatorname{tr}(B^{q})\right]^{1/q}\leq S\left(\left(\frac{M}{m}\right)^{p}\right)\operatorname{tr}\left(B^{q}\sharp_{1/p}A^{p}\right). \tag{9.22}$$

PROOF. The inequality (9.15) can be written as

$$\left[\operatorname{tr}P^{1/2}C^{p}P^{1/2}\right]^{1/p}\left[\operatorname{tr}(P)\right]^{1/q}\leq S\left(\left(\frac{M}{m}\right)^{p}\right)\operatorname{tr}\left(P^{1/2}CP^{1/2}\right). \tag{9.23}$$

Now, from (9.21) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^{p}I\leq B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\leq M^{p}I$ and by taking the power $\frac{1}{p}$ we get $mI\leq\left(B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\right)^{\frac{1}{p}}\leq MI.$

By writing the inequality (9.23) for $C=\left(B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\right)^{\frac{1}{p}}$ and $P=B^{q}$ then we get

$$\begin{aligned}&\left[\operatorname{tr}\left(B^{q/2}\left[\left(B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\right)^{\frac{1}{p}}\right]^{p}B^{q/2}\right)\right]^{1/p}\left[\operatorname{tr}(B^{q})\right]^{1/q}\\&\leq S\left(\left(\frac{M}{m}\right)^{p}\right)\operatorname{tr}\left(B^{q/2}\left(B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\right)^{\frac{1}{p}}B^{q/2}\right),\end{aligned}$$

i.e.,

$$\left[\operatorname{tr}(A^{p})\right]^{1/p}\left[\operatorname{tr}(B^{q})\right]^{1/q}\leq S\left(\left(\frac{M}{m}\right)^{p}\right)\operatorname{tr}\left(B^{q/2}\left(B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\right)^{\frac{1}{p}}B^{q/2}\right),$$

and the inequality (9.22) is proved. ∎

COROLLARY 9.6. *Let* A *and* B *be two positive invertible operators and* $m,M>0$ *such that* $B^{2}\in\mathcal{B}_{1}(H)$ *and*

$$m^{2}B^{2}\leq A^{2}\leq M^{2}B^{2}. \tag{9.24}$$

Then

$$\operatorname{tr}\left(A^{2}\right)\operatorname{tr}\left(B^{2}\right)\leq S^{2}\left(\left(\frac{M}{m}\right)^{2}\right)\left[\operatorname{tr}\left(B^{2}\sharp A^{2}\right)\right]^{2}. \tag{9.25}$$

REMARK 9.1. We remark that the condition (9.24) can be written as

$$kB^{2}\leq A^{2}\leq KB^{2} \tag{9.26}$$

where $0<k<K,$ then by (9.25) we have

$$\operatorname{tr}\left(A^{2}\right)\operatorname{tr}\left(B^{2}\right)\leq S^{2}\left(\frac{K}{k}\right)\left[\operatorname{tr}\left(B^{2}\sharp A^{2}\right)\right]^{2}. \tag{9.27}$$

9.3. Some Results Via Liao-Wu-Zhao. We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h>0. \tag{9.28}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\geq 1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative reverse of Young inequality in terms of Kantorovich's constant holds

$$\left(a^{1-\nu}b^{\nu}\leq\right)(1-\nu)a+\nu b\leq K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{9.29}$$

where $a, b>0$, $\nu\in[0,1]$ and $R=\max\{1-\nu,\nu\}$.

This inequality was obtained by Liao et al. in [**99**].

THEOREM 9.7 (Dragomir, 2015, [**64**]). *Let C be an operator with the property (9.14) for some constants m, M with $M>m>0$ and $P\in\mathcal{B}_1(H)$, $P>0$. Then for any $p, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ we have*

$$\left(\frac{\operatorname{tr}(PC^p)}{\operatorname{tr}(P)}\right)^{1/p}\leq K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(\left(\frac{M}{m}\right)^p\right)\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}. \tag{9.30}$$

In particular, we have

$$\operatorname{tr}\left(PC^2\right)\operatorname{tr}(P)\leq K\left(\left(\frac{M}{m}\right)^2\right)[\operatorname{tr}(PC)]^2. \tag{9.31}$$

PROOF. Assume that $\nu\in(0,1)$ and $R=\max\{1-\nu,\nu\}$. Let $a, b\in[m,M]\subset(0,\infty)$, then $\frac{m}{M}\leq\frac{a}{b}\leq\frac{M}{m}$ with $\frac{m}{M}<1<\frac{M}{m}$. If $\frac{a}{b}\in\left[\frac{m}{M},1\right)$ then $K^R\left(\frac{a}{b}\right)\leq K^R\left(\frac{m}{M}\right)=K^R\left(\frac{M}{m}\right)$. If $\frac{a}{b}\in\left(1,\frac{M}{m}\right]$ then also $K^R\left(\frac{a}{b}\right)\leq K^R\left(\frac{M}{m}\right)$. Therefore for any $a, b\in[m,M]$ we have by inequality (9.29) that

$$(1-\nu)a+\nu b\leq K^R\left(\frac{M}{m}\right)a^{1-\nu}b^{\nu}. \tag{9.32}$$

Now, if C is an operator with $mI\leq C\leq MI$ then for $p>1$ we have $m^pI\leq C^p\leq M^pI$. Using the functional calculus we get from (9.32) for $\nu=\frac{1}{p}$ that

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}C^p\leq K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}C,$$

namely, the vector inequality,

$$\left(1-\frac{1}{p}\right)d\langle y,y\rangle+\frac{1}{p}\langle C^py,y\rangle\leq K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}\langle Cy,y\rangle,$$

for any $y\in H$ and $d\in[m^p,M^p]$.

Now, by employing a similar argument to the one in the proof of Theorem 9.4 we deduce the desired result (9.30). The details are omitted. ∎

We have:

COROLLARY 9.8. *Let A and B be two positive invertible operators, $p, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m, M>0$ such that $B^q\in\mathcal{B}_1(H)$ and the condition (9.21) holds. Then*

$$[\operatorname{tr}(A^p)]^{1/p}[\operatorname{tr}(B^q)]^{1/q}\leq K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(\left(\frac{M}{m}\right)^p\right)\operatorname{tr}\left(B^q\sharp_{1/p}A^p\right). \tag{9.33}$$

If $B^2 \in \mathcal{B}_1(H)$ *and the condition (9.24) is valid, then*

$$\operatorname{tr}\left(A^2\right)\operatorname{tr}\left(B^2\right) \le K\left(\left(\frac{M}{m}\right)^2\right)\left[\operatorname{tr}\left(B^2\sharp A^2\right)\right]^2. \tag{9.34}$$

9.4. Some Logarithmic Inequalities. In the recent paper [**61**] we obtained the following logarithmic reverse of Young's inequality:

$$0 \le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le \nu(1-\nu)(a-b)(\ln a - \ln b) \tag{9.35}$$

where $a, b > 0$, $\nu \in [0,1]$.

THEOREM 9.9 (Dragomir, 2015, [**64**]). *Let* C *be a positive operator and* $P \in \mathcal{B}_1(H)$, $P > 0$. *Then for any* $\nu \in [0,1]$ *we have*

$$\begin{aligned} 0 &\le \left[\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{\nu} - \frac{\operatorname{tr}(PC^{\nu})}{\operatorname{tr}(P)}\right]\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1-\nu} \\ &\le \nu(1-\nu)\left[\frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)}\right], \end{aligned} \tag{9.36}$$

and, in particular

$$\begin{aligned} 0 &\le \left[\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1/2} - \frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right]\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1/2} \\ &\le \frac{1}{4}\left[\frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)}\right]. \end{aligned} \tag{9.37}$$

PROOF. The inequality (9.35) may be written as

$$0 \le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le \nu(1-\nu)(a\ln a + b\ln b - a\ln b - b\ln a) \tag{9.38}$$

for any $a, b > 0$, $\nu \in [0,1]$.

Fix $b > 0$, $\nu \in [0,1]$. By using the functional calculus for the operator C we have

$$\begin{aligned} 0 &\le (1-\nu)\langle Cx, x\rangle + \nu b\langle x, x\rangle - b^{\nu}\left\langle C^{1-\nu}x, x\right\rangle \\ &\le \nu(1-\nu)\left(\langle C\ln Cx, x\rangle + b\ln b\langle x, x\rangle - \ln b\langle Cx, x\rangle - b\langle \ln Cx, x\rangle\right) \end{aligned} \tag{9.39}$$

for any $b > 0$, $\nu \in [0,1]$ and $x \in H$.

Now, let $x = P^{1/2}e$ where $e \in H$. Then by (9.39) we get

$$\begin{aligned} 0 &\le (1-\nu)\left\langle P^{1/2}CP^{1/2}e, e\right\rangle + \nu b\langle Pe, e\rangle - b^{\nu}\left\langle P^{1/2}C^{1-\nu}P^{1/2}e, e\right\rangle \\ &\le \nu(1-\nu)\left[\left\langle P^{1/2}(C\ln C)P^{1/2}e, e\right\rangle + \langle Pe, e\rangle b\ln b\right. \\ &\left. - \left\langle P^{1/2}CP^{1/2}e, e\right\rangle \ln b - \left\langle P^{1/2}(\ln C)P^{1/2}e, e\right\rangle b\right] \end{aligned} \tag{9.40}$$

for any $b > 0$, $\nu \in [0,1]$ and $e \in H$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. If we take in (9.40) $e = e_i$, $i \in I$ and summing over $i \in I$, then we get

$$\begin{aligned}
0 &\le (1-\nu)\sum_{i\in I}\left\langle P^{1/2}CP^{1/2}e_i, e_i\right\rangle + \nu b\sum_{i\in I}\langle Pe_i, e_i\rangle \\
&\quad - b^{\nu}\sum_{i\in I}\left\langle P^{1/2}C^{1-\nu}P^{1/2}e_i, e_i\right\rangle \\
&\le \nu(1-\nu)\left[\sum_{i\in I}\left\langle P^{1/2}(C\ln C)P^{1/2}e_i, e_i\right\rangle + b\ln b\sum_{i\in I}\langle Pe_i, e_i\rangle\right. \\
&\quad \left. - \ln b\sum_{i\in I}\left\langle P^{1/2}CP^{1/2}e_i, e_i\right\rangle - b\sum_{i\in I}\left\langle P^{1/2}(\ln C)P^{1/2}e_i, e_i\right\rangle\right].
\end{aligned} \tag{9.41}$$

Using the properties of the trace, we have from (9.41) that

$$\begin{aligned}
0 &\le (1-\nu)\operatorname{tr}(PC) + \nu b\operatorname{tr}(P) - b^{\nu}\operatorname{tr}\left(PC^{1-\nu}\right) \\
&\le \nu(1-\nu)\left[\operatorname{tr}(PC\ln C) + b\ln b\operatorname{tr}(P) - \ln b\operatorname{tr}(PC) - b\operatorname{tr}(P\ln C)\right],
\end{aligned}$$

which by division with $\operatorname{tr}(P) > 0$ produces

$$\begin{aligned}
0 &\le (1-\nu)\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} + \nu b - b^{\nu}\frac{\operatorname{tr}(PC^{1-\nu})}{\operatorname{tr}(P)} \\
&\le \nu(1-\nu)\left[\frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)} + b\ln b - \ln b\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - b\frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)}\right],
\end{aligned} \tag{9.42}$$

for any $b > 0$, $\nu \in [0,1]$.

This is an inequality of interest in itself.

Now, if we take in (9.42) $b = \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}$, then we get

$$\begin{aligned}
0 &\le \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{\nu}\frac{\operatorname{tr}(PC^{1-\nu})}{\operatorname{tr}(P)} \\
&\le \nu(1-\nu)\left[\frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)} + \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\ln\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)\right. \\
&\quad \left. -\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\ln\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)}\right] \\
&= \nu(1-\nu)\left[\frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)}\right],
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
0 &\le \left[\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1-\nu} - \frac{\operatorname{tr}(PC^{1-\nu})}{\operatorname{tr}(P)}\right]\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{\nu} \\
&\le \nu(1-\nu)\left[\frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)}\right],
\end{aligned}$$

for any $\nu \in [0,1]$.

Now, by replacing ν with $1-\nu$ we get the desired result (9.36). ∎

We say that the functions $f, g : [a,b] \longrightarrow \mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a,b]$ if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \ge (\le)\, 0 \text{ for each } t,\ s \in [a,b].$$

In recent paper [**49**] we obtained the following result: Let A be a selfadjoint operators on the Hilbert space H with $\operatorname{Sp}(A) \subseteq J$ and assume that the continuous functions $f, g : J \to \mathbb{R}$ are synchronous on J. If $P, Q \in \mathcal{B}_1(H)$ with $P, Q > 0$, then

$$\frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)} + \frac{\operatorname{tr}[Qf(A)g(A)]}{\operatorname{tr}(Q)} \geq \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Qg(A)]}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Qf(A)]}{\operatorname{tr}(Q)} \tag{9.43}$$

and, in particular

$$\frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)} \geq \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}. \tag{9.44}$$

Now, if we take in (9.44) $f(t) = t$, $g(t) = \ln t$, $t > 0$ and $A = C > 0$ then we get

$$0 \leq \frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)} \tag{9.45}$$

for $P \in \mathcal{B}_1(H)$, $P > 0$.

Therefore, the inequalities (9.36) and (9.37) provide refinements for (9.45).

In [**47**] we obtained amongst other the following Grüss type trace inequality

$$\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \leq \frac{1}{4}(M-m)(K-k) \tag{9.46}$$

provided that $k1_H \leq A \leq K1_H$, $m1_H \leq C \leq M1_H$ and $P \in \mathcal{B}_1(H)$, $P > 0$.

Therefore, if we take $A = \ln C$, then we get from (9.46) that

$$\frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \leq \frac{1}{4}(M-m)(\ln M - \ln m) \tag{9.47}$$

provided that $0 < m1_H \leq C \leq M1_H$ and $P \in \mathcal{B}_1(H)$, $P > 0$.

COROLLARY 9.10. *Let C be an operator such that $m1_H \leq C \leq M1_H$ for some constants $0 < m < M$ and $P \in \mathcal{B}_1(H)$, $P > 0$. Then for any $\nu \in [0,1]$ we have*

$$0 \leq \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PC^\nu)}{\operatorname{tr}(P)}\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1-\nu} \leq \frac{1}{4}\nu(1-\nu)(M-m)(\ln M - \ln m), \tag{9.48}$$

and, in particular

$$0 \leq \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1/2} \leq \frac{1}{16}(M-m)(\ln M - \ln m). \tag{9.49}$$

REMARK 9.2. The inequality (9.48) is equivalent to

$$0 \leq \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{\nu} - \frac{\operatorname{tr}(PC^\nu)}{\operatorname{tr}(P)} \leq \frac{1}{4}\nu(1-\nu)(M-m)(\ln M - \ln m)\left(\frac{\operatorname{tr}(P)}{\operatorname{tr}(PC)}\right)^{1-\nu}. \tag{9.50}$$

Since $\frac{\operatorname{tr}(P)}{\operatorname{tr}(PC)} \leq \frac{1}{m}$, then we get from (9.50) that

$$0 \leq \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{\nu} - \frac{\operatorname{tr}(PC^\nu)}{\operatorname{tr}(P)} \leq \frac{1}{4}\nu(1-\nu)\frac{(M-m)(\ln M - \ln m)}{m^{1-\nu}}, \tag{9.51}$$

and in particular

$$0 \le \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1/2} - \frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)} \le \frac{1}{16}\frac{(M-m)(\ln M - \ln m)}{m^{1/2}}, \tag{9.52}$$

provided that $m1_H \le C \le M1_H$ for some constants $0 < m < M$ and $P \in \mathcal{B}_1(H)$, $P > 0$.

COROLLARY 9.11. *Let A and B be two positive invertible operators, $p, q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$ and $m, M > 0$ such that $B^q \in \mathcal{B}_1(H)$ and (9.21) is valid. Then*

$$\begin{aligned} 0 &\le [\operatorname{tr}(A^p)]^{1/p}[\operatorname{tr}(B^q)]^{1/q} - \operatorname{tr}\left(B^q \sharp_{1/p} A^p\right) \\ &\le \frac{(M^p - m^p)(\ln M - \ln m)}{4qm^{p/q}} \operatorname{tr}(B^q). \end{aligned} \tag{9.53}$$

In particular, we have

$$\begin{aligned} 0 &\le \left[\operatorname{tr}\left(A^2\right)\right]^{1/2}\left[\operatorname{tr}\left(B^2\right)\right]^{1/2} - \operatorname{tr}\left(B^2 \sharp A^2\right) \\ &\le \frac{(M^2 - m^2)(\ln M - \ln m)}{8m} \operatorname{tr}\left(B^2\right) \end{aligned} \tag{9.54}$$

provided the condition (9.24) is valid.

PROOF. From the inequality (9.50) we have

$$\begin{aligned} 0 &\le \left(\frac{\operatorname{tr}\left(P^{1/2}CP^{1/2}\right)}{\operatorname{tr}(P)}\right)^{\nu} - \frac{\operatorname{tr}\left(P^{1/2}C^{\nu}P^{1/2}\right)}{\operatorname{tr}(P)} \\ &\le \frac{1}{4}\nu(1-\nu)(M-m)(\ln M - \ln m)\left(\frac{\operatorname{tr}(P)}{\operatorname{tr}\left(P^{1/2}CP^{1/2}\right)}\right)^{1-\nu}, \end{aligned} \tag{9.55}$$

provided that $m1_H \le C \le M1_H$ for some constants $0 < m < M$ and $P \in \mathcal{B}_1(H)$, $P > 0$.

Now, from (9.21) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^p I \le B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \le M^p I$.

By writing the inequality (9.55) for $C = B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}$, $P = B^q$ the bounds m^p, M^p and $\nu = \frac{1}{p}$, then we get

$$\begin{aligned} 0 &\le \left(\frac{\operatorname{tr}(A^p)}{\operatorname{tr}(B^q)}\right)^{\frac{1}{p}} - \frac{\operatorname{tr}\left(B^q \sharp_{1/p} A^p\right)}{\operatorname{tr}(B^q)} \\ &\le \frac{1}{4pq}(M^p - m^p)(\ln M^p - \ln m^p)\left(\frac{\operatorname{tr}(B^q)}{\operatorname{tr}(A^p)}\right)^{\frac{1}{q}}. \end{aligned}$$

By multiplying this with $\operatorname{tr}(B^q) > 0$ we get

$$\begin{aligned} 0 &\le [\operatorname{tr}(A^p)]^{\frac{1}{p}} \operatorname{tr}(B^q)^{\frac{1}{q}} - \operatorname{tr}\left(B^q \sharp_{1/p} A^p\right) \\ &\le \frac{1}{4q}(M^p - m^p)(\ln M - \ln m)\left(\frac{\operatorname{tr}(B^q)}{\operatorname{tr}(A^p)}\right)^{\frac{1}{q}} \operatorname{tr}(B^q). \end{aligned} \tag{9.56}$$

Since $m^p \operatorname{tr}(B^q) \le \operatorname{tr}(A^p) \le M^p \operatorname{tr}(B^q)$, then $\frac{\operatorname{tr}(B^q)}{\operatorname{tr}(A^p)} \le \frac{1}{m^p}$ and by (9.56) we get the desired result (9.53). ∎

9.5. Some Exponential Inequalities. In paper [**61**] we also obtained the following multiplicative reverse of Young's inequality

$$1 \leq \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \leq \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{9.57}$$

where $a, b > 0$, $\nu \in [0,1]$.

For a numerical comparison of the several bounds in the Young's inequality, see

By using a similar argument to the one in the proof of Theorem 9.4 we can prove the following result as well:

THEOREM 9.12 (Dragomir, 2015, [**64**]). *Let C be an operator with the property (9.14) for some constants m, M with $M > m > 0$ and $P \in \mathcal{B}_1(H)$, $P \geq 0$ with $\operatorname{tr}(P) > 0$. Then for any $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ we have*

$$\left(\frac{\operatorname{tr}(PC^p)}{\operatorname{tr}(P)}\right)^{1/p} \leq \exp\left[\frac{4}{pq}\left(K\left(\left(\frac{M}{m}\right)^p\right)-1\right)\right]\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}. \tag{9.58}$$

In particular, we have

$$\operatorname{tr}\left(PC^2\right)\operatorname{tr}(P) \leq \exp\left[2\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right][\operatorname{tr}(PC)]^2. \tag{9.59}$$

We also have:

COROLLARY 9.13. *Let A and B be two positive invertible operators, $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m, M > 0$ such that $B^q \in \mathcal{B}_1(H)$ and $m^pB^q \leq A^p \leq M^pB^q$. Then*

$$[\operatorname{tr}(A^p)]^{1/p}[\operatorname{tr}(B^q)]^{1/q} \leq \exp\left[\frac{4}{pq}\left(K\left(\left(\frac{M}{m}\right)^p\right)-1\right)\right]\operatorname{tr}\left(B^q\sharp_{1/p}A^p\right). \tag{9.60}$$

In particular, if $m^2B^2 \leq A^2 \leq M^2B^2$, then

$$\operatorname{tr}\left(A^2\right)\operatorname{tr}\left(B^2\right) \leq \exp\left[2\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right]\left[\operatorname{tr}\left(B^2\sharp A^2\right)\right]^2. \tag{9.61}$$

In [**57**] we obtained the following inequalities that improve the corresponding results of Furuichi and Minculete from [**84**]:

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \leq \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right] \tag{9.62}$$

for any $a, b > 0$ and $\nu \in [0,1]$.

THEOREM 9.14 (Dragomir, 2015, [**64**]). *Let C be an operator with the property (9.14) for some constants m, M with $M > m > 0$ and $P \in \mathcal{B}_1(H)$, $P > 0$. Then for any $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ we have*

$$\left(\frac{\operatorname{tr}(PC^p)}{\operatorname{tr}(P)}\right)^{1/p} \leq \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p-1\right)^2\right]\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \tag{9.63}$$

and, in particular,

$$\text{tr}\left(PC^2\right)\text{tr}\left(P\right)\leq\exp\left[\frac{1}{pq}\left(\left(\frac{M}{m}\right)^p-1\right)^2\right]\left[\text{tr}\left(PC\right)\right]^2. \tag{9.64}$$

PROOF. If $a,b\in[m,M]\subset(0,\infty)$ and since

$$0<\frac{\max\{a,b\}}{\min\{a,b\}}-1\leq\frac{M}{m}-1,$$

hence

$$\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\leq\left(\frac{M}{m}-1\right)^2.$$

Therefore, by (9.62) we get

$$(1-\nu)\,a+\nu b\leq a^{1-\nu}b^{\nu}\exp\left[\frac{1}{2}\nu\,(1-\nu)\left(\frac{M}{m}-1\right)^2\right], \tag{9.65}$$

for any $a,b\in[m,M]$ and $\nu\in(0,1)$.

Now, if C is an operator with $mI\leq C\leq MI$ then for $p>1$ we have $m^pI\leq C^p\leq M^pI$. Using the functional calculus we get from (9.65) for $\nu=\frac{1}{p}$ that

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}C^p\leq\exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p-1\right)^2\right]d^{1-\frac{1}{p}}C,$$

namely, the vector inequality,

$$\left(1-\frac{1}{p}\right)d\left\langle y,y\right\rangle+\frac{1}{p}\left\langle C^py,y\right\rangle\leq\exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p-1\right)^2\right]d^{1-\frac{1}{p}}\left\langle Cy,y\right\rangle,$$

for any $y\in H$ and $d\in[m^p,M^p]$.

This is an inequality of interest in itself.

Now, by employing a similar argument to the one in the proof of Theorem 9.4 we deduce the desired result (9.63). The details are omitted. ∎

Finally, we have:

COROLLARY 9.15. *Let A and B be two positive invertible operators, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m,M>0$ such that $B^q\in\mathcal{B}_1(H)$ and $m^pB^q\leq A^p\leq M^pB^q$. Then*

$$\left[\text{tr}\left(A^p\right)\right]^{1/p}\left[\text{tr}\left(B^q\right)\right]^{1/q}\leq\exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p-1\right)^2\right]\text{tr}\left(B^q\sharp_{1/p}A^p\right). \tag{9.66}$$

In particular, if $m^2B^2\leq A^2\leq M^2B^2$, then

$$\text{tr}\left(A^2\right)\text{tr}\left(B^2\right)\leq\exp\left[\frac{1}{4}\left(\left(\frac{M}{m}\right)^2-1\right)^2\right]\left[\text{tr}\left(B^2\sharp A^2\right)\right]^2. \tag{9.67}$$

10. ADDITIVE REVERSES

10.1. Operator Inequalities. We consider the function $f_\nu : [0,\infty) \to [0,\infty)$ defined for $\nu \in (0,1)$ by

$$f_\nu(x) = 1 - \nu + \nu x - x^\nu. \tag{10.1}$$

The following lemma holds.

LEMMA 10.1 (Dragomir, 2015, [**67**]). *For any* $x \in [m,M] \subset [0,\infty)$ *we have*

$$\max_{x\in[m,M]} f_\nu(x) = \Delta_\nu(m,M) := \begin{cases} f_\nu(m) \text{ if } M < 1, \\ \max\{f_\nu(m), f_\nu(M)\} \text{ if } m \le 1 \le M, \\ f_\nu(M) \text{ if } 1 < m \end{cases} \tag{10.2}$$

and

$$\min_{x\in[m,M]} f_\nu(x) = \delta_\nu(m,M) := \begin{cases} f_\nu(M) \text{ if } M < 1, \\ 0 \text{ if } m \le 1 \le M, \\ f_\nu(m) \text{ if } 1 < m. \end{cases} \tag{10.3}$$

PROOF. The function f_ν is differentiable and

$$f_\nu'(x) = \nu\left(1 - x^{\nu-1}\right) = \nu\frac{x^{1-\nu}-1}{x^{1-\nu}},$$

which shows that the function f_ν is decreasing on $[0,1]$ and increasing on $[1,\infty)$, $f_\nu(0) = 1-\nu$, $f_\nu(1) = 0$ and the equation $f_\nu(x) = 1-\nu$ for $x > 0$ has the unique solution $x_\nu = \nu^{\frac{1}{\nu-1}} > 1$.

Therefore, by considering the 3 possible situations for the location of the interval $[m,M]$ and the number 1 we get the desired bounds (10.2) and (10.3). ∎

REMARK 10.1. We have the inequalities

$$0 \le f_\nu(x) \le 1-\nu \text{ for any } x \in \left[0, \nu^{\frac{1}{\nu-1}}\right]$$

and

$$1-\nu \le f_\nu(x) \text{ for any } x \in \left[\nu^{\frac{1}{\nu-1}}, \infty\right).$$

THEOREM 10.2 (Dragomir, 2015, [**67**]). *Assume that* A, B *are positive invertible operators and the constants* $M > m > 0$ *are such that*

$$mA \le B \le MA. \tag{10.4}$$

Let $\nu \in [0,1]$, *then we have the inequalities*

$$\delta_\nu(m,M)A \le A\nabla_\nu B - A\sharp_\nu B \le \Delta_\nu(m,M)A, \tag{10.5}$$

where $\Delta_\nu(m,M)$ *and* $\delta_\nu(m,M)$ *are defined by (10.2) and (10.3), respectively.*

PROOF. From Lemma 10.1 we have the double inequality

$$\delta_\nu(m,M) \le 1-\nu+\nu x - x^\nu \le \Delta_\nu(m,M) \tag{10.6}$$

for any $x \in [m,M]$.

If X is an operator such that $mI \le X \le MI$, then by (10.6) and the continuous functional calculus, we have

$$\delta_\nu(m,M)I \le (1-\nu)I + \nu X - X^\nu \le \Delta_\nu(m,M)I. \tag{10.7}$$

If the condition (10.4) holds, then by multiplying in both sides with $A^{-1/2}$ we get $mI \leq A^{-1/2}BA^{-1/2} \leq MI$ and by taking $X = A^{-1/2}BA^{-1/2}$ in (10.8) we get

$$\begin{aligned}\delta_{\nu}(m,M)I &\leq (1-\nu)I + \nu A^{-1/2}BA^{-1/2} - \left(A^{-1/2}BA^{-1/2}\right)^{\nu} \\ &\leq \Delta_{\nu}(m,M)I.\end{aligned} \tag{10.8}$$

Now, if we multiply (10.8) in both sides with $A^{1/2}$ we get the desired result (10.5). ∎

COROLLARY 10.3. *For two positive operators* A, B *and positive real numbers* m, m', M, M' *put* $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$.
If

(i) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$,
then

$$f_{\nu}(h')A \leq A\nabla_{\nu}B - A\sharp_{\nu}B \leq f_{\nu}(h)A. \tag{10.9}$$

If

(ii) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$,
then

$$f_{\nu}\left((h')^{-1}\right)A \leq A\nabla_{\nu}B - A\sharp_{\nu}B \leq f_{\nu}\left(h^{-1}\right)A. \tag{10.10}$$

PROOF. If *(i)* is valid, then we have

$$A < \frac{M'}{m'}A = h'A \leq B \leq hA = \frac{M}{m}A,$$

and by (10.5) we have for $1 < h' \leq h$

$$f_{\nu}(h')A \leq A\nabla_{\nu}B - A\sharp_{\nu}B \leq f_{\nu}(h)A,$$

and the inequality (10.9) is proved.

If *(ii)* is valid, then we have

$$\frac{1}{h}A \leq B \leq \frac{1}{h'}A < A$$

and by (10.5) for $\frac{1}{h} \leq \frac{1}{h'} < 1$ we also have

$$f_{\nu}\left(\frac{1}{h'}\right)A \leq A\nabla_{\nu}B - A\sharp_{\nu}B \leq f_{\nu}\left(\frac{1}{h}\right)A,$$

and the inequality (10.10) is proved. ∎

We have the following simpler bounds:

COROLLARY 10.4. *With the assumptions of Theorem 10.2 we have*

$$r \times \begin{cases} \left(1-\sqrt{M}\right)^2 A \text{ if } M<1, \\ 0 \text{ if } m \le 1 \le M, \\ \left(\sqrt{m}-1\right)^2 A \text{ if } 1<m, \end{cases} \tag{10.11}$$

$$\le A\nabla_\nu B - A\sharp_\nu B$$

$$\le R \times \begin{cases} \left(1-\sqrt{m}\right)^2 A \text{ if } M<1, \\ \max\left\{\left(1-\sqrt{m}\right)^2, \left(\sqrt{M}-1\right)^2\right\} A \text{ if } m \le 1 \le M, \\ \left(\sqrt{M}-1\right)^2 A \text{ if } 1<m, \end{cases},$$

where $\nu \in [0,1]$, $r=\min\{1-\nu,\nu\}$ *and* $R=\max\{1-\nu,\nu\}$.

PROOF. From the inequality (8.1) we have for $b=t$ and $a=1$ that

$$r\left(\sqrt{t}-1\right)^2 \le f_\nu(t) = 1-\nu+\nu t - t^\nu \le R\left(\sqrt{t}-1\right)^2$$

for any $t \in [0,1]$.

Then we have

$$\Delta_\nu(m,M) \le R \times \begin{cases} \left(1-\sqrt{m}\right)^2 \text{ if } M<1, \\ \max\left\{\left(1-\sqrt{m}\right)^2, \left(\sqrt{M}-1\right)^2\right\} \text{ if } m \le 1 \le M, \\ \left(\sqrt{M}-1\right)^2 \text{ if } 1<m \end{cases}$$

and

$$\delta_\nu(m,M) \ge r \times \begin{cases} \left(1-\sqrt{M}\right)^2 \text{ if } M<1, \\ 0 \text{ if } m \le 1 \le M, \\ \left(\sqrt{m}-1\right)^2 \text{ if } 1<m, \end{cases}$$

which by Theorem 10.2 proves the corollary. ∎

REMARK 10.2. With the assumptions of Corollary 10.3, we have, in the case (i), that

$$r\left(\sqrt{h'}-1\right)^2 A \le A\nabla_\nu B - A\sharp_\nu B \le R\left(\sqrt{h}-1\right)^2 A, \tag{10.12}$$

and in the case (ii), that

$$r\frac{\left(1-\sqrt{h'}\right)^2}{h'}A \le A\nabla_\nu B - A\sharp_\nu B \le R\frac{\left(1-\sqrt{h}\right)^2}{h}A. \tag{10.13}$$

The following bounds in terms of Specht's ratio can be stated as well:

COROLLARY 10.5. *With the assumptions of Theorem 10.2 we have*

$$\begin{cases} [S(M^r)-1]\,M^\nu A \textit{ if } M<1, \\ 0 \textit{ if } m\le 1\le M, \\ [S(m^r)-1]\,m^\nu A \textit{ if } 1<m, \end{cases} \tag{10.14}$$

$$\le A\nabla_\nu B - A\sharp_\nu B$$

$$\le \begin{cases} [S(m)-1]\,m^\nu A \textit{ if } M<1, \\ \max\{[S(m)-1]\,m^\nu, [S(M)-1]\,M^\nu\}\,A \textit{ if } m\le 1\le M, \\ [S(M)-1]\,M^\nu A \textit{ if } 1<m. \end{cases}$$

PROOF. From the inequality (10.1) we have for $a=1$ and $b=t$ that

$$S(t^r)\,t^\nu \le 1-\nu+\nu t \le S(t)\,t^\nu, \tag{10.15}$$

where $t>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$.

By subtracting t^ν in the inequality (10.15) we get

$$(0\le)\,[S(t^r)-1]\,t^\nu \le f_\nu(t) \le [S(t)-1]\,t^\nu, \tag{10.16}$$

for any $t>0$, $\nu\in[0,1]$.

Then we have

$$\Delta_\nu(m,M) \le \begin{cases} [S(m)-1]\,m^\nu \text{ if } M<1, \\ \max\{[S(m)-1]\,m^\nu, [S(M)-1]\,M^\nu\} \text{ if } m\le 1\le M, \\ [S(M)-1]\,M^\nu \text{ if } 1<m \end{cases}$$

and

$$\delta_\nu(m,M) \ge \begin{cases} [S(M^r)-1]\,M^\nu \text{ if } M<1, \\ 0 \text{ if } m\le 1\le M, \\ [S(m^r)-1]\,m^\nu \text{ if } 1<m, \end{cases}$$

which by Theorem 10.2 proves the corollary. ∎

REMARK 10.3. With the assumptions of Corollary 10.3, we have in the case (i), that

$$\left[S\left((h')^r\right)-1\right](h')^\nu A \le A\nabla_\nu B - A\sharp_\nu B \le [S(h)-1]\,h^\nu A, \tag{10.17}$$

and in the case (ii), that

$$\frac{S((h')^r)-1}{(h')^r}A \le A\nabla_\nu B - A\sharp_\nu B \le \frac{S(h)-1}{h^\nu}A. \tag{10.18}$$

We have

$$(0<)\left[K^{r}(t)-1\right]t^{\nu}\leq 1-\nu+\nu t-t^{\nu}\leq\left[K^{R}(t)-1\right]t^{\nu} \tag{10.19}$$

where $t>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

We then have the following bounds in terms of Kantorovich's constant:

COROLLARY 10.6. *With the assumptions of Theorem 10.2 we have*

$$\begin{cases}\left[K^{r}(M)-1\right]M^{\nu}A \text{ if } M<1,\\ 0 \text{ if } m\leq 1\leq M,\\ \left[K^{r}(m)-1\right]m^{\nu}A \text{ if } 1<m,\end{cases} \tag{10.20}$$

$$\leq A\nabla_{\nu}B-A\sharp_{\nu}B$$

$$\leq\begin{cases}\left[K^{R}(m)-1\right]m^{\nu}A \text{ if } M<1,\\ \max\left\{\left[K^{R}(m)-1\right]m^{\nu},\left[K^{R}(M)-1\right]M^{\nu}\right\}A \text{ if } m\leq 1\leq M,\\ \left[K^{R}(M)-1\right]M^{\nu}A \text{ if } 1<m.\end{cases}$$

REMARK 10.4. With the assumptions of Corollary 10.3, we have in the case (i), that

$$\left[K^{r}(h')-1\right](h')^{\nu}A\leq A\nabla_{\nu}B-A\sharp_{\nu}B\leq\left[K^{R}(h)-1\right]h^{\nu}A, \tag{10.21}$$

and in the case (ii), that

$$\frac{K^{r}(h')-1}{(h')^{r}}A\leq A\nabla_{\nu}B-A\sharp_{\nu}B\leq\frac{K^{R}(h)-1}{h^{\nu}}A. \tag{10.22}$$

Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Assume that the positive invertible operators A, B satisfy the condition

$$mA^{p}\leq B^{q}\leq MA^{p}. \tag{10.23}$$

Then by replacing A with A^{p}, B with B^{q} and $\nu=\frac{1}{q}$ in (10.5) we have

$$\delta_{\frac{1}{q}}(m,M)A^{p}\leq\frac{1}{p}A^{p}+\frac{1}{q}B^{q}-A^{p}\sharp_{\frac{1}{q}}B^{q}\leq\Delta_{\frac{1}{q}}(m,M)A^{p}, \tag{10.24}$$

where $\Delta_{\frac{1}{q}}(m,M)$ and $\delta_{\frac{1}{q}}(m,M)$ are defined by (10.2) and (10.3) respectively.

If the positive invertible operators A, B satisfy the condition (10.23), then from (10.11) we get for

(10.25)
$$r_{p,q} \times \begin{cases} \left(1-\sqrt{M}\right)^2 A^p \text{ if } M<1, \\ 0 \text{ if } m \le 1 \le M, \\ \left(\sqrt{m}-1\right)^2 A^p \text{ if } 1<m \end{cases}$$

$$\le \frac{1}{p}A^p + \frac{1}{q}B^q - A^p \sharp_{\frac{1}{q}} B^q$$

$$\le R_{p,q} \times \begin{cases} \left(1-\sqrt{m}\right)^2 A^p \text{ if } M<1, \\ \max\left\{\left(1-\sqrt{m}\right)^2, \left(\sqrt{M}-1\right)^2\right\} A^p \text{ if } m \le 1 \le M, \\ \left(\sqrt{M}-1\right)^2 A^p \text{ if } 1<m, \end{cases},$$

from (10.14) we get

(10.26)
$$\begin{cases} \left[S\left(M^{r_{p,q}}\right)-1\right] M^{\nu} A^p \text{ if } M<1, \\ 0 \text{ if } m \le 1 \le M, \\ \left[S\left(m^{r_{p,q}}\right)-1\right] m^{\nu} A^p \text{ if } 1<m \end{cases}$$

$$\le \frac{1}{p}A^p + \frac{1}{q}B^q - A^p \sharp_{\frac{1}{q}} B^q$$

$$\le \begin{cases} \left[S(m)-1\right] m^{\nu} A^p \text{ if } M<1, \\ \max\left\{\left[S(m)-1\right] m^{\nu}, \left[S(M)-1\right] M^{\nu}\right\} A^p \text{ if } m \le 1 \le M, \\ \left[S(M)-1\right] M^{\nu} A^p \text{ if } 1<m, \end{cases}$$

while from (10.20) we get

$$
(10.27)\qquad \begin{cases} \left[K^{r_{p,q}}(M)-1\right]M^{\nu}A^{p} \text{ if } M<1, \\ \\ 0 \text{ if } m\le 1\le M, \\ \\ \left[K^{r_{p,q}}(m)-1\right]m^{\nu}A^{p} \text{ if } 1<m \end{cases}
$$

$$
\le \frac{1}{p}A^{p}+\frac{1}{q}B^{q}-A^{p}\sharp_{\frac{1}{q}}B^{q}
$$

$$
\le \begin{cases} \left[K^{R_{p,q}}(m)-1\right]m^{\nu}A^{p} \text{ if } M<1, \\ \\ \max\left\{\left[K^{R_{p,q}}(m)-1\right]m^{\nu},\left[K^{R_{p,q}}(M)-1\right]M^{\nu}\right\}A^{p} \text{ if } m\le 1\le M, \\ \\ \left[K^{R_{p,q}}(M)-1\right]M^{\nu}A^{p} \text{ if } 1<m, \end{cases}
$$

where $r_{p,q}=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ and $R_{p,q}=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

If $p=q=2$ and if we assume that

$$
(10.28)\qquad mA^{2}\le B^{2}\le MA^{2},
$$

then by (10.24) we get

$$
(10.29)\qquad \delta_{\frac{1}{2}}(m,M)A^{2}\le \frac{1}{2}\left(A^{2}+B^{2}\right)-A^{2}\sharp B^{2}\le \Delta_{\frac{1}{2}}(m,M)A^{2}.
$$

Assume that A and B satisfy the conditions

$$
(10.30)\qquad m_{1}I\le A\le M_{1}I,\ m_{2}I\le B\le M_{2}I
$$

for some $0<m_{1}<M_{1}$ and $0<m_{2}<M_{2}$. We have from (10.30) that

$$
m_{1}^{p}I\le A^{p}\le M_{1}^{p}I.
$$

Then by (10.30) we also have

$$
m_{1}^{p}M_{2}^{-q}I\le m_{1}^{p}B^{-q}\le B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\le M_{1}^{p}B^{-q}\le M_{1}^{p}m_{2}^{-q}I,
$$

which implies that

$$
m_{1}M_{2}^{-\frac{q}{p}}I\le \left(B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\right)^{\frac{1}{p}}\le M_{1}m_{2}^{-\frac{q}{p}}I.
$$

Now, on using the inequality (10.24) for $m=m_{1}M_{2}^{-\frac{q}{p}}$ and $M=M_{1}m_{2}^{-\frac{q}{p}}$, we get

$$
(10.31)\qquad \delta_{\nu}\left(m_{1}M_{2}^{-\frac{q}{p}},M_{1}m_{2}^{-\frac{q}{p}}\right)A^{p}\le \frac{1}{p}A^{p}+\frac{1}{q}B^{q}-A^{p}\sharp_{\frac{1}{q}}B^{q}
$$
$$
\le \Delta_{\nu}\left(m_{1}M_{2}^{-\frac{q}{p}},M_{1}m_{2}^{-\frac{q}{p}}\right)A^{p},
$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

In particular, we have

$$
(10.32)\qquad \delta_{\frac{1}{2}}\left(m_{1}M_{2}^{-1},M_{1}m_{2}^{-1}\right)A^{2}\le \frac{1}{2}\left(A^{2}+B^{2}\right)-A^{2}\sharp B^{2}
$$
$$
\le \Delta_{\frac{1}{2}}\left(m_{1}M_{2}^{-1},M_{1}m_{2}^{-1}\right)A^{2},
$$

provided that A and B satisfy the conditions (10.30).

Further inequalities in terms of Specht's ratio and Kantorovich's constant may be obtained by using (10.26) and (10.27) respectively, however the details are not presented here.

10.2. Inequalities Related to McCarthy's. By the use of the spectral resolution of $P \geq 0$ and the Hölder inequality, C. A. McCarthy [**109**] proved that

$$\langle Px, x\rangle^p \leq \langle P^p x, x\rangle, \ p \in (1, \infty) \tag{10.33}$$

and

$$\langle P^p x, x\rangle \leq \langle Px, x\rangle^p, \ p \in (0, 1) \tag{10.34}$$

for any $x \in H$ with $\|x\| = 1$.

From the previous section, for positive numbers a, b with $\frac{b}{a} \in [m, M] \subset (0, \infty)$ and $\nu \in [0, 1]$ we can state the following scalar inequalities

$$\delta_\nu (m, M)\, a \leq (1 - \nu)\, a + \nu b - a^{1-\nu} b^\nu \leq \Delta_\nu (m, M)\, a, \tag{10.35}$$

where $\Delta_\nu (m, M)$ and $\delta_\nu (m, M)$ are defined by (10.2) and (10.3) respectively.

We also have the scalar inequalities

$$r \times \begin{cases} \left(1 - \sqrt{M}\right)^2 a \text{ if } M < 1, \\ \\ 0 \text{ if } m \leq 1 \leq M, \\ \\ \left(\sqrt{m} - 1\right)^2 a \text{ if } 1 < m, \end{cases} \tag{10.36}$$

$$\leq (1 - \nu)\, a + \nu b - a^{1-\nu} b^\nu$$

$$\leq R \times \begin{cases} \left(1 - \sqrt{m}\right)^2 a \text{ if } M < 1, \\ \\ \max \left\{ \left(1 - \sqrt{m}\right)^2, \left(\sqrt{M} - 1\right)^2 \right\} a \text{ if } m \leq 1 \leq M, \\ \\ \left(\sqrt{M} - 1\right)^2 a \text{ if } 1 < m, \end{cases},$$

$$\begin{cases} \left[S\left(M^r\right) - 1\right] M^\nu a \text{ if } M < 1, \\ \\ 0 \text{ if } m \leq 1 \leq M, \\ \\ \left[S\left(m^r\right) - 1\right] m^\nu a \text{ if } 1 < m \end{cases} \tag{10.37}$$

$$\leq (1 - \nu)\, a + \nu b - a^{1-\nu} b^\nu$$

$$\leq \begin{cases} \left[S(m) - 1\right] m^\nu a \text{ if } M < 1, \\ \\ \max \left\{ \left[S(m) - 1\right] m^\nu, \left[S(M) - 1\right] M^\nu \right\} a \text{ if } m \leq 1 \leq M, \\ \\ \left[S(M) - 1\right] M^\nu a \text{ if } 1 < m \end{cases}$$

and

$$(10.38)\qquad \begin{cases} [K^r(M)-1]\,M^\nu a \text{ if } M<1,\\ 0 \text{ if } m\le 1\le M,\\ [K^r(m)-1]\,m^\nu a \text{ if } 1<m \end{cases}$$

$$\le (1-\nu)\,a+\nu b-a^{1-\nu}b^\nu$$

$$\le \begin{cases} \left[K^R(m)-1\right]m^\nu a \text{ if } M<1,\\ \max\left\{\left[K^R(m)-1\right]m^\nu,\left[K^R(M)-1\right]M^\nu\right\}a \text{ if } m\le 1\le M,\\ \left[K^R(M)-1\right]M^\nu a \text{ if } 1<m, \end{cases}$$

where $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

THEOREM 10.7 (Dragomir, 2015, [**67**]). *Let P and operator such that*

$$(10.39)\qquad zI\le P\le ZI$$

for some constants $Z>z>0$.

Then for any $x\in H$ with $\|x\|=1$ we have

$$(10.40)\qquad 0\le 1-\frac{\langle P^\lambda x,x\rangle}{\langle Px,x\rangle^\lambda}\le\max\left\{f_\lambda\left(\frac{z}{Z}\right),f_\lambda\left(\frac{Z}{z}\right)\right\},$$

where $\lambda\in[0,1]$ and the function $f_\lambda:[0,\infty)\to[0,\infty)$ is defined by

$$(10.41)\qquad f_\lambda(t)=1-\lambda+\lambda t-t^\lambda.$$

PROOF. If $u,v\in[z,Z]$ then $\frac{u}{v}\in\left[\frac{z}{Z},\frac{Z}{z}\right]$ and by (10.35) we have

$$0\le(1-\lambda)\,v+\lambda u-v^{1-\lambda}u^\lambda\le\max\left\{f_\lambda\left(\frac{z}{Z}\right),f_\lambda\left(\frac{Z}{z}\right)\right\}v$$

for any $\lambda\in[0,1]$.

Fix $v\in[z,Z]$, then by using the functional calculus for the operator P with $zI\le P\le ZI$ we have

$$(10.42)\qquad 0\le(1-\lambda)\,vI+\lambda P-v^{1-\lambda}P^\lambda\le\max\left\{f_\lambda\left(\frac{z}{Z}\right),f_\lambda\left(\frac{Z}{z}\right)\right\}v$$

for any $\lambda\in[0,1]$.

The inequality (10.42) implies that

$$(10.43)\qquad 0\le(1-\lambda)\,v+\lambda\langle Px,x\rangle-v^{1-\lambda}\langle P^\lambda x,x\rangle\le\max\left\{f_\lambda\left(\frac{z}{Z}\right),f_\lambda\left(\frac{Z}{z}\right)\right\}v,$$

for any $x\in H$ with $\|x\|=1$, for any $\lambda\in[0,1]$ and for any $v\in[z,Z]$.

If we take in (10.43) $v=\langle Px,x\rangle\in[z,Z]$, for $x\in H$ with $\|x\|=1$, then we have

$$0\le\langle Px,x\rangle-\langle Px,x\rangle^{1-\lambda}\langle P^\lambda x,x\rangle\le\max\left\{f_\lambda\left(\frac{z}{Z}\right),f_\lambda\left(\frac{Z}{z}\right)\right\}\langle Px,x\rangle,$$

which, by division with $\langle Px,x\rangle^{1-\lambda}>0$ produces

$$0\le\langle Px,x\rangle^{\lambda}-\left\langle P^{\lambda}x,x\right\rangle\le\max\left\{f_{\lambda}\left(\frac{z}{Z}\right),f_{\lambda}\left(\frac{Z}{z}\right)\right\}\langle Px,x\rangle^{\lambda}$$

that is equivalent to the desired result (10.40). ∎

REMARK 10.5. If $1<\frac{Z}{z}\le\lambda^{\frac{1}{\lambda-1}}$ with $\lambda\in(0,1)$ then by Remark 10.1 we have that $\max\left\{f_{\lambda}\left(\frac{z}{Z}\right),f_{\lambda}\left(\frac{Z}{z}\right)\right\}\le1-\lambda$ and by (10.40) we get

$$\lambda\langle Px,x\rangle^{\lambda}\le\left\langle P^{\lambda}x,x\right\rangle\left(\le\langle Px,x\rangle^{\lambda}\right)\tag{10.44}$$

for any $x\in H$ with $\|x\|=1$.

COROLLARY 10.8. *With the assumptions of Theorem 10.7 and if* $T=\max\{\lambda,1-\lambda\}$ *for* $\lambda\in(0,1)$, *then we have*

$$0\le1-\frac{\left\langle P^{\lambda}x,x\right\rangle}{\langle Px,x\rangle^{\lambda}}\le\begin{cases}T\left(\sqrt{\frac{Z}{z}}-1\right)^{2},\\[2mm] \left[S\left(\frac{Z}{z}\right)-1\right]\left(\frac{Z}{z}\right)^{\lambda},\\[2mm] \left[K^{T}\left(\frac{Z}{z}\right)-1\right]\left(\frac{Z}{z}\right)^{\lambda}\end{cases}$$

for any $x\in H$ *with* $\|x\|=1$.

We have:

THEOREM 10.9 (Dragomir, 2015, [**67**]). *Let* A *and* B *be two positive invertible operators,* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $m,M>0$ *such that*

$$m^{p}B^{q}\le A^{p}\le M^{p}B^{q}.\tag{10.45}$$

Then we have

$$0\le1-\frac{\left\langle B^{q}\sharp_{1/p}A^{p}x,x\right\rangle}{\langle A^{p}x,x\rangle^{1/p}\langle B^{q}x,x\rangle^{1/q}}\le\max\left\{f_{\frac{1}{p}}\left(\left(\frac{m}{M}\right)^{p}\right),f_{\frac{1}{p}}\left(\left(\frac{M}{m}\right)^{p}\right)\right\},\tag{10.46}$$

where the function $f_{\frac{1}{p}}:[0,\infty)\to[0,\infty)$ *is defined by (10.41) for* $\lambda=\frac{1}{p}$.

PROOF. From the inequality (10.40) for $x=\frac{y}{\|y\|}$, $y\neq0$ we have

$$0\le1-\frac{\left\langle P^{\lambda}y,y\right\rangle}{\langle y,y\rangle^{1-\lambda}\langle Py,y\rangle^{\lambda}}\le\max\left\{f_{\lambda}\left(\frac{z}{Z}\right),f_{\lambda}\left(\frac{Z}{z}\right)\right\},\tag{10.47}$$

provided that P satisfy the condition (10.39).

Now, from (10.45) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^{p}I\le B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\le M^{p}I$.

By writing the inequality (10.47) for $P=B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}$, $z=m^{p}$, $Z=M^{p}$, $\lambda=\frac{1}{p}$ and $y=B^{\frac{q}{2}}x$, with $x\in H$, $x\neq0$, we have

$$\begin{aligned}0&\le1-\frac{\left\langle\left(B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\right)^{\frac{1}{p}}B^{\frac{q}{2}}x,B^{\frac{q}{2}}x\right\rangle}{\left\langle B^{\frac{q}{2}}x,B^{\frac{q}{2}}x\right\rangle^{\frac{1}{q}}\left\langle\left(B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\right)B^{\frac{q}{2}}x,B^{\frac{q}{2}}x\right\rangle^{\frac{1}{p}}}\\&\le\max\left\{f_{\frac{1}{p}}\left(\left(\frac{m}{M}\right)^{p}\right),f_{\frac{1}{p}}\left(\left(\frac{M}{m}\right)^{p}\right)\right\}\end{aligned}$$

that is equivalent to

$$0 \leq 1 - \frac{\left\langle B^{\frac{q}{2}} \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \right)^{\frac{1}{p}} B^{\frac{q}{2}} x, x \right\rangle}{\langle B^q x, x \rangle^{\frac{1}{q}} \langle A^p x, x \rangle^{\frac{1}{p}}} \leq \max \left\{ f_{\frac{1}{p}} \left(\left(\frac{m}{M} \right)^p \right), f_{\frac{1}{p}} \left(\left(\frac{M}{m} \right)^p \right) \right\}$$

with $x \in H$, $x \neq 0$.

This is equivalent to the desired result (10.46). ∎

COROLLARY 10.10. *With the assumptions of Theorem 10.9 we have for $x \in H$, $x \neq 0$, that*

$$0 \leq 1 - \frac{\left\langle B^q \sharp_{1/p} A^p x, x \right\rangle}{\langle A^p x, x \rangle^{1/p} \langle B^q x, x \rangle^{1/q}} \leq \begin{cases} T_{p,q} \left(\left(\frac{M}{m} \right)^{\frac{p}{2}} - 1 \right)^2, \\ \left[S \left(\left(\frac{M}{m} \right)^p \right) - 1 \right] \frac{M}{m}, \\ \left[K^{T_{p,q}} \left(\left(\frac{M}{m} \right)^p \right) - 1 \right] \frac{M}{m}, \end{cases}$$

where $T_{p,q} = \max \left\{ \frac{1}{p}, \frac{1}{q} \right\}$.

10.3. Trace Inequalities. We have the following trace inequality:

THEOREM 10.11 (Dragomir, 2015, [**67**]). *Let C be an operator with the property that*

$$zI \leq C \leq ZI \tag{10.48}$$

for some constants z, Z with $Z > z > 0$ and $P \in \mathcal{B}_1(H)$, $P \geq 0$ with $\operatorname{tr}(P) > 0$. Then for any $\lambda \in [0,1]$ we have

$$0 \leq 1 - \frac{\operatorname{tr}\left(PC^\lambda\right)}{\operatorname{tr}^{1-\lambda}(P)\operatorname{tr}^\lambda(PC)} \leq \max \left\{ f_\lambda \left(\frac{z}{Z} \right), f_\lambda \left(\frac{Z}{z} \right) \right\} \tag{10.49}$$

and the function f_λ is defined by (10.41).

PROOF. As in the proof of Theorem 10.7, we have

$$0 \leq (1-\lambda) vI + \lambda C - v^{1-\lambda} C^\lambda \leq \max \left\{ f_\lambda \left(\frac{z}{Z} \right), f_\lambda \left(\frac{Z}{z} \right) \right\} v$$

for any $\lambda \in [0,1]$.

This inequality implies that

$$0 \leq (1-\lambda) v \langle x, x \rangle + \lambda \langle Cx, x \rangle - v^{1-\lambda} \left\langle C^\lambda x, x \right\rangle \leq \max \left\{ f_\lambda \left(\frac{z}{Z} \right), f_\lambda \left(\frac{Z}{z} \right) \right\} v \langle x, x \rangle, \tag{10.50}$$

for any $x \in H$, for any $\lambda \in [0,1]$ and for any $v \in [z, Z]$.

Now, if we take in (10.50) $x = P^{1/2} e$, where $e \in H$, then

$$0 \leq (1-\lambda) v \langle Pe, e \rangle + \lambda \left\langle P^{1/2} C P^{1/2} e, e \right\rangle - v^{1-\lambda} \left\langle P^{1/2} C^\lambda P^{1/2} e, e \right\rangle \leq \max \left\{ f_\lambda \left(\frac{z}{Z} \right), f_\lambda \left(\frac{Z}{z} \right) \right\} v \langle Pe, e \rangle, \tag{10.51}$$

for any $e \in H$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. If we take in (10.51) $e = e_i$, $i \in I$ and by summing over $i \in I$, then we get

$$(10.52)\qquad \begin{aligned} 0 &\le (1-\lambda) v \sum_{i\in I} \langle Pe_i, e_i\rangle + \lambda \sum_{i\in I} \left\langle P^{1/2} C P^{1/2} e_i, e_i\right\rangle \\ &- v^{1-\lambda} \sum_{i\in I} \left\langle P^{1/2} C^{\lambda} P^{1/2} e_i, e_i\right\rangle \\ &\le \max\left\{ f_\lambda\left(\frac{z}{Z}\right), f_\lambda\left(\frac{Z}{z}\right)\right\} v \sum_{i\in I} \langle Pe_i, e_i\rangle, \end{aligned}$$

and by the properties of trace we have

$$\begin{aligned} 0 &\le (1-\lambda) v \operatorname{tr}(P) + \lambda \operatorname{tr}(PC) - v^{1-\lambda} \operatorname{tr}\left(PC^{\lambda}\right) \\ &\le \max\left\{ f_\lambda\left(\frac{z}{Z}\right), f_\lambda\left(\frac{Z}{z}\right)\right\} v \operatorname{tr}(P), \end{aligned}$$

for any $\lambda \in [0,1]$ and for any $v \in [z, Z]$.

This inequality can be written as

$$(10.53)\qquad \begin{aligned} 0 &\le (1-\lambda) v + \lambda \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - v^{1-\lambda} \frac{\operatorname{tr}\left(PC^{\lambda}\right)}{\operatorname{tr}(P)} \\ &\le \max\left\{ f_\lambda\left(\frac{z}{Z}\right), f_\lambda\left(\frac{Z}{z}\right)\right\} v, \end{aligned}$$

for any $\lambda \in [0,1]$ and for any $v \in [z, Z]$.

Now, if we take in (10.53) $v = \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \in [z, Z]$, then we get

$$\begin{aligned} 0 &\le (1-\lambda) \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} + \lambda \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1-\lambda} \frac{\operatorname{tr}\left(PC^{\lambda}\right)}{\operatorname{tr}(P)} \\ &\le \max\left\{ f_\lambda\left(\frac{z}{Z}\right), f_\lambda\left(\frac{Z}{z}\right)\right\} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}, \end{aligned}$$

namely

$$\begin{aligned} 0 &\le \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1-\lambda} \frac{\operatorname{tr}\left(PC^{\lambda}\right)}{\operatorname{tr}(P)} \\ &\le \max\left\{ f_\lambda\left(\frac{z}{Z}\right), f_\lambda\left(\frac{Z}{z}\right)\right\} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}, \end{aligned}$$

and by multiplying with $\frac{\operatorname{tr}(P)}{\operatorname{tr}(PC)} > 0$ we get the desired result (10.49). ∎

In particular, we have:

COROLLARY 10.12. *With the assumptions of Theorem 10.11 and if* $T = \max\{\lambda, 1-\lambda\}$ *for* $\lambda \in (0,1)$, *then we have*

$$(10.54)\qquad 0 \le 1 - \frac{\operatorname{tr}\left(PC^{\lambda}\right)}{\operatorname{tr}^{1-\lambda}(P)\operatorname{tr}^{\lambda}(PC)} \le \begin{cases} T\left(\sqrt{\frac{Z}{z}} - 1\right)^2, \\ \left[S\left(\frac{Z}{z}\right) - 1\right]\left(\frac{Z}{z}\right)^{\lambda}, \\ \left[K^T\left(\frac{Z}{z}\right) - 1\right]\left(\frac{Z}{z}\right)^{\lambda}. \end{cases}$$

The following reverse of Hölder's trace inequality may be stated:

THEOREM 10.13 (Dragomir, 2015, [**67**]). *Let A and B be two positive invertible operators, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m, M>0$ such that*

$$m^p B^q \leq A^p \leq M^p B^q. \tag{10.55}$$

If $B^q \in \mathcal{B}_1(H)$, then

$$0 \leq 1-\frac{\operatorname{tr}\left(B^q \sharp_{1/p} A^p\right)}{\operatorname{tr}^{1/p}\left(A^p\right) \operatorname{tr}^{1/q}\left(B^q\right)} \leq \max \left\{f_{\frac{1}{p}}\left(\left(\frac{m}{M}\right)^p\right), f_{\frac{1}{p}}\left(\left(\frac{M}{m}\right)^p\right)\right\}. \tag{10.56}$$

PROOF. Now, from (10.55) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^p I \leq B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \leq M^p I$. By writing the inequality (10.49) for $C=B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}$, $z=m^p$, $Z=M^p$, $\lambda=\frac{1}{p}$ and $P=B^q$ we get the desired result (10.56). ∎

Finally, we have

COROLLARY 10.14. *With the assumptions of Theorem 10.13 and if $T_{p,q}=\max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, then we have*

$$0 \leq 1-\frac{\operatorname{tr}\left(B^q \sharp_{1/p} A^p\right)}{\operatorname{tr}^{1/p}\left(A^p\right) \operatorname{tr}^{1/q}\left(B^q\right)} \leq \begin{cases} T_{p,q}\left(\left(\frac{M}{m}\right)^{\frac{p}{2}}-1\right)^2, \\ \left[S\left(\left(\frac{M}{m}\right)^p\right)-1\right] \frac{M}{m}, \\ \left[K^{T_{p,q}}\left(\left(\frac{M}{m}\right)^p\right)-1\right] \frac{M}{m}. \end{cases} \tag{10.57}$$

10.4. Other Upper and Lower Bounds. In [**57**] we proved the following reverses of Young's inequality

$$0 \leq (1-\nu) a+\nu b-a^{1-\nu} b^\nu \leq \nu(1-\nu)(a-b)(\ln a-\ln b) \tag{10.58}$$

and

$$1 \leq \frac{(1-\nu) a+\nu b}{a^{1-\nu} b^\nu} \leq \exp \left[4 \nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{10.59}$$

for any $a, b>0$ and $\nu \in[0,1]$, where K is Kantorovich's constant.

The inequality (10.59) is equivalent to

$$\begin{aligned} 0 & \leq (1-\nu) a+\nu b-a^{1-\nu} b^\nu \\ & \leq\left(\exp \left[4 \nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right]-1\right) a^{1-\nu} b^\nu \end{aligned} \tag{10.60}$$

for any $a, b>0$ and $\nu \in[0,1]$.

Therefore, by (10.2), (10.58) and (10.60) we have

$$\Delta_\nu(m, M)=\begin{cases} f_\nu(m) \text{ if } M<1, \\ \max \left\{f_\nu(m), f_\nu(M)\right\} \text{ if } m \leq 1 \leq M, \\ f_\nu(M) \text{ if } 1<m \end{cases} \tag{10.61}$$

$$\leq \nu(1-\nu) \times \begin{cases} (m-1) \ln m \text{ if } M<1, \\ \max \{(m-1) \ln m,(M-1) \ln M\} \text{ if } m \leq 1 \leq M, \\ (M-1) \ln M \text{ if } 1<m, \end{cases}$$

and

$$\Delta_\nu(m,M) \le \begin{cases} (\exp[4\nu(1-\nu)(K(m)-1)]-1)\,m^\nu \text{ if } M<1, \\ \\ \max\{(\exp[4\nu(1-\nu)(K(m)-1)]-1)\,m^\nu, \\ (\exp[4\nu(1-\nu)(K(M)-1)]-1)\,M^\nu\} \text{ if } m\le 1\le M, \\ \\ (\exp[4\nu(1-\nu)(K(M)-1)]-1)\,M^\nu \text{ if } 1<m. \end{cases} \tag{10.62}$$

In [**61**] we also obtained the following refinements and reverses of Young's inequality

$$\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\min\{a,b\} \le (1-\nu)a+\nu b-a^{1-\nu}b^\nu \le \frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\max\{a,b\} \tag{10.63}$$

and

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \le \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^\nu} \le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right] \tag{10.64}$$

for any $a,b>0$ and $\nu\in[0,1]$.

The inequality (10.64) is equivalent to

$$\left(\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right]-1\right)a^{1-\nu}b^\nu \le (1-\nu)a+\nu b-a^{1-\nu}b^\nu \le \left(\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right]-1\right)a^{1-\nu}b^\nu \tag{10.65}$$

for any $a,b>0$ and $\nu\in[0,1]$.

Therefore, by (10.63) and (10.65) we have the upper bounds

$$\Delta_\nu(m,M) \le \frac{1}{2}\nu(1-\nu)\begin{cases} (\ln m)^2 \text{ if } M<1, \\ \\ \max\left\{(\ln m)^2,(\ln M)^2 M\right\} \text{ if } m\le 1\le M, \\ \\ (\ln M)^2 M \text{ if } 1<m \end{cases} \tag{10.66}$$

and

$$\Delta_\nu(m,M) \le \begin{cases} \left(\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{1}{m}-1\right)^2\right]-1\right)m^\nu \text{if } M<1, \\ \\ \max\left\{\left(\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{1}{m}-1\right)^2\right]-1\right)m^\nu, \right. \\ \left.\left(\exp\left[\frac{1}{2}\nu(1-\nu)(M-1)^2\right]-1\right)M^\nu\right\} \text{ if } m\le 1\le M, \\ \\ \left(\exp\left[\frac{1}{2}\nu(1-\nu)(M-1)^2\right]-1\right)M^\nu \text{ if } 1<m. \end{cases} \tag{10.67}$$

From (10.3), (10.63) and (10.65) we have the lower bounds

$$\delta_{\nu}(m,M):=\begin{cases} f_{\nu}(M) \text{ if } M<1, \\ 0 \text{ if } m\leq 1\leq M, \\ f_{\nu}(m) \text{ if } 1<m. \end{cases} \tag{10.68}$$

$$\geq \frac{1}{2}\nu(1-\nu)\times\begin{cases} (\ln M)^2 M \text{ if } M<1, \\ 0 \text{ if } m\leq 1\leq M, \\ (\ln m)^2 \text{ if } 1<m \end{cases}$$

and

$$\delta_{\nu}(m,M)\geq\begin{cases} \left(\exp\left[\frac{1}{2}\nu(1-\nu)(1-M)^2\right]-1\right)M^{\nu} \text{ if } M<1, \\ 0 \text{ if } m\leq 1\leq M, \\ \left(\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{1}{m}\right)^2\right]-1\right)m^{\nu} \text{ if } 1<m. \end{cases} \tag{10.69}$$

Assume that A, B are positive invertible operators and the constants $M>m>0$ are such that $mA\leq B\leq MA$. If we use the second inequality in (10.5), then we have the following upper bounds for the difference $A\nabla_{\nu}B-A\sharp_{\nu}B$:

$$A\nabla_{\nu}B-A\sharp_{\nu}B \tag{10.70}$$

$$\leq\nu(1-\nu)\times\begin{cases} [(m-1)\ln m]A \text{ if } M<1, \\ \max\{(m-1)\ln m,(M-1)\ln M\}A \\ \text{if } m\leq 1\leq M, \\ [(M-1)\ln M]A \text{ if } 1<m, \end{cases}$$

$$A\nabla_{\nu}B-A\sharp_{\nu}B \tag{10.71}$$

$$\leq\begin{cases} (\exp[4\nu(1-\nu)(K(m)-1)]-1)Am^{\nu} \text{ if } M<1, \\ \max\{(\exp[4\nu(1-\nu)(K(m)-1)]-1)m^{\nu}, \\ (\exp[4\nu(1-\nu)(K(M)-1)]-1)M^{\nu}\}A \text{ if } m\leq 1\leq M, \\ (\exp[4\nu(1-\nu)(K(M)-1)]-1)AM^{\nu} \text{ if } 1<m, \end{cases}$$

$$A\nabla_{\nu}B-A\sharp_{\nu}B \tag{10.72}$$

$$\leq\frac{1}{2}\nu(1-\nu)\begin{cases} \left[(\ln m)^2\right]A \text{ if } M<1, \\ \max\left\{(\ln m)^2,(\ln M)^2 M\right\}A \text{ if } m\leq 1\leq M, \\ \left[(\ln M)^2 M\right]A \text{ if } 1<m \end{cases}$$

and

$$A\nabla_\nu B - A\sharp_\nu B \le \begin{cases} \left(\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{1}{m}-1\right)^2\right]-1\right) Am^\nu \text{if } M<1, \\ \max\left\{\left(\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{1}{m}-1\right)^2\right]-1\right) m^\nu, \right. \\ \left.\left(\exp\left[\frac{1}{2}\nu(1-\nu)(M-1)^2\right]-1\right) M^\nu\right\} A \text{ if } m\le 1\le M, \\ \left(\exp\left[\frac{1}{2}\nu(1-\nu)(M-1)^2\right]-1\right) AM^\nu \text{ if } 1<m. \end{cases} \tag{10.73}$$

If we use the first inequality in (10.5), then we have the following lower bounds for the difference $A\nabla_\nu B - A\sharp_\nu B$:

$$A\nabla_\nu B - A\sharp_\nu B \ge \frac{1}{2}\nu(1-\nu)\times \begin{cases} \left[(\ln M)^2 M\right] A\text{if } M<1, \\ 0 \text{ if } m\le 1\le M, \\ \left[(\ln m)^2\right] A \text{ if } 1<m \end{cases} \tag{10.74}$$

and

$$A\nabla_\nu B - A\sharp_\nu B \ge \begin{cases} \left(\exp\left[\frac{1}{2}\nu(1-\nu)(1-M)^2\right]-1\right) AM^\nu \text{ if } M<1, \\ 0 \text{ if } m\le 1\le M, \\ \left(\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{1}{m}\right)^2\right]-1\right) Am^\nu \text{ if } 1<m. \end{cases} \tag{10.75}$$

The interested reader may state other inequalities by using Theorems 10.7-10.13, however the details are nor presented here.

11. MULTIPLICATIVE REVERSES

In [**57**] we proved the following multiplicative reverse of Young's inequality

$$1 \le \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^\nu} \le \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{11.1}$$

for any $a, b>0$ and $\nu\in[0,1]$, where K is Kantorovich's constant.

In [**61**] we also obtained the following multiplicative refinement and reverse of Young's inequality

$$\begin{aligned} &\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \\ &\le \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^\nu} \\ &\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right] \end{aligned} \tag{11.2}$$

for any $a, b>0$ and $\nu\in[0,1]$.

11.1. Multiplicative Reverses. We consider the function $g_\nu : (0,\infty) \to (0,\infty)$ defined for $\nu \in (0,1)$ by

$$g_\nu(x) = \frac{1-\nu+\nu x}{x^\nu} = (1-\nu)x^{-\nu} + \nu x^{1-\nu}. \tag{11.3}$$

For $[m,M] \subset (0,\infty)$ define the quantities

$$\Gamma_\nu(m,M) := \begin{cases} g_\nu(m) \text{ if } M<1, \\ \\ \max\{g_\nu(m), g_\nu(M)\} \text{ if } m \le 1 \le M, \\ \\ g_\nu(M) \text{ if } 1<m \end{cases} \tag{11.4}$$

$$= \begin{cases} (1-\nu)m^{-\nu} + \nu m^{1-\nu} \text{ if } M<1, \\ \\ \max\{(1-\nu)m^{-\nu} + \nu m^{1-\nu}, (1-\nu)M^{-\nu} + \nu M^{1-\nu}\} \\ \text{if } m \le 1 \le M, \\ \\ (1-\nu)M^{-\nu} + \nu M^{1-\nu} \text{ if } 1<m \end{cases}$$

and

$$\gamma_\nu(m,M) := \begin{cases} g_\nu(M) \text{ if } M<1, \\ \\ 1 \text{ if } m \le 1 \le M, \\ \\ g_\nu(m) \text{ if } 1<m. \end{cases} \tag{11.5}$$

$$= \begin{cases} (1-\nu)M^{-\nu} + \nu M^{1-\nu} \text{ if } M<1, \\ \\ 1 \text{ if } m \le 1 \le M, \\ \\ (1-\nu)m^{-\nu} + \nu m^{1-\nu} \text{ if } 1<m. \end{cases}$$

The following lemma holds.

LEMMA 11.1 (Dragomir, 2015, [**48**]). *For any* $x \in [m,M] \subset (0,\infty)$ *we have*

$$\max_{x\in[m,M]} g_\nu(x) = \Gamma_\nu(m,M) \tag{11.6}$$

and

$$\min_{x\in[m,M]} g_\nu(x) = \gamma_\nu(m,M). \tag{11.7}$$

PROOF. The function g_ν is differentiable and

$$g_\nu'(x) = (1-\nu)\nu x^{-\nu-1}(x-1),$$

which shows that the function g_ν is decreasing on $(0,1)$ and increasing on $[1,\infty)$. We have $g_\nu(1) = 1$, $\lim_{x\to 0+} g_\nu(x) = +\infty$, $\lim_{x\to\infty} g_\nu(x) = +\infty$ and $g_\nu\left(\frac{1}{x}\right) = g_{1-\nu}(x)$ for any $x>0$ and $\nu \in (0,1)$.

Therefore, by considering the 3 possible situations for the location of the interval $[m,M]$ and the number 1 we get the desired bounds (11.6) and (11.7). ∎

The following result provides a multiplicative refinement and reverse for the operator Young's inequality:

THEOREM 11.2 (Dragomir, 2015, [**48**]). *Assume that* A, B *are positive invertible operators and the constants* $M > m > 0$ *are such that*

$$mA \leq B \leq MA. \tag{11.8}$$

Let $\nu \in [0,1]$, *then we have the inequalities*

$$\gamma_\nu (m, M) A\sharp_\nu B \leq A\nabla_\nu B \leq \Gamma_\nu (m, M) A\sharp_\nu B, \tag{11.9}$$

where $\Gamma_\nu (m, M)$ *and* $\gamma_\nu (m, M)$ *are defined by (11.4) and (11.5), respectively.*

PROOF. From Lemma 11.1 we have the double inequality

$$\gamma_\nu (m, M) x^\nu \leq 1 - \nu + \nu x \leq \Gamma_\nu (m, M) x^\nu \tag{11.10}$$

for any $x \in [m, M]$.

If X is an operator such that $mI \leq X \leq MI$, then by (11.10) and the continuous functional calculus, we have

$$\gamma_\nu (m, M) X^\nu \leq (1 - \nu) I + \nu X \leq \Gamma_\nu (m, M) X^\nu. \tag{11.11}$$

If the condition (11.8) holds, then by multiplying in both sides with $A^{-1/2}$ we get $mI \leq A^{-1/2}BA^{-1/2} \leq MI$ and by taking $X = A^{-1/2}BA^{-1/2}$ in (11.11) we get

$$\begin{aligned} \gamma_\nu (m, M) \left(A^{-1/2}BA^{-1/2}\right)^\nu &\leq (1 - \nu) I + \nu A^{-1/2}BA^{-1/2} \\ &\leq \Gamma_\nu (m, M) \left(A^{-1/2}BA^{-1/2}\right)^\nu . \end{aligned} \tag{11.12}$$

Now, if we multiply (11.12) in both sides with $A^{1/2}$ we get the desired result (11.9). ∎

COROLLARY 11.3. *For two positive operators* A, B *and positive real numbers* m, m', M, M' *put* $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$. *Let* $\nu \in (0,1)$.
If

(i) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI,$
then

$$g_\nu (h') A\sharp_\nu B \leq A\nabla_\nu B \leq g_\nu (h) A\sharp_\nu B. \tag{11.13}$$

If

(ii) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI,$
then

$$g_{1-\nu} (h') A\sharp_\nu B \leq A\nabla_\nu B \leq g_{1-\nu} (h) A\sharp_\nu B. \tag{11.14}$$

PROOF. If *(i)* is valid, then we have

$$A < \frac{M'}{m'}A = h'A \leq B \leq hA = \frac{M}{m}A,$$

and by (11.6) for $1 < h' \leq h$ we have

$$g_\nu (h') A\sharp_\nu B \leq A\nabla_\nu B \leq g_\nu (h) A\sharp_\nu B,$$

and the inequality (11.13) is proved.

If *(ii)* is valid, then we have

$$\frac{1}{h}A \leq B \leq \frac{1}{h'}A < A$$

and by (11.13) for $\frac{1}{h} \leq \frac{1}{h'} < 1$ we also have

$$g_{\nu}\left(\frac{1}{h'}\right) A\sharp_{\nu}B \leq A\nabla_{\nu}B \leq g_{\nu}\left(\frac{1}{h}\right) A\sharp_{\nu}B,$$

and since

$$g_{\nu}\left(\frac{1}{h}\right) = g_{1-\nu}\left(h\right),\ g_{\nu}\left(\frac{1}{h'}\right) = g_{1-\nu}\left(h'\right),$$

the inequality (11.14) is proved. ∎

REMARK 11.1. By making use of (11.9) we have the following upper and lower bounds in terms of Specht's ratio S

$$\begin{cases} S\left(M^{r}\right) A\sharp_{\nu}B \text{ if } M < 1, \\ \\ A\sharp_{\nu}B \text{ if } m \leq 1 \leq M, \\ \\ S\left(m^{r}\right) A\sharp_{\nu}B \text{ if } 1 < m. \end{cases} \tag{11.15}$$

$$\leq A\nabla_{\nu}B$$

$$\leq \begin{cases} S\left(m\right) A\sharp_{\nu}B \text{ if } M < 1, \\ \\ \max\left\{S\left(m\right), S\left(M\right)\right\} A\sharp_{\nu}B \text{ if } m \leq 1 \leq M, \\ \\ S\left(M\right) A\sharp_{\nu}B \text{ if } 1 < m \end{cases},$$

where $\nu \in [0,1]$, $r = \min\left\{1-\nu, \nu\right\}$.

With the assumptions of Corollary 11.3, we have, either in the case (i) or in the case (ii) that

$$S\left(\left(h'\right)^{r}\right) A\sharp_{\nu}B \leq A\nabla_{\nu}B \leq S\left(h\right) A\sharp_{\nu}B \tag{11.16}$$

where $\nu \in [0,1]$, $r = \min\left\{1-\nu, \nu\right\}$.

REMARK 11.2. By making use of (11.9) we have the following upper and lower bounds in terms of Kantorovich's constant K

$$\begin{cases} K^{r}\left(M\right) A\sharp_{\nu}B \text{ if } M < 1, \\ \\ A\sharp_{\nu}B \text{ if } m \leq 1 \leq M, \\ \\ K^{r}\left(m\right) A\sharp_{\nu}B \text{ if } 1 < m. \end{cases} \tag{11.17}$$

$$\leq A\nabla_{\nu}B$$

$$\leq \begin{cases} K^{R}\left(m\right) A\sharp_{\nu}B \text{ if } M < 1, \\ \\ \max\left\{K^{R}\left(m\right), K^{R}\left(M\right)\right\} A\sharp_{\nu}B \text{ if } m \leq 1 \leq M, \\ \\ K^{R}\left(M\right) A\sharp_{\nu}B \text{ if } 1 < m, \end{cases},$$

where $\nu \in [0,1]$, $r = \min\left\{1-\nu, \nu\right\}$ and $R = \max\left\{1-\nu, \nu\right\}$.

With the assumptions of Corollary 11.3, we have, either in the case (i) or in the case (ii) that

$$K^{r}(h')\,A\sharp_{\nu}B\leq A\nabla_{\nu}B\leq K^{R}(h)\,A\sharp_{\nu}B, \tag{11.18}$$

where $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

REMARK 11.3. By making use of (11.9) and (11.1) we have the following exponential upper bound

$$A\nabla_{\nu}B\leq\begin{cases}\exp\left[4\nu(1-\nu)(K(m)-1)\right]A\sharp_{\nu}B \text{ if } M<1,\\ \\ \max\left\{\exp\left[4\nu(1-\nu)(K(m)-1)\right],\right.\\ \left.\exp\left[4\nu(1-\nu)(K(M)-1)\right]\right\}A\sharp_{\nu}B \text{ if } m\leq 1\leq M,\\ \\ \exp\left[4\nu(1-\nu)(K(M)-1)\right]A\sharp_{\nu}B \text{ if } 1<m.\end{cases} \tag{11.19}$$

With the assumptions of Corollary 11.3, we have either in the case (i) or in the case (ii) that

$$A\nabla_{\nu}B\leq\exp\left[4\nu(1-\nu)(K(h)-1)\right]A\sharp_{\nu}B,$$

where $\nu\in[0,1]$.

REMARK 11.4. By making use of (11.9) and (11.1) we have the following exponential lower and upper bounds

$$\begin{cases}\exp\left[\frac{1}{2}\nu(1-\nu)(1-M)^{2}\right]A\sharp_{\nu}B \text{ if } M<1,\\ \\ A\sharp_{\nu}B \text{ if } m\leq 1\leq M,\\ \\ \exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{1}{m}\right)^{2}\right]A\sharp_{\nu}B \text{ if } 1<m.\end{cases} \tag{11.20}$$

$$\leq A\nabla_{\nu}B$$

$$\leq\begin{cases}\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{1}{M}-1\right)^{2}\right]A\sharp_{\nu}B \text{ if } M<1,\\ \\ \max\left\{\exp\left[\frac{1}{2}\nu(1-\nu)(m-1)^{2}\right],\right.\\ \left.\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{1}{M}-1\right)^{2}\right]\right\}A\sharp_{\nu}B \text{ if } m\leq 1\leq M,\\ \\ \exp\left[\frac{1}{2}\nu(1-\nu)(m-1)^{2}\right]A\sharp_{\nu}B \text{ if } 1<m.\end{cases}$$

With the assumptions of Corollary 11.3, we have either in the case (i) or in the case (ii) that

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{h'-1}{h'}\right)^{2}\right]A\sharp_{\nu}B \leq A\nabla_{\nu}B\leq\exp\left[\frac{1}{2}\nu(1-\nu)(h-1)^{2}\right]A\sharp_{\nu}B, \tag{11.21}$$

where $\nu\in[0,1]$.

Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Assume that the positive invertible operators A, B satisfy the condition

$$mA^{p}\leq B^{q}\leq MA^{p}. \tag{11.22}$$

Then by replacing A with A^p, B with B^q and $\nu = \frac{1}{q}$ in (11.9) we have

$$\gamma_{\frac{1}{q}}(m,M)\, A^p \sharp_{\frac{1}{q}} B^q \leq \frac{1}{p}A^p + \frac{1}{q}B^q \leq \Gamma_{\frac{1}{q}}(m,M)\, A^p \sharp_{\frac{1}{q}} B^q, \tag{11.23}$$

where $\Gamma_{\frac{1}{q}}(m,M)$ and $\gamma_{\frac{1}{q}}(m,M)$ are defined by (11.4) and (11.5), respectively.

Assume that A and B satisfy the conditions

$$m_1 I \leq A \leq M_1 I,\ m_2 I \leq B \leq M_2 I \tag{11.24}$$

for some $0 < m_1 < M_1$ and $0 < m_2 < M_2$. We have from (11.24) that

$$m_1^p I \leq A^p \leq M_1^p I.$$

Then by (11.24) we also have

$$m_1^p M_2^{-q} I \leq m_1^p B^{-q} \leq B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \leq M_1^p B^{-q} \leq M_1^p m_2^{-q} I,$$

which implies that

$$m_1 M_2^{-\frac{q}{p}} I \leq \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} \leq M_1 m_2^{-\frac{q}{p}} I.$$

Now, on using the inequality (11.23) for $m = m_1 M_2^{-\frac{q}{p}}$ and $M = M_1 m_2^{-\frac{q}{p}}$, we get

$$\begin{aligned} &\gamma_{\frac{1}{q}}\left(m_1 M_2^{-\frac{q}{p}}, M_1 m_2^{-\frac{q}{p}}\right) A^p \sharp_{\frac{1}{q}} B^q \\ &\leq \frac{1}{p}A^p + \frac{1}{q}B^q \\ &\leq \Gamma_{\frac{1}{q}}\left(m_1 M_2^{-\frac{q}{p}}, M_1 m_2^{-\frac{q}{p}}\right) A^p \sharp_{\frac{1}{q}} B^q, \end{aligned} \tag{11.25}$$

where $\Gamma_{\frac{1}{q}}(\cdot,\cdot)$ and $\gamma_{\frac{1}{q}}(\cdot,\cdot)$ are defined by (11.4) and (11.5), respectively.

Further bounds may be stated as in Remarks 11.1-11.4, however the details are not provided here.

11.2. Inequalities Related to McCarthy's.

By the use of the spectral resolution of $P \geq 0$ and the Hölder inequality, C. A. McCarthy [**109**] proved that

$$\langle Px, x\rangle^p \leq \langle P^p x, x\rangle,\ p \in (1,\infty) \tag{11.26}$$

and

$$\langle P^p x, x\rangle \leq \langle Px, x\rangle^p,\ p \in (0,1) \tag{11.27}$$

for any $x \in H$ with $\|x\| = 1$.

From the previous section, for positive numbers a, b with $\frac{b}{a} \in [m,M] \subset (0,\infty)$ and $\nu \in [0,1]$ we can state the following scalar inequalities

$$\gamma_\nu(m,M)\, a^{1-\nu} b^\nu \leq (1-\nu)\, a + \nu b \leq \Gamma_\nu(m,M)\, a^{1-\nu} b^\nu, \tag{11.28}$$

where $\gamma_\nu(m,M)$ and $\Gamma_\nu(m,M)$ are defined by (11.4) and (11.5), respectively.

This inequality can be written explicitly as

$$\begin{cases} g_{\nu}(M)\, a^{1-\nu}b^{\nu} \text{ if } M<1, \\ a^{1-\nu}b^{\nu} \text{ if } m\le 1\le M, \\ g_{\nu}(m)\, a^{1-\nu}b^{\nu} \text{ if } 1<m. \end{cases} \tag{11.29}$$

$$\le (1-\nu)\, a+\nu b$$

$$\le \begin{cases} g_{\nu}(m)\, a^{1-\nu}b^{\nu} \text{ if } M<1, \\ \max\{g_{\nu}(m), g_{\nu}(M)\}\, a^{1-\nu}b^{\nu} \text{ if } m\le 1\le M, \\ g_{\nu}(M)\, a^{1-\nu}b^{\nu} \text{ if } 1<m, \end{cases},$$

where

$$g_{\nu}(x)=\frac{1-\nu+\nu x}{x^{\nu}}=(1-\nu)\, x^{-\nu}+\nu x^{1-\nu},\ x>0. \tag{11.30}$$

We have the following reverse of McCarthy's inequality:

THEOREM 11.4 (Dragomir, 2015, [**48**]). *Let P and operator such that*

$$zI\le P\le ZI \tag{11.31}$$

for some constants $Z>z>0$.

Then for any $x\in H$ with $\|x\|=1$ we have

$$(1\le)\,\frac{\langle Px,x\rangle^{\lambda}}{\langle P^{\lambda}x,x\rangle}\le \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\}, \tag{11.32}$$

where $\lambda\in[0,1]$ and the function $g_{\lambda}:(0,\infty)\to(0,\infty)$ is defined by (11.30).

PROOF. If $u,v\in[z,Z]$ then $\frac{u}{v}\in\left[\frac{z}{Z},\frac{Z}{z}\right]$ and by (11.29) we have

$$\left(v^{1-\lambda}u^{\lambda}\le\right)(1-\lambda)\, v+\lambda u\le \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} v^{1-\lambda}u^{\lambda}$$

for any $\lambda\in[0,1]$.

Fix $v\in[z,Z]$, then by using the functional calculus for the operator P with $zI\le P\le ZI$ we have

$$(1-\lambda)\, vI+\lambda P\le \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} v^{1-\lambda}P^{\lambda} \tag{11.33}$$

for any $\lambda\in[0,1]$.

The inequality (11.33) implies that

$$(1-\lambda)\, vI+\lambda\langle Px,x\rangle\le \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} v^{1-\lambda}\left\langle P^{\lambda}x,x\right\rangle \tag{11.34}$$

for any $x\in H$ with $\|x\|=1$, for any $\lambda\in[0,1]$ and for any $v\in[z,Z]$.

If we take in (11.34) $v=\langle Px,x\rangle\in[z,Z]$, for $x\in H$ with $\|x\|=1$, then we have

$$\langle Px,x\rangle\le \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\}\langle Px,x\rangle^{1-\lambda}\left\langle P^{\lambda}x,x\right\rangle,$$

which, by division with $\langle Px, x\rangle^{1-\lambda} > 0$ produces the desired result (11.32). ∎

COROLLARY 11.5. *With the assumptions of Theorem 11.4 and if* $T = \max\{\lambda, 1-\lambda\}$ *for* $\lambda \in (0,1)$, *then we have*

$$(1 \leq) \frac{\langle Px, x\rangle^{\lambda}}{\langle P^{\lambda}x, x\rangle} \leq \begin{cases} S\left(\frac{Z}{z}\right), \\ K^{T}\left(\frac{Z}{z}\right), \\ \exp\left[4\lambda(1-\lambda)\left(K\left(\frac{Z}{z}\right)-1\right)\right], \\ \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{Z}{z}-1\right)^{2}\right] \end{cases}$$

for any $x \in H$ *with* $\|x\| = 1$.

We have:

THEOREM 11.6 (Dragomir, 2015, [**48**]). *Let* A *and* B *be two positive invertible operators,* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $m, M > 0$ *such that*

$$m^{p}B^{q} \leq A^{p} \leq M^{p}B^{q}. \tag{11.35}$$

Then we have

$$(1 \leq) \frac{\langle A^{p}x, x\rangle^{1/p}\langle B^{q}x, x\rangle^{1/q}}{\left\langle B^{q}\sharp_{1/p}A^{p}x, x\right\rangle} \leq \max\left\{g_{\frac{1}{q}}\left(\left(\frac{M}{m}\right)^{p}\right), g_{\frac{1}{p}}\left(\left(\frac{M}{m}\right)^{p}\right)\right\}, \tag{11.36}$$

where the function $g_{\lambda} : (0,\infty) \to (0,\infty)$ *is defined by (11.30).*

PROOF. From the inequality (11.32) for $x = \frac{y}{\|y\|}$, $y \neq 0$ we have

$$(1 \leq) \frac{\langle y, y\rangle^{1-\lambda}\langle Py, y\rangle^{\lambda}}{\langle P^{\lambda}y, y\rangle} \leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\}, \tag{11.37}$$

provided that P satisfy the condition (11.31).

Now, from (11.35) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^{p}I \leq B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}} \leq M^{p}I$. By writing the inequality (11.37) for $P = B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}$, $z = m^{p}$, $Z = M^{p}$, $\lambda = \frac{1}{p}$ and $y = B^{\frac{q}{2}}x$, with $x \in H$, $x \neq 0$, we have

$$\begin{aligned}(1 \leq) &\frac{\left\langle B^{\frac{q}{2}}x, B^{\frac{q}{2}}x\right\rangle^{\frac{1}{q}}\left\langle\left(B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\right)B^{\frac{q}{2}}x, B^{\frac{q}{2}}x\right\rangle^{\frac{1}{p}}}{\left\langle\left(B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\right)^{\frac{1}{p}}B^{\frac{q}{2}}x, B^{\frac{q}{2}}x\right\rangle} \\ &\leq \max\left\{g_{\frac{1}{q}}\left(\left(\frac{M}{m}\right)^{p}\right), g_{\frac{1}{p}}\left(\left(\frac{M}{m}\right)^{p}\right)\right\}\end{aligned}$$

that is equivalent to

$$(1 \leq) \frac{\langle B^{q}x, x\rangle^{\frac{1}{q}}\langle A^{p}x, x\rangle^{\frac{1}{p}}}{\left\langle B^{\frac{q}{2}}\left(B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\right)^{\frac{1}{p}}B^{\frac{q}{2}}x, x\right\rangle} \leq \max\left\{g_{\frac{1}{q}}\left(\left(\frac{M}{m}\right)^{p}\right), g_{\frac{1}{p}}\left(\left(\frac{M}{m}\right)^{p}\right)\right\}$$

with $x \in H$, $x \neq 0$. ∎

COROLLARY 11.7. *With the assumptions of Theorem 11.6 we have for* $x \in H$, $x \neq 0$, *that*

$$(1 \leq) \frac{\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q}}{\langle B^q \sharp_{1/p} A^p x, x\rangle} \leq \begin{cases} S\left(\left(\frac{M}{m}\right)^p\right), \\ K^{T_{p,q}}\left(\left(\frac{M}{m}\right)^p\right), \\ \exp\left[\frac{4}{pq}\left(K\left(\left(\frac{M}{m}\right)^p\right) - 1\right)\right], \\ \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p - 1\right)^2\right], \end{cases}$$

where $T_{p,q} = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

11.3. Trace Inequalities. We have the following trace inequality:

THEOREM 11.8 (Dragomir, 2015, [**48**]). *Let* C *be an operator with the property that*

$$zI \leq C \leq ZI \tag{11.38}$$

for some constants z, Z *with* $Z > z > 0$ *and* $P \in \mathcal{B}_1(H)$, $P \geq 0$ *with* $\operatorname{tr}(P) > 0$. *Then for any* $\lambda \in [0,1]$ *we have*

$$(1 \leq) \frac{\operatorname{tr}^{1-\lambda}(P)\operatorname{tr}^{\lambda}(PC)}{\operatorname{tr}(PC^{\lambda})} \leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} \tag{11.39}$$

and the function g_λ *is defined by (11.30).*

PROOF. From the proof of Theorem 11.4 we have

$$(1-\lambda) vI + \lambda \langle Cx, x\rangle \leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} v^{1-\lambda}\langle C^{\lambda}x, x\rangle$$

for any $x \in H$ with $\|x\| = 1$, for any $\lambda \in [0,1]$ and for any $v \in [z, Z]$.

This inequality implies that

$$(1-\lambda) v \langle x, x\rangle + \lambda \langle Cx, x\rangle \leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} v^{1-\lambda}\langle C^{\lambda}x, x\rangle \tag{11.40}$$

for any $x \in H$, for any $\lambda \in [0,1]$ and for any $v \in [z, Z]$.

Now, if we take in (11.40) $x = P^{1/2}e$, where $e \in H$, then

$$\begin{aligned}(1-\lambda) v \langle Pe, e\rangle &+ \lambda \langle P^{1/2}CP^{1/2}e, e\rangle \\ &\leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} v^{1-\lambda}\langle P^{1/2}C^{\lambda}P^{1/2}e, e\rangle\end{aligned} \tag{11.41}$$

for any $e \in H$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. If we take in (11.41) $e = e_i$, $i \in I$ and by summing over $i \in I$, then we get

$$\begin{aligned}(1-\lambda) v \sum_{i\in I}\langle Pe_i, e_i\rangle &+ \lambda \sum_{i\in I}\langle P^{1/2}CP^{1/2}e_i, e_i\rangle \\ &\leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} v^{1-\lambda}\sum_{i\in I}\langle P^{1/2}C^{\lambda}P^{1/2}e_i, e_i\rangle,\end{aligned} \tag{11.42}$$

and by the properties of trace we have

$$(1-\lambda) v \operatorname{tr}(P) + \lambda \operatorname{tr}(PC) \leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} v^{1-\lambda}\operatorname{tr}(PC^{\lambda}),$$

for any $\lambda \in [0,1]$ and for any $v \in [z,Z]$.

This inequality can be written as

$$(1-\lambda) v + \lambda \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_\lambda\left(\frac{Z}{z}\right)\right\} v^{1-\lambda} \frac{\operatorname{tr}\left(PC^\lambda\right)}{\operatorname{tr}(P)}, \tag{11.43}$$

for any $\lambda \in [0,1]$ and for any $v \in [z,Z]$.

Now, if we take in (11.43) $v = \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \in [z,Z]$, then we get

$$\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_\lambda\left(\frac{Z}{z}\right)\right\} \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1-\lambda} \frac{\operatorname{tr}\left(PC^\lambda\right)}{\operatorname{tr}(P)},$$

that is equivalent to the desired result (11.39). ∎

In particular, we have:

COROLLARY 11.9. *With the assumptions of Theorem 11.8 and if* $T = \max\{\lambda, 1-\lambda\}$ *for* $\lambda \in (0,1)$, *then we have*

$$(1 \leq) \frac{\operatorname{tr}^{1-\lambda}(P) \operatorname{tr}^\lambda(PC)}{\operatorname{tr}(PC^\lambda)} \leq \begin{cases} S\left(\frac{Z}{z}\right), \\ K^T\left(\frac{Z}{z}\right), \\ \exp\left[4\lambda(1-\lambda)\left(K\left(\frac{Z}{z}\right)-1\right)\right], \\ \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{Z}{z}-1\right)^2\right]. \end{cases}$$

The following reverse of Hölder's trace inequality may be stated:

THEOREM 11.10 (Dragomir, 2015, [**48**]). *Let* A *and* B *be two positive invertible operators,* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $m, M > 0$ *such that*

$$m^p B^q \leq A^p \leq M^p B^q. \tag{11.44}$$

If $B^q \in \mathcal{B}_1(H)$, *then*

$$(1 \leq) \frac{\operatorname{tr}^{1/p}(A^p) \operatorname{tr}^{1/q}(B^q)}{\operatorname{tr}\left(B^q \sharp_{1/p} A^p\right)} \leq \max\left\{g_{\frac{1}{q}}\left(\left(\frac{M}{m}\right)^p\right), g_{\frac{1}{p}}\left(\left(\frac{M}{m}\right)^p\right)\right\}. \tag{11.45}$$

PROOF. Now, from (11.44) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^p I \leq B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \leq M^p I$. By writing the inequality (11.39) for $C = B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}$, $z = m^p$, $Z = M^p$, $\lambda = \frac{1}{p}$ and $P = B^q$ we get the desired result (11.45). ∎

Finally, we have:

COROLLARY 11.11. *With the assumptions of Theorem 11.10 and if* $T_{p,q} = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, *then we have*

$$(1 \leq) \frac{\operatorname{tr}^{1/p}(A^p) \operatorname{tr}^{1/q}(B^q)}{\operatorname{tr}\left(B^q \sharp_{1/p} A^p\right)} \leq \begin{cases} S\left(\left(\frac{M}{m}\right)^p\right), \\ K^{T_{p,q}}\left(\left(\frac{M}{m}\right)^p\right), \\ \exp\left[\frac{4}{pq}\left(K\left(\left(\frac{M}{m}\right)^p\right)-1\right)\right], \\ \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p-1\right)^2\right]. \end{cases} \tag{11.46}$$

12. HERMITE-HADAMARD TYPE INEQUALITIES

12.1. Some Basic Facts. The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}, a < b. \tag{12.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [**110**]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [**6**]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [**110**]. Since (12.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x, y \in X,\ x \neq y$. Define the segment

$$[x, y] := \{(1-t)x + ty,\ t \in [0, 1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x, y) : [0, 1] \to \mathbb{R},\ g(x, y)(t) := f[(1-t)x + ty],\ t \in [0, 1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$.

For any convex function defined on a segment $[x, y] \subset X$, we have the *Hermite-Hadamard integral inequality* (see [**30**, p. 2], [**31**, p. 2])

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]dt \leq \frac{f(x)+f(y)}{2}, \tag{12.2}$$

which can be derived from the classical Hermite-Hadamard inequality (12.1) for the convex function $g(x, y) : [0, 1] \to \mathbb{R}$.

Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \leq p < \infty$) is a convex function, then for any $x, y \in X$ we have the following norm inequality from (12.2) (see [**118**, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p \leq \int_0^1 \|(1-t)x + ty\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2}. \tag{12.3}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* if

$$f\left(\left(1-\lambda\right)A + \lambda B\right) \leq (\geq) \left(1-\lambda\right) f\left(A\right) + \lambda f\left(B\right) \tag{OC}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e.10., $A \leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [**88**] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$

is operator concave on $(0,\infty)$. The exponential function$f(t)=e^t$ is neither operator convex nor operator monotone.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [**88**] and the references therein.

We recall the following result concerning a Hermite-Hadamard type inequality for operator convex functions [**43**] (see also [**44**, p. 60]):

THEOREM 12.1. *Let $f:I\to\mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality in the operator order*

$$f\left(\frac{A+B}{2}\right)\le\frac{1}{2}\left[f\left(\frac{3A+B}{4}\right)+f\left(\frac{A+3B}{4}\right)\right]\le\int_0^1 f\left((1-t)A+tB\right)dt\le\frac{1}{2}\left[f\left(\frac{A+B}{2}\right)+\frac{f(A)+f(B)}{2}\right]\le\frac{f(A)+f(B)}{2}. \tag{12.4}$$

12.2. Inequalities for Operator Convex Functions. The following representation result holds.

LEMMA 12.2 (Dragomir, 2014, [**62**]). *Let $f:I\to\mathbb{R}$ be a continuous function on the interval I. Then for any selfadjoint operators A and B with spectra in I and for any $\lambda\in[0,1]$ we have the equality*

$$\int_0^1 f\left[(1-t)A+tB\right]dt=(1-\lambda)\int_0^1 f\left[(1-t)\left((1-\lambda)A+\lambda B\right)+tB\right]dt+\lambda\int_0^1 f\left[(1-t)A+t\left((1-\lambda)A+\lambda B\right)\right]dt. \tag{12.5}$$

PROOF. For $\lambda=0$ and $\lambda=1$ the equality (12.5) is obvious.

Let $\lambda\in(0,1)$. Observe that

$$\int_0^1 f\left[(1-t)\left(\lambda B+(1-\lambda)A\right)+tB\right]dt=\int_0^1 f\left[\left((1-t)\lambda+t\right)B+(1-t)(1-\lambda)A\right]dt$$

and

$$\int_0^1 f\left[t\left(\lambda B+(1-\lambda)A\right)+(1-t)A\right]dt=\int_0^1 f\left[t\lambda B+(1-\lambda t)A\right]dt.$$

If we make the change of variable $u:=(1-t)\lambda+t$ then we have $1-u=(1-t)(1-\lambda)$ and $du=(1-\lambda)du$. Then

$$\int_0^1 f\left[\left((1-t)\lambda+t\right)B+(1-t)(1-\lambda)A\right]dt=\frac{1}{1-\lambda}\int_\lambda^1 f\left[uB+(1-u)A\right]du.$$

If we make the change of variable $u:=\lambda t$ then we have $du=\lambda dt$ and

$$\int_0^1 f\left[t\lambda B+(1-\lambda t)A\right]dt=\frac{1}{\lambda}\int_0^\lambda f\left[uB+(1-u)A\right]du.$$

Therefore

$$
\begin{aligned}
&(1-\lambda)\int_0^1 f[(1-t)(\lambda B+(1-\lambda)A)+tB]\,dt\\
&+\lambda\int_0^1 f[t(\lambda B+(1-\lambda)A)+(1-t)A]\,dt\\
&=\int_\lambda^1 f[uB+(1-u)A]\,du+\int_0^\lambda f[uB+(1-u)A]\,du\\
&=\int_0^1 f[uB+(1-u)A]\,du
\end{aligned}
$$

and the identity (12.5) is proved. ∎

THEOREM 12.3 (Dragomir, 2014, [**62**]). *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and for any $\lambda \in [0,1]$ we have the inequalities*

$$
\begin{aligned}
f\left(\frac{A+B}{2}\right) &\le (1-\lambda) f\left[\frac{(1-\lambda)A+(\lambda+1)B}{2}\right]+\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right]\\
&\le \int_0^1 f[(1-t)A+tB]\,dt\\
&\le \frac{1}{2}[f((1-\lambda)A+\lambda B)+(1-\lambda)f(B)+\lambda f(A)]\\
&\le \frac{f(A)+f(B)}{2}.
\end{aligned}
\tag{12.6}
$$

PROOF. Since $f : I \to \mathbb{R}$ is an operator convex function on the interval I, then by Theorem 12.1 we have

$$
\begin{aligned}
f\left[\frac{(1-\lambda)A+(\lambda+1)B}{2}\right] &\le \int_0^1 f[(1-t)((1-\lambda)A+\lambda B)+tB]\,dt\\
&\le \frac{1}{2}[f((1-\lambda)A+\lambda B)+f(B)]
\end{aligned}
\tag{12.7}
$$

and

$$
\begin{aligned}
f\left[\frac{(2-\lambda)A+\lambda B}{2}\right] &\le \int_0^1 f[(1-t)A+t((1-\lambda)A+\lambda B)]\,dt\\
&\le \frac{1}{2}[f(A)+f((1-\lambda)A+\lambda B)]\int_0^1 h(t)\,dt.
\end{aligned}
\tag{12.8}
$$

Now, if we multiply the inequality (12.7) by $1-\lambda \geq 0$ and (12.8) by $\lambda \geq 0$ and add the obtained inequalities, then we get

$$
\begin{aligned}
(12.9) \qquad & (1-\lambda) f\left[\frac{(1-\lambda) A+(\lambda+1) B}{2}\right]+\lambda f\left[\frac{(2-\lambda) A+\lambda B}{2}\right] \\
& \leq (1-\lambda) \int_0^1 f[(1-t)((1-\lambda) A+\lambda B)+t B] \, dt \\
& +\lambda \int_0^1 f[(1-t) A+t((1-\lambda) A+\lambda B)] \, dt \\
& \leq \frac{1}{2}(1-\lambda)[f((1-\lambda) A+\lambda B)+f(B)] \\
& +\frac{1}{2} \lambda[f(A)+f((1-\lambda) A+\lambda B)] \\
& =\frac{1}{2}[f((1-\lambda) A+\lambda B)+(1-\lambda) f(B)+\lambda f(A)]
\end{aligned}
$$

and by (12.5) we obtain the second and the third inequalities in (12.6).

The first and the last inequality in (12.6) are obvious by operator convexity of f. ∎

REMARK 12.1. If we take $\lambda=\frac{1}{2}$, then we get from (12.6) the inequality (12.4).

Some examples are as follows:

REMARK 12.2. Utilising different instances of operator convex or concave functions, we can provide inequalities of interest.

If $r \in[-1,0] \cup[1,2]$ then we have the inequalities for powers of operators

$$
\begin{aligned}
(12.10) \qquad \left(\frac{A+B}{2}\right)^r & \leq(1-\lambda)\left[\frac{(1-\lambda) A+(\lambda+1) B}{2}\right]^r+\lambda f\left[\frac{(2-\lambda) A+\lambda B}{2}\right]^r \\
& \leq \int_0^1((1-t) A+t B)^r \, dt \\
& \leq \frac{1}{2}\left[((1-\lambda) A+\lambda B)^r+(1-\lambda) B^r+\lambda A^r\right] \\
& \leq \frac{A^r+B^r}{2}
\end{aligned}
$$

for any two selfadjoint operators A and B with spectra in $(0, \infty)$ and $\lambda \in[0,1]$.

If $r \in(0,1)$ the inequalities in (12.10) hold with " $\geq$ " instead of " $\leq$ ".

We also have the following inequalities for logarithm

$$
\begin{aligned}
(12.11) \qquad & \ln \left(\frac{A+B}{2}\right) \\
& \geq(1-\lambda) \ln \left[\frac{(1-\lambda) A+(\lambda+1) B}{2}\right]+\lambda \ln \left[\frac{(2-\lambda) A+\lambda B}{2}\right] \\
& \geq \int_0^1 \ln ((1-t) A+t B) \, dt \\
& \geq \frac{1}{2}[\ln ((1-\lambda) A+\lambda B)+(1-\lambda) \ln B+\lambda \ln A] \\
& \geq \frac{\ln (A)+\ln (B)}{2}
\end{aligned}
$$

for any two selfadjoint operators A and B with spectra in $(0, \infty)$ and $\lambda \in[0,1]$.

If A and B are selfadjoint operators with $A \leq B$ and $P \in \mathcal{B}_1(H)$ with $P \geq 0$, then

$$\operatorname{tr}(PA) \leq \operatorname{tr}(PB). \tag{12.12}$$

Now, if A is a selfadjoint operator, then we know that

$$|\langle Ax, x\rangle| \leq \langle |A| x, x\rangle \text{ for any } x \in H.$$

This inequality follows by Jensen's inequality for the convex function $f(t) = |t|$ defined on a closed interval containing the spectrum of A.

If $\{e_i\}_{i\in I}$ is an orthonormal basis of H, then

$$\begin{aligned} |\operatorname{tr}(PA)| &= \left|\sum_{i\in I} \left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle\right| \leq \sum_{i\in I} \left|\left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle\right| \\ &\leq \sum_{i\in I} \left\langle |A| P^{1/2}e_i, P^{1/2}e_i\right\rangle = \operatorname{tr}(P|A|), \end{aligned} \tag{12.13}$$

for any A a selfadjoint operator and $P \in \mathcal{B}_1(H)$ with $P \geq 0$.

COROLLARY 12.4. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I, for any $P \in \mathcal{B}_1(H)$ with $P \geq 0$, $\operatorname{tr}(P) = 1$ and for any $\lambda \in [0,1]$ we have the inequalities*

$$\begin{aligned} &\operatorname{tr}\left[Pf\left(\frac{A+B}{2}\right)\right] \\ &\leq (1-\lambda)\operatorname{tr}\left(Pf\left[\frac{(1-\lambda)A + (\lambda+1)B}{2}\right]\right) \\ &+ \lambda \operatorname{tr}\left(Pf\left[\frac{(2-\lambda)A + \lambda B}{2}\right]\right) \\ &\leq \int_0^1 \operatorname{tr}(Pf[(1-t)A + tB])\,dt \\ &\leq \frac{1}{2}\left[\operatorname{tr}[Pf((1-\lambda)A + \lambda B)] + (1-\lambda)\operatorname{tr}[Pf(B)] + \lambda \operatorname{tr}[Pf(A)]\right] \\ &\leq \frac{1}{2}\left(\operatorname{tr}[Pf(A)] + \operatorname{tr}[Pf(B)]\right). \end{aligned} \tag{12.14}$$

The proof follows by the property (12.12) and the inequality (12.6).

Similar particular inequalities of interest may be stated if one chooses various operator convex functions. However the details are not presented here.

12.3. Inequalities for Matrices and Convex Functions. Let M_n denote the space of $n \times n$ matrices with complex elements. Let H_n denote the $n \times n$ *Hermitian matrices*, i.e. the subset of M_n consisting of all matrices $A \in H_n$ such that $A^* = A$. There is a natural partial order on H_n : a matrix $A \in H_n$ is said to be *positive semi-definite* in case

$$\langle v, Av\rangle \geq 0 \text{ for all } v \in \mathbb{C}^n, \tag{12.15}$$

in which case we write $A \geq 0$. A is said to be *positive definite* in case the inequality (12.15) is strict for all $v \neq 0$ in $\mathbb{C}^n$, in which case we write $A > 0$. Notice that in the finite-dimensional case we have $A > 0$ if and only if $A \geq 0$ and A is invertible. Let H_n^+ denote the $n \times n$ positive definite matrices.

A function $f : (0, \infty) \to \mathbb{R}$ is said to be *operator monotone* in case when whenever for all n and all $A, B \in H_n^+$

$$A \geq B \Rightarrow f(A) \geq f(B).$$

The following result is well know.

THEOREM 12.5. *Let $f:\mathbb{R}\left(\mathbb{R}^{+}\right)\rightarrow\mathbb{R}$ be continuous and let n be a natural number. If f is monotone increasing, so is $A\mapsto\operatorname{tr}\left[f\left(A\right)\right]$ on $H_{n}\left(H_{n}^{+}\right)$. Likewise, if f is convex, so is $A\mapsto\operatorname{tr}\left[f\left(A\right)\right]$ on $H_{n}\left(H_{n}^{+}\right)$, and strictly so if f is strictly convex.*

The following Hermite-Hadamard type trace inequality holds.

THEOREM 12.6 (Dragomir, 2014, [**62**]). *Let $f:\mathbb{R}\left(\mathbb{R}^{+}\right)\rightarrow\mathbb{R}$ be continuous convex. Then for any $A,B\in H_{n}\left(H_{n}^{+}\right)$ we have the inequality*

$$\operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\right]\leq\int_{0}^{1}\operatorname{tr}\left(f\left[\left(1-t\right)A+tB\right]\right)dt\leq\frac{1}{2}\left(\operatorname{tr}\left[f\left(A\right)\right]+\operatorname{tr}\left[f\left(B\right)\right]\right). \tag{12.16}$$

The proof follows by the Hermite-Hadamard type inequality (12.2) applied for the convex function $A\mapsto\operatorname{tr}\left[f\left(A\right)\right]$ defined on $H_{n}\left(H_{n}^{+}\right)$.

We can prove the following improvement of (12.16).

THEOREM 12.7 (Dragomir, 2014, [**62**]). *Let $f:\mathbb{R}\left(\mathbb{R}^{+}\right)\rightarrow\mathbb{R}$ be continuous convex. Then for any $A,B\in H_{n}\left(H_{n}^{+}\right)$ and $\lambda\in[0,1]$ we have the inequality*

$$\begin{aligned}&\operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\right]\\&\leq\left(1-\lambda\right)\operatorname{tr}\left(f\left[\frac{\left(1-\lambda\right)A+\left(\lambda+1\right)B}{2}\right]\right)+\lambda\operatorname{tr}\left(f\left[\frac{\left(2-\lambda\right)A+\lambda B}{2}\right]\right)\\&\leq\int_{0}^{1}\operatorname{tr}\left(f\left[\left(1-t\right)A+tB\right]\right)dt\\&\leq\frac{1}{2}\left[\operatorname{tr}\left[f\left(\left(1-\lambda\right)A+\lambda B\right)\right]+\left(1-\lambda\right)\operatorname{tr}\left[f\left(B\right)\right]+\lambda\operatorname{tr}\left[f\left(A\right)\right]\right]\\&\leq\frac{1}{2}\left(\operatorname{tr}\left[f\left(A\right)\right]+\operatorname{tr}\left[f\left(B\right)\right]\right).\end{aligned} \tag{12.17}$$

PROOF. Utilising Lemma 12.2 we have the equality

$$\begin{aligned}&\int_{0}^{1}\operatorname{tr}\left(f\left[\left(1-t\right)A+tB\right]\right)dt\\&=\left(1-\lambda\right)\int_{0}^{1}\operatorname{tr}\left(f\left[\left(1-t\right)\left(\left(1-\lambda\right)A+\lambda B\right)+tB\right]\right)dt\\&+\lambda\int_{0}^{1}\operatorname{tr}\left(f\left[\left(1-t\right)A+t\left(\left(1-\lambda\right)A+\lambda B\right)\right]\right)dt\end{aligned} \tag{12.18}$$

for any $A,B\in H_{n}\left(H_{n}^{+}\right)$ and $\lambda\in[0,1]$.

Utilizing a similar argument to the one in the proof of Theorem 12.3 we obtain the desired result (12.17). We omit the details. ∎

It is known that if A and B are *commuting*, i.e. $AB=BA$, then the exponential function satisfies the property

$$\exp\left(A\right)\exp\left(B\right)=\exp\left(B\right)\exp\left(A\right)=\exp\left(A+B\right).$$

Also, if X is invertible and $a,b\in\mathbb{R}$ with $a<b$ then

$$\int_{a}^{b}\exp\left(tX\right)dt=X^{-1}\left[\exp\left(bX\right)-\exp\left(aX\right)\right].$$

Moreover, if A and B are commuting and $B-A$ is invertible, then

$$\begin{aligned}\int_0^1 \exp\left((1-s)A+sB\right)ds &= \int_0^1 \exp\left(s(B-A)\right)\exp(A)\,ds \\ &= \left(\int_0^1 \exp\left(s(B-A)\right)ds\right)\exp(A) \\ &= (B-A)^{-1}\left[\exp(B-A)-I_n\right]\exp(A) \\ &= (B-A)^{-1}\left[\exp(B)-\exp(A)\right].\end{aligned}$$

If we write the inequality (12.17) for the convex function $f(t)=\exp t$, then for any commuting $A,B\in H_n$ with $B-A$ is invertible we have

$$\begin{aligned}&\operatorname{tr}\left[\exp\left(\frac{A+B}{2}\right)\right] \\ &\le (1-\lambda)\operatorname{tr}\left(\exp\left[\frac{(1-\lambda)A+(\lambda+1)B}{2}\right]\right) \\ &+\lambda\operatorname{tr}\left(\exp\left[\frac{(2-\lambda)A+\lambda B}{2}\right]\right) \\ &\le \operatorname{tr}\left((B-A)^{-1}\left[\exp(B)-\exp(A)\right]\right) \\ &\le \frac{1}{2}\left[\operatorname{tr}\left[\exp\left((1-\lambda)A+\lambda B\right)\right]+(1-\lambda)\operatorname{tr}\left[\exp(B)\right]+\lambda\operatorname{tr}\left[\exp(A)\right]\right] \\ &\le \frac{1}{2}\left(\operatorname{tr}\left[\exp(A)\right]+\operatorname{tr}\left[\exp(B)\right]\right),\end{aligned} \tag{12.19}$$

for any $\lambda\in[0,1]$.

If we apply the inequality (12.17) for the convex function $f(t)=t^r$, $r\in(-\infty,0)\cup[1,\infty)$ then we have the matrix power trace inequalities

$$\begin{aligned}&\operatorname{tr}\left[\left(\frac{A+B}{2}\right)^r\right] \\ &\le (1-\lambda)\operatorname{tr}\left(\left[\frac{(1-\lambda)A+(\lambda+1)B}{2}\right]^r\right)+\lambda\operatorname{tr}\left(\left[\frac{(2-\lambda)A+\lambda B}{2}\right]^r\right) \\ &\le \int_0^1 \operatorname{tr}\left(\left[(1-t)A+tB\right]^r\right)dt \\ &\le \frac{1}{2}\left[\operatorname{tr}\left[\left((1-\lambda)A+\lambda B\right)^r\right]+(1-\lambda)\operatorname{tr}\left(B^r\right)+\lambda\operatorname{tr}\left(A^r\right)\right] \\ &\le \frac{1}{2}\left(\operatorname{tr}\left(A^r\right)+\operatorname{tr}\left(B^r\right)\right),\end{aligned} \tag{12.20}$$

for any $A,B\in H_n^+$ and for any $\lambda\in[0,1]$

If $r\in(0,1)$ then the inequalities in (12.20) reverse.

If we choose in (12.20) $r=2$ and take into account that

$$\int_0^1 \operatorname{tr}\left(\left[(1-t)A+tB\right]^2\right)dt=\frac{1}{3}\operatorname{tr}\left(A^2+AB+B^2\right),$$

then from (12.20) we have the quadratic inequality

$$\begin{aligned} (12.21)\qquad & \operatorname{tr}\left[\left(\frac{A+B}{2}\right)^{2}\right] \\ & \leq (1-\lambda)\operatorname{tr}\left(\left[\frac{(1-\lambda)A+(\lambda+1)B}{2}\right]^{2}\right)+\lambda\operatorname{tr}\left(\left[\frac{(2-\lambda)A+\lambda B}{2}\right]^{2}\right) \\ & \leq \frac{1}{3}\operatorname{tr}\left(A^{2}+AB+B^{2}\right) \\ & \leq \frac{1}{2}\left[\operatorname{tr}\left[((1-\lambda)A+\lambda B)^{2}\right]+(1-\lambda)\operatorname{tr}\left(B^{2}\right)+\lambda\operatorname{tr}\left(A^{2}\right)\right] \\ & \leq \frac{1}{2}\left(\operatorname{tr}\left(A^{2}\right)+\operatorname{tr}\left(B^{2}\right)\right), \end{aligned}$$

for any $A, B \in H_n$ and for any $\lambda \in [0,1]$.

12.4. Some Quasilinearity Properties. Consider $f : \mathbb{R}\ (\mathbb{R}^+) \to \mathbb{R}$ and continuous convex function and $A, B \in H_n\ (H_n^+)$. We denote by $[A, B]$ the closed matrix segment defined by the family of matrices $\{(1-t)A + tB,\ t \in [0,1]\}$. We also define the trace functional

$$(12.22)\qquad \Upsilon_f(A,B;t) := (1-t)\operatorname{tr}[f(A)] + t\operatorname{tr}[f(B)] - \operatorname{tr}[f((1-t)A+tB)] \geq 0$$

for any $t \in [0,1]$.

The following result concerning a trace quasilinearity property for the functional $\Upsilon_f(\cdot,\cdot;t)$ may be stated:

THEOREM 12.8 (Dragomir, 2014, [**62**]). *Let $f : \mathbb{R}\ (\mathbb{R}^+) \to \mathbb{R}$ be a continuous convex function and $A, B \in H_n\ (H_n^+)$. Then for any $C \in [A, B]$ we have*

$$(12.23)\qquad 0 \leq \Upsilon_f(A,C;t) + \Upsilon_f(C,B;t) \leq \Upsilon_f(A,B;t)$$

for each $t \in [0,1]$, i.e., the functional $\Upsilon_f(\cdot,\cdot;t)$ is superadditive as a function of matrix interval.

If $[C, D] \subset [A, B]$, then

$$(12.24)\qquad 0 \leq \Upsilon_f(C,D;t) \leq \Upsilon_f(A,B;t)$$

for each $t \in [0,1]$, i.e., the functional $\Upsilon_f(\cdot,\cdot;t)$ is operator nondecreasing as a function of matrix interval.

PROOF. Let $C = (1-s)A + sB$ with $s \in (0,1)$. For $t \in (0,1)$ we have

$$\begin{aligned} \Upsilon_f(C,B;t) &= (1-t)\operatorname{tr}[f((1-s)A+sB)] + t\operatorname{tr}[f(B)] \\ &\quad - \operatorname{tr}[f((1-t)[(1-s)A+sB]+tB)] \end{aligned}$$

and

$$\begin{aligned} \Upsilon_f(A,C;t) &= (1-t)\operatorname{tr}[f(A)] + t\operatorname{tr}[f((1-s)A+sB)] \\ &\quad - \operatorname{tr}[f((1-t)A+t[(1-s)A+sB])] \end{aligned}$$

giving that

$$\begin{aligned} (12.25)\qquad & \Upsilon_f(A,C;t) + \Upsilon_f(C,B;t) - \Upsilon_f(A,B;t) \\ & = \operatorname{tr}[f((1-s)A+sB)] + \operatorname{tr}[f((1-t)A+tB)] \\ & - \operatorname{tr}[f((1-t)(1-s)A+[(1-t)s+t]B)] - \operatorname{tr}[f((1-ts)A+tsB)]. \end{aligned}$$

Now, for a convex function $\varphi : I \subset \mathbb{R} \to \mathbb{R}$, where I is an interval, and any real numbers t_1, t_2, s_1 and s_2 from I and with the properties that $t_1 \leq s_1$ and $t_2 \leq s_2$ we have that

$$\frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2} \leq \frac{\varphi(s_1) - \varphi(s_2)}{s_1 - s_2}. \tag{12.26}$$

Indeed, since φ is convex on I then for any $a \in I$ the function $\psi : I \backslash \{a\} \to \mathbb{R}$

$$\psi(t) := \frac{\varphi(t) - \varphi(a)}{t - a}$$

is monotonic nondecreasing on $I \backslash \{a\}$. Utilising this property repeatedly we have

$$\begin{aligned} \frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2} &\leq \frac{\varphi(s_1) - \varphi(t_2)}{s_1 - t_2} = \frac{\varphi(t_2) - \varphi(s_1)}{t_2 - s_1} \\ &\leq \frac{\varphi(s_2) - \varphi(s_1)}{s_2 - s_1} = \frac{\varphi(s_1) - \varphi(s_2)}{s_1 - s_2}, \end{aligned}$$

which proves the inequality (12.26).

Consider the function $\varphi : [0,1] \to \mathbb{R}$ given by $\varphi(t) := \operatorname{tr}[f((1-t)A + tB)]$. Since f is convex on I it follows that φ is convex on $[0,1]$. Now, if we consider, for given $t, s \in (0,1)$, $t_1 := ts < s =: s_1$ and $t_2 := t < t + (1-t)s =: s_2$, then $\varphi(t_1) = \operatorname{tr}[f((1-ts)A + tsB)]$ and $\varphi(t_2) = \operatorname{tr}[f((1-t)A + tB)]$ giving that

$$\frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2} = \operatorname{tr}\left[\frac{f((1-ts)A + tsB) - f((1-t)A + tB)}{t(s-1)}\right].$$

Also

$$\varphi(s_1) = \operatorname{tr}[f((1-s)A + sB)]$$

and

$$\varphi(s_2) = \operatorname{tr}[f((1-t)(1-s)A + [(1-t)s + t]B)]$$

giving that

$$\begin{aligned} &\frac{\varphi(s_1) - \varphi(s_2)}{s_1 - s_2} \\ &= \operatorname{tr}\left[\frac{f((1-s)A + sB) - f((1-t)(1-s)A + [(1-t)s + t]B)}{t(s-1)}\right]. \end{aligned}$$

Utilising the inequality (12.26) and multiplying with $t(s-1) < 0$ we deduce the following inequality

$$\begin{aligned} &\operatorname{tr}[f((1-ts)A + tsB)] - \operatorname{tr}[f((1-t)A + tB)] \\ &\geq \operatorname{tr}[f((1-s)A + sB)] - \operatorname{tr}[f((1-t)(1-s)A + [(1-t)s + t]B)]. \end{aligned} \tag{12.27}$$

Finally, by (12.25) and (12.27) we get the desired result (12.23).

Applying repeatedly the superadditivity property we have for $[C, D] \subset [A, B]$ that

$$\Upsilon_f(A, C; t) + \Upsilon_f(C, D; t) + \Upsilon_f(D, B; t) \leq \Upsilon_f(A, B; t)$$

giving that

$$0 \leq \Upsilon_f(A, C; t) + \Upsilon_f(D, B; t) \leq \Upsilon_f(A, B; t) - \Upsilon_f(C, D; t),$$

which proves (12.24). ∎

For $t = \frac{1}{2}$ we consider the functional

$$\begin{aligned}\Upsilon_f(A,B) &:= \Upsilon_f\left(A,B;\frac{1}{2}\right)\\ &= \frac{\operatorname{tr}[f(A)] + \operatorname{tr}[f(B)]}{2} - \operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\right] \geq 0,\end{aligned}$$

which obviously inherits the superadditivity and monotonicity properties of the functional $\Upsilon_f(\cdot,\cdot;t)$.

We are able then to state the following

COROLLARY 12.9. *Let $f : \mathbb{R}(\mathbb{R}^+) \to \mathbb{R}$ be a continuous convex function and $A, B \in H_n$ (H_n^+). Then we have the following bounds*

$$\inf_{C\in[A,B]}\left[\operatorname{tr}\left[f\left(\frac{A+C}{2}\right)\right] + \operatorname{tr}\left[f\left(\frac{C+B}{2}\right)\right] - \operatorname{tr}[f(C)]\right] = \operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\right] \tag{12.28}$$

and

$$\begin{aligned}&\sup_{C,D\in[A,B]}\left[\frac{\operatorname{tr}[f(C)] + \operatorname{tr}[f(D)]}{2} - \operatorname{tr}\left[f\left(\frac{C+D}{2}\right)\right]\right]\\ &= \frac{\operatorname{tr}[f(A)] + \operatorname{tr}[f(B)]}{2} - \operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\right].\end{aligned} \tag{12.29}$$

PROOF. By the superadditivity of the functional $\Upsilon_f(\cdot,\cdot)$ we have for each $C \in [A,B]$ that

$$\begin{aligned}&\frac{\operatorname{tr}[f(A)] + \operatorname{tr}[f(B)]}{2} - \operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\right]\\ &\geq \frac{\operatorname{tr}[f(A)] + \operatorname{tr}[f(C)]}{2} - \operatorname{tr}\left[f\left(\frac{A+C}{2}\right)\right]\\ &+ \frac{\operatorname{tr}[f(C)] + \operatorname{tr}[f(B)]}{2} - \operatorname{tr}\left[f\left(\frac{C+B}{2}\right)\right],\end{aligned}$$

which is equivalent with

$$\operatorname{tr}\left[f\left(\frac{A+C}{2}\right)\right] + \operatorname{tr}\left[f\left(\frac{C+B}{2}\right)\right] - \operatorname{tr}[f(C)] \geq \operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\right]. \tag{12.30}$$

Since the equality case in (12.30) is realized for either $C = A$ or $C = B$ we get the desired bound (12.28).

The bound (12.29) is obvious by the monotonicity of the functional $\Upsilon_f(\cdot,\cdot)$ as a function of matrix interval. ∎

Consider now the following functional

$$\Omega_f(A,B;t) := \operatorname{tr}[f(A)] + \operatorname{tr}[f(B)] - \operatorname{tr}[f((1-t)A + tB)] - \operatorname{tr}[f((1-t)B + tA)],$$

where, as above, $f : \mathbb{R}(\mathbb{R}^+) \to \mathbb{R}$ is a continuous convex function and $A, B \in H_n$ (H_n^+) while $t \in [0,1]$.

We notice that

$$\Omega_f(A,B;t) = \Omega_f(B,A;t) = \Omega_f(A,B;1-t)$$

and

$$\Omega_f(A,B;t) = \Upsilon_f(A,B;t) + \Upsilon_f(A,B;1-t) \geq 0$$

for any $A, B \in H_n$ (H_n^+) and $t \in [0,1]$.

Therefore, we can state the following result as well:

COROLLARY 12.10. *Let* $f : \mathbb{R}\,(\mathbb{R}^+) \to \mathbb{R}$ *be a continuous convex function and* $A, B \in H_n$ (H_n^+)*. The functional* $\Omega_f(\cdot,\cdot;t)$ *is superadditive and nondecreasing as a function of matrix interval.*

In particular, if $C \in [A, B]$ then we have the inequality

$$\begin{aligned}(12.31)\qquad &\frac{1}{2}\left[\operatorname{tr}\left[f\left((1-t)A+tB\right)\right]+\operatorname{tr}\left[f\left((1-t)B+tA\right)\right]\right]\\ &\leq \frac{1}{2}\left[\operatorname{tr}\left[f\left((1-t)A+tC\right)\right]+\operatorname{tr}\left[f\left((1-t)C+tA\right)\right]\right]\\ &+\frac{1}{2}\left[\operatorname{tr}\left[f\left((1-t)C+tB\right)\right]+\operatorname{tr}\left[f\left((1-t)B+tC\right)\right]\right]-\operatorname{tr}\left[f\left(C\right)\right].\end{aligned}$$

Also, if $C, D \in [A, B]$ then we have the inequality

$$\begin{aligned}(12.32)\qquad &\operatorname{tr}\left[f\left(A\right)\right]+\operatorname{tr}\left[f\left(B\right)\right]-\operatorname{tr}\left[f\left((1-t)A+tB\right)\right]-\operatorname{tr}\left[f\left((1-t)B+tA\right)\right]\\ &\geq \operatorname{tr}\left[f\left(C\right)\right]+\operatorname{tr}\left[f\left(D\right)\right]-\operatorname{tr}\left[f\left((1-t)C+tD\right)\right]-\operatorname{tr}\left[f\left((1-t)C+tD\right)\right]\end{aligned}$$

for any $t \in [0, 1]$.

Perhaps the most interesting functional we can consider is the following one:

$$(12.33)\qquad \Phi_f(A,B)=\frac{\operatorname{tr}\left[f\left(A\right)\right]+\operatorname{tr}\left[f\left(B\right)\right]}{2}-\int_0^1 \operatorname{tr}\left[f\left((1-t)A+tB\right)\right]dt.$$

Notice that, by the second Hermite-Hadamard trace inequality for convex functions we have that $\Phi_f(A, B) \geq 0$.

We also observe that

$$(12.34)\qquad \Phi_f(A,B)=\int_0^1 \Upsilon_f(A,B;t)\,dt=\int_0^1 \Upsilon_f(A,B;1-t)\,dt.$$

Utilising this representation, we can state the following result as well:

COROLLARY 12.11. *Let* $f : \mathbb{R}\,(\mathbb{R}^+) \to \mathbb{R}$ *be a continuous convex function and* $A, B \in H_n$ (H_n^+)*. The functional* $\Phi_f(\cdot,\cdot)$ *is superadditive and nondecreasing as a function of matrix interval. Moreover, we have the bounds*

$$\begin{aligned}(12.35)\qquad &\inf_{C\in[A,B]}\left[\int_0^1\left[\operatorname{tr}\left[f\left((1-t)A+tC\right)\right]+\operatorname{tr}\left[f\left((1-t)C+tB\right)\right]\right]dt-\operatorname{tr}\left[f\left(C\right)\right]\right]\\ &=\int_0^1 \operatorname{tr}\left[f\left((1-t)A+tB\right)\right]dt\end{aligned}$$

and

$$\begin{aligned}(12.36)\qquad &\sup_{C,D\in[A,B]}\left[\frac{\operatorname{tr}\left[f\left(C\right)\right]+\operatorname{tr}\left[f\left(D\right)\right]}{2}-\int_0^1 \operatorname{tr}\left[f\left((1-t)C+tD\right)\right]dt\right]\\ &=\frac{\operatorname{tr}\left[f\left(A\right)\right]+\operatorname{tr}\left[f\left(B\right)\right]}{2}-\int_0^1 \operatorname{tr}\left[f\left((1-t)A+tB\right)\right]dt.\end{aligned}$$

REMARK 12.3. The above inequalities can be applied to various concrete convex functions of interest.

If we use the inequality (12.29), then we have

$$\sup_{C,D\in[A,B]}\left[\frac{\operatorname{tr}(C^r)+\operatorname{tr}(D^r)}{2}-\operatorname{tr}\left[\left(\frac{C+D}{2}\right)^r\right]\right] = \frac{\operatorname{tr}(A^r)+\operatorname{tr}(B^r)}{2}-\operatorname{tr}\left[\left(\frac{A+B}{2}\right)^r\right], \tag{12.37}$$

where $r\in(-\infty,0)\cup[1,\infty)$ and $A,B\in H_n^+$.

If $r\in(0,1)$, then

$$\sup_{C,D\in[A,B]}\left[\operatorname{tr}\left[\left(\frac{C+D}{2}\right)^r\right]-\frac{\operatorname{tr}(C^r)+\operatorname{tr}(D^r)}{2}\right] = \operatorname{tr}\left[\left(\frac{A+B}{2}\right)^r\right]-\frac{\operatorname{tr}(A^r)+\operatorname{tr}(B^r)}{2}, \tag{12.38}$$

for any $A,B\in H_n^+$.

We have the logarithmic bounds

$$\sup_{C,D\in[A,B]}\left[\operatorname{tr}\left[\ln\left(\frac{C+D}{2}\right)\right]-\frac{\operatorname{tr}[\ln(C)]+\operatorname{tr}[\ln(D)]}{2}\right] = \operatorname{tr}\left[\ln\left(\frac{A+B}{2}\right)\right]-\frac{\operatorname{tr}[\ln(A)]+\operatorname{tr}[\ln(B)]}{2} \tag{12.39}$$

for any $A,B\in H_n^+$.

The following bound for the exponential also holds

$$\sup_{C,D\in[A,B]}\left[\frac{\operatorname{tr}[\exp(C)]+\operatorname{tr}[\exp(D)]}{2}-\operatorname{tr}\left[\exp\left(\frac{C+D}{2}\right)\right]\right] = \frac{\operatorname{tr}[\exp(A)]+\operatorname{tr}[\exp(B)]}{2}-\operatorname{tr}\left[\exp\left(\frac{A+B}{2}\right)\right]. \tag{12.40}$$

for any $A,B\in H_n$.

If we use the inequality (12.36), then we get the following bounds

$$\sup_{C,D\in[A,B]}\left[\frac{\operatorname{tr}(C^r)+\operatorname{tr}(D^r)}{2}-\int_0^1\operatorname{tr}\left[((1-t)C+tD)^r\right]dt\right] = \frac{\operatorname{tr}(A^r)+\operatorname{tr}(B^r)}{2}-\int_0^1\operatorname{tr}\left[((1-t)A+tB)^r\right]dt, \tag{12.41}$$

where $r\in(-\infty,0)\cup[1,\infty)$ and $A,B\in H_n^+$.

If $r\in(0,1)$, then

$$\sup_{C,D\in[A,B]}\left[\int_0^1\operatorname{tr}\left[((1-t)C+tD)^r\right]dt-\frac{\operatorname{tr}(C^r)+\operatorname{tr}(D^r)}{2}\right] = \int_0^1\operatorname{tr}\left[((1-t)A+tB)^r\right]dt-\frac{\operatorname{tr}(A^r)+\operatorname{tr}(B^r)}{2}, \tag{12.42}$$

for any $A,B\in H_n^+$.

We also have the bound for the logarithm

$$\sup_{C,D\in[A,B]}\left[\int_0^1 \operatorname{tr}\left[\ln\left((1-t)C+tD\right)\right]dt-\frac{\operatorname{tr}\left[\ln(C)\right]+\operatorname{tr}\left[\ln(D)\right]}{2}\right] \tag{12.43}$$
$$=\int_0^1 \operatorname{tr}\left[\ln\left((1-t)A+tB\right)\right]dt-\frac{\operatorname{tr}\left[\ln(A)\right]+\operatorname{tr}\left[\ln(B)\right]}{2},$$

for any $A, B \in H_n^+$.

The following bound for the exponential also holds

$$\sup_{C,D\in[A,B]}\left[\frac{\operatorname{tr}\left[\exp(C)\right]+\operatorname{tr}\left[\exp(D)\right]}{2}-\int_0^1 \operatorname{tr}\left[\exp\left((1-t)C+tD\right)\right]dt\right] \tag{12.44}$$
$$=\frac{\operatorname{tr}\left[\exp(A)\right]+\operatorname{tr}\left[\exp(B)\right]}{2}-\int_0^1 \operatorname{tr}\left[\exp\left((1-t)A+tB\right)\right]dt,$$

for any $A, B \in H_n$.

13. JENSEN TYPE INEQUALITIES FOR CONVEX FUNCTIONS

13.1. Some Preliminary Facts. Let A be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(A)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{R}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral* (see for instance [**91**, p. 257]):

$$\langle f(A)x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d\left(\langle E_\lambda x, y\rangle\right), \tag{13.1}$$

and

$$\|f(A)x\|^2 = \int_{m-0}^{M} |f(\lambda)|^2\, d\,\|E_\lambda x\|^2, \tag{13.2}$$

for any $x, y \in H$.

The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ while } g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$ for any $x \in H$.

The following result that provides an operator version for the Jensen inequality may be found for instance in Mond & Pečarić [**112**] (see also [**88**, p. 5]):

THEOREM 13.1. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If h is a convex function on $[m, M]$, then*

$$h\left(\langle Ax, x\rangle\right) \le \langle h(A)x, x\rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 13.1 we have the following Hölder-McCarthy inequality:

THEOREM 13.2 (Hölder-McCarthy, 1967, [**109**]). *Let A be a selfadjoint positive operator on a Hilbert space H. Then for all $x \in H$ with $\|x\| = 1$,*

(i) $\langle A^r x, x\rangle \ge \langle Ax, x\rangle^r$ *for all* $r > 1$;

(ii) $\langle A^r x, x\rangle \le \langle Ax, x\rangle^r$ *for all* $0 < r < 1$;

(iii) If A is invertible, then $\langle A^r x, x\rangle \ge \langle Ax, x\rangle^r$ *for all* $r < 0$.

The following reverse for (MP) that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [**88**, p. 57]:

THEOREM 13.3. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m,M]$ for some scalars m, M with $m<M$. If h is a convex function on $[m,M]$, then*

$$\langle h(A)x,x\rangle \leq \frac{M-\langle Ax,x\rangle}{M-m}h(m)+\frac{\langle Ax,x\rangle-m}{M-m}h(M) \tag{LR}$$

for each $x\in H$ with $\|x\|=1$.

13.2. Some Trace Inequalities for Convex Functions. Consider the orthonormal basis $\mathcal{E}:=\{e_i\}_{i\in I}$ in the complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$ and for a nonzero operator $B\in\mathcal{B}_2(H)$ let introduce the subset of indices from I defined by

$$I_{\mathcal{E},B}:=\{i\in I:\ Be_i\neq 0\}.$$

We observe that $I_{\mathcal{E},B}$ is non-empty for any nonzero operator B and if $\ker(B)=0$, i.e. B is injective, then $I_{\mathcal{E},B}=I$. We also have for $B\in\mathcal{B}_2(H)$ that

$$\operatorname{tr}\left(|B|^2\right)=\operatorname{tr}(B^*B)=\sum_{i\in I}\langle B^*Be_i,e_i\rangle=\sum_{i\in I}\|Be_i\|^2=\sum_{i\in I_{\mathcal{E},B}}\|Be_i\|^2.$$

THEOREM 13.4 (Dragomir, 2014, [**52**]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m<M$. If f is a continuous convex function on $[m,M]$, $\mathcal{E}:=\{e_i\}_{i\in I}$ is an orthonormal basis in H and $B\in\mathcal{B}_2(H)\setminus\{0\}$, then $\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\in[m,M]$ and*

$$\begin{aligned} f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)&\operatorname{tr}\left(|B|^2\right) \\ &\leq J_{\varepsilon}(f;A,B)\leq \operatorname{tr}\left(|B|^2f(A)\right) \\ &\leq \frac{1}{M-m}\left(f(m)\operatorname{tr}\left[|B|^2(M1_H-A)\right]+f(M)\operatorname{tr}\left[|B|^2(A-m1_H)\right]\right), \end{aligned} \tag{13.3}$$

where

$$J_{\varepsilon}(f;A,B):=\sum_{i\in I_{\mathcal{E},B}}f\left(\frac{\langle B^*ABe_i,e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2. \tag{13.4}$$

PROOF. Since $Sp(A)\subseteq[m,M]$, then $m\|y\|^2\leq\langle Ay,y\rangle\leq M\|y\|^2$ for any $y\in H$. Therefore

$$m\|Be_i\|^2\leq\langle ABe_i,Be_i\rangle\leq M\|Be_i\|^2,$$

for any $i\in I$, which implies that

$$m\sum_{i\in I}\|Be_i\|^2\leq\sum_{i\in I}\langle ABe_i,Be_i\rangle\leq M\sum_{i\in I}\|Be_i\|^2$$

and we conclude that $\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\in[m,M]$.

By Jensen's inequality (MP) we have

$$f\left(\frac{\langle Ay,y\rangle}{\|y\|^2}\right)\leq\frac{\langle f(A)y,y\rangle}{\|y\|^2} \tag{13.5}$$

for any $y\in H\setminus\{0\}$.

Let F be a finite part of $I_{\mathcal{E},B}$. Then for any $i \in F$ we have from (13.5) that

$$f\left(\frac{\langle ABe_i, Be_i\rangle}{\|Be_i\|^2}\right) \le \frac{\langle f(A)\, Be_i, Be_i\rangle}{\|Be_i\|^2},$$

which is equivalent to

$$f\left(\frac{\langle B^*ABe_i, e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2 \le \langle B^* f(A)\, Be_i, e_i\rangle . \tag{13.6}$$

Summing over $i \in F$ we get

$$\sum_{i\in F} f\left(\frac{\langle B^*ABe_i, e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2 \le \sum_{i\in F}\langle B^* f(A)\, Be_i, e_i\rangle . \tag{13.7}$$

Using Jensen's discrete inequality for finite sums and for the positive weights w_i

$$f\left(\frac{\sum_{i\in F} w_i u_i}{\sum_{i\in F} w_i}\right) \le \frac{\sum_{i\in F} w_i f(u_i)}{\sum_{i\in F} w_i},$$

we have

$$f\left(\frac{\sum_{i\in F}\frac{\langle B^*ABe_i,e_i\rangle}{\|Be_i\|^2}\|Be_i\|^2}{\sum_{i\in F}\|Be_i\|^2}\right) \le \frac{\sum_{i\in F} f\left(\frac{\langle B^*ABe_i,e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2}{\sum_{i\in F}\|Be_i\|^2},$$

which is equivalent to

$$f\left(\frac{\sum_{i\in F}\langle B^*ABe_i, e_i\rangle}{\sum_{i\in F}\|Be_i\|^2}\right)\sum_{i\in F}\|Be_i\|^2 \le \sum_{i\in F} f\left(\frac{\langle B^*ABe_i, e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2 . \tag{13.8}$$

Therefore, for any F a finite part of $I_{\mathcal{E},B}$ we have from (13.7) that

$$\begin{aligned} f\left(\frac{\sum_{i\in F}\langle B^*ABe_i, e_i\rangle}{\sum_{i\in F}\|Be_i\|^2}\right)\sum_{i\in F}\|Be_i\|^2 &\le \sum_{i\in F} f\left(\frac{\langle B^*ABe_i, e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2 \\ &\le \sum_{i\in F}\langle B^* f(A)\, Be_i, e_i\rangle . \end{aligned} \tag{13.9}$$

By the continuity of f we then have from (13.9) that

$$\begin{aligned} & f\left(\frac{\sum_{i\in I_{\mathcal{E},B}}\langle B^*ABe_i, e_i\rangle}{\sum_{i\in I_{\mathcal{E},B}}\|Be_i\|^2}\right)\sum_{i\in I_{\mathcal{E},B}}\|Be_i\|^2 \\ & \le \sum_{i\in I_{\mathcal{E},B}} f\left(\frac{\langle B^*ABe_i, e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2 \le \sum_{i\in I_{\mathcal{E},B}}\langle B^* f(A)\, Be_i, e_i\rangle \end{aligned} \tag{13.10}$$

and since $B \in \mathcal{B}_2(H)\setminus\{0\}$, then also

$$\sum_{i\in I_{\mathcal{E},B}}\|Be_i\|^2 = \sum_{i\in I}\|Be_i\|^2 = \operatorname{tr}\left(|B|^2\right),$$

$$\sum_{i\in I_{\mathcal{E},B}}\langle B^*ABe_i, e_i\rangle = \sum_{i\in I}\langle B^*ABe_i, e_i\rangle = \operatorname{tr}\left(|B|^2 A\right)$$

and

$$\sum_{i\in I_{\mathcal{E},B}}\langle B^* f(A)\, Be_i, e_i\rangle = \sum_{i\in I}\langle B^* f(A)\, Be_i, e_i\rangle = \operatorname{tr}\left(|B|^2 f(A)\right).$$

From (13.10) we then get the first and the second inequality in (13.3).

From (LR) we also have

$$\langle f(A)y,y\rangle \leq \frac{1}{M-m}\left[\langle (M1_H-A)y,y\rangle f(m)+\langle (A-m1_H)y,y\rangle f(M)\right] \tag{13.11}$$

for any $y\in H$.

This implies that

$$\begin{aligned}\langle f(A)Be_i,Be_i\rangle \\ \leq \frac{1}{M-m}\left[\langle (M1_H-A)Be_i,Be_i\rangle f(m)+\langle (A-m1_H)Be_i,Be_i\rangle f(M)\right]\end{aligned} \tag{13.12}$$

for any $i\in I$.

By summation we have

$$\begin{aligned}&\sum_{i\in I}\langle f(A)Be_i,Be_i\rangle \\ &\leq \frac{1}{M-m}\left[f(m)\sum_{i\in I}\langle (M1_H-A)Be_i,Be_i\rangle + f(M)\sum_{i\in I}\langle (A-m1_H)Be_i,Be_i\rangle\right]\end{aligned}$$

and the last part of (13.3) is proved. ∎

REMARK 13.1. We observe that the quantities

$$J_s(f;A,B)=\sup_{\varepsilon}J_{\varepsilon}(f;A,B) \text{ and } J_i(f;A,B)=\inf_{\varepsilon}J_{\varepsilon}(f;A,B)$$

are finite and satisfy the bounds

$$\begin{aligned}f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\operatorname{tr}\left(|B|^2\right) &\leq J_i(f;A,B) \\ &\leq J_s(f;A,B)\leq \operatorname{tr}\left(|B|^2 f(A)\right).\end{aligned} \tag{13.13}$$

We have the following version for nonnegative operators $P\geq 0$, i.e. P satisfies the condition $\langle Px,x\rangle \geq 0$ for any $x\in H$.

COROLLARY 13.5. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq [m,M]$ for some scalars m, M with $m<M$. If f is a continuous convex function on $[m,M]$, $\mathcal{E}:=\{e_i\}_{i\in I}$ is an orthonormal basis in H and $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\geq 0$ then $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\in[m,M]$ and*

$$\begin{aligned}&f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\operatorname{tr}(P) \\ &\leq K_{\varepsilon}(f;A,P)\leq \operatorname{tr}(Pf(A)) \\ &\leq \frac{1}{M-m}\left(f(m)\operatorname{tr}[P(M1_H-A)]+f(M)\operatorname{tr}[P(A-m1_H)]\right),\end{aligned} \tag{13.14}$$

where

$$K_{\varepsilon}(f;A,P):=\sum_{i\in I_{\mathcal{E},P}} f\left(\frac{\langle P^{1/2}AP^{1/2}e_i,e_i\rangle}{\langle Pe_i,e_i\rangle}\right)\langle Pe_i,e_i\rangle$$

and

$$I_{\mathcal{E},P}:=\left\{i\in I:\ P^{1/2}e_i\neq 0\right\}$$

Moreover, the quantities

$$K_i(f;A,P):=\inf_{\varepsilon}K_{\varepsilon}(f;A,P) \textit{ and } K_s(f;A,P):=\sup_{\varepsilon}K_{\varepsilon}(f;A,P)$$

are finite and satisfy the bounds

$$f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\operatorname{tr}(P)\le K_i(f;A,P)\le K_s(f;A,P)\le\operatorname{tr}(Pf(A)). \tag{13.15}$$

The finite dimensional case is of interest.

Let $\mathcal{M}_n(\mathbb{C})$ be the space of all square matrices of order n with complex elements.

COROLLARY 13.6. *Let $A\in\mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m<M$. If f is a continuous convex function on $[m,M]$, $\mathcal{E}:=\{e_i\}_{i\in\{1,\dots,n\}}$ is an orthonormal basis in $\mathbb{C}^n$, then $\frac{1}{n}\operatorname{tr}(A)\in[m,M]$ and*

$$\begin{aligned} nf\left(\frac{\operatorname{tr}(A)}{n}\right)&\le J_{\varepsilon}(f;A)\le\operatorname{tr}(f(A)) \\ &\le\frac{1}{M-m}\left[f(m)\operatorname{tr}(MI_n-A)+f(M)\operatorname{tr}(A-mI_n)\right], \end{aligned} \tag{13.16}$$

where

$$J_{\varepsilon}(f;A):=\sum_{i=1}^{n}f(\langle Ae_i,e_i\rangle),$$

and I_n is the identity matrix in $\mathcal{M}_n(\mathbb{C})$.

REMARK 13.2. The second inequality in (13.16), namely

$$\sum_{i=1}^{n}f(\langle Ae_i,e_i\rangle)\le\operatorname{tr}(f(A))$$

for any $\{e_i\}_{i\in\{1,\dots,n\}}$ an orthonormal basis in $\mathbb{C}^n$, is known in literature as Peierls Inequality. For a different proof and some applications, see, for instance [**12**].

13.3. Some Functional Properties. If we denote by $\mathcal{B}_1^+(H)$ the convex cone of nonnegative operators from $\mathcal{B}_1(H)$ we can consider the functional $\sigma_{f,A}:\mathcal{B}_1^+(H)\setminus\{0\}\to[0,\infty)$ defined by

$$\sigma_{f,A}(P):=\operatorname{tr}(Pf(A))-\operatorname{tr}(P)f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\ge 0, \tag{13.17}$$

where A is a selfadjoint operator on the Hilbert space H with $Sp(A)\subseteq[m,M]$ for some scalars m, M $(m<M)$ and f is a continuous convex function on $[m,M]$.

One can easily observe that, if f is a continuous strictly convex function on $[m,M]$, then the inequality is strict in (13.17).

THEOREM 13.7 (Dragomir, 2014, [**52**]). *Let A be a selfadjoint operator on the Hilbert space H with $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m<M$ and f is a continuous convex function on $[m,M]$.*

(i) For any $P,Q\in\mathcal{B}_1^+(H)\setminus\{0\}$ we have

$$\sigma_{f,A}(P+Q)\ge\sigma_{f,A}(P)+\sigma_{f,A}(Q)\,(\ge 0), \tag{13.18}$$

i.e. $\sigma_{f,A}(\cdot)$ is a superadditive functional on $\mathcal{B}_1^+(H)\setminus\{0\}$;

(ii) For any $P,Q\in\mathcal{B}_1^+(H)\setminus\{0\}$ with $P\ge Q$ we have

$$\sigma_{f,A}(P)\ge\sigma_{f,A}(Q)\,(\ge 0), \tag{13.19}$$

i.e. $\sigma_{f,A}(\cdot)$ is a monotonic nondecreasing functional on $\mathcal{B}_1^+(H)\setminus\{0\}$;

(iii) If there exists the real numbers $\gamma,\ \Gamma > 0$ *such that* $\Gamma Q \geq P \geq \gamma Q$ *with* $P,\ Q \in \mathcal{B}_1^+(H) \setminus \{0\}$ *, then*

$$\Gamma \sigma_{f,A}(Q) \geq \sigma_{f,A}(P) \geq \gamma \sigma_{f,A}(Q)\ (\geq 0)\,. \tag{13.20}$$

PROOF. (i) Let $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$. Then we have

$$\begin{aligned} \sigma_{f,A}(P+Q) &= \operatorname{tr}((P+Q) f(A)) - \operatorname{tr}(P+Q) f\left(\frac{\operatorname{tr}((P+Q)A)}{\operatorname{tr}(P+Q)}\right) \\ &= \operatorname{tr}(Pf(A)) + \operatorname{tr}(Pf(A)) \\ &\quad - [\operatorname{tr}(P) + \operatorname{tr}(Q)] f\left(\frac{\operatorname{tr}(PA) + \operatorname{tr}(QA)}{\operatorname{tr}(P) + \operatorname{tr}(Q)}\right). \end{aligned} \tag{13.21}$$

By the convexity of f we have

$$\begin{aligned} f\left(\frac{\operatorname{tr}(PA) + \operatorname{tr}(QA)}{\operatorname{tr}(P) + \operatorname{tr}(Q)}\right) &= f\left(\frac{\operatorname{tr}(P)\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \operatorname{tr}(Q)\frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}}{\operatorname{tr}(P) + \operatorname{tr}(Q)}\right) \\ &\leq \frac{\operatorname{tr}(P) f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) + \operatorname{tr}(Q) f\left(\frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}\right)}{\operatorname{tr}(P) + \operatorname{tr}(Q)}. \end{aligned} \tag{13.22}$$

Making use of (13.21) and (13.22) we have

$$\begin{aligned} \sigma_{f,A}(P+Q) &\geq \operatorname{tr}(Pf(A)) + \operatorname{tr}(Pf(A)) \\ &\quad - [\operatorname{tr}(P) + \operatorname{tr}(Q)] \frac{\operatorname{tr}(P) f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) + \operatorname{tr}(Q) f\left(\frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}\right)}{\operatorname{tr}(P) + \operatorname{tr}(Q)} \\ &= \operatorname{tr}(Pf(A)) + \operatorname{tr}(Pf(A)) \\ &\quad - \operatorname{tr}(P) f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \operatorname{tr}(Q) f\left(\frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}\right) \\ &= \sigma_{f,A}(P) + \sigma_{f,A}(Q) \end{aligned}$$

and the inequality (13.18) is proved.

(ii) Let $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$ with $P \geq Q$. Then on applying the superadditivity property of $\sigma_{f,A}$ for $P - Q \geq 0$ and $Q \geq 0$ we have

$$\sigma_{f,A}(P) = \sigma_{f,A}(P - Q + Q) \geq \sigma_{f,A}(P - Q) + \sigma_{f,A}(Q) \geq \sigma_{f,A}(Q)$$

and the inequality (13.19) is proved.

(iii) If $P \geq \gamma Q$, then by the monotonicity property of $\sigma_{f,A}$ we have

$$\sigma_{f,A}(P) \geq \sigma_{f,A}(\gamma Q) = \gamma \sigma_{f,A}(Q)$$

and a similar inequality for Γ. ∎

We have the following particular case of interest:

COROLLARY 13.8. *Let* $A \in \mathcal{M}_n(\mathbb{C})$ *be a Hermitian matrix and assume that* $Sp(A) \subseteq [m, M]$ *for some scalars* m, M *with* $m < M$*. If* f *is a continuous convex function on* $[m, M]$ *, there exists the real numbers* $\gamma, \Gamma > 0$ *such that* $\Gamma I_n \geq P \geq \gamma I_n$ *with* P *positive definite, where*

I_n is the identity matrix, then

$$\Gamma\left[\operatorname{tr}(f(A)) - nf\left(\frac{\operatorname{tr}(A)}{n}\right)\right] \geq \operatorname{tr}(Pf(A)) - \operatorname{tr}(P) f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \geq \gamma\left[\operatorname{tr}(f(A)) - nf\left(\frac{\operatorname{tr}(A)}{n}\right)\right] (\geq 0). \tag{13.23}$$

The following result also holds:

THEOREM 13.9 (Dragomir, 2014, [**52**]). *Let A be a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$ and f is a continuous convex function on $[m, M]$. For $p \geq 1$, the functional $\psi_{p,f,A} : \mathcal{B}_1^+(H) \setminus \{0\} \to [0, \infty)$ defined by*

$$\psi_{p,f,A}(P) := [\operatorname{tr}(P)]^{1-\frac{1}{p}} \sigma_{f,A}(P)$$

is superadditive on $\mathcal{B}_1^+(H) \setminus \{0\}$.

PROOF. First of all we observe that the following elementary inequality holds:

$$(\alpha + \beta)^p \geq (\leq) \alpha^p + \beta^p \tag{13.24}$$

for any $\alpha, \beta \geq 0$ and $p \geq 1$ $(0 < p < 1)$.

Indeed, if we consider the function $f_p : [0, \infty) \to \mathbb{R}$, $f_p(t) = (t+1)^p - t^p$ we have $f_p'(t) = p\left[(t+1)^{p-1} - t^{p-1}\right]$. Observe that for $p > 1$ and $t > 0$ we have that $f_p'(t) > 0$ showing that f_p is strictly increasing on the interval $[0, \infty)$. Now for $t = \frac{\alpha}{\beta}$ $(\beta > 0, \alpha \geq 0)$ we have $f_p(t) > f_p(0)$ giving that $\left(\frac{\alpha}{\beta} + 1\right)^p - \left(\frac{\alpha}{\beta}\right)^p > 1$, i.e., the desired inequality (13.24).

For $p \in (0, 1)$ we have that f_p is strictly decreasing on $[0, \infty)$ which proves the second case in (13.24).

Now, since $\sigma_{f,A}(\cdot)$ is superadditive on $\mathcal{B}_1^+(H) \setminus \{0\}$ and $p \geq 1$ then by (13.24) we have

$$\sigma_{f,A}^p(P + Q) \geq [\sigma_{f,A}(P) + \sigma_{f,A}(Q)]^p \geq \sigma_{f,A}^p(P) + \sigma_{f,A}^p(Q) \tag{13.25}$$

for any $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Utilising (13.25) and the additivity property of $\operatorname{tr}(\cdot)$ on $\mathcal{B}_1^+(H) \setminus \{0\}$ we have

$$\begin{aligned} \frac{\sigma_{f,A}^p(P+Q)}{\operatorname{tr}(P+Q)} &\geq \frac{\sigma_{f,A}^p(P) + \sigma_{f,A}^p(Q)}{\operatorname{tr}(P) + \operatorname{tr}(Q)} \\ &= \frac{\operatorname{tr}(P) \frac{\sigma_{f,A}^p(P)}{\operatorname{tr}(P)} + \operatorname{tr}(Q) \frac{\sigma_{f,A}^p(Q)}{\operatorname{tr}(Q)}}{\operatorname{tr}(P) + \operatorname{tr}(Q)} \\ &= \frac{\operatorname{tr}(P) \left(\frac{\sigma_{f,A}(P)}{\operatorname{tr}^{1/p}(P)}\right)^p + \operatorname{tr}(Q) \left(\frac{\sigma_{f,A}(Q)}{\operatorname{tr}^{1/q}(Q)}\right)^p}{\operatorname{tr}(P) + \operatorname{tr}(Q)} =: I, \end{aligned} \tag{13.26}$$

for any $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Since for $p \geq 1$ the power function $g(t) = t^p$ is convex, then

$$\begin{aligned} I &\geq \left(\frac{\operatorname{tr}(P) \frac{\sigma_{f,A}(P)}{\operatorname{tr}^{1/p}(P)} + \operatorname{tr}(Q) \frac{\sigma_{f,A}(Q)}{\operatorname{tr}^{1/q}(Q)}}{\operatorname{tr}(P) + \operatorname{tr}(Q)}\right)^p \\ &= \left(\frac{\operatorname{tr}^{1-1/p}(P) \sigma_{f,A}(P) + \operatorname{tr}^{1-1/q}(Q) \sigma_{f,A}(Q)}{\operatorname{tr}(P+Q)}\right)^p \end{aligned} \tag{13.27}$$

for any $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

By combining (13.26) with (13.27) we get

$$\frac{\sigma_{f,A}^{p}(P+Q)}{\operatorname{tr}(P+Q)} \geq \left(\frac{\operatorname{tr}^{1-1/p}(P)\,\sigma_{f,A}(P)+\operatorname{tr}^{1-1/q}(Q)\,\sigma_{f,A}(Q)}{\operatorname{tr}(P+Q)}\right)^{p},$$

which is equivalent to

$$\frac{\sigma_{f,A}(P+Q)}{\operatorname{tr}^{1/p}(P+Q)} \geq \frac{\operatorname{tr}^{1-1/p}(P)\,\sigma_{f,A}(P)+\operatorname{tr}^{1-1/q}(Q)\,\sigma_{f,A}(Q)}{\operatorname{tr}(P+Q)}, \tag{13.28}$$

for any $P, Q \in \mathcal{B}_1^{+}(H) \setminus \{0\}$.

Finally, if we multiply (13.28) by $\operatorname{tr}(P+Q)>0$ we get

$$\psi_{p,f,A}(P+Q) \geq \psi_{p,f,A}(P)+\psi_{p,f,A}(Q)$$

for any $P, Q \in \mathcal{B}_1^{+}(H) \setminus \{0\}$ and the proof is complete. ∎

COROLLARY 13.10. *With the assumptions of Theorem 13.9, the two parameters* $p,\, q \geq 1$ *functional* $\psi_{p,q,f,A}: \mathcal{B}_1^{+}(H) \setminus \{0\} \to [0,\infty)$ *defined by*

$$\psi_{p,q,f,A}(P) := [\operatorname{tr}(P)]^{q\left(1-\frac{1}{p}\right)}\,\sigma_{f,A}^{q}(P)$$

is superadditive on $\mathcal{B}_1^{+}(H) \setminus \{0\}$.

PROOF. Observe that $\psi_{p,q,f,A}(P) = \left[\psi_{p,f,A}(P)\right]^{q}$ for $P \in \mathcal{B}_1^{+}(H) \setminus \{0\}$. Therefore, by Theorem 13.9 and the inequality (13.24) for $q \geq 1$ we have that

$$\begin{aligned}
\psi_{p,q,f,A}(P+Q) &= \left[\psi_{p,f,A}(P+Q)\right]^{q} \\
&\geq \left[\psi_{p,f,A}(P)+\psi_{p,f,A}(Q)\right]^{q} \\
&\geq \left[\psi_{p,f,A}(P)\right]^{q}+\left[\psi_{p,f,A}(Q)\right]^{q} \\
&= \psi_{p,q,f,A}(P)+\psi_{p,q,f,A}(Q)
\end{aligned}$$

for any $P, Q \in \mathcal{B}_1^{+}(H) \setminus \{0\}$ and the statement is proved. ∎

REMARK 13.3. If we consider the functional

$$\tilde{\psi}_{p,f,A}(P) := [\operatorname{tr}(P)]^{p-1}\,\sigma_{f,A}^{p}(P)$$

then, for $p \geq 1$, $\tilde{\psi}_{p,f,A}(\cdot)$ is superadditive on $\mathcal{B}_1^{+}(H) \setminus \{0\}$.

COROLLARY 13.11. *With the assumptions of Theorem 13.9 and for parameter* $p \geq 1$, *if there exists the real numbers* $\gamma,\, \Gamma > 0$ *such that* $\Gamma Q \geq P \geq \gamma Q$ *with* $P, Q \in \mathcal{B}_1^{+}(H) \setminus \{0\}$, *then*

$$\begin{aligned}
\Gamma^{2-\frac{1}{p}}\,[\operatorname{tr}(Q)]^{1-\frac{1}{p}}\,\sigma_{f,A}(Q) &\geq [\operatorname{tr}(P)]^{1-\frac{1}{p}}\,\sigma_{f,A}(P) \\
&\geq \gamma^{2-\frac{1}{p}}\,[\operatorname{tr}(Q)]^{1-\frac{1}{p}}\,\sigma_{f,A}(Q)\,(\geq 0).
\end{aligned} \tag{13.29}$$

The case of finite-dimensional spaces is as follows:

COROLLARY 13.12. *Let* $A \in \mathcal{M}_n(\mathbb{C})$ *be a Hermitian matrix and assume that* $Sp(A) \subseteq [m, M]$ *for some scalars* $m,\, M$ *with* $m < M$. *If* f *is a continuous convex function on* $[m, M]$, *there exists the real numbers* $\gamma, \Gamma > 0$ *such that* $\Gamma I_n \geq P \geq \gamma I_n$ *with* P *positive definite, then*

$$\Gamma^{2-\frac{1}{p}} n^{1-\frac{1}{p}} \left[\operatorname{tr}(f(A)) - nf\left(\frac{\operatorname{tr}(A)}{n}\right) \right] \tag{13.30}$$
$$\geq [\operatorname{tr}(P)]^{1-\frac{1}{p}} \left[\operatorname{tr}(Pf(A)) - \operatorname{tr}(P) f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \right]$$
$$\geq \gamma^{2-\frac{1}{p}} n^{1-\frac{1}{p}} \left[\operatorname{tr}(f(A)) - nf\left(\frac{\operatorname{tr}(A)}{n}\right) \right] (\geq 0)$$

for any $p \geq 1$.

The following result also holds:

THEOREM 13.13 (Dragomir, 2014, [**52**]). *Let* A *be a selfadjoint operator on the Hilbert space* H *with* $Sp(A) \subseteq [m, M]$ *for some scalars* m, M *with* $m < M$ *and* f *is a continuous strictly convex function on* $[m, M]$. *For* $p \in (0,1)$, *the functional* $\chi_{p,f,A} : \mathcal{B}_1^+(H) \setminus \{0\} \to [0, \infty)$ *defined by*

$$\chi_{p,f,A}(P) := \frac{[\operatorname{tr}(P)]^{1-\frac{1}{p}}}{\sigma_{f,A}(P)}$$

is subadditive on $\mathcal{B}_1^+(H) \setminus \{0\}$.

PROOF. Let $s := -p \in (-1, 0)$. For $s < 0$ we have the following inequality

$$(\alpha + \beta)^s \leq \alpha^s + \beta^s \tag{13.31}$$

for any $\alpha, \beta > 0$.

Indeed, by the convexity of the function $f_s(t) = t^s$ on $(0, \infty)$ with $s < 0$ we have that

$$(\alpha + \beta)^s \leq 2^{s-1}(\alpha^s + \beta^s)$$

for any $\alpha, \beta > 0$ and since, obviously, $2^{s-1}(\alpha^s + \beta^s) \leq \alpha^s + \beta^s$, then (13.31) holds true.

Taking into account that $\sigma_{f,A}(\cdot)$ is superadditive and $s \in (-1, 0)$ we have

$$\sigma_{f,A}^s(P+Q) \leq [\sigma_{f,A}(P) + \sigma_{f,A}(Q)]^s \leq \sigma_{f,A}^s(P) + \sigma_{f,A}^s(Q) \tag{13.32}$$

for any $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Since $\operatorname{tr}(\cdot)$ is additive on $\mathcal{B}_1^+(H) \setminus \{0\}$, then by (13.33) we have

$$\frac{\sigma_{f,A}^s(P+Q)}{\operatorname{tr}(P+Q)} \leq \frac{\sigma_{f,A}^s(P) + \sigma_{f,A}^s(Q)}{\operatorname{tr}(P) + \operatorname{tr}(Q)} \tag{13.33}$$
$$= \frac{\operatorname{tr}(P)\left(\frac{\sigma_{f,A}(P)}{\operatorname{tr}^{1/s}(P)}\right)^s + \operatorname{tr}(Q)\left(\frac{\sigma_{f,A}(Q)}{\operatorname{tr}^{1/s}(Q)}\right)^s}{\operatorname{tr}(P) + \operatorname{tr}(Q)}$$
$$= \frac{\operatorname{tr}(P)\left(\frac{\operatorname{tr}^{1/s}(P)}{\sigma_{f,A}(P)}\right)^{-s} + \operatorname{tr}(Q)\left(\frac{\operatorname{tr}^{1/s}(Q)}{\sigma_{f,A}(Q)}\right)^{-s}}{\operatorname{tr}(P) + \operatorname{tr}(Q)} =: J$$

for any $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

By the concavity of the function $g(t) = t^{-s}$ with $s \in (-1, 0)$ we also have

$$J \leq \left[\frac{\operatorname{tr}(P)\frac{\operatorname{tr}^{1/s}(P)}{\sigma_{f,A}(P)} + \operatorname{tr}(Q)\frac{\operatorname{tr}^{1/s}(Q)}{\sigma_{f,A}(Q)}}{\operatorname{tr}(P) + \operatorname{tr}(Q)} \right]^{-s} \tag{13.34}$$

for any $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Making use of (13.33) and (13.34) we get

$$\frac{\sigma_{f,A}^{s}(P+Q)}{\operatorname{tr}(P+Q)} \leq \left[\frac{\operatorname{tr}(P)\frac{\operatorname{tr}^{1/s}(P)}{\sigma_{f,A}(P)}+\operatorname{tr}(Q)\frac{\operatorname{tr}^{1/s}(Q)}{\sigma_{f,A}(Q)}}{\operatorname{tr}(P)+\operatorname{tr}(Q)}\right]^{-s}$$

for any $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$, and by taking the power $-1/s > 0$ we get

$$\frac{\sigma_{f,A}^{-1}(P+Q)}{\operatorname{tr}^{-1/s}(P+Q)} \leq \frac{\frac{\operatorname{tr}^{1+1/s}(P)}{\sigma_{f,A}(P)}+\frac{\operatorname{tr}^{1+1/s}(Q)}{\sigma_{f,A}(Q)}}{\operatorname{tr}(P)+\operatorname{tr}(Q)},$$

which is equivalent to

$$\frac{\operatorname{tr}^{1+1/s}(P+Q)}{\sigma_{f,A}(P+Q)} \leq \frac{\operatorname{tr}^{1+1/s}(P)}{\sigma_{f,A}(P)}+\frac{\operatorname{tr}^{1+1/s}(Q)}{\sigma_{f,A}(Q)}$$

for any $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

This completes the proof. ∎

The following result may be stated as well:

COROLLARY 13.14. *With the assumptions of Theorem 13.13, the two parameters* $0 < p$, $q < 1$ *functional* $\chi_{p,q,f,A} : \mathcal{B}_1^+(H) \setminus \{0\} \to [0,\infty)$ *defined by*

$$\chi_{p,q,f,A}(P) = \frac{\operatorname{tr}^{q\left(1-\frac{1}{p}\right)}(P)}{\sigma_{f,A}^{q}(P)}$$

is subadditive on $\mathcal{B}_1^+(H) \setminus \{0\}$.

REMARK 13.4. If we consider the functional $\tilde{\chi}_{p,f,A}(P) = \frac{\operatorname{tr}^{p-1}(P)}{\sigma_{f,A}^{p}(P)}$ for $0 < p < 1$, then $\tilde{\chi}_{p,f,A}(\cdot)$ is also subadditive on $\mathcal{B}_1^+(H) \setminus \{0\}$.

13.4. Some Examples. We consider the power function $f : (0,\infty) \to (0,\infty)$, $f(t) = t^r$ with $t \in \mathbb{R} \setminus \{0\}$. For $r \in (-\infty, 0) \cup [1,\infty)$, f is convex while for $r \in (0,1)$, f is concave.

Let $r \geq 1$ and A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 \leq m < M$. If $\mathcal{E} := \{e_i\}_{i\in I}$ is an orthonormal basis in H and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ then

$$\begin{aligned} &[\operatorname{tr}(PA)]^r [\operatorname{tr}(P)]^{1-r} \\ &\leq K_{\varepsilon}(r; A, P) \leq \operatorname{tr}(PA^r) \\ &\leq \frac{1}{M-m}\left(m^r \operatorname{tr}[P(M1_H - A)] + M^r \operatorname{tr}[P(A - m1_H)]\right), \end{aligned} \tag{13.35}$$

where

$$K_{\varepsilon}(r; A, P) := \sum_{i\in I_{\mathcal{E},P}} \left\langle P^{1/2}AP^{1/2}e_i, e_i\right\rangle^r \langle Pe_i, e_i\rangle^{1-r}.$$

Moreover, the quantities

$$K_i(r; A, P) := \inf_{\varepsilon} K_{\varepsilon}(r; A, P) \text{ and } K_s(r; A, P) := \sup_{\varepsilon} K_{\varepsilon}(r; A, P)$$

are finite and satisfy the bounds

$$[\operatorname{tr}(PA)]^r [\operatorname{tr}(P)]^{1-r} \leq K_i(r; A, P) \leq K_s(r; A, P) \leq \operatorname{tr}(PA^r). \tag{13.36}$$

Now, if we take $A = P$, $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then by (13.35) we have

$$\left[\operatorname{tr}\left(P^2\right)\right]^r [\operatorname{tr}(P)]^{1-r} \leq K_{\varepsilon}(r; P) \leq \operatorname{tr}\left(P^{r+1}\right) \tag{13.37}$$

where

$$K_{\varepsilon}(r;P) := \sum_{i\in I_{\mathcal{E},P}} \left\langle P^2 e_i, e_i \right\rangle^r \left\langle P e_i, e_i \right\rangle^{1-r}.$$

If we consider the functional $\sigma_{r,A} : \mathcal{B}_1^+(H) \setminus \{0\} \to [0,\infty)$ defined by

$$\sigma_{r,A}(P) := \operatorname{tr}(PA^r) - [\operatorname{tr}(PA)]^r [\operatorname{tr}(P)]^{1-r} \geq 0, \tag{13.38}$$

where A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m,M] \subset [0,\infty)$, then $\sigma_{r,A}(\cdot)$ is superadditive, monotonic nondecreasing and if there exists the real numbers $\gamma, \Gamma > 0$ such that $\Gamma Q \geq P \geq \gamma Q$ with $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$\Gamma \sigma_{r,A}(Q) \geq \sigma_{r,A}(P) \geq \gamma \sigma_{r,A}(Q)\,(\geq 0). \tag{13.39}$$

Consider the convex function $f : (0,\infty) \to (0,\infty)$, $f(t) = -\ln t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m,M]$ for some scalars m, M with $0 < m < M$. If $\mathcal{E} := \{e_i\}_{i\in I}$ is an orthonormal basis in H and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ then

$$\begin{aligned} \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P)} &\geq L_{\varepsilon}(A,P) \geq \exp[\operatorname{tr}(P\ln A)] \\ &\geq m^{\frac{\operatorname{tr}[P(M1_H - A)]}{M-m}} M^{\frac{\operatorname{tr}[P(A-m1_H)]}{M-m}}, \end{aligned} \tag{13.40}$$

where

$$L_{\varepsilon}(A,P) := \prod_{i\in I_{\mathcal{E},P}} \left(\frac{\left\langle P^{1/2}AP^{1/2}e_i, e_i\right\rangle}{\langle Pe_i, e_i\rangle}\right)^{\langle Pe_i, e_i\rangle}.$$

Moreover, the quantities

$$L_i(A,P) := \inf_{\varepsilon} L_{\varepsilon}(A,P) \text{ and } L_s(A,P) := \sup_{\varepsilon} L_{\varepsilon}(A,P)$$

are finite and satisfy the bounds

$$\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P)} \geq L_s(A,P) \geq L_i(A,P) \geq \exp[\operatorname{tr}(P\ln A)]. \tag{13.41}$$

Now, if we take $A = P$, $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then by (13.40) we get

$$\left(\frac{\operatorname{tr}(P^2)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P)} \geq L_{\varepsilon}(P) \geq \exp[\operatorname{tr}(P\ln P)] \tag{13.42}$$

where

$$L_{\varepsilon}(P) := \prod_{i\in I_{\mathcal{E},P}} \left(\frac{\langle P^2 e_i, e_i\rangle}{\langle Pe_i, e_i\rangle}\right)^{\langle Pe_i, e_i\rangle}.$$

Consider the functional $\delta_A : \mathcal{B}_1^+(H) \setminus \{0\} \to (0,\infty)$ defined by

$$\delta_A(P) := \frac{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P)}}{\exp(\operatorname{tr}(P\ln A))} \geq 1,$$

where A is a selfadjoint operator on the Hilbert space H and such that $Sp(A) \subseteq [m,M]$ for some scalars m, M with $0 < m < M$.

Observe that

$$\sigma_{-\ln,A}(P) := \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P)} - \ln[\exp(\operatorname{tr}(P\ln A))] = \ln[\delta_A(P)]$$

for $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Utilising the properties of $\sigma_{-\ln,A}(\cdot)$ we conclude that $\delta_A(\cdot)$ is supermultiplicative, i.e.

$$\delta_A(P+Q) \geq \delta_A(P)\,\delta_A(Q) \geq 1$$

for any $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$. The functional $\delta_A(\cdot)$ is also monotonic nondecreasing on $\mathcal{B}_1(H) \setminus \{0\}$.

Consider the convex function $f(t) = t\ln t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If $\mathcal{E} := \{e_i\}_{i\in I}$ is an orthonormal basis in H and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ then

$$\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(PA)} \leq I_{\mathcal{E}}(A,P) \leq \exp\left[\operatorname{tr}(PA\ln A)\right] \leq m^{\frac{m\operatorname{tr}\left[P\left(M1_H - A\right)\right]}{M-m}} M^{\frac{M\operatorname{tr}\left[P\left(A - m1_H\right)\right]}{M-m}}, \tag{13.43}$$

where

$$I_{\mathcal{E}}(A,P) := \prod_{i\in I_{\mathcal{E},P}} \left(\frac{\left\langle P^{1/2}AP^{1/2}e_i, e_i\right\rangle}{\langle Pe_i, e_i\rangle}\right)^{\left\langle P^{1/2}AP^{1/2}e_i, e_i\right\rangle}.$$

Moreover, the quantities

$$I_i(A,P) := \inf_{\mathcal{E}} I_{\mathcal{E}}(A,P) \text{ and } I_s(A,P) := \sup_{\mathcal{E}} I_{\mathcal{E}}(A,P)$$

are finite and satisfy the bounds

$$\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(PA)} \leq I_i(A,P) \leq I_s(A,P) \leq \exp\left[\operatorname{tr}(PA\ln A)\right]. \tag{13.44}$$

Now, if we take $A = P$, $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then by (13.43) we get

$$\left(\frac{\operatorname{tr}(P^2)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}\left(P^2\right)} \leq I_{\mathcal{E}}(P) \leq \exp\left[\operatorname{tr}\left(P^2\ln P\right)\right] \tag{13.45}$$

where

$$I_{\mathcal{E}}(P) := \prod_{i\in I_{\mathcal{E},P}} \left(\frac{\langle P^2e_i, e_i\rangle}{\langle Pe_i, e_i\rangle}\right)^{\langle P^2e_i, e_i\rangle}.$$

Observe that for $f(t) = t\ln t$ we have

$$\begin{aligned}\sigma_{(\cdot)\ln(\cdot),A}(P) &= \operatorname{tr}(PA\ln A) - \operatorname{tr}(PA)\ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\ &= \ln\left[\frac{\exp\left[\operatorname{tr}(PA\ln A)\right]}{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(PA)}}\right]\end{aligned}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Consider the functional $\lambda_A : \mathcal{B}_1^+(H) \setminus \{0\} \to (0, \infty)$ defined by

$$\lambda_A(P) := \frac{\exp\left[\operatorname{tr}(PA\ln A)\right]}{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(PA)}} \geq 1.$$

Utilising the properties of $\sigma_{(\cdot)\ln(\cdot),A}(\cdot)$ we can conclude that $\lambda_A(\cdot)$ is supermultiplicative and monotonic nondecreasing on $\mathcal{B}_1^+(H) \setminus \{0\}$.

14. REVERSES OF JENSEN'S INEQUALITY

14.1. New Inequalities for Convex Functions. We recall the *gradient inequality* for the convex function $f:[m,M]\to\mathbb{R}$, namely

$$f(\varsigma)-f(\tau)\geq\delta_f(\tau)(\varsigma-\tau) \tag{14.1}$$

for any $\varsigma,\tau\in[m,M]$ where $\delta_f(\tau)\in\left[f'_-(\tau),f'_+(\tau)\right]$, (for $\tau=m$ we take $\delta_f(\tau)=f'_+(m)$ and for $\tau=M$ we take $\delta_f(\tau)=f'_-(M)$). Here $f'_+(m)$ and $f'_-(M)$ are the lateral derivatives of the convex function f.

The following result holds:

THEOREM 14.1 (Dragomir, 2014, [**55**]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m<M$. If f is a continuous convex function on $[m,M]$ and $B\in\mathcal{B}_2(H)\setminus\{0\}$, then we have $\frac{\operatorname{tr}(|B|^2A)}{\operatorname{tr}(|B|^2)}\in[m,M]$,*

$$\begin{aligned}\delta_f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)&\frac{\operatorname{tr}\left(|B^*|^2A\right)-\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\\&\leq\frac{\operatorname{tr}\left(|B^*|^2f(A)\right)}{\operatorname{tr}\left(|B|^2\right)}-f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right),\end{aligned} \tag{14.2}$$

where

$$\delta_f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\in\left[f'_-\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right),f'_+\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\right]$$

and the Jensen's inequality

$$f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\leq\frac{\operatorname{tr}\left(|B|^2f(A)\right)}{\operatorname{tr}\left(|B|^2\right)}. \tag{14.3}$$

PROOF. Let $\mathcal{E}:=\{e_i\}_{i\in I}$ be an orthonormal basis in H.

Utilising the gradient inequality (14.1) we get

$$f(\varsigma)-f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\geq\delta_f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(\varsigma-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \tag{14.4}$$

for any $\varsigma\in[m,M]$, since obviously, by $Sp(A)\subseteq[m,M]$ we have

$$m\|Be_i\|^2\leq\langle ABe_i,Be_i\rangle\leq M\|Be_i\|^2,$$

for $i\in I$, which, by summation shows that

$$\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\in[m,M].$$

The inequality (14.4) implies in the operator order of $\mathcal{B}(H)$ that

$$f(A)-f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)1_H\geq\delta_f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(A-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right), \tag{14.5}$$

which can be written as

$$\langle f(A)y,y\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\langle y,y\rangle \geq \delta_f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(\langle Ay,y\rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\langle y,y\rangle\right), \tag{14.6}$$

for any $y \in H$. This inequality is also of interest in itself.

Taking in (14.6) $y = Be_i$ we get

$$\langle f(A)Be_i,Be_i\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\langle Be_i,Be_i\rangle \geq \delta_f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(\langle ABe_i,Be_i\rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\langle Be_i,Be_i\rangle\right),$$

which is equivalent to

$$\langle B^*f(A)Be_i,e_i\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left\langle |B|^2 e_i,e_i\right\rangle \geq \delta_f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(\langle B^*ABe_i,e_i\rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\left\langle |B|^2 e_i,e_i\right\rangle\right), \tag{14.7}$$

for any $i \in I$.

Summing in (14.7) we get

$$\sum_{i\in I}\langle B^*f(A)Be_i,e_i\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\sum_{i\in I}\left\langle |B|^2 e_i,e_i\right\rangle \geq \delta_f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(\sum_{i\in I}\langle B^*ABe_i,e_i\rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\sum_{i\in I}\left\langle |B|^2 e_i,e_i\right\rangle\right). \tag{14.8}$$

However

$$\sum_{i\in I}\langle B^*f(A)Be_i,e_i\rangle = \sum_{i\in I}\langle BB^*f(A)e_i,e_i\rangle = \sum_{i\in I}\left\langle |B^*|^2 f(A)e_i,e_i\right\rangle = \operatorname{tr}\left(|B^*|^2 f(A)\right)$$

and

$$\sum_{i\in I}\langle B^*ABe_i,e_i\rangle = \sum_{i\in I}\langle BB^*Ae_i,e_i\rangle = \operatorname{tr}\left(|B^*|^2 A\right).$$

By (14.8) we get

$$\operatorname{tr}\left(|B^*|^2 f(A)\right) - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\operatorname{tr}\left(|B|^2\right) \geq \delta_f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(\operatorname{tr}\left(|B^*|^2 A\right) - \operatorname{tr}\left(|B|^2 A\right)\right), \tag{14.9}$$

and the inequality (14.2) is thus proved.

Taking in (14.6) $y = B^*e_i$ we also get

$$\langle f(A) B^* e_i, B^* e_i \rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \langle B^* e_i, B^* e_i \rangle$$
$$\geq \delta_f \left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \left(\langle AB^* e_i, B^* e_i \rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} \langle B^* e_i, B^* e_i \rangle\right),$$

which is equivalent to

$$\langle Bf(A) B^* e_i, e_i \rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \langle BB^* e_i, e_i \rangle \tag{14.10}$$
$$\geq \delta_f \left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \left(\langle BAB^* e_i, e_i \rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} \langle BB^* e_i, e_i \rangle\right),$$

for any $i \in I$.

Summing in (14.10) we get

$$\sum_{i \in I} \langle Bf(A) B^* e_i, e_i \rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \sum_{i \in I} \langle BB^* e_i, e_i \rangle \tag{14.11}$$
$$\geq \delta_f \left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \left(\sum_{i \in I} \langle BAB^* e_i, e_i \rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} \sum_{i \in I} \langle BB^* e_i, e_i \rangle\right).$$

Since

$$\sum_{i \in I} \langle Bf(A) B^* e_i, e_i \rangle = \operatorname{tr}(Bf(A) B^*) = \operatorname{tr}(B^* Bf(A)) = \operatorname{tr}\left(|B|^2 f(A)\right),$$

$$\sum_{i \in I} \langle BB^* e_i, e_i \rangle = \operatorname{tr}(BB^*) = \operatorname{tr}(B^* B) = \operatorname{tr}\left(|B|^2\right)$$

and

$$\sum_{i \in I} \langle BAB^* e_i, e_i \rangle = \operatorname{tr}(BAB^*) = \operatorname{tr}(B^* BA) = \operatorname{tr}\left(|B|^2 A\right),$$

then by (14.11) we get

$$\operatorname{tr}\left(|B|^2 f(A)\right) - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \operatorname{tr}\left(|B|^2\right) \geq 0$$

and the inequality (14.3) is obtained. ∎

COROLLARY 14.2. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuous convex function on $[m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$ then $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \in [m, M]$ and*

$$f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)}. \tag{14.12}$$

The proof follows by either (14.2) or (14.3) on choosing $B = P^{1/2}$, $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$.

The following lemma is of interest in itself:

LEMMA 14.3 (Dragomir, 2014, [**55**]). *Let S be a selfadjoint operator such that $\gamma 1_H \leq S \leq \Gamma 1_H$ for some real constants $\Gamma \geq \gamma$. Then for any $B \in \mathcal{B}_2(H) \setminus \{0\}$ we have*

$$0 \leq \frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2 \leq \frac{1}{2}(\Gamma-\gamma)\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left|S - \frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right|\right) \leq \frac{1}{2}(\Gamma-\gamma)\left[\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2} \leq \frac{1}{4}(\Gamma-\gamma)^2. \tag{14.13}$$

PROOF. The first inequality follows by Jensen's inequality (14.3) for the convex function $f(t) = t^2$.

Now, observe that

$$\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left(S - \frac{\Gamma+\gamma}{2}1_H\right)\left(S - \frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right)\right) = \frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2 S\left(S - \frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right)\right) - \frac{\Gamma+\gamma}{2}\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left(S - \frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right)\right) = \frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2 \tag{14.14}$$

since, obviously

$$\operatorname{tr}\left(|B|^2\left(S - \frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right)\right) = 0.$$

Now, since $\gamma 1_H \leq S \leq \Gamma 1_H$ then

$$\left|S - \frac{\Gamma+\gamma}{2}1_H\right| \leq \frac{1}{2}(\Gamma-\gamma).$$

Taking the modulus in (14.14) and using the properties of trace, we have

$$\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2 = \frac{1}{\operatorname{tr}\left(|B|^2\right)}\left|\operatorname{tr}\left(|B|^2\left(S - \frac{\Gamma+\gamma}{2}1_H\right)\left(S - \frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right)\right)\right| \leq \frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left|\left(S - \frac{\Gamma+\gamma}{2}1_H\right)\left(S - \frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right)\right|\right) \leq \frac{1}{2}(\Gamma-\gamma)\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left|S - \frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right|\right), \tag{14.15}$$

which proves the first part of (14.13).

By Schwarz inequality for trace we also have

$$\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left|S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right|\right) \tag{14.16}$$
$$\leq\left[\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left(S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right)^2\right)\right]^{1/2}$$
$$=\left[\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)}-\left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2}.$$

From (14.15) and (14.16) we get

$$\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)}-\left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2$$
$$\leq\frac{1}{2}(\Gamma-\gamma)\left[\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)}-\left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2},$$

which implies that

$$\left[\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)}-\left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2}\leq\frac{1}{2}(\Gamma-\gamma).$$

By (14.16) we then obtain

$$\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left|S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right|\right)$$
$$\leq\left[\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)}-\left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2}\leq\frac{1}{2}(\Gamma-\gamma)$$

that proves the last part of (14.13). ∎

REMARK 14.1. Let S be a selfadjoint operator such that $\gamma 1_H \leq S \leq \Gamma 1_H$ for some real constants $\Gamma \geq \gamma$. Then for any $P \in \mathcal{B}_1(H)\setminus\{0\}$, $P \geq 0$ we have

$$0\leq\frac{\operatorname{tr}\left(PS^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2 \tag{14.17}$$
$$\leq\frac{1}{2}(\Gamma-\gamma)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(P\left|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right|\right)$$
$$\leq\frac{1}{2}(\Gamma-\gamma)\left[\frac{\operatorname{tr}\left(PS^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}\leq\frac{1}{4}(\Gamma-\gamma)^2.$$

The following result provides reverses for the inequalities (14.2) and (14.3) above:

THEOREM 14.4 (Dragomir, 2014, [**55**]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m<M$. If f is a*

continuously differentiable convex function on $[m, M]$ *and* $B \in \mathcal{B}_2(H) \setminus \{0\}$, *then we have*

$$\frac{\operatorname{tr}\left(|B^*|^2 f(A)\right)}{\operatorname{tr}\left(|B|^2\right)} - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \leq \frac{\operatorname{tr}\left(|B^*|^2 f'(A) A\right)}{\operatorname{tr}\left(|B|^2\right)} - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} \cdot \frac{\operatorname{tr}\left(|B^*|^2 f'(A)\right)}{\operatorname{tr}\left(|B|^2\right)} \tag{14.18}$$

and

$$0 \leq \frac{\operatorname{tr}\left(|B|^2 f(A)\right)}{\operatorname{tr}\left(|B|^2\right)} - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \leq \frac{\operatorname{tr}\left(|B|^2 f'(A) A\right)}{\operatorname{tr}\left(|B|^2\right)} - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} \cdot \frac{\operatorname{tr}\left(|B|^2 f'(A)\right)}{\operatorname{tr}\left(|B|^2\right)} =: \mathcal{K}(f', B, A). \tag{14.19}$$

Moreover, we have

$$\begin{aligned} \mathcal{K}(f', B, A) &\leq \begin{cases} \frac{1}{2}\left[f'(M) - f'(m)\right] \dfrac{\operatorname{tr}\left(|B|^2 \left|A - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H\right|\right)}{\operatorname{tr}\left(|B|^2\right)} \\ \\ \frac{1}{2}(M - m) \dfrac{\operatorname{tr}\left(|B|^2 \left|f'(A) - \frac{\operatorname{tr}\left(|B|^2 f'(A)\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H\right|\right)}{\operatorname{tr}\left(|B|^2\right)} \end{cases} \\ &\leq \begin{cases} \frac{1}{2}\left[f'(M) - f'(m)\right] \left[\frac{\operatorname{tr}\left(|B|^2 A^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M - m) \left[\frac{\operatorname{tr}\left(|B|^2 [f'(A)]^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 f'(A)\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2} \end{cases} \\ &\leq \frac{1}{4}\left[f'(M) - f'(m)\right](M - m). \end{aligned} \tag{14.20}$$

PROOF. By the gradient inequality we have

$$f(\tau) - f(\varsigma) \leq f'(\tau)(\tau - \varsigma) \tag{14.21}$$

for any $\tau, \varsigma \in [m, M]$.

This inequality implies in the operator order

$$f(A) - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) 1_H \leq f'(A)\left(A - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H\right)$$

that is equivalent to

$$\langle f(A) y, y\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\langle y, y\rangle \leq \langle f'(A) A y, y\rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\langle f'(A) y, y\rangle \tag{14.22}$$

for any $y \in H$, which is of interest in itself as well.

Let $\mathcal{E} := \{e_i\}_{i \in I}$ be an orthonormal basis in H.

If we take in (14.22) $y = Be_i$ and sum, then we get

$$\sum_{i\in I}\langle f(A)Be_i,Be_i\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\sum_{i\in I}\langle Be_i,Be_i\rangle$$
$$\leq \sum_{i\in I}\langle f'(A)ABe_i,Be_i\rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\sum_{i\in I}\langle f'(A)Be_i,Be_i\rangle,$$

which is equivalent to

$$\sum_{i\in I}\langle B^*f(A)Be_i,e_i\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\sum_{i\in I}\langle B^*Be_i,e_i\rangle$$
$$\leq \sum_{i\in I}\langle B^*f'(A)ABe_i,e_i\rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\sum_{i\in I}\langle B^*f'(A)Be_i,e_i\rangle$$

and the inequality (14.18) is obtained.

If we take in (14.22) $y = B^*e_i$ and sum, then we get

$$\sum_{i\in I}\langle f(A)B^*e_i,B^*e_i\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\sum_{i\in I}\langle B^*e_i,B^*e_i\rangle$$
$$\leq \sum_{i\in I}\langle f'(A)AB^*e_i,B^*e_i\rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\sum_{i\in I}\langle f'(A)B^*e_i,B^*e_i\rangle$$

that is equivalent to

$$\sum_{i\in I}\langle Bf(A)B^*e_i,e_i\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\sum_{i\in I}\langle BB^*e_i,e_i\rangle \tag{14.23}$$
$$\leq \sum_{i\in I}\langle Bf'(A)AB^*e_i,e_i\rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\sum_{i\in I}\langle Bf'(A)B^*e_i,e_i\rangle$$

and the inequality (14.19) is obtained.

Now, since f is continuously convex on $[m,M]$, then f' is monotonic nondecreasing on $[m,M]$ and $f'(m)\leq f'(t)\leq f'(M)$ for any $t\in[m,M]$. We also observe that

$$\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left[f'(A)-\frac{f'(m)+f'(M)}{2}1_H\right]\left[A-\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right]\right) \tag{14.24}$$
$$=\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2 f'(A)\left[A-\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right]\right)$$
$$-\frac{f'(m)+f'(M)}{2}\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left[A-\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right]\right)$$
$$=\mathcal{K}(f',B,A).$$

Since

$$\left|f'(A)-\frac{f'(m)+f'(M)}{2}1_H\right|\leq\frac{1}{2}\left[f'(M)-f'(m)\right]1_H,$$

then by taking the modulus in (14.24) and utilizing the properties of trace we have

$$
\begin{aligned}
(14.25)\qquad & 0 \le \mathcal{K}\left(f', B, A\right) \\
& \le \frac{1}{\operatorname{tr}\left(|B|^2\right)} \\
& \times \operatorname{tr}\left(|B|^2\left|\left[f'(A) - \frac{f'(m)+f'(M)}{2}1_H\right]\left[A - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right]\right|\right) \\
& \le \frac{1}{2}\left[f'(M) - f'(m)\right]\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left|A - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right|\right),
\end{aligned}
$$

and the first inequality in the first branch of (14.20) is proved.

We have $m1_H \le A \le M1_H$ and by applying Lemma 14.3 we can state that

$$
\begin{aligned}
(14.26)\qquad & \frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left|A - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right|\right) \\
& \le \left[\frac{\operatorname{tr}\left(|B|^2 A^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2} \le \frac{1}{2}(M-m).
\end{aligned}
$$

Making use of (14.25) and (14.26) we deduce the second and the third inequalities in the first branch of (14.20).

We observe that $\mathcal{K}\left(f', B, A\right)$ can be also represented as

$$
\begin{aligned}
& \mathcal{K}\left(f', B, A\right) \\
& = \frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left[f'(A) - \frac{\operatorname{tr}\left(|B|^2 f'(A)\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right]\left(A - \frac{m+M}{2}1_H\right)\right).
\end{aligned}
$$

Applying a similar argument as above for this representation, we get the second branch of the inequality (14.20).

The proof is complete. ∎

COROLLARY 14.5. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable* *convex function on* $[m,M]$ *and* $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\geq 0$, *then we have*

$$0\leq\frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)}-f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\tag{14.27}$$
$$\leq\frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}$$
$$\leq\begin{cases}\frac{1}{2}\left[f'(M)-f'(m)\right]\frac{\operatorname{tr}\left(P\left|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)}\\ \\ \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(P\left|f'(A)-\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)}\end{cases}$$
$$\leq\begin{cases}\frac{1}{2}\left[f'(M)-f'(m)\right]\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}\\ \\ \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2}\end{cases}$$
$$\leq\frac{1}{4}\left[f'(M)-f'(m)\right](M-m).$$

REMARK 14.2. Let $\mathcal{M}_n(\mathbb{C})$ be the space of all square matrices of order n with complex elements and $A\in\mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $Sp(A)\subseteq[m,M]$ for some scalars m,M with $m<M$. If f is a continuously differentiable convex function on $[m,M]$, then by taking $P=I_n$, the identity matrix, in (14.27) we get

$$0\leq\frac{\operatorname{tr}(f(A))}{n}-f\left(\frac{\operatorname{tr}(A)}{n}\right)\tag{14.28}$$
$$\leq\frac{\operatorname{tr}(f'(A)A)}{n}-\frac{\operatorname{tr}(A)}{n}\cdot\frac{\operatorname{tr}(f'(A))}{n}$$
$$\leq\begin{cases}\frac{1}{2}\left[f'(M)-f'(m)\right]\frac{\operatorname{tr}\left(\left|A-\frac{\operatorname{tr}(A)}{n}I_n\right|\right)}{n}\\ \\ \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(\left|f'(A)-\frac{\operatorname{tr}(f'(A))}{n}I_n\right|\right)}{n}\end{cases}$$
$$\leq\begin{cases}\frac{1}{2}\left[f'(M)-f'(m)\right]\left[\frac{\operatorname{tr}\left(A^2\right)}{n}-\left(\frac{\operatorname{tr}(A)}{n}\right)^2\right]^{1/2}\\ \\ \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left([f'(A)]^2\right)}{n}-\left(\frac{\operatorname{tr}(f'(A))}{n}\right)^2\right]^{1/2}\end{cases}$$
$$\leq\frac{1}{4}\left[f'(M)-f'(m)\right](M-m).$$

14.2. Some Examples. We consider the power function $f:(0,\infty)\to(0,\infty)$, $f(t)=t^r$ with $t\in\mathbb{R}\setminus\{0\}$. For $r\in(-\infty,0)\cup[1,\infty)$, f is convex while for $r\in(0,1)$, f is concave. Denote $\mathcal{B}_1^+(H):=\{P\text{ with }P\in\mathcal{B}_1(H)\text{ and }P\geq 0\}$.

Let $r \geq 1$ and A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 \leq m < M$. If $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^r \\
&\leq r\left[\frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(PA^{r-1})}{\operatorname{tr}(P)}\right] \\
&\leq \begin{cases} \frac{1}{2}r\left(M^{r-1} - m^{r-1}\right)\frac{\operatorname{tr}\left(P\left|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \\ \\ \frac{1}{2}r(M-m)\frac{\operatorname{tr}\left(P\left|A^{r-1} - \frac{\operatorname{tr}\left(PA^{r-1}\right)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \end{cases} \\
&\leq \begin{cases} \frac{1}{2}r\left(M^{r-1} - m^{r-1}\right)\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}r(M-m)\left[\frac{\operatorname{tr}\left(PA^{2(r-1)}\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}\left(PA^{r-1}\right)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4}r\left(M^{r-1} - m^{r-1}\right)(M-m).
\end{aligned} \tag{14.29}
$$

Consider the convex function $f : (0, \infty) \to (0, \infty)$, $f(t) = -\ln t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$
\begin{aligned}
0 &\leq \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)} \\
&\leq \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(PA^{-1})}{\operatorname{tr}(P)} - 1 \\
&\leq \begin{cases} \frac{M-m}{2mM}\frac{\operatorname{tr}\left(P\left|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \\ \\ \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(P\left|A^{-1} - \frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \end{cases} \\
&\leq \begin{cases} \frac{M-m}{2mM}\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(PA^{-2}\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\
&\leq \frac{(M-m)^2}{4mM}.
\end{aligned} \tag{14.30}
$$

Consider the convex function $f(t) = t\ln t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If

$P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$0 \leq \frac{\operatorname{tr}(PA\ln A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \tag{14.31}$$
$$\leq \frac{\operatorname{tr}(PA\ln(eA))}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(P\ln(eA))}{\operatorname{tr}(P)}$$
$$\leq \begin{cases} \frac{1}{2}\ln\left(\frac{M}{m}\right) \frac{\operatorname{tr}\left(P\left|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} 1_H\right|\right)}{\operatorname{tr}(P)} \\ \\ \frac{1}{2}(M-m) \frac{\operatorname{tr}\left(P\left|\ln(eA) - \frac{\operatorname{tr}(P\ln(eA))}{\operatorname{tr}(P)} 1_H\right|\right)}{\operatorname{tr}(P)} \end{cases}$$
$$\leq \begin{cases} \frac{1}{2}\ln\left(\frac{M}{m}\right) \left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m) \left[\frac{\operatorname{tr}\left(P[\ln(eA)]^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(P\ln(eA))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases}$$
$$\leq \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right).$$

15. OTHER REVERSE INEQUALITIES FOR CONVEX FUNCTIONS

15.1. General Results. We recall the *gradient inequality* for the convex function $f : [m, M] \to \mathbb{R}$, namely

$$f(\varsigma) - f(\tau) \geq \delta_f(\tau)(\varsigma - \tau) \tag{15.1}$$

for any $\varsigma, \tau \in [m, M]$ where $\delta_f(\tau) \in \left[f'_-(\tau), f'_+(\tau)\right]$, (for $\tau = m$ we take $\delta_f(\tau) = f'_+(m)$ and for $\tau = M$ we take $\delta_f(\tau) = f'_-(M)$). Here $f'_+(m)$ and $f'_-(M)$ are the lateral derivatives of the convex function f.

The following result holds:

THEOREM 15.1 (Dragomir, 2014, [**53**]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuos convex function on $[m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$ is such that $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \in (m, M)$ then we have*

$$0 \leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \tag{15.2}$$
$$\leq \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{M-m} \Psi_f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}; m, M\right)$$
$$\leq \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{M-m} \sup_{t \in (m,M)} \Psi_f(t; m, M)$$
$$\leq \left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right) \frac{f'_-(M) - f'_+(m)}{M-m}$$
$$\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right],$$

where $\Psi_f(\cdot;m,M):(m,M)\to\mathbb{R}$ *is defined by*

$$\Psi_f(t;m,M)=\frac{f(M)-f(t)}{M-t}-\frac{f(t)-f(m)}{t-m}.$$

We also have

$$
\begin{aligned}
(15.3)\qquad 0 &\le \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)}-f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\\
&\le \frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{M-m}\Psi_f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)};m,M\right)\\
&\le \frac{1}{4}(M-m)\Psi_f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)};m,M\right)\\
&\le \frac{1}{4}(M-m)\sup_{t\in(m,M)}\Psi_f(t;m,M)\\
&\le \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right],
\end{aligned}
$$

for any $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\ge 0$ *such that* $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\in(m,M)$.

PROOF. Since f is convex, then we have

$$f(t)=f\left(\frac{m(M-t)+M(t-m)}{M-m}\right)\le\frac{(M-t)f(m)+(t-m)f(M)}{M-m}$$

for any $t\in[m,M]$.

This scalar inequality implies, by utilizing the spectral representation of continuous functions of selfadjoint operators, the following inequality

$$(15.4)\qquad f(A)\le\frac{f(m)(M1_M-A)+f(M)(A-m1_H)}{M-m}$$

in the operator order of $\mathcal{B}(H)$.

Utilising the properties of the trace and the inequality (15.4), we have

$$
\begin{aligned}
(15.5)\qquad & \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\
&= \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}\left(P\frac{m(M1_H-A)+M(A-1_Hm)}{M-m}\right)}{\operatorname{tr}(P)}\right) \\
&\le \frac{\operatorname{tr}\left(P\frac{f(m)(M1_M-A)+f(M)(A-m1_H)}{M-m}\right)}{\operatorname{tr}(P)} \\
&- f\left(\frac{\operatorname{tr}\left(P\frac{m(M1_H-A)+M(A-1_Hm)}{M-m}\right)}{\operatorname{tr}(P)}\right) \\
&= \frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)f(m)+\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)f(M)}{M-m} \\
&- f\left(\frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)m+\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)M}{M-m}\right) \\
&=: B(f,P,A,m,M)
\end{aligned}
$$

for any $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \ge 0$.

By denoting

$$\Delta_f(t;m,M) := \frac{(t-m)f(M)+(M-t)f(m)}{M-m} - f(t), \quad t \in [m,M]$$

we have

$$
\begin{aligned}
(15.6)\qquad \Delta_f(t;m,M) &= \frac{(t-m)f(M)+(M-t)f(m)-(M-m)f(t)}{M-m} \\
&= \frac{(t-m)f(M)+(M-t)f(m)-(M-t+t-m)f(t)}{M-m} \\
&= \frac{(t-m)[f(M)-f(t)]-(M-t)[f(t)-f(m)]}{M-m} \\
&= \frac{(M-t)(t-m)}{M-m}\Psi_f(t;m,M)
\end{aligned}
$$

for any $t \in (m,M)$.

Therefore

$$(15.7)\qquad B(f,P,A,m,M) = \frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{M-m}\Psi_f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)};m,M\right),$$

provided that $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \in (m,M)$.

If $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \in (m, M)$, then

$$\begin{aligned} (15.8) \qquad & \Psi_f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}; m, M\right) \\ & \leq \sup_{t\in(m,M)} \Psi_f(t; m, M) \\ & = \sup_{t\in(m,M)} \left[\frac{f(M) - f(t)}{M-t} - \frac{f(t) - f(m)}{t-m}\right] \\ & \leq \sup_{t\in(m,M)} \left[\frac{f(M) - f(t)}{M-t}\right] + \sup_{t\in(m,M)} \left[-\frac{f(t) - f(m)}{t-m}\right] \\ & = \sup_{t\in(m,M)} \left[\frac{f(M) - f(t)}{M-t}\right] - \inf_{t\in(m,M)} \left[\frac{f(t) - f(m)}{t-m}\right] \\ & = f'_-(M) - f'_+(m), \end{aligned}$$

which by (15.5) and (15.7) produces the second, third and fourth inequalities in (15.2).

Since, obviously

$$\frac{1}{M-m}\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right) \leq \frac{1}{4}(M-m),$$

then the last part of (15.2) also holds.

The second part of the theorem is clear and the details are omitted. ∎

The following result also holds:

THEOREM 15.2 (Dragomir, 2014, [**53**]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuos convex function on $[m, M]$ then for all $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$ we have that $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \in [m, M]$ and*

$$\begin{aligned} (15.9) \qquad 0 & \leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\ & \leq 2\max\left\{\frac{M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m}, \frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m}{M-m}\right\}\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ & \leq 2\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]. \end{aligned}$$

PROOF. Since $m1_H \leq A \leq M1_H$, it follows that $m\operatorname{tr}(P) \leq \operatorname{tr}(PA) \leq M\operatorname{tr}(P)$ for any $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$, which shows that $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \in [m, M]$.

Further on, we recall the following result (see for instance [**37**]) that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} (15.10) \qquad & n \min_{i\in\{1,\dots,n\}} \{p_i\}\left[\frac{1}{n}\sum_{i=1}^n f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right] \\ & \leq \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \\ & \leq n \max_{i\in\{1,\dots,n\}} \{p_i\}\left[\frac{1}{n}\sum_{i=1}^n f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right], \end{aligned}$$

where $f : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}} \subset C$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^n p_i > 0$.

For $n = 2$ we deduce from (15.10) that

$$
\begin{aligned}
(15.11)\qquad & 2\min\{t,1-t\}\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right] \\
&\le tf(x) + (1-t)f(y) - f(tx+(1-t)y) \\
&\le 2\max\{t,1-t\}\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right]
\end{aligned}
$$

for any $x, y \in C$ and $t \in [0,1]$.

If we use the second inequality in (15.11) for the convex function $f : I \to \mathbb{R}$ where m, $M \in \mathbb{R}$, $m < M$ with $[m, M] = I$, we have for $x = m$, $y = M$ and $t = \frac{M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m}$ that

$$
\begin{aligned}
B(f,P,A,m,M) &= \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) f(m) + \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right) f(M)}{M-m} \\
&\quad - f\left(\frac{m\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) + M\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{M-m}\right) \\
&\le 2\max\left\{\frac{M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m}, \frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m}{M-m}\right\} \\
&\quad \times \left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right].
\end{aligned}
$$

Making use of (15.5) we deduce the first inequality in (15.9).

Since

$$
\max\left\{\frac{M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m}, \frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m}{M-m}\right\} \le 1,
$$

the last part of (15.9) is also proved. ∎

15.2. Some Examples. For $p > 1$ and $0 < m < M < \infty$ consider the convex function $f(t) = t^p$ defined on $[m, M]$. Then $\Psi_p(\cdot; m, M) : (m, M) \to \mathbb{R}$ is defined by

$$
\begin{aligned}
\Psi_p(t; m, M) &= \frac{M^p - t^p}{M-t} - \frac{t^p - m^p}{t-m} \\
&= \frac{t(M^p - m^p) - t^p(M-m) - mM(M^{p-1} - m^{p-1})}{(M-t)(t-m)}.
\end{aligned}
$$

Let A be a nonnegative selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $0 \le m < M$. If $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \ge 0$ such that

$\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \in (m, M)$, then we have from (15.2) that

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p \\
&\leq \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{M - m} \Psi_p\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}; m, M\right) \\
&\leq \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{M - m} \sup_{t \in (m,M)} \Psi_p(t; m, M) \\
&\leq \left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right) \frac{M^p - m^p}{M - m} \\
&\leq \frac{1}{4} p (M - m)\left(M^{p-1} - m^{p-1}\right)
\end{aligned} \tag{15.12}
$$

and from (15.3) that

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p \\
&\leq \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{M - m} \Psi_p\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}; m, M\right) \\
&\leq \frac{1}{4}(M - m) \Psi_p\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}; m, M\right) \\
&\leq \frac{1}{4}(M - m) \sup_{t \in (m,M)} \Psi_p(t; m, M) \\
&\leq \frac{1}{4} p (M - m)\left(M^{p-1} - m^{p-1}\right).
\end{aligned} \tag{15.13}
$$

For $p = 2$, we have

$$
\Psi_2(t; m, M) = \frac{M^2 - t^2}{M - t} - \frac{t^2 - m^2}{t - m} = M - m
$$

and by (15.12) we get

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2 \leq \left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right) \\
&\leq \frac{1}{4}(M - m)^2
\end{aligned} \tag{15.14}
$$

for any $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$.

Making use of the inequality (15.9) we have

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p \\
&\leq 2 \max\left\{\frac{M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M - m}, \frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m}{M - m}\right\}\left[\frac{m^p + M^p}{2} - \left(\frac{m + M}{2}\right)^p\right] \\
&\leq 2\left[\frac{m^p + M^p}{2} - \left(\frac{m + M}{2}\right)^p\right],
\end{aligned} \tag{15.15}
$$

for any positive operator A with $\operatorname{Sp}(A) \subseteq [m, M]$ and for any $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$.

In particular, for $p = 2$ we get

$$0 \le \frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2 \le \frac{1}{2}(M-m)\max\left\{M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}, \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right\} \le \frac{1}{2}(M-m)^2. \tag{15.16}$$

Since

$$\max\left\{M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}, \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right\} = \frac{1}{2}(M-m) + \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - \frac{1}{2}(m+M)\right|,$$

then the second inequality in (15.16) is not as good as the second inequality in (15.14).

For $p = -1$ and $0 < m < M < \infty$ consider the convex function $f(t) = t^{-1}$ defined on $[m, M]$. Then $\Psi_{-1}(\cdot; m, M) : (m, M) \to \mathbb{R}$ is defined by

$$\Psi_{-1}(t; m, M) = \frac{M^{-1} - t^{-1}}{M - t} - \frac{t^{-1} - m^{-1}}{t - m} = \frac{M - m}{mMt}.$$

The definition of $\Psi_{-1}(\cdot; m, M)$ can be extended to the closed interval $[m, M]$. We also have that

$$\sup_{t\in(m,M)} \Psi_{-1}(t; m, M) = \frac{M - m}{m^2 M}.$$

From the inequality (15.2) we get

$$0 \le \frac{\operatorname{tr}(PA^{-1})}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)} \le \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{mM}\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)} \le \frac{1}{m^2 M}\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right) \le \frac{1}{4}\frac{(M-m)^2(M+m)}{m^2M^2}, \tag{15.17}$$

while from (15.3) we get

$$0 \le \frac{\operatorname{tr}(PA^{-1})}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)} \le \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{mM}\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)} \le \frac{1}{4}\frac{(M-m)^2}{mM}\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)} \le \frac{1}{4}\frac{(M-m)^2}{m^2 M}. \tag{15.18}$$

for any positive definite operator A with $\operatorname{Sp}(A) \subseteq [m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \ge 0$.

From the inequality (15.9) we have

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(PA^{-1})}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)} \\
&\leq \frac{(M-m)^2}{mM(m+M)} \max\left\{ \frac{M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m}, \frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m}{M-m} \right\} \\
&\leq \frac{(M-m)^2}{mM(m+M)},
\end{aligned} \tag{15.19}
$$

for any positive definite operator A with $\operatorname{Sp}(A) \subseteq [m, M]$ and any $P \in \mathcal{B}_1(H) \setminus \{0\}, P \geq 0.$

In order to compare the upper bounds provided by (15.18) and (15.19) consider the difference

$$
\begin{aligned}
\Delta(m, M) &:= \frac{1}{4}\frac{(M-m)^2}{m^2 M} - \frac{(M-m)^2}{mM(m+M)} \\
&= \frac{(M-m)^2}{mM}\left(\frac{1}{4m} - \frac{1}{m+M}\right) = \frac{(M-m)^2(M-3m)}{4m^2 M(m+M)},
\end{aligned}
$$

where $0 < m < M.$

We observe that if $M < 3m$, then the upper bound provided by (15.18) is better than the bound provided by (15.19). The conclusion is the other way around if $M \geq 3m.$

If we consider the convex function $f(t) = -\ln t$ defined on $[m, M] \subset (0, \infty)$, then $\Psi_{-\ln}(\cdot; m, M) : (m, M) \to \mathbb{R}$ is defined by

$$
\begin{aligned}
\Psi_{-\ln}(t; m, M) &= \frac{-\ln M + \ln t}{M-t} - \frac{-\ln t + \ln m}{t-m} \\
&= \frac{(M-m)\ln t - (M-t)\ln m - (t-m)\ln M}{(M-t)(t-m)} \\
&= \ln\left(\frac{t^{M-m}}{m^{M-t}M^{t-m}}\right)^{\frac{1}{(M-t)(t-m)}}.
\end{aligned}
$$

Utilising the inequality (15.2) we have

$$
\begin{aligned}
0 &\leq \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)} \\
&\leq \frac{1}{M-m}\ln\left(\frac{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{M-m}}{m^{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}} M^{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}}\right) \\
&\leq \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{M-m} \sup_{t\in(m,M)} \Psi_{-\ln}(t; m, M) \\
&\leq \frac{1}{Mm}\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right) \leq \frac{(M-m)^2}{4mM},
\end{aligned} \tag{15.20}
$$

for any positive definite operator A with $\operatorname{Sp}(A) \subseteq [m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}, P \geq 0.$

From (15.3) we have

$$
\begin{aligned}
0 &\leq \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)} \\
&\leq \frac{1}{M-m}\ln\left(\frac{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{M-m}}{m^{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}M^{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}}\right) \\
&\leq \frac{1}{4}\frac{(M-m)}{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}\ln\left(\frac{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{M-m}}{m^{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}M^{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}}\right) \\
&\leq \frac{1}{4}(M-m)\sup_{t\in(m,M)}\Psi_{-\ln}(t;m,M) \\
&\leq \frac{(M-m)^2}{4mM},
\end{aligned}
\tag{15.21}
$$

for any positive definite operator A with $\operatorname{Sp}(A) \subseteq [m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$.

From the inequality (15.9) we get

$$
\begin{aligned}
0 &\leq \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)} \\
&\leq \max\left\{\frac{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m}, \frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}{M-m}\right\}\ln\left(\frac{\left(\frac{m+M}{2}\right)^2}{mM}\right) \\
&\leq \ln\left(\frac{\left(\frac{m+M}{2}\right)^2}{mM}\right),
\end{aligned}
\tag{15.22}
$$

for any positive definite operator A with $\operatorname{Sp}(A) \subseteq [m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$.

We observe that, since $\ln x \leq x - 1$ for any $x > 0$, then

$$
\ln\left(\frac{\left(\frac{m+M}{2}\right)^2}{mM}\right) \leq \frac{\left(\frac{m+M}{2}\right)^2}{mM} - 1 = \frac{(M-m)^2}{4mM},
$$

which shows that the absolute upper bound for

$$
\ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)}
$$

provided by the inequality (15.22) is better than the one provided by (15.21).

15.3. Reverses of Hölder's Inequality. We have the following result:

THEOREM 15.3 (Dragomir, 2014, [**53**]). *Assume that* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* S *be a positive operator that commutes with* Q, *a positive definite operator and such that there exists the constants* $k, K > 0$ *with*

$$
k1_H \leq SQ^{1-q} \leq K1_H. \tag{15.23}
$$

If $S^p, Q^q \in \mathcal{B}_1(H)$, *then we have*

$$
0 \leq [\operatorname{tr}(S^p)]^{1/p}[\operatorname{tr}(Q^q)]^{1/q} - \operatorname{tr}(SQ) \leq B_p(k, K)\operatorname{tr}(Q^q), \tag{15.24}
$$

where

$$B_p(k,K) = \begin{cases} \frac{1}{4^{1/p}} p^{1/p} (K-k)^{1/p} \left(K^{p-1} - k^{p-1}\right)^{1/p} \\ \\ 2^{1/p} \left[\frac{k^p + K^p}{2} - \left(\frac{k+K}{2}\right)^p\right]^{1/p}. \end{cases} \tag{15.25}$$

PROOF. If we write the inequality

$$0 \le \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p \le \frac{1}{4} p (M-m) \left(M^{p-1} - m^{p-1}\right) \tag{15.26}$$

for the operators $P = Q^q$ and $A = SQ^{1-q}$ then we get

$$\begin{aligned} 0 &\le \frac{\operatorname{tr}\left(Q^q \left(SQ^{1-q}\right)^p\right)}{\operatorname{tr}(Q^q)} - \left(\frac{\operatorname{tr}\left(Q^q SQ^{1-q}\right)}{\operatorname{tr}(Q^q)}\right)^p \\ &\le \frac{1}{4} p (K-k) \left(K^{p-1} - k^{p-1}\right). \end{aligned} \tag{15.27}$$

Observe that, by the properties of trace we have

$$\operatorname{tr}\left(Q^q S Q^{1-q}\right) = \operatorname{tr}\left(SQ^{1-q} Q^q\right) = \operatorname{tr}(SQ).$$

It is known, see for instance [**121**, p. 356-358], that if A and B are two *commuting bounded selfadjoint operators* on the complex Hilbert space H, then there exists a bounded selfadjoint operator T on H and two bounded functions φ and ψ such that $A = \varphi(T)$ and $B = \psi(T)$. Moreover, if $\{E_\lambda\}$ is the spectral family over the closed interval $[0,1]$ for the selfadjoint operator T, then $T = \int_{0-}^{1} \lambda dE_\lambda$, where the integral is taken in the Riemann-Stieltjes sense, the functions φ and ψ are summable with respect with $\{E_\lambda\}$ on $[0,1]$ and

$$A = \varphi(T) = \int_{0-}^{1} \varphi(\lambda)\, dE_\lambda \text{ and } B = \psi(T) = \int_{0-}^{1} \psi(\lambda)\, dE_\lambda.$$

Now, if A and B are as above with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq J$ an interval of real numbers, then for any continuous functions $f, g : J \to \mathbb{C}$ we have the representations

$$f(A) = \int_{0-}^{1} (f \circ \varphi)(\lambda)\, dE_\lambda \text{ and } g(B) = \int_{0-}^{1} (g \circ \psi)(\lambda)\, dE_\lambda.$$

If we apply the above property to the commuting selfadjoint operators S and Q, then we have two positive functions φ and ψ such that $S = \varphi(T)$ and $Q = \psi(T)$. Moreover, using the integral representation for functions of selfadjoint operators, we have

$$\begin{aligned} Q^q \left(SQ^{1-q}\right)^p &= [\psi(T)]^q \left(\varphi(T) [\psi(T)]^{1-q}\right)^p \\ &= \int_{0-}^{1} [\psi(\lambda)]^q \left(\varphi(\lambda) [\psi(\lambda)]^{1-q}\right)^p dE_\lambda \\ &= \int_{0-}^{1} [\psi(\lambda)]^q [\varphi(\lambda)]^p [\psi(\lambda)]^{(1-q)p}\, dE_\lambda \\ &= \int_{0-}^{1} [\varphi(\lambda)]^p [\psi(\lambda)]^{q+p-qp}\, dE_\lambda = \int_{0-}^{1} [\varphi(\lambda)]^p\, dE_\lambda = S^p. \end{aligned}$$

Therefore, the inequality (15.27) is equivalent to

$$0 \le \frac{\operatorname{tr}(S^p)}{\operatorname{tr}(Q^q)} - \left(\frac{\operatorname{tr}(SQ)}{\operatorname{tr}(Q^q)}\right)^p \le \frac{1}{4} p (K-k) \left(K^{p-1} - k^{p-1}\right), \tag{15.28}$$

which is of interest in itself.

From this inequality we have

$$\operatorname{tr}\left(S^{p}\right)\left[\operatorname{tr}\left(Q^{q}\right)\right]^{p-1} \leq(\operatorname{tr}(S Q))^{p}+\frac{1}{4} p(K-k)\left(K^{p-1}-k^{p-1}\right)\left[\operatorname{tr}\left(Q^{q}\right)\right]^{p} .$$

Taking the power $1/p \in (0,1)$ and using the property that

$$(\alpha+\beta)^{r} \leq \alpha^{r}+\beta^{r}, \text{ where } \alpha, \beta \geq 0 \text{ and } r \in(0,1),$$

we get

$$\begin{aligned}
&\left[\operatorname{tr}\left(S^{p}\right)\right]^{1 / p}\left[\operatorname{tr}\left(Q^{q}\right)\right]^{(p-1) / p} \\
&\leq\left[(\operatorname{tr}(S Q))^{p}+\frac{1}{4} p(K-k)\left(K^{p-1}-k^{p-1}\right)\left[\operatorname{tr}\left(Q^{q}\right)\right]^{p}\right]^{1 / p} \\
&\leq \operatorname{tr}(S Q)+\frac{1}{4^{1 / p}} p^{1 / p}(K-k)^{1 / p}\left(K^{p-1}-k^{p-1}\right)^{1 / p}\left[\operatorname{tr}\left(Q^{q}\right)\right],
\end{aligned}$$

i.e.

$$\begin{aligned}
&\left[\operatorname{tr}\left(S^{p}\right)\right]^{1 / p}\left[\operatorname{tr}\left(Q^{q}\right)\right]^{1 / q}-\operatorname{tr}(S Q) \\
&\leq \frac{1}{4^{1 / p}} p^{1 / p}(K-k)^{1 / p}\left(K^{p-1}-k^{p-1}\right)^{1 / p}\left[\operatorname{tr}\left(Q^{q}\right)\right]
\end{aligned}$$

The second part follows from the inequality

$$0 \leq \frac{\operatorname{tr}\left(P A^{p}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(P A)}{\operatorname{tr}(P)}\right)^{p} \leq 2\left[\frac{m^{p}+M^{p}}{2}-\left(\frac{m+M}{2}\right)^{p}\right],$$

and the details are omitted. ∎

REMARK 15.1. We observe that under the previous assumptions, from any upper bound for the difference

$$0 \leq \frac{\operatorname{tr}\left(P A^{p}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(P A)}{\operatorname{tr}(P)}\right)^{p}$$

we can deduce in a similar way an upper bound for the Hölder's difference

$$0 \leq\left[\operatorname{tr}\left(S^{p}\right)\right]^{1 / p}\left[\operatorname{tr}\left(Q^{q}\right)\right]^{1 / q}-\operatorname{tr}(S Q) .$$

Also, if the commutativity property of the operators S and Q is dropped, then we can prove that

$$0 \leq\left[\operatorname{tr}\left(Q^{q}\left(S Q^{1-q}\right)^{p}\right)\right]^{1 / p}\left[\operatorname{tr}\left(Q^{q}\right)\right]^{1 / q}-\operatorname{tr}(S Q) \leq B_{p}(k, K) \operatorname{tr}\left(Q^{q}\right) \tag{15.29}$$

with the same $B_p(k,K)$. However, the noncommutative case of the second inequality in (15.24) is an open question for the author.

The following reverse of Schwarz inequality holds:

COROLLARY 15.4. *Let S be a positive operator that commutes with Q, a positive definite operator and such that there exists the constants $k,\ K>0$ with*

$$k 1_{H} \leq S Q^{-1} \leq K 1_{H}. \tag{15.30}$$

If $S^2, Q^2 \in \mathcal{B}_1(H)$, then we have

$$0 \leq\left[\operatorname{tr}\left(S^{2}\right)\right]^{1 / 2}\left[\operatorname{tr}\left(Q^{2}\right)\right]^{1 / 2}-\operatorname{tr}(S Q) \leq \frac{\sqrt{2}}{2}(K-k) \operatorname{tr}\left(Q^{2}\right) . \tag{15.31}$$

REMARK 15.2. If we take $p = q = 2$ in (15.29) and drop the commutativity assumption, then we get

$$0 \leq \left[\operatorname{tr}\left(QSQ^{-1}S\right)\right]^{1/2}\left[\operatorname{tr}\left(Q^2\right)\right]^{1/2} - \operatorname{tr}(SQ) \leq \frac{\sqrt{2}}{2}(K-k)\operatorname{tr}\left(Q^2\right),$$

provided that (15.30) holds true.

Also, if we use the inequality (15.14), then we have

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(QSQ^{-1}S\right)\operatorname{tr}\left(Q^2\right) - \left[\operatorname{tr}(SQ)\right]^2 \\ &\leq \left(K\operatorname{tr}\left(Q^2\right) - \operatorname{tr}(SQ)\right)\left(\operatorname{tr}(SQ) - k\operatorname{tr}\left(Q^2\right)\right) \leq \frac{1}{4}(K-k)^2\left[\operatorname{tr}\left(Q^2\right)\right]^2 \end{aligned} \tag{15.32}$$

provided that (15.30) holds true.

16. SLATER'S TYPE INEQUALITIES

16.1. Some Preliminary Facts. Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \geq f(a) + (x-a)\varphi(a) \text{ for any } x, a \in I.$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \leq \varphi(x) \leq f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

The following result is well known in the literature as *Slater inequality:*

THEOREM 16.1 (Slater, 1981, [**127**]). *If $f : I \to \mathbb{R}$ is a nonincreasing (nondecreasing) convex function, $x_i \in I$, $p_i \geq 0$ with $P_n := \sum_{i=1}^n p_i > 0$ and $\sum_{i=1}^n p_i\varphi(x_i) \neq 0$, where $\varphi \in \partial f$, then*

$$\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) \leq f\left(\frac{\sum_{i=1}^n p_i x_i \varphi(x_i)}{\sum_{i=1}^n p_i \varphi(x_i)}\right). \tag{16.1}$$

As pointed out in [**35**, p. 208], the monotonicity assumption for the derivative φ can be replaced with the condition

$$\frac{\sum_{i=1}^n p_i x_i \varphi(x_i)}{\sum_{i=1}^n p_i \varphi(x_i)} \in I, \tag{16.2}$$

which is more general and can hold for suitable points in I and for not necessarily monotonic functions.

The following result that provides a reverse of the Jensen inequality has been obtained in [**39**]:

THEOREM 16.2 (Dragomir, 2008, [**39**]). *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$, then*

$$(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle \langle f'(A)x, x\rangle, \tag{16.3}$$

for any $x \in H$ with $\|x\| = 1$.

Perhaps more convenient reverses of (16.3) are the following inequalities that have been obtained in the same paper [**39**]:

THEOREM 16.3 (Dragomir, 2008, [**39**]). *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$, then*

$$\begin{aligned}(0 \leq) &\langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \\ &\leq \begin{cases} \frac{1}{2}(M-m)\left[\|f'(A)x\|^2 - \langle f'(A)x, x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}(f'(M) - f'(m))\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \end{cases} \\ &\leq \frac{1}{4}(M-m)(f'(M) - f'(m)),\end{aligned} \tag{16.4}$$

for any $x \in H$ with $\|x\| = 1$.

We also have the inequality

$$\begin{aligned}(0 \leq) &\langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \\ &\leq \frac{1}{4}(M-m)(f'(M) - f'(m)) \\ &- \begin{cases} [\langle Mx - Ax, Ax - mx\rangle \langle f'(M)x - f'(A)x, f'(A)x - f'(m)x\rangle]^{\frac{1}{2}}, \\ \\ \left|\langle Ax, x\rangle - \frac{M+m}{2}\right| \left|\langle f'(A)x, x\rangle - \frac{f'(M)+f'(m)}{2}\right| \end{cases} \\ &\leq \frac{1}{4}(M-m)(f'(M) - f'(m)),\end{aligned} \tag{16.5}$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, if $m > 0$ and $f'(m) > 0$, then we also have

$$\begin{aligned}(0 \leq) &\langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \\ &\leq \begin{cases} \frac{1}{4}\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}} \langle Ax, x\rangle \langle f'(A)x, x\rangle, \\ \\ \left(\sqrt{M} - \sqrt{m}\right)\left(\sqrt{f'(M)} - \sqrt{f'(m)}\right)[\langle Ax, x\rangle \langle f'(A)x, x\rangle]^{\frac{1}{2}}, \end{cases}\end{aligned} \tag{16.6}$$

for any $x \in H$ with $\|x\| = 1$.

In [**41**] we obtained the following operator version for Slater's inequality as well as a reverse of it:

THEOREM 16.4 (Dragomir, 2008, [**41**]). *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A*

is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(A)$ is a positive invertible operator on H then

$$0 \leq f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) - \langle f(A)x, x\rangle \leq f'\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right)\left[\frac{\langle Af'(A)x, x\rangle - \langle Ax, x\rangle\langle f'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right], \tag{16.7}$$

for any $x \in H$ with $\|x\| = 1$.

For other similar results, see [**41**].

16.2. Slater Type Trace Inequalities. We denote by

$$\mathcal{B}_1^+(H) := \{P : P \in \mathcal{B}_1(H) \text{ and } P \geq 0\}.$$

The following result holds:

THEOREM 16.5. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(A)$ is a positive invertible operator on H, then*

$$0 \leq f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \leq f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right), \tag{16.8}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

PROOF. Since f is convex and differentiable on $\mathring{I}$, then we have

$$f'(s)(t-s) \leq f(t) - f(s) \leq f'(t)(t-s) \tag{16.9}$$

for any $t, s \in [m, M]$.

Now, if we fix $t \in [m, M]$, then we have

$$tf'(A) - Af'(A) \leq f(t) \cdot 1_H - f(A) \leq f'(t)\, t \cdot 1_H - f'(t) A \tag{16.10}$$

for any $t \in [m, M]$.

Using the inequality (16.10), then we have

$$t \operatorname{tr}[Pf'(A)] - \operatorname{tr}[PAf'(A)] \leq f(t) \operatorname{tr}(P) - \operatorname{tr}[Pf(A)] \leq f'(t)\, t \operatorname{tr}(P) - f'(t) \operatorname{tr}(PA) \tag{16.11}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Now, since A is selfadjoint with $m1_H \leq A \leq M1_H$ and $f'(A)$ is positive, then

$$mf'(A) \leq Af'(A) \leq Mf'(A).$$

By taking the trace, then we get

$$m \operatorname{tr}[Pf'(A)] \leq \operatorname{tr}[PAf'(A)] \leq M \operatorname{tr}[Pf'(A)],$$

which shows that

$$t_0 := \frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]} \in [m, M].$$

Observe that since $f'(A)$ is a positive invertible operator on H, then $\operatorname{tr}[Pf'(A)] > 0$ for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Finally, if we put $t = t_0$ in the equation (16.11), then we get

$$\begin{aligned} (16.12) \qquad & \frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]} \operatorname{tr}[Pf'(A)] - \operatorname{tr}[PAf'(A)] \\ & \le f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \operatorname{tr}(P) - \operatorname{tr}[Pf(A)] \\ & \le f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]} \operatorname{tr}(P) \\ & - f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \operatorname{tr}(PA), \end{aligned}$$

which is equivalent to the desired result (16.8). ∎

REMARK 16.1. It is important to observe that, the condition that $f'(A)$ is a positive invertible operator on H can be replaced with the more general assumption that

$$(16.13) \qquad \frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]} \in \mathring{I} \text{ and } \operatorname{tr}[Pf'(A)] \neq 0$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, which may be easily verified for particular convex functions f in various examples as follows.

Also, as pointed out by the referee, if $\langle f'(A)x, x\rangle > 0$ for any $x \in H$, $x \neq 0$, then $\operatorname{tr}[Pf'(A)] > 0$ for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ and the inequality (16.8) is valid as well.

REMARK 16.2. Now, if the function is concave on $\mathring{I}$ and the condition (16.13) holds, then we have the inequalities

$$\begin{aligned} (16.14) \qquad 0 & \le \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \\ & \le f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right), \end{aligned}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Utilising the inequality (16.14) for the concave function $f : (0, \infty) \to \mathbb{R}$, $f(t) = \ln t$, then we can state that

$$(16.15) \qquad 0 \le \frac{\operatorname{tr}(P \ln A)}{\operatorname{tr}(P)} - \ln\left(\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA^{-1})}\right) \le \frac{\operatorname{tr}(PA^{-1})}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - 1$$

for any positive invertible operator A and P with $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Utilising the inequality (16.8) for the convex function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t^{-1}$, then we can state that

$$(16.16) \qquad 0 \le \frac{\operatorname{tr}(PA^{-2})}{\operatorname{tr}(PA^{-1})} - \frac{\operatorname{tr}(PA^{-1})}{\operatorname{tr}(P)} \le \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PA^{-2})}{\operatorname{tr}(PA^{-1})} - \frac{\operatorname{tr}(PA^{-1})}{\operatorname{tr}(PA^{-2})},$$

for any positive invertible operator $A$and P with $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

If we take $B = A^{-1}$ in (16.16), then we get the equivalent inequality

$$(16.17) \qquad 0 \le \frac{\operatorname{tr}(PB^2)}{\operatorname{tr}(PB)} - \frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)} \le \frac{\operatorname{tr}(PB^2)}{\operatorname{tr}(PB)} \frac{\operatorname{tr}(PB^{-1})}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PB)}{\operatorname{tr}(PB^2)},$$

for any positive invertible operator B and P with $P \in \mathcal{B}_1(H) \setminus \{0\}$.

If we write the inequality (16.8) for the convex function $f(t)=\exp(\alpha t)$ with $\alpha\in\mathbb{R}\setminus\{0\}$, then we get

$$0\leq\exp\left(\alpha\frac{\operatorname{tr}[PA\exp(\alpha A)]}{\operatorname{tr}[P\exp(\alpha A)]}\right)-\frac{\operatorname{tr}[P\exp(\alpha A)]}{\operatorname{tr}(P)}\leq\alpha\exp\left(\alpha\frac{\operatorname{tr}[PA\exp(\alpha A)]}{\operatorname{tr}[P\exp(\alpha A)]}\right)\left(\frac{\operatorname{tr}[PA\exp(\alpha A)]}{\operatorname{tr}[P\exp(\alpha A)]}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right),\tag{16.18}$$

for any selfadjoint operator A and $P\in\mathcal{B}_1^+(H)\setminus\{0\}$.

16.3. Further Reverses. We use the following Grüss' type inequalities [**51**]:

LEMMA 16.6. *Let S be a selfadjoint operator with $m1_H\leq S\leq M1_H$ and $f:[m,M]\to\mathbb{C}$ a continuous function of bounded variation on $[m,M]$. For any $C\in\mathcal{B}(H)$ and $P\in\mathcal{B}_1^+(H)\setminus\{0\}$ we have the inequality*

$$\begin{aligned}&\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\\&\leq\frac{1}{2}\bigvee_m^M(f)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\&\leq\frac{1}{2}\bigvee_m^M(f)\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2},\end{aligned}\tag{16.19}$$

where $\bigvee_m^M(f)$ is the total variation of f on the interval.

If the function $f:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[m,M]$, i.e.

$$|f(t)-f(s)|\leq L|t-s|$$

for any $t,s\in[m,M]$, then

$$\begin{aligned}&\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\\&\leq L\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\&\leq L\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}\end{aligned}\tag{16.20}$$

for any $C\in\mathcal{B}(H)$ and $P\in\mathcal{B}_1^+(H)\setminus\{0\}$.

PROOF. For the sake of completeness we give here a simple proof.

We observe that, for any $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}\frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left[P\left(A-\lambda 1_H\right)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right)\right] & \\ =\frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left[PA\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right)\right] & \\ -\frac{\lambda}{\operatorname{tr}(P)} \operatorname{tr}\left[P\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right)\right] & \\ =\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} . &\end{aligned} \tag{16.21}$$

Taking the modulus in (16.21) and utilising the properties of the trace, we have

$$\begin{aligned}\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| & \\ =\frac{1}{\operatorname{tr}(P)}\left|\operatorname{tr}\left[P\left(A-\lambda 1_H\right)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right)\right]\right| & \\ =\frac{1}{\operatorname{tr}(P)}\left|\operatorname{tr}\left[\left(A-\lambda 1_H\right)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right]\right| & \\ \leq\left\|A-\lambda 1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) &\end{aligned} \tag{16.22}$$

for any $\lambda \in \mathbb{C}$.

From the inequality (16.22) we have

$$\begin{aligned}\left|\frac{\operatorname{tr}(Pf(S) C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| & \\ \leq\left\|f(S)-\lambda 1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) &\end{aligned} \tag{16.23}$$

for any $\lambda \in \mathbb{C}$.

From (16.23) we get

$$\begin{aligned}\left|\frac{\operatorname{tr}(Pf(S) C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| & \\ \leq\left\|f(S)-\frac{f(m)+f(M)}{2} 1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) . &\end{aligned} \tag{16.24}$$

Since f is of bounded variation on $[m, M]$, then we have

$$\begin{aligned}\left|f(t)-\frac{f(m)+f(M)}{2}\right| & =\left|\frac{f(t)-f(m)+f(t)-f(M)}{2}\right| \\ & \leq \frac{1}{2}[|f(t)-f(m)|+|f(M)-f(t)|] \leq \frac{1}{2} \bigvee_m^M(f)\end{aligned} \tag{16.25}$$

for any $t \in [m, M]$.

From (16.25) we get in the order $\mathcal{B}(H)$ that

$$\left|f(S)-\frac{f(m)+f(M)}{2} 1_H\right| \leq \frac{1}{2} \bigvee_m^M(f) 1_H,$$

which implies that

$$\left\|f(S)-\frac{f(m)+f(M)}{2}1_H\right\| \le \frac{1}{2}\bigvee_m^M (f). \tag{16.26}$$

Making use of (16.25) and (16.26) we get the first inequality in (16.19).

The second part is obvious by the Schwarz inequality for traces

$$\frac{\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)}{\operatorname{tr}(P)} \le \left(\frac{\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right|^2\right)}{\operatorname{tr}(P)}\right)^{1/2},$$

and by noticing that

$$\frac{\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right|^2\right)}{\operatorname{tr}(P)} = \frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2 \tag{16.27}$$

for any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

From (16.23) we also have

$$\begin{aligned} &\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\le \left\|f(S)-f\left(\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}\right)1_H\right\| \frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \end{aligned} \tag{16.28}$$

any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Since

$$|f(t)-f(s)| \le L|t-s|$$

for any $t, s \in [m, M]$, then we have in the order $\mathcal{B}(H)$ that

$$|f(S)-f(s)1_H| \le L|S-s1_H|$$

for any $s \in [m, M]$. In particular, we have

$$\left|f(S)-f\left(\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}\right)1_H\right| \le L\left|S-\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}1_H\right|,$$

which implies that

$$\left\|f(S)-f\left(\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}\right)1_H\right\| \le L\left\|S-\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}1_H\right\|$$

and by (16.28) we get the first inequality in (16.20).

The second part is obvious. ∎

We also have the following reverse of Schwarz inequality [**51**]:

LEMMA 16.7. *If C is a selfadjoint operator with $k1_H \leq C \leq K1_H$ for some real numbers $k < K$, then*

$$0 \leq \frac{\operatorname{tr}(PC^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2 \leq \frac{1}{2}(K-k)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \leq \frac{1}{2}(K-k)\left[\frac{\operatorname{tr}(PC^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \leq \frac{1}{4}(K-k)^2, \tag{16.29}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

PROOF. If we take in (16.19) $f(t) = t$ and $S = C$ we get

$$\left|\frac{\operatorname{tr}(PC^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right| \leq \frac{1}{2}(K-k)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \leq \frac{1}{2}(K-k)\left[\frac{\operatorname{tr}(PC^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}. \tag{16.30}$$

Since by (16.27) we have

$$\frac{\operatorname{tr}(PC^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2 \geq 0,$$

then by (16.30) we get

$$0 \leq \frac{\operatorname{tr}(PC^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2 \leq \frac{1}{2}(K-k)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \leq \frac{1}{2}(K-k)\left[\frac{\operatorname{tr}(PC^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}. \tag{16.31}$$

Utilising the inequality between the first and last term in (16.31) we also have

$$\left[\frac{\operatorname{tr}(PC^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \leq \frac{1}{2}(K-k),$$

which proves the last part of (16.29). ∎

THEOREM 16.8. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(A)$ is a positive invertible operator on H, or*

$$\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]} \in \mathring{I}, \ \operatorname{tr}[Pf'(A)] \neq 0$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, *then*

$$0 \leq f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \leq \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) L(P, A, f'(A)), \tag{16.32}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, *where*

$$\begin{aligned} L(P, A, f'(A)) &:= \frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}[Pf'(A)]}{\operatorname{tr}(P)} \\ &\leq \begin{cases} \frac{1}{2}(f'(M) - f'(m)) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \\ \frac{1}{2}(M - m) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(f'(A) - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \end{cases} \\ &\leq \begin{cases} \frac{1}{2}(f'(M) - f'(m)) \left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \frac{1}{2}(M - m) \left[\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\ &\leq \frac{1}{4}(f'(M) - f'(m))(M - m). \end{aligned}$$

PROOF. Utilising Lemma 16.6 and Lemma 16.7 we have

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \\ &\leq \frac{1}{2}(f'(M) - f'(m)) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \\ &\leq \frac{1}{2}(f'(M) - f'(m)) \left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ &\leq \frac{1}{4}(f'(M) - f'(m))(M - m) \end{aligned} \tag{16.33}$$

and

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \\ &\leq \frac{1}{2}(M - m) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(f'(A) - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \\ &\leq \frac{1}{2}(M - m) \left[\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ &\leq \frac{1}{4}(f'(M) - f'(m))(M - m) \end{aligned} \tag{16.34}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

The positivity of

$$\frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}$$

follows by Čebyšev's trace inequality for synchronous functions of selfadjoint operators, see [**49**]. ∎

The case of convex and monotonic functions is as follows:

COROLLARY 16.9. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(m) > 0$, then*

$$0 \le f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \le \frac{f'(M)}{f'(m)} L(P, A, f'(A)), \tag{16.35}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

The proof follows by (16.32) observing that

$$0 \le \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \le \frac{f'(M)}{f'(m)}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

If we consider the monotonic nondecreasing convex function $f(t) = t^p$ with $p \ge 1$ and $t \ge 0$, then by (16.35) we have the sequence of inequalities

$$\begin{aligned} 0 &\le \left(\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(PA^{p-1})}\right)^p - \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} \\ &\le p\left(\frac{M}{m}\right)^{p-1}\left(\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PA^{p-1})}{\operatorname{tr}(P)}\right) \\ &\le \frac{1}{2}p^2\left(\frac{M}{m}\right)^{p-1} \\ &\times \begin{cases} (M^{p-1} - m^{p-1}) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \\ \\ (M - m) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(A^{p-1} - \frac{\operatorname{tr}\left(PA^{p-1}\right)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \end{cases} \\ &\le \frac{1}{2}p^2\left(\frac{M}{m}\right)^{p-1} \\ &\times \begin{cases} (M^{p-1} - m^{p-1}) \left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ (M - m) \left[\frac{\operatorname{tr}\left(PA^{2(p-1)}\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}\left(PA^{p-1}\right)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\ &\le \frac{1}{4}p^2\left(\frac{M}{m}\right)^{p-1}\left(M^{p-1} - m^{p-1}\right)(M - m) \end{aligned} \tag{16.36}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ and A with $\operatorname{Sp}(A) \subseteq [m, M] \subset (0, \infty)$.

THEOREM 16.10. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and twice differentiable function on $\mathring{I}$ whose second derivative f'' is bounded on $\mathring{I}$, i.e. there is a positive constant K such that $0 \le f''(t) \le K$ for any $t \in \mathring{I}$. If A is a selfadjoint operator on the Hilbert space H*

with $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$ *and* $f'(A)$ *is a positive invertible operator on* H, *or*

$$\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]} \in \mathring{I}, \ \operatorname{tr}[Pf'(A)] \neq 0$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, *then*

$$\begin{aligned}
(16.37) \qquad 0 &\leq f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \\
&\leq K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\
&\times \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \\
&\leq K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\
&\times \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \\
&\leq \frac{1}{2}(M-m)K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)
\end{aligned}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

PROOF. From (16.32) we have

$$\begin{aligned}
(16.38) \qquad 0 &\leq f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \\
&\leq \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) L(P, A, f'(A)),
\end{aligned}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

From (16.20) we also have

$$\begin{aligned}
(16.39) \qquad &(0 \leq) L(P, A, f'(A)) \\
&\leq K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\
&\leq K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}
\end{aligned}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Therefore, by (16.38) and (16.39) we get

$$
\begin{aligned}
0 &\le f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)-\frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \\
&\le K\left\|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\
&\times\frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]}f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \\
&\le K\left\|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\|\left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\
&\times\frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]}f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)
\end{aligned}
$$

that proves the second and third inequalities in (16.37).

The last part follows by Lemma 16.7. ∎

The inequality (16.37) can be also written for the convex function $f(t)=t^p$ with $p\ge 1$ and $t\ge 0,$ however the details are not presented here.

17. LIPSCHITZ TYPE INEQUALITIES

17.1. Some Basic Facts. One of the central problems in perturbation theory is to find bounds for

$$\|f(A)-f(B)\|$$

in terms of $\|A-B\|$ for different classes of measurable functions f for which the function of operator can be defined.

By the help of power series $f(z)=\sum_{n=0}^{\infty}a_nz^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely, $f_a(z):=\sum_{n=0}^{\infty}|a_n|z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $a_n\ge 0,$ then $f_a=f.$

We notice that if

$$
\begin{aligned}
f(z) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^n=\ln\frac{1}{1+z},\ z\in D(0,1); \\
g(z) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}=\cos z,\ z\in\mathbb{C}; \\
h(z) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}=\sin z,\ z\in\mathbb{C}; \\
l(z) &= \sum_{n=0}^{\infty}(-1)^n z^n=\frac{1}{1+z},\ z\in D(0,1);
\end{aligned}
\tag{17.1}
$$

where $D(0,1)$ is the open disk centered in 0 and of radius $1,$ then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(z) = \sum_{n=1}^{\infty} \frac{1}{n!} z^n = \ln \frac{1}{1-z},\ z \in D(0,1); \tag{17.2}$$

$$g_a(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z,\ z \in \mathbb{C};$$

$$h_a(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z,\ z \in \mathbb{C};$$

$$l_a(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z},\ z \in D(0,1).$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n,\ z \in \mathbb{C}; \tag{17.3}$$

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1},\ z \in D(0,1);$$

$$\sin^{-1}(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} z^{2n+1},\ z \in D(0,1);$$

$$\tanh^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1},\ z \in D(0,1);$$

$${}_2F_1(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha,\beta,\gamma>0,$$
$$z \in D(0,1);$$

where Γ is *Gamma function*.

We recall the following result that provides a quasi-Lipschitzian condition for functions defined by power series and operator norm $\|\cdot\|$ [**45**]:

THEOREM 17.1. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R>0$. If $T,V \in \mathcal{B}(H)$ are such that $\|T\|, \|V\| < R$, then*

$$\|f(T) - f(V)\| \le f_a'(\max\{\|T\|, \|V\|\}) \|T-V\|. \tag{17.4}$$

If $\|T\|, \|V\| \le M < R$, then from (17.4) we have the simpler inequality

$$\|f(T) - f(V)\| \le f_a'(M) \|T-V\| \tag{17.5}$$

In the recent paper [**46**] we improved the inequality (17.4) as follows:

THEOREM 17.2. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R>0$. If $T,V \in \mathcal{B}(H)$ are such that $\|T\|, \|V\| < R$, then*

$$\|f(T) - f(V)\| \le \|T-V\| \int_0^1 f_a'(\|(1-t)T + tV\|)\, dt. \tag{17.6}$$

In order to obtain similar results for the trace of bounded linear operators on complex infinite dimensional Hilbert spaces we need some preparations as follows.

17.2. Trace Inequalities. We have the following representation result:

THEOREM 17.3 (Dragomir, 2014, [**63**]). *Let* $f(z) := \sum_{n=0}^{\infty} a_n z^n$ *be a power series with complex coefficients and convergent on the open disk* $D(0,R)$, $R>0$. *If* $T,V \in \mathcal{B}_1(H)$ *are such that* $\operatorname{tr}(|T|), \operatorname{tr}(|V|) < R$, *then* $f(V)$, $f(T)$, $f'((1-t)T+tV) \in \mathcal{B}_1(H)$ *for any* $t\in[0,1]$ *and*

$$\operatorname{tr}[f(V)] - \operatorname{tr}[f(T)] = \int_0^1 \operatorname{tr}((V-T) f'((1-t)T+tV))\, dt. \tag{17.7}$$

PROOF. We use the identity

$$A^n - B^n = \sum_{j=0}^{n-1} A^{n-1-j}(A-B)B^j \tag{17.8}$$

that holds for any $A,B\in\mathcal{B}(H)$ and $n\geq 1$.

For $T,V\in\mathcal{B}(H)$ we consider the function $\varphi:[0,1]\to\mathcal{B}(H)$ defined by $\varphi(t) = [(1-t)T+tV]^n$. For $t\in(0,1)$ and $\varepsilon\neq 0$ with $t+\varepsilon\in(0,1)$ we have from (17.8) that

$$\begin{aligned} &\varphi(t+\varepsilon)-\varphi(t) \\ &= [(1-t-\varepsilon)T+(t+\varepsilon)V]^n - [(1-t)T+tV]^n \\ &= \varepsilon \sum_{j=0}^{n-1} [(1-t-\varepsilon)T+(t+\varepsilon)V]^{n-1-j}(V-T)[(1-t)T+tV]^j . \end{aligned}$$

Dividing with $\varepsilon\neq 0$ and taking the limit over $\varepsilon\to 0$ we have in the norm topology of $\mathcal{B}$ that

$$\begin{aligned} \varphi'(t) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\varphi(t+\varepsilon)-\varphi(t)] \\ &= \sum_{j=0}^{n-1}[(1-t)T+tV]^{n-1-j}(V-T)[(1-t)T+tV]^j . \end{aligned} \tag{17.9}$$

Integrating on $[0,1]$ we get from (17.9) that

$$\int_0^1 \varphi'(t)\,dt = \sum_{j=0}^{n-1}\int_0^1 [(1-t)T+tV]^{n-1-j}(V-T)[(1-t)T+tV]^j\,dt$$

and since

$$\int_0^1 \varphi'(t)\,dt = \varphi(1)-\varphi(0) = V^n - T^n$$

then we get the following equality of interest in itself

$$V^n - T^n = \sum_{j=0}^{n-1}\int_0^1 [(1-t)T+tV]^{n-1-j}(V-T)[(1-t)T+tV]^j\,dt, \tag{17.10}$$

for any $T,V\in\mathcal{B}(H)$ and $n\geq 1$.

If $T, V \in \mathcal{B}_1(H)$ and we take the trace in (17.10) we get

$$\begin{aligned}\text{(17.11)} \quad & \operatorname{tr}(V^n) - \operatorname{tr}(T^n) \\ &= \sum_{j=0}^{n-1} \int_0^1 \operatorname{tr}\left([(1-t)T + tV]^{n-1-j}(V-T)[(1-t)T+tV]^j\right) dt \\ &= \sum_{j=0}^{n-1} \int_0^1 \operatorname{tr}\left([(1-t)T + tV]^{n-1}(V-T)\right) dt \\ &= n \int_0^1 \operatorname{tr}\left([(1-t)T + tV]^{n-1}(V-T)\right) dt \\ &= n \int_0^1 \operatorname{tr}\left((V-T)[(1-t)T + tV]^{n-1}\right) dt\end{aligned}$$

for any $n \geq 1$.

Let $m \geq 1$. Then by (17.11) we have have

$$\begin{aligned}\text{(17.12)} \quad & \operatorname{tr}\left(\sum_{n=0}^{m} a_n V^n\right) - \operatorname{tr}\left(\sum_{n=0}^{m} a_n T^n\right) \\ &= \sum_{n=0}^{m} a_n \left[\operatorname{tr}(V^n) - \operatorname{tr}(T^n)\right] \\ &= \sum_{n=1}^{m} a_n \left[\operatorname{tr}(V^n) - \operatorname{tr}(T^n)\right] \\ &= \sum_{n=1}^{m} n a_n \int_0^1 \operatorname{tr}\left((V-T)[(1-t)T + tV]^{n-1}\right) dt \\ &= \int_0^1 \operatorname{tr}\left((V-T)\sum_{n=1}^{m} n a_n [(1-t)T + tV]^{n-1}\right) dt\end{aligned}$$

for any $T, V \in \mathcal{B}_1(H)$.

Since $\operatorname{tr}(|T|), \operatorname{tr}(|V|) < R$ with $T, V \in \mathcal{B}_1(H)$ then the series $\sum_{n=0}^{\infty} a_n V^n$, $\sum_{n=0}^{\infty} a_n T^n$ and $\sum_{n=1}^{\infty} n a_n [(1-t)T + tV]^{n-1}$ are convergent in $\mathcal{B}_1(H)$ and

$$\sum_{n=0}^{\infty} a_n V^n = f(V), \quad \sum_{n=0}^{\infty} a_n T^n = f(T)$$

and

$$\sum_{n=1}^{\infty} n a_n [(1-t)T + tV]^{n-1} = f'((1-t)T + tV)$$

where $t \in [0,1]$. Moreover, we have

$$f(V),\ f(T),\ f'((1-t)T + tV) \in \mathcal{B}_1(H)$$

for any $t \in [0,1]$.

By taking the limit over $m \to \infty$ in (17.12) we get the desired result (17.7). ∎

In addition to the power identity (17.11) we have other equalities as follows

$$\text{(17.13)} \quad \operatorname{tr}[\exp(V)] - \operatorname{tr}[\exp(T)] = \int_0^1 \operatorname{tr}((V-T)\exp((1-t)T + tV))\, dt,$$

$$\text{tr}\left[\sin\left(V\right)\right]-\text{tr}\left[\sin\left(T\right)\right]=\int_0^1\text{tr}\left(\left(V-T\right)\cos\left(\left(1-t\right)T+tV\right)\right)dt, \tag{17.14}$$

and

$$\text{tr}\left[\sinh\left(V\right)\right]-\text{tr}\left[\sinh\left(T\right)\right]=\int_0^1\text{tr}\left(\left(V-T\right)\cosh\left(\left(1-t\right)T+tV\right)\right)dt. \tag{17.15}$$

for any $T,V\in\mathcal{B}_1\left(H\right)$.

If $T,V\in\mathcal{B}_1\left(H\right)$ with $\text{tr}\left(\left|T\right|\right),\text{tr}\left(\left|V\right|\right)<1$ then

$$\begin{aligned}&\text{tr}\left[\left(1_H-V\right)^{-1}\right]-\text{tr}\left[\left(1_H-T\right)^{-1}\right]\\&=\int_0^1\text{tr}\left(\left(V-T\right)\left(1_H-\left(1-t\right)T-tV\right)^{-2}\right)dt,\end{aligned} \tag{17.16}$$

and

$$\begin{aligned}&\text{tr}\left[\ln\left(1_H-V\right)^{-1}\right]-\text{tr}\left[\ln\left(1_H-T\right)^{-1}\right]\\&=\int_0^1\text{tr}\left(\left(V-T\right)\left(1_H-\left(1-t\right)T-tV\right)^{-1}\right)dt.\end{aligned} \tag{17.17}$$

We have the following result:

COROLLARY 17.4. *With the assumptions in Theorem 17.3 we have the inequalities*

$$\begin{aligned}&\left|\text{tr}\left[f\left(V\right)\right]-\text{tr}\left[f\left(T\right)\right]\right|\\&\le\min\left\{\left\|V-T\right\|\int_0^1\left\|f'\left(\left(1-t\right)T+tV\right)\right\|_1dt,\right.\\&\left.\left\|V-T\right\|_1\int_0^1\left\|f'\left(\left(1-t\right)T+tV\right)\right\|dt\right\}\\&\le\min\left\{\left\|V-T\right\|\int_0^1f_a'\left(\left\|\left(1-t\right)T+tV\right\|_1\right)dt,\right.\\&\left.\left\|V-T\right\|_1\int_0^1f_a'\left(\left\|\left(1-t\right)T+tV\right\|\right)dt\right\},\end{aligned} \tag{17.18}$$

where $\left\|\cdot\right\|$ is the operator norm and $\left\|\cdot\right\|_1$ is the 1-norm introduced for trace class operators.

PROOF. From (17.7), we have by taking the modulus

$$\left|\text{tr}\left[f\left(V\right)\right]-\text{tr}\left[f\left(T\right)\right]\right|\le\int_0^1\left|\text{tr}\left(\left(V-T\right)f'\left(\left(1-t\right)T+tV\right)\right)\right|dt. \tag{17.19}$$

Utilising the properties of trace, we get

$$\left|\text{tr}\left(\left(V-T\right)f'\left(\left(1-t\right)T+tV\right)\right)\right|\le\left\|V-T\right\|\left\|f'\left(\left(1-t\right)T+tV\right)\right\|_1$$

and

$$\left|\text{tr}\left(\left(V-T\right)f'\left(\left(1-t\right)T+tV\right)\right)\right|\le\left\|V-T\right\|_1\left\|f'\left(\left(1-t\right)T+tV\right)\right\|$$

for any $t\in\left[0,1\right]$.

By integrating these inequalities, we get the first part of (17.18).

We have, by the use of $\|\cdot\|_1$ properties that

$$\begin{aligned}\|f'((1-t)T+tV)\|_1 &= \left\|\sum_{n=1}^{\infty} na_n[(1-t)T+tV]^{n-1}\right\|_1 \\ &\le \sum_{n=1}^{\infty} n|a_n|\left\|[(1-t)T+tV]^{n-1}\right\|_1 \\ &\le \sum_{n=1}^{\infty} n|a_n|\|(1-t)T+tV\|_1^{n-1} \\ &= f_a'(\|(1-t)T+tV\|_1)\end{aligned}$$

for any $T,V\in\mathcal{B}_1(H)$ with $\|T\|_1, \|V\|_1<R.$

This proves the first part of the second inequality.

Since $\|X\|\le\|X\|_1$ for any $X\in\mathcal{B}_1(H)$ then $\|(1-t)T+tV\|<R$ for any $T,V\in\mathcal{B}_1(H)$ with $\|T\|_1, \|V\|_1<R$ which shows that $f_a'(\|(1-t)T+tV\|)$ is well defined.

The second part of the second inequality follows in a similar way and the details are omitted. ∎

REMARK 17.1. We observe that f_a' is monotonic nondecreasing and convex on the interval $[0,R)$ and since the function $\psi(t):=\|(1-t)T+tV\|$ is convex on $[0,1]$ we have that $f_a'\circ\psi$ is also convex on $[0,1]$. Utilising the Hermite-Hadamard inequality for convex functions (see for instance [**70**, p. 2]) we have the sequence of inequalities

$$\begin{aligned}&\int_0^1 f_a'(\|(1-t)T+tV\|)\,dt \\ &\le \frac{1}{2}\left[f_a'\left(\left\|\frac{T+V}{2}\right\|\right)+\frac{f_a'(\|T\|)+f_a'(\|V\|)}{2}\right] \\ &\le \frac{1}{2}[f_a'(\|T\|)+f_a'(\|V\|)]\le\max\{f_a'(\|T\|),f_a'(\|V\|)\}.\end{aligned}\tag{17.20}$$

We also have

$$\begin{aligned}&\int_0^1 f_a'(\|(1-t)T+tV\|)\,dt \\ &\le \int_0^1 f_a'((1-t)\|T\|+t\|V\|)\,dt \\ &\le \frac{1}{2}\left[f_a'\left(\frac{\|T\|+\|V\|}{2}\right)+\frac{f_a'(\|T\|)+f_a'(\|V\|)}{2}\right] \\ &\le \frac{1}{2}[f_a'(\|T\|)+f_a'(\|V\|)]\le\max\{f_a'(\|T\|),f_a'(\|V\|)\}.\end{aligned}\tag{17.21}$$

We observe that if $\|V\|\ne\|T\|$, then by the change of variable $s=(1-t)\|T\|+t\|V\|$ we have

$$\begin{aligned}\int_0^1 f_a'((1-t)\|T\|+t\|V\|)\,dt &= \frac{1}{\|V\|-\|T\|}\int_{\|T\|}^{\|V\|} f_a'(s)\,ds \\ &= \frac{f_a(\|V\|)-f_a(\|T\|)}{\|V\|-\|T\|}.\end{aligned}$$

If $\|V\| = \|T\|$, then

$$\int_0^1 f_a'\left(\left(1-t\right)\|T\| + t\|V\|\right) dt = f_a'\left(\|T\|\right).$$

Utilising these observations we then get the following divided difference inequality for $T \neq V$

$$\int_0^1 f_a'\left(\|(1-t)T + tV\|\right) dt \leq \begin{cases} \frac{f_a(\|V\|) - f_a(\|T\|)}{\|V\| - \|T\|} & \text{if } \|V\| \neq \|T\|, \\ f_a'\left(\|T\|\right) & \text{if } \|V\| = \|T\|. \end{cases} \tag{17.22}$$

Similar comments apply for the 1-norm $\|\cdot\|_1$ when $T, V \in \mathcal{B}_1(H)$.

If we use the first part in the inequalities (17.18) and the above remarks, then we get the following string of inequalities

$$\begin{aligned} &\left|\operatorname{tr}\left[f(V)\right] - \operatorname{tr}\left[f(T)\right]\right| \\ &\leq \|V - T\| \int_0^1 \left\|f'\left((1-t)T + tV\right)\right\|_1 dt \\ &\leq \|V - T\| \int_0^1 f_a'\left(\|(1-t)T + tV\|_1\right) dt \\ &\leq \|V - T\| \times \begin{cases} \frac{1}{2}\left[f_a'\left(\left\|\frac{T+V}{2}\right\|_1\right) + \frac{f_a'\left(\|T\|_1\right) + f_a'\left(\|V\|_1\right)}{2}\right], \\ \begin{cases} \frac{f_a\left(\|V\|_1\right) - f_a\left(\|T\|_1\right)}{\|V\|_1 - \|T\|_1} & \text{if } \|V\|_1 \neq \|T\|_1, \\ f_a'\left(\|T\|_1\right) & \text{if } \|V\|_1 = \|T\|_1, \end{cases} \end{cases} \\ &\leq \frac{1}{2}\|V - T\|\left[f_a'\left(\|T\|_1\right) + f_a'\left(\|V\|_1\right)\right] \\ &\leq \|V - T\| \max\left\{f_a'\left(\|T\|_1\right), f_a'\left(\|V\|_1\right)\right\} \end{aligned} \tag{17.23}$$

provided $T, V \in \mathcal{B}_1(H)$ with $\|T\|_1, \|V\|_1 < R$.

If $\|T\|_1, \|V\|_1 \leq M < R$, then we have from (17.23) the simple inequality

$$\left|\operatorname{tr}\left[f(V)\right] - \operatorname{tr}\left[f(T)\right]\right| \leq \|V - T\| f_a'(M).$$

A similar sequence of inequalities can also be stated by swapping the norm $\|\cdot\|$ with $\|\cdot\|_1$ in (17.23). We omit the details.

If we use the inequality (17.23) for the exponential function, then we have the inequalities

$$
\begin{aligned}
&\left|\operatorname{tr}[\exp(V)]-\operatorname{tr}[\exp(T)]\right| \\
&\leq \|V-T\| \int_0^1 \|\exp((1-t)T+tV)\|_1\,dt \\
&\leq \|V-T\| \int_0^1 \exp(\|(1-t)T+tV\|_1)\,dt \\
&\leq \|V-T\| \times \begin{cases} \frac{1}{2}\left[\exp\left(\left\|\frac{T+V}{2}\right\|_1\right)+\frac{\exp\left(\|T\|_1\right)+\exp\left(\|V\|_1\right)}{2}\right], \\ \begin{cases} \frac{\exp\left(\|V\|_1\right)-\exp\left(\|T\|_1\right)}{\|V\|_1-\|T\|_1} & \text{if } \|V\|_1 \neq \|T\|_1, \\ \exp(\|T\|_1) & \text{if } \|V\|_1 = \|T\|_1, \end{cases} \end{cases} \\
&\leq \frac{1}{2}\|V-T\|\left[\exp(\|T\|_1)+\exp(\|V\|_1)\right] \\
&\leq \|V-T\| \max\{\exp(\|T\|_1), \exp(\|V\|_1)\}
\end{aligned} \tag{17.24}
$$

for any $T, V \in \mathcal{B}_1(H)$.

If $\|T\|_1, \|V\|_1 < 1$, then we have the inequalities

$$
\begin{aligned}
&\left|\operatorname{tr}\left[\ln(1_H-V)^{-1}\right]-\operatorname{tr}\left[\ln(1_H-T)^{-1}\right]\right| \\
&\leq \|V-T\| \int_0^1 \left\|(1_H-(1-t)T-tV)^{-1}\right\|_1 dt \\
&\leq \|V-T\| \int_0^1 (1-\|(1-t)T+tV\|_1)^{-1}\,dt \\
&\leq \|V-T\| \times \begin{cases} \frac{1}{2}\left[\left(1-\left\|\frac{T+V}{2}\right\|_1\right)^{-1}+\frac{\left(1-\|T\|_1\right)^{-1}+\left(1-\|V\|_1\right)^{-1}}{2}\right], \\ \begin{cases} \frac{\ln\left(1-\|V\|_1\right)^{-1}-\ln\left(1-\|T\|_1\right)^{-1}}{\|V\|_1-\|T\|_1} & \text{if } \|V\|_1 \neq \|T\|_1, \\ (1-\|T\|_1)^{-1} & \text{if } \|V\|_1 = \|T\|_1, \end{cases} \end{cases} \\
&\leq \frac{1}{2}\|V-T\|\left[(1-\|T\|_1)^{-1}+(1-\|V\|_1)^{-1}\right] \\
&\leq \|V-T\| \max\left\{(1-\|T\|_1)^{-1}, (1-\|V\|_1)^{-1}\right\}.
\end{aligned} \tag{17.25}
$$

The following result for the Hilbert-Schmidt norm $\|\cdot\|_2$ also holds:

THEOREM 17.5 (Dragomir, 2014, [**63**]). *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R>0$. If $T, V \in \mathcal{B}_2(H)$ are such that $\operatorname{tr}\left(|T|^2\right), \operatorname{tr}\left(|V|^2\right) < R^2$, then $f(V)$, $f(T)$, $f'((1-t)T+tV) \in \mathcal{B}_2(H)$ for any $t \in [0,1]$ and*

$$
\operatorname{tr}[f(V)]-\operatorname{tr}[f(T)] = \int_0^1 \operatorname{tr}\left((V-T)f'((1-t)T+tV)\right)dt. \tag{17.26}
$$

Moreover, we have the inequalities

$$
\begin{aligned}
(17.27)\qquad & \left|\operatorname{tr}\left[f\left(V\right)\right]-\operatorname{tr}\left[f\left(T\right)\right]\right| \\
& \leq \left\|V-T\right\|_{2}\int_{0}^{1}\left\|f'\left(\left(1-t\right)T+tV\right)\right\|_{2}dt, \\
& \leq \left\|V-T\right\|_{2}\int_{0}^{1}f_{a}'\left(\left\|\left(1-t\right)T+tV\right\|_{2}\right)dt \\
& \leq \left\|V-T\right\|_{2}\times\begin{cases}\frac{1}{2}\left[f_{a}'\left(\left\|\frac{T+V}{2}\right\|_{2}\right)+\frac{f_{a}'\left(\left\|T\right\|_{2}\right)+f_{a}'\left(\left\|V\right\|_{2}\right)}{2}\right], \\ \begin{cases}\frac{f_{a}\left(\left\|V\right\|_{2}\right)-f_{a}\left(\left\|T\right\|_{2}\right)}{\left\|V\right\|_{2}-\left\|T\right\|_{2}} & \text{if } \left\|V\right\|_{2}\neq\left\|T\right\|_{2}, \\ f_{a}'\left(\left\|T\right\|_{2}\right) & \text{if } \left\|V\right\|_{2}=\left\|T\right\|_{2},\end{cases}\end{cases} \\
& \leq \frac{1}{2}\left\|V-T\right\|_{2}\left[f_{a}'\left(\left\|T\right\|_{2}\right)+f_{a}'\left(\left\|V\right\|_{2}\right)\right] \\
& \leq \left\|V-T\right\|_{2}\max\left\{f_{a}'\left(\left\|T\right\|_{2}\right),f_{a}'\left(\left\|V\right\|_{2}\right)\right\}.
\end{aligned}
$$

PROOF. The proof of the first part of the theorem follows in a similar manner to the one from Theorem 17.3.

Taking the modulus in (17.26) and using the Schwarz inequality for trace we have

$$
\begin{aligned}
(17.28)\qquad \left|\operatorname{tr}\left[f\left(V\right)\right]-\operatorname{tr}\left[f\left(T\right)\right]\right| & \leq \int_{0}^{1}\left|\operatorname{tr}\left(\left(V-T\right)f'\left(\left(1-t\right)T+tV\right)\right)\right|dt \\
& \leq \int_{0}^{1}\left\|V-T\right\|_{2}\left\|f'\left(\left(1-t\right)T+tV\right)\right\|_{2}dt.
\end{aligned}
$$

The rest follows in a similar manner as in the case of 1-norm and the details are omitted. ∎

We notice that similar examples to (17.24) and (17.25) may be stated where both norms $\|\cdot\|$ and $\|\cdot\|_1$ are replaced by $\|\cdot\|_2$.

We also observe that, if $T,V\in\mathcal{B}_2(H)$ with $\|T\|_2,\|V\|_2\leq K<R$, then we have from (17.23) the simple inequality

$$
\left|\operatorname{tr}\left[f\left(V\right)\right]-\operatorname{tr}\left[f\left(T\right)\right]\right|\leq\left\|V-T\right\|_{2}f_{a}'\left(K\right).
$$

17.3. Norm Inequalities. We have the following norm inequalities:

THEOREM 17.6 (Dragomir, 2014, [**63**]). *Let* $f(z):=\sum_{n=0}^{\infty}a_nz^n$ *be a power series with complex coefficients and convergent on the open disk* $D(0,R)$, $R>0$.

(i) If $T,V\in\mathcal{B}_1(H)$ *are such that* $\operatorname{tr}(|T|),\operatorname{tr}(|V|)<R$*, then we have the norm inequalities*

$$
(17.29)\qquad \left\|f\left(V\right)-f\left(T\right)\right\|_{1}\leq\begin{cases}\left\|V-T\right\|_{1}\int_{0}^{1}f_{a}'\left(\left\|\left(1-t\right)T+tV\right\|\right)dt, \\ \left\|V-T\right\|\int_{0}^{1}f_{a}'\left(\left\|\left(1-t\right)T+tV\right\|_{1}\right)dt.\end{cases}
$$

(ii) If $T,V\in\mathcal{B}_2(H)$ *are such that* $\operatorname{tr}\left(|T|^2\right),\operatorname{tr}\left(|V|^2\right)<R^2$*, then we also have the norm inequalities*

$$
(17.30)\qquad \left\|f\left(V\right)-f\left(T\right)\right\|_{2}\leq\begin{cases}\left\|V-T\right\|_{2}\int_{0}^{1}f_{a}'\left(\left\|\left(1-t\right)T+tV\right\|\right)dt, \\ \left\|V-T\right\|\int_{0}^{1}f_{a}'\left(\left\|\left(1-t\right)T+tV\right\|_{2}\right)dt.\end{cases}
$$

PROOF. We use the equality

$$V^n - T^n = \sum_{j=0}^{n-1} \int_0^1 \left[(1-t)T + tV\right]^{n-1-j} (V-T) \left[(1-t)T + tV\right]^j dt, \tag{17.31}$$

for any $T, V \in \mathcal{B}(H)$ and $n \geq 1$.

(i) If $T, V \in \mathcal{B}_1(H)$ are such that $\operatorname{tr}(|T|), \operatorname{tr}(|V|) < R$, then by taking the $\|\cdot\|_1$ norm and using its properties we have successively

$$\begin{aligned}
&\|V^n - T^n\|_1 \\
&\leq \sum_{j=0}^{n-1} \int_0^1 \left\| \left[(1-t)T + tV\right]^{n-1-j} (V-T) \left[(1-t)T + tV\right]^j \right\|_1 dt \\
&\leq \sum_{j=0}^{n-1} \int_0^1 \left\| \left[(1-t)T + tV\right]^{n-1-j} (V-T) \right\|_1 \left\| \left[(1-t)T + tV\right]^j \right\| dt \\
&\leq \sum_{j=0}^{n-1} \int_0^1 \|V-T\|_1 \left\| \left[(1-t)T + tV\right]^{n-1-j} \right\| \left\| \left[(1-t)T + tV\right]^j \right\| dt \\
&\leq \|V-T\|_1 \sum_{j=0}^{n-1} \|(1-t)T + tV\|^{n-1-j} \|(1-t)T + tV\|^j dt \\
&= n \|V-T\|_1 \int_0^1 \|(1-t)T + tV\|^{n-1} dt
\end{aligned} \tag{17.32}$$

for any $n \geq 1$.

Let $m \geq 1$. By (17.32) we have

$$\begin{aligned}
&\left\| \sum_{n=0}^m a_n V^n - \sum_{n=0}^m a_n T^n \right\|_1 \\
&= \left\| \sum_{n=1}^m a_n (V^n - T^n) \right\|_1 \\
&\leq \sum_{n=1}^m |a_n| \|V^n - T^n\|_1 \\
&\leq \|V-T\|_1 \sum_{n=1}^m |a_n| n \int_0^1 \|(1-t)T + tV\|^{n-1} dt \\
&= \|V-T\|_1 \int_0^1 \left(\sum_{n=1}^m n |a_n| \|(1-t)T + tV\|^{n-1} \right) dt.
\end{aligned} \tag{17.33}$$

Also, we observe that

$$\|(1-t)T + tV\| \leq \|(1-t)T + tV\|_1 \leq (1-t)\|T\|_1 + t\|V\|_1 < R$$

for any $t \in [0,1]$, which implies that the series $\sum_{n=1}^\infty n|a_n| \|(1-t)T + tV\|^{n-1}$ is convergent and

$$\sum_{n=1}^\infty n|a_n| \|(1-t)T + tV\|^{n-1} = f_a'(\|(1-t)T + tV\|)$$

for any $t \in [0,1]$.

Since the series $\sum_{n=0}^{\infty} a_n V^n$ and $\sum_{n=0}^{\infty} a_n T^n$ are convergent in $(\mathcal{B}_1(H), \|\cdot\|_1)$, then by letting $m \to \infty$ in the inequality (17.33) we get the first inequality in (17.29).

We also have

$$\begin{aligned}
&\|V^n - T^n\|_1 \\
&\leq \sum_{j=0}^{n-1} \int_0^1 \left\| [(1-t)T + tV]^{n-1-j} (V-T) [(1-t)T + tV]^j \right\|_1 dt \\
&\leq \sum_{j=0}^{n-1} \int_0^1 \left\| [(1-t)T + tV]^{n-1-j} (V-T) \right\| \left\| [(1-t)T + tV]^j \right\|_1 dt \\
&\leq \sum_{j=0}^{n-1} \int_0^1 \|V-T\| \left\| [(1-t)T + tV]^{n-1-j} \right\| \left\| [(1-t)T + tV]^j \right\|_1 dt \\
&\leq \|V-T\| \sum_{j=0}^{n-1} \|(1-t)T + tV\|^{n-1-j} \|(1-t)T + tV\|_1^j \, dt \\
&\leq \|V-T\| \sum_{j=0}^{n-1} \|(1-t)T + tV\|_1^{n-1-j} \|(1-t)T + tV\|_1^j \, dt \\
&= n \|V-T\| \int_0^1 \|(1-t)T + tV\|_1^{n-1} dt
\end{aligned}$$

for any $n \geq 1$, which by a similar argument produces the second inequality in (17.29).

(ii) Follows in a similar way by utilizing the inequality $\|TA\|_2 \leq \|T\| \|A\|_2$ that holds for $T \in \mathcal{B}(H)$ and $A \in \mathcal{B}_2(H)$. The details are omitted. ∎

REMARK 17.2. From the first inequality in (17.29) we have the sequence of inequalities

$$\begin{aligned}
&\|f(V) - f(T)\|_1 \\
&\leq \|V-T\|_1 \int_0^1 f_a'(\|(1-t)T + tV\|) \, dt \\
&\leq \|V-T\|_1 \times \begin{cases} \frac{1}{2}\left[f_a'\left(\left\| \frac{T+V}{2} \right\| \right) + \frac{f_a'(\|T\|) + f_a'(\|V\|)}{2} \right], \\ \begin{cases} \frac{f_a(\|V\|) - f_a(\|T\|)}{\|V\| - \|T\|} & \text{if } \|V\| \neq \|T\|, \\ f_a'(\|T\|) & \text{if } \|V\| = \|T\|, \end{cases} \end{cases} \\
&\leq \frac{1}{2} \|V-T\|_1 \left[f_a'(\|T\|) + f_a'(\|V\|) \right] \\
&\leq \|V-T\|_1 \max\{ f_a'(\|T\|), f_a'(\|V\|) \}
\end{aligned} \tag{17.34}$$

for $T, V \in \mathcal{B}_1(H)$ such that $\operatorname{tr}(|T|), \operatorname{tr}(|V|) < R$ and a similar result by swapping in the right hand side of (17.34) the norm $\|\cdot\|$ with $\|\cdot\|_1$.

In particular, if $\operatorname{tr}(|T|), \operatorname{tr}(|V|) \leq M < R$, then we have the simpler inequality

$$\|f(V) - f(T)\|_1 \leq f_a'(M) \|V-T\|_1. \tag{17.35}$$

If $T,V\in\mathcal{B}_2(H)$ are such that $\operatorname{tr}\left(|T|^2\right),\operatorname{tr}\left(|V|^2\right)<R^2$, then we have the norm inequalities

$$\begin{aligned}
&\|f(V)-f(T)\|_2 \\
&\le \|V-T\|_2\int_0^1 f_a'(\|(1-t)T+tV\|)\,dt \\
&\le \|V-T\|_2\times\begin{cases}\frac{1}{2}\left[f_a'\left(\left\|\frac{T+V}{2}\right\|\right)+\frac{f_a'(\|T\|)+f_a'(\|V\|)}{2}\right], \\[2ex] \begin{cases}\frac{f_a(\|V\|)-f_a(\|T\|)}{\|V\|-\|T\|} & \text{if } \|V\|\ne\|T\|, \\[2ex] f_a'(\|T\|) & \text{if } \|V\|=\|T\|,\end{cases}\end{cases} \\
&\le \frac{1}{2}\|V-T\|_2\left[f_a'(\|T\|)+f_a'(\|V\|)\right] \\
&\le \|V-T\|_2\max\left\{f_a'(\|T\|),f_a'(\|V\|)\right\}
\end{aligned}\tag{17.36}$$

and a similar result by swapping in the right hand side of (17.34) the norm $\|\cdot\|$ with $\|\cdot\|_2$.

In particular, if $\operatorname{tr}\left(|T|^2\right),\operatorname{tr}\left(|V|^2\right)\le K^2<R^2$, then we have the simpler inequality

$$\|f(V)-f(T)\|_2\le f_a'(K)\|V-T\|_2.\tag{17.37}$$

17.4. Applications for Jensen's Difference. We have the following representation:

LEMMA 17.7 (Dragomir, 2014, [**63**]). *Let $f(z):=\sum_{n=0}^{\infty}a_nz^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R>0$. If either $T,V\in\mathcal{B}_1(H)$ with $\|T\|_1,\|V\|_1<R$, or $T,V\in\mathcal{B}_2(H)$ with $\|T\|_2,\|V\|_2<R$ then $f(V),f(T),f\left(\frac{V+T}{2}\right)\in\mathcal{B}_1(H)$ or $f(V),f(T),f\left(\frac{V+T}{2}\right)\in\mathcal{B}_2(H)$, respectively and*

$$\begin{aligned}
&\frac{\operatorname{tr}[f(V)]+\operatorname{tr}[f(T)]}{2}-\operatorname{tr}\left[f\left(\frac{V+T}{2}\right)\right] \\
&=\frac{1}{4}\int_0^1\operatorname{tr}\left((V-T)\left[f'\left((1-t)\frac{V+T}{2}+tV\right)-f'\left((1-t)\frac{V+T}{2}+tT\right)\right]\right)dt.
\end{aligned}\tag{17.38}$$

PROOF. The first part of the theorem follows by Theorem 17.3.

From the identity (17.7) we have

$$\begin{aligned}
&\operatorname{tr}[f(V)]-\operatorname{tr}\left[f\left(\frac{V+T}{2}\right)\right] \\
&=\int_0^1\operatorname{tr}\left(\left(V-\frac{V+T}{2}\right)f'\left((1-t)\frac{V+T}{2}+tV\right)\right)dt \\
&=\frac{1}{2}\int_0^1\operatorname{tr}\left((V-T)f'\left((1-t)\frac{V+T}{2}+tV\right)\right)dt
\end{aligned}\tag{17.39}$$

and

$$\text{(17.40)} \qquad \operatorname{tr}\left[f\left(T\right)\right]-\operatorname{tr}\left[f\left(\frac{V+T}{2}\right)\right]$$
$$=\int_0^1 \operatorname{tr}\left(\left(T-\frac{V+T}{2}\right)f'\left(\left(1-t\right)\frac{V+T}{2}+tT\right)\right)dt$$
$$=\frac{1}{2}\int_0^1 \operatorname{tr}\left(\left(T-V\right)f'\left(\left(1-t\right)\frac{V+T}{2}+tT\right)\right)dt$$
$$=-\frac{1}{2}\int_0^1 \operatorname{tr}\left(\left(V-T\right)f'\left(\left(1-t\right)\frac{V+T}{2}+tT\right)\right)dt$$

for $T,V\in\mathcal{B}_1\left(H\right)$ with $\left\|T\right\|_1,\left\|V\right\|_1<R$.

If we add the above inequalities (17.39) and (17.40) and divide by 2 we get the desired result (17.38). ∎

THEOREM 17.8 (Dragomir, 2014, [**63**]). *Let $f\left(z\right):=\sum_{n=0}^{\infty}a_nz^n$ be a power series with complex coefficients and convergent on the open disk $D\left(0,R\right),R>0$. If $T,V\in\mathcal{B}_1\left(H\right)$ with $\left\|T\right\|_1,\left\|V\right\|_1<R$, then*

$$\text{(17.41)} \qquad \left|\frac{\operatorname{tr}\left[f\left(V\right)\right]+\operatorname{tr}\left[f\left(T\right)\right]}{2}-\operatorname{tr}\left[f\left(\frac{V+T}{2}\right)\right]\right|$$
$$\leq\frac{1}{2}\left\|V-T\right\|^2\int_0^1\left|t-\frac{1}{2}\right|f_a''\left(\left\|\left(1-t\right)T+tV\right\|_1\right)dt$$
$$\leq\frac{1}{24}\left\|V-T\right\|^2\left[f_a''\left(\left\|\frac{V+T}{2}\right\|_1\right)+\frac{f_a''\left(\left\|V\right\|_1\right)+f_a''\left(\left\|T\right\|_1\right)}{2}\right]$$
$$\leq\frac{1}{12}\left\|V-T\right\|^2\left[f_a''\left(\left\|V\right\|_1\right)+f_a''\left(\left\|T\right\|_1\right)\right]$$
$$\leq\frac{1}{6}\left\|V-T\right\|^2\max\left\{f_a''\left(\left\|V\right\|_1\right),f_a''\left(\left\|T\right\|_1\right)\right\}.$$

PROOF. Taking the modulus in (17.38), we have

$$\text{(17.42)} \qquad \left|\frac{\operatorname{tr}\left[f\left(V\right)\right]+\operatorname{tr}\left[f\left(T\right)\right]}{2}-\operatorname{tr}\left[f\left(\frac{V+T}{2}\right)\right]\right|$$
$$\leq\frac{1}{4}\int_0^1\left|\operatorname{tr}\left(\left(V-T\right)\left[f'\left(\left(1-t\right)\frac{V+T}{2}+tV\right)-f'\left(\left(1-t\right)\frac{V+T}{2}+tT\right)\right]\right)\right|dt$$

for $T,V\in\mathcal{B}_1\left(H\right)$ with $\left\|T\right\|_1,\left\|V\right\|_1<R$.

Using the properties of trace we have

$$\text{(17.43)} \qquad \left|\operatorname{tr}\left(\left(V-T\right)\left[f'\left(\left(1-t\right)\frac{V+T}{2}+tV\right)-f'\left(\left(1-t\right)\frac{V+T}{2}+tT\right)\right]\right)\right|$$
$$\leq\left\|V-T\right\|\left\|\left[f'\left(\left(1-t\right)\frac{V+T}{2}+tV\right)-f'\left(\left(1-t\right)\frac{V+T}{2}+tT\right)\right]\right\|_1$$

for $T,V\in\mathcal{B}_1\left(H\right)$ with $\left\|T\right\|_1,\left\|V\right\|_1<R$ and $t\in\left[0,1\right]$.

From (17.29) we have

$$\begin{aligned}
&\left\|f(A)-f(B)\right\|_1 \\
&\leq \|A-B\| \int_0^1 f_a'\left(\|(1-t)B+tA\|_1\right)dt \\
&\leq \frac{1}{2}\|A-B\|\left[f_a'\left(\left\|\frac{A+B}{2}\right\|_1\right)+\frac{f_a'\left(\|A\|_1\right)+f_a'\left(\|B\|_1\right)}{2}\right] \\
&\leq \frac{1}{2}\|A-B\|\left[f_a'\left(\|A\|_1\right)+f_a'\left(\|B\|_1\right)\right] \\
&\leq \|A-B\|\max\left\{f_a'\left(\|A\|_1\right),f_a'\left(\|B\|_1\right)\right\}
\end{aligned} \tag{17.44}$$

for $A,B\in\mathcal{B}_1(H)$ with $\|A\|_1,\|B\|_1<R$.

Applying the second and third inequalities in (17.44) for f' and $A=(1-t)\frac{V+T}{2}+tV$, $B=(1-t)\frac{V+T}{2}+tT$ we get

$$\begin{aligned}
&\left\|\left[f'\left((1-t)\frac{V+T}{2}+tV\right)-f'\left((1-t)\frac{V+T}{2}+tT\right)\right]\right\|_1 \\
&\leq \frac{1}{2}t\|V-T\|\left[f_a''\left(\left\|\frac{V+T}{2}\right\|_1\right)\right. \\
&\left.+\frac{f_a''\left(\left\|(1-t)\frac{V+T}{2}+tV\right\|_1\right)+f_a''\left(\left\|(1-t)\frac{V+T}{2}+tT\right\|_1\right)}{2}\right] \\
&\leq \frac{1}{2}t\|V-T\| \\
&\times\left[f_a''\left(\left\|(1-t)\frac{V+T}{2}+tV\right\|_1\right)+f_a''\left(\left\|(1-t)\frac{V+T}{2}+tT\right\|_1\right)\right]
\end{aligned} \tag{17.45}$$

for $T,V\in\mathcal{B}_1(H)$ with $\|T\|_1,\|V\|_1<R$ and $t\in[0,1]$.

Since f_a'' is convex and monotonic nondecreasing and $\|\cdot\|_1$ is convex, then

$$\begin{aligned}
&\frac{f_a''\left(\left\|(1-t)\frac{V+T}{2}+tV\right\|_1\right)+f_a''\left(\left\|(1-t)\frac{V+T}{2}+tT\right\|_1\right)}{2} \\
&\leq (1-t)f_a''\left(\left\|\frac{V+T}{2}\right\|_1\right)+t\frac{f_a''\left(\|V\|_1\right)+f_a''\left(\|T\|_1\right)}{2}
\end{aligned} \tag{17.46}$$

for $T,V\in\mathcal{B}_1(H)$ with $\|T\|_1,\|V\|_1<R$ and $t\in[0,1]$.

From (17.45) and (17.46) we get

$$\begin{aligned}
&\left\|\left[f'\left((1-t)\frac{V+T}{2}+tV\right)-f'\left((1-t)\frac{V+T}{2}+tT\right)\right]\right\|_1 \\
&\leq \frac{1}{2}t\|V-T\| \\
&\times\left[f_a''\left(\left\|(1-t)\frac{V+T}{2}+tV\right\|_1\right)+f_a''\left(\left\|(1-t)\frac{V+T}{2}+tT\right\|_1\right)\right] \\
&\leq t\|V-T\|\left[(1-t)f_a''\left(\left\|\frac{V+T}{2}\right\|_1\right)+t\frac{f_a''\left(\|V\|_1\right)+f_a''\left(\|T\|_1\right)}{2}\right]
\end{aligned} \tag{17.47}$$

for $T,V\in\mathcal{B}_1(H)$ with $\|T\|_1,\|V\|_1<R$ and $t\in[0,1]$.

Integrating (17.47) over t on $[0,1]$ we get

$$\int_0^1 \left\| \left[f'\left((1-t)\frac{V+T}{2} + tV \right) - f'\left((1-t)\frac{V+T}{2} + tT \right) \right] \right\|_1 dt$$
$$\le \frac{1}{2}\|V-T\|$$
$$\times \left[\int_0^1 t f_a''\left(\left\| (1-t)\frac{V+T}{2} + tV \right\|_1 \right) dt + \int_0^1 t f_a''\left(\left\| (1-t)\frac{V+T}{2} + tT \right\|_1 dt \right) \right]$$
$$\le \|V-T\| \left[f_a''\left(\left\| \frac{V+T}{2} \right\|_1 \right) \int_0^1 t(1-t)\,dt + \frac{f_a''(\|V\|_1) + f_a''(\|T\|_1)}{2} \int_0^1 t^2 dt \right]$$
$$= \frac{1}{6}\|V-T\| \left[f_a''\left(\left\| \frac{V+T}{2} \right\|_1 \right) + \frac{f_a''(\|V\|_1) + f_a''(\|T\|_1)}{2} \right],$$

which together with (17.42) and (17.43) produce the inequality

$$\left| \frac{\operatorname{tr}[f(V)] + \operatorname{tr}[f(T)]}{2} - \operatorname{tr}\left[f\left(\frac{V+T}{2} \right) \right] \right| \tag{17.48}$$
$$\le \frac{1}{8}\|V-T\|^2$$
$$\times \left[\int_0^1 t f_a''\left(\left\| (1-t)\frac{V+T}{2} + tV \right\|_1 \right) dt + \int_0^1 t f_a''\left(\left\| (1-t)\frac{V+T}{2} + tT \right\|_1 dt \right) \right]$$
$$\le \frac{1}{24}\|V-T\|^2 \left[f_a''\left(\left\| \frac{V+T}{2} \right\|_1 \right) + \frac{f_a''(\|V\|_1) + f_a''(\|T\|_1)}{2} \right].$$

Now, observe that

$$\int_0^1 t f_a''\left(\left\| (1-t)\frac{V+T}{2} + tV \right\|_1 \right) dt = \int_0^1 t f_a''\left(\left\| \frac{1-t}{2}T + \frac{1+t}{2}V \right\|_1 \right) dt \tag{17.49}$$

and

$$\int_0^1 t f_a''\left(\left\| (1-t)\frac{V+T}{2} + tT \right\|_1 dt \right) = \int_0^1 t f_a''\left(\left\| \frac{1-t}{2}V + \frac{1+t}{2}T \right\|_1 dt \right). \tag{17.50}$$

Using the change of variable $u = \frac{1+t}{2}$, then we get

$$\int_0^1 t f_a''\left(\left\| \frac{1-t}{2}T + \frac{1+t}{2}V \right\|_1 \right) dt \tag{17.51}$$
$$= 2\int_{\frac{1}{2}}^1 (2u-1) f_a''(\|(1-u)T + uV\|)\,du.$$

Also, by changing the variable $v = \frac{1-t}{2}$, we get

$$\int_0^1 t f_a''\left(\left\| \frac{1-t}{2}V + \frac{1+t}{2}T \right\|_1 dt \right) \tag{17.52}$$
$$= 2\int_0^{\frac{1}{2}} (1-2v) f_a''(\|(1-v)T + vV\|)\,dv.$$

Utilising the equalities (17.49)-(17.52) we obtain

$$\begin{aligned}&\int_0^1 tf_a''\left(\left\|(1-t)\frac{V+T}{2}+tV\right\|_1\right)dt+\int_0^1 tf_a''\left(\left\|(1-t)\frac{V+T}{2}+tT\right\|_1 dt\right)\\&=2\int_{\frac{1}{2}}^1(2t-1)\,f_a''(\|(1-t)\,T+tV\|)\,dt\\&+2\int_0^{\frac{1}{2}}(1-2t)\,f_a''(\|(1-t)\,T+tV\|)\,dt\\&=2\int_0^1|2t-1|\,f_a''(\|(1-t)\,T+tV\|)\,dt\\&=4\int_0^1\left|t-\frac{1}{2}\right|f_a''(\|(1-t)\,T+tV\|)\,dt\end{aligned}$$

for $T,V\in\mathcal{B}_1(H)$ with $\|T\|_1,\|V\|_1<R.$

Making use of (17.48) we deduce the first two inequalities in (17.41).

The rest is obvious. ∎

COROLLARY 17.9. *Let* $f(z):=\sum_{n=0}^{\infty}a_nz^n$ *be a power series with complex coefficients and convergent on the open disk* $D(0,R)\,,R>0$*. If* $T,V\in\mathcal{B}_1(H)$ *with* $\|T\|_1,\|V\|_1\le M<R$*, then*

$$\left|\frac{\operatorname{tr}[f(V)]+\operatorname{tr}[f(T)]}{2}-\operatorname{tr}\left[f\left(\frac{V+T}{2}\right)\right]\right|\le\frac{1}{8}\|V-T\|^2 f_a''(M)\,. \tag{17.53}$$

The constant $\frac{1}{8}$ *is best possible in (17.53).*

PROOF. From the first inequality in (17.41) we have

$$\begin{aligned}&\left|\frac{\operatorname{tr}[f(V)]+\operatorname{tr}[f(T)]}{2}-\operatorname{tr}\left[f\left(\frac{V+T}{2}\right)\right]\right|\\&\le\frac{1}{2}\|V-T\|^2\int_0^1\left|t-\frac{1}{2}\right|f_a''(\|(1-t)\,T+tV\|_1)\,dt\\&\le\frac{1}{2}\|V-T\|^2 f_a''(M)\int_0^1\left|t-\frac{1}{2}\right|dt=\frac{1}{8}\|V-T\|^2 f_a''(M)\end{aligned}$$

for $T,V\in\mathcal{B}_1(H)$ with $\|T\|_1,\|V\|_1\le M<R$, and the inequality is proved.

If we consider the scalar case and take $f(z)=z^2$, $V=a$, $T=b$ with $a,b\in\mathbb{R}$ then we get in both sides of (17.53) the same quantity $\frac{1}{4}(b-a)^2$. ∎

REMARK 17.3. A similar result holds by swapping the norm $\|\cdot\|$ with $\|\cdot\|_1$ in the right hand side of (17.41). The case of Hilbert-Schmidt norm may also be stated, however the details are not presented here.

If we write the inequality (17.41) for the exponential function, then we get

$$\begin{aligned}(17.54)\qquad &\left|\frac{\operatorname{tr}[\exp(V)]+\operatorname{tr}[\exp(T)]}{2}-\operatorname{tr}\left[\exp\left(\frac{V+T}{2}\right)\right]\right|\\ &\le \frac{1}{2}\|V-T\|^2\int_0^1\left|t-\frac{1}{2}\right|\exp\left(\|(1-t)T+tV\|_1\right)dt\\ &\le \frac{1}{24}\|V-T\|^2\left[\exp\left(\left\|\frac{V+T}{2}\right\|_1\right)+\frac{\exp(\|V\|_1)+\exp(\|T\|_1)}{2}\right]\\ &\le \frac{1}{12}\|V-T\|^2\left[\exp(\|V\|_1)+\exp(\|T\|_1)\right]\\ &\le \frac{1}{6}\|V-T\|^2\max\left\{\exp(\|V\|_1),\exp(\|T\|_1)\right\}.\end{aligned}$$

for any for $T,V\in\mathcal{B}_1(H)$.

If $T,V\in\mathcal{B}_1(H)$ with $\|V\|_1,\|T\|_1\le M$, then

$$(17.55)\qquad \left|\frac{\operatorname{tr}[\exp(V)]+\operatorname{tr}[\exp(T)]}{2}-\operatorname{tr}\left[\exp\left(\frac{V+T}{2}\right)\right]\right|\le\frac{1}{8}\|V-T\|^2\exp(M).$$

If we write the inequality (17.41) for the function $f(z)=(1-z)^{-1}$, then we get

$$\begin{aligned}(17.56)\qquad &\left|\frac{\operatorname{tr}\left[(1_H-V)^{-1}\right]+\operatorname{tr}\left[(1_H-T)^{-1}\right]}{2}-\operatorname{tr}\left[\left(1_H-\frac{V+T}{2}\right)^{-1}\right]\right|\\ &\le \|V-T\|^2\int_0^1\left|t-\frac{1}{2}\right|\left(1-\|(1-t)T+tV\|_1\right)^{-3}dt\\ &\le\frac{1}{12}\|V-T\|^2\\ &\times\left[\left(1-\left\|\frac{V+T}{2}\right\|_1\right)^{-3}+\frac{(1-\|V\|_1)^{-3}+(1-\|T\|_1)^{-3}}{2}\right]\\ &\le\frac{1}{6}\|V-T\|^2\left[(1-\|V\|_1)^{-3}+(1-\|T\|_1)^{-3}\right]\\ &\le\frac{1}{3}\|V-T\|^2\max\left\{(1-\|V\|_1)^{-3},(1-\|T\|_1)^{-3}\right\},\end{aligned}$$

for any for $T,V\in\mathcal{B}_1(H)$ with $\|V\|_1,\|T\|_1<1$.

Moreover, if $\|V\|_1,\|T\|_1\le M<1$, then

$$\begin{aligned}(17.57)\qquad &\left|\frac{\operatorname{tr}\left[(1_H-V)^{-1}\right]+\operatorname{tr}\left[(1_H-T)^{-1}\right]}{2}-\operatorname{tr}\left[\left(1_H-\frac{V+T}{2}\right)^{-1}\right]\right|\\ &\le\frac{1}{4}\|V-T\|^2(1-M)^{-3}.\end{aligned}$$

The interested reader may choose other examples of power series to get similar results. However, the details are not presented here.

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