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HANKEL FUNCTIONAL CONNECTED TO LEMNISCATE OF BERNOULLI

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ABSTRACT. The aim of present paper is to derive a higher bound (HB) of 3^{rd} order Hankel determinant for a collection of holomorphic mappings connected with exactly to the right side of the lemniscate of Bernoulli, whose polar coordinates form is $r^2 = 2\cos^2(2\theta)$. The method carried in this paper is more refined than the method adopted by the authors (see [1]), who worked on this problem earlier.

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1. ORIGINATION

Let $\mathcal{A}$ represent a collection of entire holomorphic mappings f, namely

$$f(z) = z + \sum_{i\geq 2} a_i z^i \tag{1.1}$$

in the unit disc $\mathcal{U}_d = \{z \in \mathcal{C} : |z| < 1\}$ whose subfamily is denoted by S, possessing functions which are univalent. A correspondence t_1, which is regular is subordinate to another regular correspondence t_2, expressed as $t_1 \prec t_2$, if and only if there occurs a Schwarz's function ν, analytic in $\mathcal{U}_d$ satisfying the properties $\nu(0) = 0$ and $|\nu(z)| < 1$ with $t_1(z) = t_2(\nu(z))$. In specific, if t_2 is one-to-one (univalent) in $\mathcal{U}_d$ then $t_1(0) = t_2(0)$ and $t_1(\mathcal{U}_d) \subset t_2(\mathcal{U}_d)$.

The q^{th} order Hankel functional for the regular mapping f, was defined by Pommerenke [2], which has been investigated by many authors, as follows.

$$H_{q,t}(f) = \begin{vmatrix} a_t & a_{t+1} & \cdots & a_{t+q-1} \\ a_{t+1} & a_{t+2} & \cdots & a_{t+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{t+q-1} & a_{t+q} & \cdots & a_{t+2q-2} \end{vmatrix} \tag{1.2}$$

Here $a_1 = 1$, q and t are positive integers.

In recent years, the research has concentrated on the estimation of $H_{2,2}(f)$, known as the second Hankel functional obtained for $q = 2 = t$ in (1.2), given by

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

Many authors obtained results associated with estimation of HB of the functional $H_{2,2}(f)$ for dissimilar sub-collections of univalent and multivalent holomorphic mappings. For instance $q = 3$ seems to be more tough than for $q = 2$. A small number of papers have been devoted for the study of 3rd order Hankel determinant denoted by $H_{3,1}(f)$, obtained for $q = 3$ and $t = 1$ in (1.2), gives

$$H_{3,1}(f) = a_5(a_3 - a_2^2) - a_4(a_4 - a_2a_3) + a_3(a_2a_4 - a_3^2). \tag{1.3}$$

The concept of estimation of an upper bound for $H_{3,1}(f)$ was firstly introduced and studied by Babalola [3], who tried to estimate for this functional to the classes $\mathcal{R}$, S^* and $\mathcal{K}$, obtained as follows:

(i) $f \in S^* \Rightarrow |H_{3,1}(f)| \leq 16$
(ii) $f \in \mathcal{K} \Rightarrow |H_{3,1}(f)| \leq 0.714$
(iii) $f \in \mathcal{R} \Rightarrow |H_{3,1}(f)| \leq 0.742$

By the motivation of the results derived by many authors who are working in this direction (see [4, 5, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]), in this paper, we are making an attempt to adopt more concise method than the procedure used by the authors Raza and Malik [1], who worked on this problem earlier in obtaining HB to the functional $H_{3,1}(f)$ for the member f belonging to the following class.

Definition 1.1. *A member f of $\mathcal{A}$ is in the family $\mathcal{SL}^*$, consisting of regular mappings connected to leminscate of Bernoulli*

$$\Leftrightarrow \left\{\frac{zf'(z)}{f(z)}\right\} \prec \{1+z\}^{\frac{1}{2}} = q(z), \quad z \in \mathcal{U}_d.$$

In proving our result, we require a few sharp estimates in the form of lemmas valid for mappings possessing +ve real part.

Let $\overline{\omega}$ indicate the family of members consisting of g, where g is the Caratheodòry function [14], of the form

$$g(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots, \tag{1.4}$$

which are regular with Re $g(z) > 0$ in $\mathcal{U}_d$.

Lemma 1.2. *([16]) If $g \in \overline{\omega}$, then the inequality $|c_n - \eta c_k c_{n-k}| \leq 2$, holds for the positive integers n, k, with $n > k$ and $\eta \in [0, 1]$.*

Lemma 1.3. *([17]) If $g \in \overline{\omega}$, then the inequality $|c_n - c_k c_{n-k}| \leq 2$, satisfies for the natural numbers n, k, with $n > k$*

Lemma 1.4. *([18]) If $g \in \overline{\omega}$ then $|c_\sigma| \leq 2$, for each $\sigma \geq 1$ and the external function is $g(z) = \frac{1+z}{1-z}$, $z \in \mathcal{U}_d$.*

Lemma 1.5. *([19]) If $f \in \mathcal{SL}^*$, then $|a_5| \leq \frac{1}{8}$ and this inequality becomes equality for the function $z + \sum_{j=2}^{\infty} \frac{1}{2j} z^j$.*

Lemma 1.6. *([20]) If $g \in \overline{\omega}$, then $|c_3 - 2c_1c_2 + c_1^3| \leq 2$.*

In order to obtain our result, we referred to the sophisticated procedure prepared by Libera and Zlotkiewicz [21], widely performed by a large number of authors.

2. MAIN RESULT

Theorem 2.1. *If $f \in \mathcal{SL}^*$ given in (1.1) then $|H_{3,1}(f)| \leq \dfrac{43}{576}$.*

Proof. For the function $f \in \mathcal{SL}^*$, from its Definition given in 1.1,

$$\frac{zf'(z)}{f(z)} \prec q(z) = \{1 + z\}^{\frac{1}{2}}. \tag{2.1}$$

By the principle of subordination, there occurs a Schwarz's function $\nu(z)$ with $|\nu(z)| < 1$

$$\left[\frac{zf'(z)}{f(z)}\right] = [q\{\nu(z)\}], \ z \in \mathcal{U}_d. \tag{2.2}$$

Construct a mapping

$$\nu(z) = \frac{g(z) - 1}{g(z) + 1} \Leftrightarrow g(z) = \frac{1 + \nu(z)}{1 - \nu(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{2.3}$$

From (2.1), (2.2) and (2.3), we obtain

$$\left\{\frac{zf'(z)}{f(z)}\right\} = [q\{\nu(z)\}] = \left\{1 + \frac{g(z) - 1}{g(z) + 1}\right\}^{\frac{1}{2}} = \left\{\frac{2g(z)}{g(z) + 1}\right\}^{\frac{1}{2}}. \tag{2.4}$$

Considering the expressions for f, f' and g in (2.4), which simplifies to

$$\begin{aligned}
&1 + a_2 z + \left[2a_3 - a_2^2\right] z^2 + \left[3a_4 - 3a_2a_3 + a_2^3\right] z^3 \\
&\quad + \left[4a_5 - 4a_2a_4 - 2a_3^2 + 4a_3a_2^2 - a_2^4\right] z^4 + \dots \\
&= 1 + \frac{1}{4}c_1 z + \left[\frac{1}{4}c_2 - \frac{5}{32}c_1^2\right] z^2 + \left[\frac{1}{4}c_3 - \frac{5}{16}c_1c_2 + \frac{13}{128}c_1^3\right] z^3 \\
&\quad + \left[\frac{1}{4}c_4 - \frac{5}{32}c_2^2 - \frac{5}{16}c_1c_3 + \frac{39}{128}c_1^2c_2 - \frac{141}{2048}c_1^4\right] z^4 + \dots
\end{aligned} \tag{2.5}$$

Equating the coefficients of z, z^2, z^3 and z^4 in (2.5), after simplifying, we obtain

$$a_2 = \frac{c_1}{4};\ a_3 = \frac{1}{8}\left(c_2 - \frac{3}{8}c_1^2\right);\ a_4 = \frac{1}{48}\left(4c_3 - \frac{7}{2}c_1c_2 + \frac{13}{16}c_1^3\right);$$

$$a_5 = \frac{1}{96}\left(6c_4 - 3c_2^2 - \frac{11}{2}c_1c_3 + \frac{17}{4}c_1^2c_2 - \frac{49}{64}c_1^4\right). \tag{2.6}$$

Putting the values namely a_2, a_3, a_4 and a_5 from (2.6) in the functional given in (1.3), it simplifies to

$$\begin{aligned} H_{3,1}(f) = a_5\left[\frac{1}{8}\left(c_2 - \frac{7}{8}c_1^2\right)\right] - a_4\left[\frac{1}{384}\left(32c_3 - 40c_1c_2 + c_1^3\right)\right] \\ + a_3\left[\frac{1}{12288}\left(256c_1c_3 - 192c_2^2 - 80c_1^2c_2 + 25c_1^4\right)\right]. \end{aligned} \tag{2.7}$$

Grouping the terms in the above expression, we have

$$\begin{aligned} H_{3,1}(f) = a_5\frac{1}{8}\left[\left(c_2 - \frac{7}{8}c_1^2\right)\right] - a_4\frac{1}{384}\left[11\left(c_3 - 2c_1c_2 + c_1^3\right) + 18\left(c_3 - c_1c_2\right) + 3c_3\right] \\ + a_3\frac{1}{12288}\left[256c_1\left(c_3 - \frac{192}{256}c_2^2\right) - 80c_1^2\left(c_2 - \frac{25}{80}c_1^2\right)\right]. \end{aligned} \tag{2.8}$$

Applying the triangle inequality in the expression (2.8), we obtain

$$\begin{aligned} |H_{3,1}(f)| \le \frac{|a_5|}{8}\left[\left|c_2 - \frac{7}{8}c_1^2\right|\right] + \frac{|a_4|}{384}\left[11\left|c_3 - 2c_1c_2 + c_1^3\right| + 18\left|c_3 - c_1c_2\right| + 3\left|c_3\right|\right] \\ + \frac{|a_3|}{12288}\left[256\left|c_1\right|\left|c_3 - \frac{192}{256}c_2^2\right| + 80\left|c_1^2\right|\left|c_2 - \frac{25}{80}c_1^2\right|\right]. \end{aligned} \tag{2.9}$$

Using the lemmas 1.2, 1.3, 1.4, 1.5 and 1.6 in the above inequality, it simplifies to

$$|H_{3,1}(f)| \le \left[\left(\frac{1}{8}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{16}\right)\right] = \frac{43}{576}. \tag{2.10}$$

∎

Remark 2.2. The above result coincides with that of Raza and Malik [1].

3. CONCLUSION

A higher bound of 3^{rd} order Hankel determinant for a collection of holomorphic mappings connected with exactly to the right side of the lemniscate of Bernoulli, whose polar coordinates form is $r^2 = 2\cos^2(2\theta)$ has been estimated. The result, $|H_{3,1}(f)| \le \frac{43}{576}$, coincides with the previous study by Raza and Malik [1].

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