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A NEW RELAXED COMPLEX-VALUED b-METRIC TYPE AND FIXED POINT RESULTS

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ABSTRACT. In this paper, we study the existence and uniqueness of fixed point in complex valued b-metric spaces and introduce a new relaxed α, β Complex-valued b-metric type by relaxing the triangle inequality and determine whether the fixed point theorems are applicable in these spaces.

Key words and phrases: b-metric; Complex-valued b-metric space; Fixed point theorem.

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1. INTRODUCTION

The concept of a b-metric was initiated from the contributions of Bourbaki [2] and Bakhtin [1]. Czerwik [3] gave an axiom which was weaker than the triangular inequality and formally defined a b-metric space with a view of generalizing the Banach contraction mapping theorem. Later on, Fagin et al. [4] discussed some kind of relaxation in the triangular inequality and called this new distance measure a non-linear elastic pattern matching. In 2011, A. Azzam, B. Fisher and M. Khan introduced the notion of a complex valued metric space and called the complex-valued metric space as an extension of the classical metric space and proved some common fixed point theorems, [5]. In a similar way various authors have studied and proved the fixed point results for mappings satisfying different types of contractive conditions in the framework of complex-valued metric spaces,[6]. In 2013, Rao, et al. introduced the concept of a complex-valued b-metric space which is a generalization of the concept of a complex-valued metric space, [8] and subsequent to that A.A Mukheimer obtained common fixed point results, [7]. In this paper, we generalize the concept of a complex-valued b-metric and prove the common fixed results satisfying certain expressions in this new space.

2. PRELIMINARIES

Let $\mathbb{C}$ be the set of complex numbers and if $z_1, z_2 \in \mathbb{C}$ then define a partial ordering $\precsim$ on $\mathbb{C}$ as follows:
$z_1 \precsim z_2 \iff \text{Re}(z_1) \le \text{Re}(z_2)$ and $\text{Im}(z_1) \le \text{Im}(z_2)$

Futhermore, if $z_3 \in \mathbb{C}$, we obtain that following:
(i) If $0 \precsim z_1 \precnsim z_2$ then $|z_1| < |z_2|$
(ii)If $z_1 \precsim z_2$ and $z_2 \precsim z_3$ then $z_1 \precsim z_3$
(iii) If $a, b \in \mathbb{R}$ and $a \le b$ then $az \precsim bz$ for all $z \in \mathbb{C}$

Definition 2.1. Let X be a non-empty set. A function $d : X \times X \to \mathbb{C}$ is a complex-valued b-metric on X, [8], if there exists a real number $\alpha \ge 1$ such that the following conditions hold for all $x, y, z \in X$:

(i) $0 \precsim d(x,y)$ and $d(x,y) = 0 \iff x = y$
(ii) $d(x,y) = d(y,x)$
(iii) $d(x,y) \precsim \alpha\,[d(x,z) + d(z,y)]$

The pair (X, d) is a called a complex-valued b-metric space.

Definition 2.2. Let X be a non-empty set. A function $\rho : X \times X \to \mathbb{C}$ is a generalized α, β complex-valued b-metric on X if there exists real numbers $\alpha, \beta \ge 1$ such that the following conditions hold for all $x, y, z \in X$:

(i) $0 \precsim \rho(x,y)$ and $\rho(x,y) = 0 \iff x = y$
(ii) $\rho(x,y) = \rho(y,x)$
(iii) $\rho(x,y) \precsim \alpha\rho(x,z) + \beta\rho(z,y)$

The pair (X, ρ) is a called a α, β complex-valued b-metric space.
The following example justifies the generalization found in the definition.

Example 2.1. *Let $X = (1,3)$ and let $\rho : X \times X \to \mathbb{C}$ be a function defined by*

$$\rho(x,y) = \begin{cases} e^{|x-y|} + ie^{|x-y|}, & \textit{if } x \neq y \\ 0, & \textit{iff } x = y. \end{cases}$$

To show that the example is a generalized α,β complex-valued b-metric, we only need to verify the α,β triangle inequality:
For $x \neq y,\ z \in X$ and $\theta \in (0,1)$

$$\begin{aligned}
\rho(x,y) &\precsim (1+i)e^{|x-z|+|z-y|} \\
&= (1+i)e^{\theta|x-z|+(1-\theta)|z-y|}e^{(1-\theta)|x-z|+\theta|z-y|} \\
&\precsim \sup_{x,y,z\in X} e^{\theta|x-z|+(1-\theta)|z-y|}\left((1-\theta)(1+i)e^{|x-z|}+\theta(1+i)e^{|z-y|}\right) \\
&\precsim (1-\theta)e^2(1+i)e^{|x-z|}+\theta(1+i)e^2e^{|z-y|} \\
&= (1-\theta)e^2\rho(x,z)+\theta e^2\rho(z,y).
\end{aligned}$$

For $\theta = \frac{1}{3}$ we have constants $\alpha = \frac{2}{3}e^2$ and $\beta = \frac{1}{3}e^2$.

One introduces a topology on a generalized α,β b-metric space (X,ρ) in the usual way. The open ball $B(x,\epsilon)$ with centre $x \in X$ and radius $0 \prec \epsilon \in \mathbb{C}$ is given by

$$B(x,\epsilon) = \{y \in X : \rho(x,y) \prec \epsilon\}$$

A subset A of X is open if for every $x \in A$ there is a number $0 \prec \epsilon \in \mathbb{C}$ such that $B(x,\epsilon) \subseteq A$.

Definition 2.3. Let (X,ρ) be a generalized α,β complex-valued b-metric space, and let $\{x_n\}$ be a sequence in X and $x \in X$. Then:

(i) The sequence $\{x_n\}$ converges to $x \in X$, if for every $0 \prec \epsilon \in \mathbb{C}$ then there is $N \in \mathbb{N}$ such that $\rho(x_n,x) \prec \epsilon$. The sequence $\{x_n\}$ converges to $x \in X \iff |\rho(x_n,x)| \to 0$ as $n \to \infty$, [8].
(ii) The sequence $\{x_n\}$ is a Cauchy in (X,ρ) if for every $\epsilon \in \mathbb{C}$ there is $N \in \mathbb{N}$ such that $\rho(x_n,x_{n+m}) \prec \epsilon$, where $m \in \mathbb{N}$. The sequence $\{x_n\}$ is a Cauchy in $(X,\rho) \iff |\rho(x_n,x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$, [8].
(iii) The space (X,ρ) is complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$.

3. FIXED POINT THEOREM FOR GENERALIZED α,β COMPLEX-VALUED b-METRIC SPACES

As a consequence of (iii) of definition 2.2 , the α,β triangle inequality, we get for $n,m \in \mathbb{N}$ with $m > n$

$$\begin{aligned}
&\rho(x_n,x_m) \\
&\precsim \alpha\rho(x_n,x_{n+1})+\beta\rho(x_{n+1},x_m) \\
&\precsim \alpha\rho(x_n,x_{n+1})+\beta[\alpha\rho(x_{n+1},x_{n+2})+\beta\rho(x_{n+2},x_m)]
\end{aligned} \tag{3.1}$$

Successively applying the α,β triangle inequality, we obtain

$$\rho(x_n,x_m) \precsim \alpha\sum_{i=0}^{m-n-2}\beta^i\rho(x_{n+i},x_{n+i+1})+\beta^{m-n-1}\rho(x_{m-1},x_m). \tag{3.2}$$

This theorem is a generalization of the fixed point theorem studied by Mishra, et al., in [9].

Theorem 3.1. *Let (X,ρ) be a complete generalized α,β complex-valued b-metric space and $T: X \to X$ a mapping such that*

$$\rho(Tx,Ty) \precsim a\rho(x,Tx)+b\rho(y,Ty)+c\rho(x,y)$$

for all $x, y \in X$, where a, b, and c are non-negative real numbers satisfying $a + \beta(b + c) < 1$, then T has a unique fixed point.

Proof. We begin by proving that for $x_0 \in X$, the sequence $\{x_n\}$ generated by the recussive formula $x_n = Tx_{n-1} = T^n x_0$ is a Cauchy sequence in X. For $n \in \mathbb{N}$ we obtain

$$\begin{aligned}
\rho(x_{n+2}, x_{n+1}) &= \rho(Tx_{n+1}, Tx_n) \qquad (3.3)\\
&\precsim a\rho(x_{n+1}, Tx_{n+1}) + b\rho(x_n, Tx_n) + c\rho(x_{n+1}, x_n)\\
&\precsim a\rho(x_{n+2}, x_{n+1}) + b\rho(x_{n+1}, x_n) + c\rho(x_{n+1}, x_n)\\
&\precsim a\rho(x_{n+2}, x_{n+1}) + (b + c)\rho(x_{n+1}, x_n) \qquad (3.4)
\end{aligned}$$

$$(1 - a)\rho(x_{n+2}, x_{n+1}) \precsim (b + c)\rho(x_{n+1}, x_n)$$

$$\rho(x_{n+2}, x_{n+1}) \precsim \left(\frac{b + c}{1 - a}\right)\rho(x_{n+1}, x_n) \qquad (3.5)$$

If we let $\gamma = \left(\frac{b+c}{1-a}\right)$ then repeated use of (3.5) we get

$$\rho(x_{n+2}, x_{n+1}) \precsim \gamma^{n+1}\rho(x_1, x_0). \qquad (3.6)$$

Using (3.2) and (3.6) for $k \in \mathbb{N}$, we get

$$\begin{aligned}
\rho(x_n, x_{n+k+1}) &\precsim \alpha\sum_{i=0}^{k-1}\beta^i\rho(x_{n+i}, x_{n+i+1}) + \beta^k\rho(x_{n+k}, x_{n+k+1})\\
&\precsim \alpha\sum_{i=0}^{k-1}\beta^i\gamma^{n+i}\rho(x_0, x_1) + \beta^k\gamma^{n+k}\rho(x_0, x_1)\\
&= \gamma^n\left[\alpha\sum_{i=0}^{k-1}\beta^i\gamma^i + \beta^k\gamma^k\right]\rho(x_0, x_1)\\
&= \gamma^n\left[\alpha\frac{1 - \beta^k\gamma^k}{1 - \beta\gamma} + \beta^k\gamma^k\right]\rho(x_0, x_1)\\
&= \frac{\gamma^n}{(1 - \beta\gamma)}\left[\alpha - \beta^k\gamma^k(\alpha + \beta\gamma - 1)\right]\rho(x_0, x_1)\\
&\prec \gamma^n\frac{\alpha}{(1 - \beta\gamma)}\rho(x_0, x_1). \qquad (3.7)
\end{aligned}$$

Now,

$$|\rho(x_n, x_{n+k+1})| \leq \gamma^n\frac{\alpha}{(1 - \beta\gamma)}\rho(x_0, x_1) \qquad (3.8)$$

Since $a + \beta(b + c) < 1$ for $\beta \geq 1$ then $\beta\gamma < 1$ and $\gamma < 1$. Taking the limit $n \to \infty$ we get $\gamma^n \to 0$, which implies that $|\rho(x_n, x_{n+k+1})| \to 0$ as $n \to \infty$ thus the sequence $\{x_n\}$ is a Cauchy sequence. Since X is a complete α, β complex-valued b-metric space the sequence converges to $x^* \in X$. We show that x^* is a fixed point of T. Using the α, β triangle inequality, we have

$$\begin{aligned}
\rho(x^*, Tx^*) &\precsim \alpha\rho(x^*, x_{n+1}) + \beta\rho(x_{n+1}, Tx^*)\\
&\precsim \alpha\rho(x^*, x_{n+1}) + \beta\rho(Tx_n, Tx^*)\\
&\precsim \alpha\rho(x^*, x_{n+1}) + \beta[a\rho(x_n, Tx_n) + b\rho(x^*, Tx^*) + c\rho(x_n, x^*)] \qquad (3.9)
\end{aligned}$$

$$(1-b\beta)\rho(x^*,Tx^*) \precsim \alpha\rho(x^*,x_{n+1}) + a\beta\rho(x_n,Tx_n) + c\beta\rho(x_n,x^*)$$

$$\rho(x^*,Tx^*) \precsim \frac{1}{(1-b\beta)}\left[\alpha\rho(x^*,x_{n+1}) + a\beta\rho(x_n,x_{n+1}) + c\beta\rho(x_n,x^*)\right] \tag{3.10}$$

Taking the absolute value of both sides, we get

$$\begin{aligned} |\rho(x^*,Tx^*)| &\leq \frac{1}{(1-b\beta)}\left|\left[\alpha\rho(x^*,x_{n+1}) + a\beta\rho(x_n,x_{n+1}) + c\beta\rho(x_n,x^*)\right]\right| \\ &\leq \frac{1}{(1-b\beta)}\left|\alpha\rho(x^*,x_{n+1})\right| + a\beta\gamma^n\left|\rho(x_0,x_1)\right| + c\beta\left|\rho(x_n,x^*)\right| \end{aligned} \tag{3.11}$$

Since x_n converges to x^* , taking limit $n\to\infty$ implies that $|\rho(x^*,Tx^*)|\to 0$ which yields $x^*=Tx^*$. To show uniqueness of the fixed point. Assume that there exists $x^{**}\in X$ such that $Tx^{**}=x^{**}$. Then

$$\rho(x^{**},x^*) = \rho(Tx^{**},Tx^*) \precsim a\rho(x^{**},Tx^{**}) + b\rho(x^*,Tx^*) + c\rho(x^*,x^{**}) \tag{3.12}$$

which implies that $\rho(x^{**},x^*)\precsim c\rho(x^*,x^{**})$. Taking the absolute value of both sides, we get $|\rho(x^{**},x^*)|\leq c\,|\rho(x^*,x^{**})|$. This implies that $\rho(x^*,x^{**})=0$. Thus $x^*=x^{**}$. ∎

Theorem 3.2. *Let (X,ρ) be a complete generalized complex-valued α,β b-metric space and let $T:X\to X$ be a mapping such that*

$$\rho(Tx,Ty) \precsim a\rho(x,Ty) + b\rho(y,Tx) \tag{3.13}$$

for every $x,y\in X$, where a,b are non-negative constants with $\beta b<\frac{1}{1+\alpha}$. Then T has a fixed point in X and has a unique fixed point if $a+b<1$.

Proof. Let $x_0\in X$ be fixed then consider the sequence generated by the formula
$x_n=Tx_{n-1}=T^nx_0$. Let $n\in\mathbb{N}$ then we get

$$\begin{aligned} \rho(x_{n+2},x_{n+1}) &= \rho(Tx_{n+1},Tx_n) \\ &\precsim a\rho(x_{n+1},Tx_n) + b\rho(x_n,Tx_{n+1}) \\ &= a\rho(x_{n+1},x_{n+1}) + b\rho(x_n,x_{n+2}) \\ &= b\rho(x_n,x_{n+2}) \\ &\precsim b\alpha\rho(x_n,x_{n+1}) + b\beta\rho(x_{n+1},x_{n+2}) \\ (1-b\beta)\rho(x_{n+2},x_{n+1}) &\precsim b\alpha\rho(x_n,x_{n+1}) \\ \rho(x_{n+2},x_{n+1}) &\precsim \tfrac{b\alpha}{1-b\beta}\rho(x_n,x_{n+1}) \end{aligned} \tag{3.14}$$

Letting $\gamma=\frac{b\alpha}{1-b\beta}$ and repeated use of (3.14) yields

$$\begin{aligned} \rho(x_{n+1},x_{n+2}) &\precsim \gamma\rho(x_n,x_{n+1}) \\ &\precsim \gamma^2\rho(x_{n-1},x_n) \\ &\;\;\vdots \\ &\precsim \gamma^{n+1}\rho(x_0,x_1) \end{aligned} \tag{3.15}$$

Let $m \in \mathbb{N}$ then

$$\begin{aligned}
&\rho(x_n, x_{n+m})\\
&\precsim \alpha\rho(x_n, x_{n+1}) + \beta\rho(x_{n+1}, x_{n+m})\\
&\precsim \alpha\rho(x_n, x_{n+1}) + \beta\left[\alpha\rho(x_{n+1}, x_{n+2}) + \beta\rho(x_{n+2}, x_{n+m})\right]\\
&\precsim \alpha\rho(x_n, x_{n+1}) + \alpha\beta\rho(x_{n+1}, x_{n+2}) + \cdots + \alpha\beta^{m-2}\rho(x_{n+m-2}, x_{n+m-1}) + \beta^{m-1}\rho(x_{n+m-1}, x_{n+m})\\
&\precsim \alpha\gamma^n\rho(x_0, x_1) + \alpha\beta\gamma^{n+1}\rho(x_0, x_1) + \cdots + \alpha\gamma^{n+m-2}\beta^{m-2}\rho(x_0, x_1) + \beta^{m-1}\gamma^{n+m-1}\rho(x_0, x_1)\\
&\precsim \alpha\gamma^n\rho(x_0, x_1)\left[1 + \beta\gamma + \cdots + \gamma^{m-2}\beta^{m-2}\right] + \beta^{m-1}\gamma^{n+m-1}\rho(x_0, x_1)\\
&\precsim \alpha\gamma^n\rho(x_0, x_1)\left[\tfrac{1-(\beta\gamma)^{m-1}}{1-\beta\gamma}\right] + \beta^{m-1}\gamma^{n+m-1}\rho(x_0, x_1)\\
&\precsim \alpha\tfrac{\gamma^n}{1-\beta\gamma}\rho(x_0, x_1)[\alpha - (\beta\gamma)^{m-1}(\alpha - 1 + \beta\gamma)]\\
&\prec \alpha\tfrac{\gamma^n}{1-\beta\gamma}\rho(x_0, x_1)
\end{aligned}$$

Since $\beta b < \frac{1}{1+\alpha}$ then $0 < \gamma < \frac{1}{\beta}$ and $\beta\gamma < 1$. Taking the limit $n \to \infty$, we get $\gamma^n \to 0$. This implies that $|\rho(x_n, x_{n+m})| \to 0$ as $n \to \infty$. The sequence $\{x_n\}$ is a Cauchy sequence in X. Since (X, ρ) is a complete complex-valued b-metric space then $\{x_n\}$ converges to $x^* \in X$.
We show that x^* is a fixed point of T.

$$\begin{aligned}
\rho(x^*, Tx^*) &\precsim \alpha\rho(x^*, x_n) + \beta\rho(x_n, Tx^*)\\
&= \alpha\rho(x^*, x_n) + \beta\rho(Tx_{n-1}, Tx^*)\\
&\precsim \alpha\rho(x^*, x_n) + \beta\left[a\rho(x_{n-1}, Tx^*) + b\rho(Tx_{n-1}, Tx^*)\right]\\
&= \alpha\rho(x^*, x_n) + \beta a\rho(x_{n-1}, Tx^*) + \beta b\rho(x_n, x^*)\\
&\precsim (\alpha + \beta b)\rho(x_n, x^*) + \beta a\alpha\rho(x_{n-1}, x^*) + a\beta^2\rho(x^*, Tx^*)
\end{aligned}$$

$$[1 - a\beta^2]\rho(x^*, Tx^*) \precsim (\alpha + \beta b)\rho(x_n, x^*) + a\beta\alpha\rho(x_{n-1}, x^*) \tag{3.16}$$

Since $\{x_n\}$ converges to x^* we get $|\rho(x^*, x_n)| \to 0$ as $n \to \infty$, and taking the absolute value of both sides of (3.16) we obtain $|\rho(x^*, Tx^*)| \le 0$ thus $\rho(x^*, Tx^*) = 0$, which implies that $Tx^* = x^*$.
To prove the uniqueness of the fixed point we assume that there $x^*, x^{**} \in X$ such that $Tx^* = x^*$ and $Tx^{**} = x^{**}$. Now

$$\rho(x^*, x^{**}) = \rho(Tx^*, Tx^{**}) \tag{3.17}$$
$$\precsim a\rho(x^*, Tx^{**}) + b\rho(x^{**}, Tx^*) \tag{3.18}$$
$$= a\rho(x^*, x^{**}) + b\rho(x^{**}, x^*) \tag{3.19}$$
$$= (a + b)\rho(x^*, x^{**}) \tag{3.20}$$

Thus we get $|\rho(x^*, x^{**})| \le |(a + b)|\,|\rho(x^*, x^{**})|$. Since $a + b < 1$ we get $|\rho(x^*, x^{**})| = 0$ thus $x^* = x^{**}$.
∎

Author Kir et. al,, studied the following fixed point theorem in b-metric spaces and we generalized the result into a α, β complex-valued b-metric spaces, [10].

Theorem 3.3. *Let (X, ρ) be a α, β complex-valued b-metric space and let $T : X \to X$ be a mapping such that*

$$\rho(Tx, Ty) \precsim \lambda[\rho(x, Tx) + \rho(y, Ty)],$$

where $\lambda \in [0, \frac{1}{2})$, for all $x, y \in X$. Then T has a unique fixed point.

Proof. We begin by showing that for $x_0 \in X$ fixed,the sequence $\{x_n\}$ where $x_n = Tx_{n-1} = T^n x_0$ is a Cauchy sequence in (X, ρ).
For $n \in \mathbb{N}$, we have

$$\begin{aligned}
\rho(x_{n+2}, x_{n+1}) &= \rho(Tx_{n+1}, Tx_n) \\
&\preccurlyeq \lambda[\rho(x_{n+1}, Tx_{n+1}) + \rho(x_n, Tx_n)] \\
&= \lambda[\rho(x_{n+1}, x_{n+2}) + \rho(x_n, Tx_n)] \\
(1-\lambda)\rho(x_{n+2}, x_{n+1}) &\preccurlyeq \lambda\rho(x_{n+1}, x_n) \\
\rho(x_{n+2}, x_{n+1}) &\preccurlyeq \tfrac{\lambda}{(1-\lambda)}\rho(x_{n+1}, x_n) \qquad (3.21)
\end{aligned}$$

Repeated use of (3.21), for $n \in \mathbb{N}$, we get

$$\rho(x_{n+2}, x_{n+1}) \preccurlyeq \left(\tfrac{\lambda}{1-\lambda}\right)^{n+1} \rho(x_1, x_0).$$

For $m, n \in X$

$$\begin{aligned}
&\rho(x_n, x_{n+m}) \\
&\preccurlyeq \alpha\rho(x_n, x_{n+1}) + \beta\rho(x_{n+1}, x_{n+m}) \\
&\preccurlyeq \alpha\rho(x_n, x_{n+1}) + \beta\alpha\rho(x_{n+1}, x_{n+2}) + \beta^2\rho(x_{n+2}, x_{n+m}) \\
&\preccurlyeq \alpha\rho(x_n, x_{n+1}) + \alpha\beta\rho(x_{n+1}, x_{n+2}) + \cdots + \alpha\beta^{m-2}\rho(x_{n+m-2}, x_{n+m-1}) \\
&+ \beta^{m-1}\rho(x_{n+m-1}, x_{n+m}) \\
&\preccurlyeq \alpha\left(\tfrac{\lambda}{1-\lambda}\right)^n \rho(x_0, x_1) + \alpha\beta\left(\tfrac{\lambda}{1-\lambda}\right)^{n+1} \rho(x_0, x_1) + \cdots + \alpha\beta^{m-2}\left(\tfrac{\lambda}{1-\lambda}\right)^{n+m-2} \rho(x_0, x_1) \\
&+ \beta^{m-1}\left(\tfrac{\lambda}{1-\lambda}\right)^{n+m-1} \rho(x_0, x_1) \\
&= \alpha\left(\tfrac{\lambda}{1-\lambda}\right)^n \rho(x_0, x_1)\left[1 + \beta\left(\tfrac{\lambda}{1-\lambda}\right) + \cdots + \beta^{m-2}\left(\tfrac{\lambda}{1-\lambda}\right)^{m-2}\right] + \beta^{m-1}\left(\tfrac{\lambda}{1-\lambda}\right)^{n+m-1} \rho(x_0, x_1)
\end{aligned}$$

Since $\lambda \in [0, \frac{1}{2})$ implies that $0 \leq \frac{\lambda}{1-\lambda} < 1$. Taking the absolute value of both sides we get $|\rho(x_n, x_m)| \to 0$ as $n \to \infty$. It follows that that the sequence $\{x_n\}$ is a Cauchy sequence in (X, ρ). Since (X, ρ) is complete there exists a $x^* \in X$ such that

$$\lim_{n\to\infty} \rho(x_n, x^*) = 0.$$

We now show that x^* is a fixed point of the mapping T.

$$\begin{aligned}
\rho(x^*, Tx^*) &\preccurlyeq \alpha\rho(x^*, x_n) + \beta\rho(x_n, Tx^*) \\
&= \alpha\rho(x^*, x_n) + \beta\rho(Tx_{n-1}, Tx^*) \\
&\preccurlyeq \alpha\rho(x^*, x_n) + \beta\lambda\rho(x_{n-1}, Tx_{n-1}) + \beta\lambda\rho(x^*, Tx^*) \\
[1-\beta\lambda]\,\rho(x^*, Tx^*) &\preccurlyeq \alpha\rho(x^*, x_n) + \beta\lambda\rho(x_{n-1}, x_n) \qquad (3.22)
\end{aligned}$$

Taking the absolute value of both sides of (3.22) and taking $n \to \infty$, we obtain $\rho(x^*, Tx^*) = 0$. This implies that x^* is a fixed point of T.

To prove the uniqueness of the fixed point we assume that there $x^*, x^{**} \in X$ such that $Tx^* = x^*$ and $Tx^{**} = x^{**}$. Now

$$\begin{aligned}
\rho(x^*, x^{**}) &= \rho(Tx^*, Tx^{**}) \\
&\preccurlyeq \lambda\left[\rho(x^*, Tx^*) + \rho(x^{**}, Tx^{**})\right] \\
&= 0
\end{aligned}$$

Thus we get $|\rho(x^*, x^{**})| \leq 0$, which implies that $x^* = x^{**}$. Hence x^* is a unique fixed point.

∎

4. CONCLUSION

In this paper we have presented a relaxed α, β complex-valued b-metric and proved some fixed point results for this new class of a generalized metric.The generalization may bring a wider applications of fixed point results. We have shown that for complex-valued b-metric spaces the mappings that have fixed points have fixed points in the generalized space.

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