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COEFFICIENT ESTIMATES OF SAKAGUCHI KIND FUNCTIONS USING LUCAS POLYNOMIALS

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ABSTRACT. By means of (p, q) Lucas polynomials, we estimate coefficient bounds and Fekete-Szego inequalities for functions belonging to this class. Several corollaries and consequences of the main results are also obtained.

Key words and phrases: Lucas polynomial; Analytic functions; Univalent functions; Bi-univalent functions.

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1. INTRODUCTION

Let $\mathcal{A}$ indicate an analytic functions family , which is normalized under the condition $f(0) = f'(0) - 1 = 0$ in $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and given by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in $\mathbb{U}$. Then we say that the function f is subordinate to g if there exists a Schwarz function $\omega(z)$, analytic in $\mathbb{U}$ with

$$\omega(0) = 0,\ |\omega(z)| < 1, (z \in \mathbb{U})$$

such that $f(z) = g(\omega(z))$
We denote this subordination by,

$$f \prec g\ (or)\ f(z) \prec g(z)$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0)$, $f(\mathbb{U}) \subset g(\mathbb{U})$
The Koebe-One Quarter theorem [11] asserts that image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disc of radius $\frac{1}{4}$, thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$ and $f(f^{-1}(w)) = w$, $(|\omega| < r_0(f), r_0(f) > \frac{1}{4})$

$$f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \dots \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent functions in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. A function $f \in \mathcal{S}$ is said to be bi-univalent in $\mathbb{U}$ if there exist a function $g \in S$ such that $g(z)$ is an univalent extension of f^{-1} to $\mathbb{U}$. Let Λ denote the class of bi-univalent functions in $\mathbb{U}$. The functions $\frac{z}{1-z}$, $-log(1 - z)$, $\frac{1}{2}\log(\frac{1+z}{1-z})$ are in the class Λ (see details in [20]). However, the familiar Koebe function is not bi-univalent . Lewin [17] investigated the class of bi-univalent functions Λ and obtained a bound $|a_2| \leq 1.51$. Motivated by the work of Lewin [17], Brannan and Clunie [9] conjectured that $|a_2| \leq \sqrt{2}$. The coefficient estimate problem for $|a_n|$ $[(n \in \mathbb{N}), n \geq 3]$ is still open [20]. Brannan and Taha [10] also worked on certain subclasses of the bi-univalent function class Λ and obtained estimates for their initial co-efficients. Various classes of bi-univalent functions were introduced and studied in recent times, the study of bi-univalent functions gained momentum mainly due to the work of Srivastava et al [20] , Motivates by this, many researchers [1],[4, 8], [13, 15], [20], [21] and [27, 29], also the references cited there in recently investigated several interesting subclasses of the class Λ and found non-sharp estimates on the fisrt two Taylor-Maclaurin co-efficients. Recently, many researchers have been exploring bi-univalent functions, few to mention Fibonacci polynomials, Lucas polynomials, Chebyshev polynomials, Pell polynomials, Lucas-Lehmer polynomials, Orthogonal polynomials and the other special polynomials and their generalizations are of great importance in a variety of branches such as Physics, Engineering , Architecture, Nature, Art , Number theory , Combinatorics and Numerical analysis. These polynomials have been studied in several papers from a theoretical point view (see for example , [23, 30] also see references therein)

We recall the following results relevant for our study as stated in [3]. Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The (p, q)-Lucas polynomials $\mathcal{L}_{p,q,n}(x)$ are defined by the recurrence relation.

$$\mathcal{L}_{p,q,n}(x) = p(x)\mathcal{L}_{p,q,n-1}(x) + q(x)\mathcal{L}_{p,q,n-2}(x) \ (n \geq 2)$$

From which the first few Lucas polynomials can be found as

$$\begin{aligned} \mathcal{L}_{p,q,0}(x) &= 2 \\ \mathcal{L}_{p,q,1}(x) &= p(x) \\ \mathcal{L}_{p,q,2}(x) &= p^2(x) + 2q(x) \\ \mathcal{L}_{p,q,3}(x) &= p^3(x) + 3p(x)q(x), \cdots \end{aligned} \tag{1.3}$$

For the special cases of $p(x)$ and $q(x)$, we can get the polynomials given $\mathcal{L}_{x,1,n}(x) \equiv \mathcal{L}_n(x)$ Lucas polynomials, $\mathcal{L}_{2x,1,n}(x) \equiv D_n(x)$ Pell -Lucas polynomials , $\mathcal{L}_{1,2x,n}(x) \equiv j_n(x)$ Jacobsthal-Lucas polynomials, $\mathcal{L}_{3x,-2,n}(x) \equiv F_n(x)$ Fermate-Lucas polynomials, $\mathcal{L}_{2x,-1,n}(x) \equiv T_n(x)$ Chebyshev polynomials first kind.

Lemma 1.1. [16] *Let $\mathcal{G}_{\{\mathcal{L}(x)\}}(z)$ be the generating function of the (p,q)-Lucas polynomial sequence $\mathcal{L}_{p,q,n}(x)$ Then,*

$$\mathcal{G}_{\{\mathcal{L}(x)\}}(z) = \sum_{n=0}^{\infty} \mathcal{L}_{p,q,n}(x) z^n = \frac{2 - p(x)z}{1 - p(x)z - q(x)z^2}$$

and

$$\begin{aligned} \mathcal{G}_{\{\mathcal{L}(x)\}}(z) = \mathcal{G}_{\{\mathcal{L}(x)\}}(z) - 1 &= 1 + \sum_{n=1}^{\infty} \mathcal{L}_{p,q,n}(x) z^n \\ &= \frac{1 + q(x)z^2}{1 - p(x)z - q(x)z^2} \end{aligned}$$

Definition 1.1. A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is said to be in the class $\mathcal{C}(t,\lambda)$ if it satisfies the subordination conditions which are as follows

$$\frac{(1-t)\left[\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)\right]}{\lambda z^2 \left[f''(z) - tf''(tz)\right] + z\left[f'(z) - tf'(tz)\right]} \prec \mathcal{G}_{\{\mathcal{L}(x)\}}(z) \ (z \in \mathbb{U}) \tag{1.4}$$

and

$$\frac{(1-t)\left[\lambda \omega^3 g'''(\omega) + (1+2\lambda)\omega^2 g''(\omega) + \omega g'(\omega)\right]}{\lambda \omega^2 \left[g''(\omega) - tg''(t\omega)\right] + \omega\left[g'(\omega) - tg'(t\omega)\right]} \prec \mathcal{G}_{\{\mathcal{L}(x)\}}(\omega) \ (\omega \in \mathbb{U}) \tag{1.5}$$

where $\mathcal{G}_{\{\mathcal{L}_{(p,q,n)}\}}(z) \in \phi$ and the function g is described as $g(\omega) = f^{-1}(\omega)$

Definition 1.2. For the case $t = 0$ the class $\mathcal{C}(t,\lambda)$ reduces to the class $\mathcal{C}(0,\lambda)$ satisfying the following conditions,

$$\frac{\left[\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)\right]}{\lambda z^2 f''(z) + zf'(z)} \prec \mathcal{G}_{\{\mathcal{L}(x)\}}(z)$$

and

$$\frac{\left[\lambda \omega^3 g'''(\omega) + (1+2\lambda)\omega^2 g''(\omega) + \omega g'(\omega)\right]}{\lambda \omega^2 g''(\omega) + \omega g'(\omega)} \prec \mathcal{G}_{\{\mathcal{L}(x)\}}(\omega)$$

Definition 1.3. For the case $\lambda = 0$ the class $\mathcal{C}(t,\lambda)$ reduces to the class $\mathcal{C}(t,0)$ satisfying the following conditions,

$$\frac{(1-t)\left[z^2 f''(z) + zf'(z)\right]}{z\left[f'(z) - tf'(tz)\right]} \prec \mathcal{G}_{\{\mathcal{L}(x)\}}(z)$$

$$\frac{(1-t)\left[\omega^2 g''(\omega)+\omega g'(\omega)\right]}{\omega\left[g'(\omega)-tg'(t\omega)\right]} \prec \mathcal{G}_{\{\mathcal{L}(x)\}}(\omega)$$

2. Coefficient Bounds For The Function Of The Class $\mathcal{C}(t,\lambda)$

Theorem 2.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{C}(t,\lambda)$. Then*

$$|a_2| \leq \frac{|p(x)|\sqrt{|p(x)|}}{\sqrt{\left|2\left\{4(T_2-2)\left\{T_2p^2(x)-(p^2(x))(T_2-2)(1+\lambda)-2q(x)(T_2-2)(1+\lambda)\right\}-3(T_3-3)p^2(x)\right\}\right|}}$$

$$|a_3| \leq \frac{|p^1(x)|}{3(1+2\lambda)(3-T_3)}+\frac{|p^2(x)|}{4(1+\lambda)^2(2-T_2)^2}$$

Proof. Let $f \in \mathcal{C}(t,\lambda)$ there exists two analytic functions $u, v : \mathbb{U} \to \mathbb{U}$ with $u(0)=v(0)$ such that $|u(z)|<1$, $|v(\omega)|<1$, we can write from (1.4) and (1.5), we have

$$\frac{(1-t)\left[\lambda z^3 f'''(z)+(1+2\lambda)z^2 f''(z)+zf'(z)\right]}{\lambda z^2\left[f''(z)-tf''(tz)\right]+z\left[f'(z)-tf'(tz)\right]}=\mathcal{G}_{\{\mathcal{L}(x)\}}(z)\ (z\in\mathbb{U}) \tag{2.1}$$

and

$$\frac{(1-t)\left[\lambda\omega^3 g'''(\omega)+(1+2\lambda)\omega^2 g''(\omega)+\omega g'(\omega)\right]}{\lambda\omega^2\left[g''(\omega)-tg''(t\omega)\right]+\omega\left[g'(\omega)-tg'(t\omega)\right]}=\mathcal{G}_{\{\mathcal{L}(x)\}}(\omega)\ (\omega\in\mathbb{U}) \tag{2.2}$$

It is fairly well known that if

$$\begin{array}{rll} & |u(z)| = & |u_1 z+u_2z^2+\cdots|<1 \\ \text{and} & |v(\omega)| = & |v_1\omega+v_2\omega^2+\cdots|<1 \\ \text{then} & |u_k|\leq 1 & \text{and } |v_k|\leq 1\ (k\in\mathbb{N}) \end{array}$$

It follows that,

$$\begin{aligned} \mathcal{G}_{\{\mathcal{L}(x)\}}(u(z)) &= 1+\mathcal{L}_{p,q,1}(x)u(z)+\mathcal{L}_{p,q,2}(x)u^2(z)+\cdots \\ &= 1+\mathcal{L}_{p,q,1}(x)u_1(z)+\left[\mathcal{L}_{p,q,1}(x)u_2+\mathcal{L}_{p,q,2}(x)u_1^2\right]z^2+\cdots \end{aligned} \tag{2.3}$$

$$\begin{aligned} \mathcal{G}_{\{\mathcal{L}(x)\}}(v(\omega)) &= 1+\mathcal{L}_{p,q,1}(x)v(\omega)+\mathcal{L}_{p,q,2}(x)v^2(\omega)+\cdots \\ &= 1+\mathcal{L}_{p,q,1}(x)v_1(\omega)+\left[\mathcal{L}_{p,q,1}(x)v_2+\mathcal{L}_{p,q,2}(x)v_1^2\right]\omega^2+\cdots \end{aligned} \tag{2.4}$$

From the equalities (2.1) and (2.2), we obtain that

$$\begin{aligned} \frac{(1-t)\left[\lambda z^3 f'''(z)+(1+2\lambda)z^2 f''(z)+zf'(z)\right]}{\lambda z^2\left[f''(z)-tf''(tz)\right]+z\left[f'(z)-tf'(tz)\right]} &= 1+\mathcal{L}_{p,q,1}(x)u_1(z) \\ &\quad+\left[\mathcal{L}_{p,q,1}(x)u_2+\mathcal{L}_{p,q,2}(x)u_1^2\right]z^2+\cdots \end{aligned} \tag{2.5}$$

$$\begin{aligned} \frac{(1-t)\left[\lambda\omega^3 g'''(\omega)+(1+2\lambda)\omega^2 g''(\omega)+\omega g'(\omega)\right]}{\lambda\omega^2\left[g''(\omega)-tg''(t\omega)\right]+\omega\left[g'(\omega)-tg'(t\omega)\right]} &= 1+\mathcal{L}_{p,q,1}(x)v_1(\omega) \\ &\quad+\left[\mathcal{L}_{p,q,1}(x)v_2+\mathcal{L}_{p,q,2}(x)v_1^2\right]\omega^2+\cdots \end{aligned} \tag{2.6}$$

It is follows that, from (2.5) and (2.6), we obtain,

$$2a_2(1+\lambda)(2-T_2)=\mathcal{L}_{p,q,1}(x)u_1 \tag{2.7}$$

$$3a_3(1+2\lambda)(3-T_3)+4a_2^2T_2(1+\lambda)^2(T_2-2)=\mathcal{L}_{p,q,1}(x)u_2+\mathcal{L}_{p,q,2}(x)u_1^2 \tag{2.8}$$

and

$$-2a_2(1+\lambda)(2-T_2)=\mathcal{L}_{p,q,1}(x)v_1 \tag{2.9}$$

$$3(1+2\lambda)(2a_2^2-a_3)(3-T_3)+4a_2^2T_2(1+\lambda)(T_2-2)=\mathcal{L}_{p,q,1}(x)v_2+\mathcal{L}_{p,q,2}(x)v_1^2 \tag{2.10}$$

From (2.7) and (2.9), we get,

$$u_1=-v_1 \tag{2.11}$$

$$\text{and } 2\left[4(1+\lambda)^2(2-T_2)^2\right]a_2^2=\mathcal{L}_{p,q,1}^2(x)(u_1^2+v_1^2) \tag{2.12}$$

By adding (2.8) to (2.10), we get,

$$2\left[3(3-T_3)+4T_2(T_2-2)\right]a_2^2=\mathcal{L}_{p,q,1}(x)(u_2+v_2)+\mathcal{L}_{p,q,2}(x)(u_1^2+v_1^2) \tag{2.13}$$

By using (2.12) in (2.13) we have,

$$a_2^2=\frac{\mathcal{L}_{p,q,1}^3(x)(u_2+v_2)}{2\left\{4(T_2-2)\left\{T_2\mathcal{L}_{p,q,1}^2(x)-\mathcal{L}_{p,q,2}(x)(T_2-2)(1+\lambda)\right\}-3(T_3-3)\mathcal{L}_{p,q,1}^2(x)\right\}} \tag{2.14}$$

Thus From (1.3) and (2.14) we get,

$$|a_2|\le\frac{|p(x)|\sqrt{|p(x)|}}{\sqrt{|2\left\{4(T_2-2)\left\{T_2p^2(x)-(p^2(x))(T_2-2)(1+\lambda)-2q(x)(T_2-2)(1+\lambda)\right\}-3(T_3-3)p^2(x)\right\}|}}$$

Next, in order to find the bound on $|a_3|$, by subtracting (2.10) from (2.8), we obtain

$$6(1+2\lambda)(3-T_3)(a_3-a_2^2)=\mathcal{L}_{p,q,1}(x)(u_2-v_2)$$

$$a_3=\frac{\mathcal{L}_{p,q,1}(x)(u_2-v_2)}{6(1+2\lambda)(3-T_3)}+a_2^2 \tag{2.15}$$

$$a_3=\frac{\mathcal{L}_{p,q,1}(x)(u_2-v_2)}{6(1+2\lambda)(3-T_3)}+\frac{\mathcal{L}_{p,q,1}^2(x)(u_1^2+v_1^2)}{8(1+\lambda)^2(2-T_2)^2}$$

$$|a_3|\le\frac{|p^1(x)|}{3(1+2\lambda)(3-T_3)}+\frac{|p^2(x)|}{4(1+\lambda)^2(2-T_2)^2}$$

This completes the proof ∎

If we take $\lambda=0$ and $t=0$ in theorem (2.1) we obtain the following corollaries respectively,

Corollary 2.2. *A function $f\in\mathcal{A}$ of the form (1.1) is in the class $\mathcal{C}(t,\lambda)$ where $\lambda=0$, we obtain,*

$$|a_2|\le\frac{|p(x)|\sqrt{|p(x)|}}{\sqrt{|2\left\{4(T_2-2)\left\{T_2p^2(x)-(p^2(x))(T_2-2)-2q(x)(T_2-2)\right\}-3(T_3-3)p^2(x)\right\}|}}$$

$$|a_3|\le\frac{|p^1(x)|}{3(3-T_3)}+\frac{|p^2(x)|}{4(2-T_2)^2}$$

Corollary 2.3. *A function $f\in\mathcal{A}$ of the form (1.1) is in the class $\mathcal{C}(t,\lambda)$ where $t=0$, we obtain*

$$|a_2|\le\frac{|p(x)|\sqrt{|p(x)|}}{\sqrt{|2\left\{-4\left\{T_2p^2(x)+p^2(x)(1+\lambda)+2q(x)(1+\lambda)\right\}+3p^2(x)\right\}|}}$$

$$|a_3|\le\frac{|p^1(x)|}{6(1+2\lambda)}+\frac{|p^2(x)|}{4(1+\lambda)^2}$$

3. Fekete-Szego Inequality For The Class $\mathcal{C}(t,\lambda)$

Fekete-Szego inequality is one of the famous problem related to coefficient of univalent analytic functions. It was first given by [12] , the classical Fekete-Szego inequality for the coefficients of $f \in \mathcal{S}$ is

$$|a_3 - \mu a_2^2| \leq 1 + 2e^{\frac{-2\mu}{1-\mu}} \; for \; \mu \in [0,1)$$

As $\mu \to 1^-$ we have the elementary inequality $|a_3 - a_2^2| \leq 1$. Moreover, the coefficient functional $\mathcal{S}_\mu(f) = a_3 - \mu a_2^2$ on the normalized analytic functions f in the unit disk $\mathbb{U}$ plays an important role in function theory. The problem of maximizing the absolute value of the functional $\mathcal{S}_\mu(f)$ is called the Fekete-Szego problem. In this section, we are ready to find the sharp bounds of Fekete-Szego functional $\mathcal{S}_\mu(f)$ defined for $f \in \mathcal{C}(t,\lambda)$ given by (1.1)

Theorem 3.1. *Let f given by (1.1) be in the class $\mathcal{C}(t,\lambda)$ and $\mu \in \mathbb{R}$. Then,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|p(x)|}{3(1+2\lambda)(3-T_3)}, & 0 \leq |h(\mu)| \leq \dfrac{1}{6(1+2\lambda)(3-T_3)} \\ 2|p(x)||h(\mu)|, & |h(\mu)| \geq \dfrac{1}{6(1+2\lambda)(3-T_3)} \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\mathcal{L}^2_{p,q,1}(x)}{\left[2\left\{\left[3(1+2\lambda)(3-T_3) - 2T_2(1+\lambda)^2(2-T_2)\right]\mathcal{L}^2_{p,q,1}(x) - 4\mathcal{L}_{p,q,2}(x)(1+\lambda)^2(2-T_2)^2\right\}\right]}$$

Proof. From (2.14) & (2.15) we conclude that

$$a_3 - \mu a_2^2 = \left[\frac{(1-\mu)\mathcal{L}^3_{p,q,1}(x)(u_2+v_2)}{\left[2\left\{\left[3(1+2\lambda)(3-T_3) - 2T_2(1+\lambda)^2(2-T_2)\right]\mathcal{L}^2_{p,q,1}(x) - 4\mathcal{L}_{p,q,2}(x)(1+\lambda)^2(2-T_2)^2\right\}\right]}\right] + \left[\frac{\mathcal{L}_{p,q,1}(x)(u_2-v_2)}{6(1+2\lambda)(3-T_3)}\right]$$

$$a_3 - \mu a_2^2 = \mathcal{L}_{p,q,1}(x)\left[\left(h(\mu) + \frac{1}{6(1+2\lambda)(3-T_3)}\right)u_2 + \left(h(\mu) - \frac{1}{6(1+2\lambda)(3-T_3)}\right)v_2\right]$$

where,

$$h(\mu) = \frac{(1-\mu)\mathcal{L}^2_{p,q,1}(x)}{\left[2\left\{\left[3(1+2\lambda)(3-T_3) - 2T_2(1+\lambda)^2(2-T_2)\right]\mathcal{L}^2_{p,q,1}(x) - 4\mathcal{L}_{p,q,2}(x)(1+\lambda)^2(2-T_2)^2\right\}\right]}$$

Then, in view of (1.3), we obtain

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|p(x)|}{3(1+2\lambda)(3-T_3)}, & 0 \leq |h(\mu)| \leq \dfrac{1}{6(1+2\lambda)(3-T_3)} \\ 2|p(x)||h(\mu)|, & |h(\mu)| \geq \dfrac{1}{6(1+2\lambda)(3-T_3)} \end{cases}$$

∎

we end this section with some corollaries.
Taking $\mu = 1$ in theorem (3.1) we get the following corollary,

Corollary 3.2. *If $f \in \mathcal{C}(t, \lambda)$ then,*

$$|a_3 - a_2^2| \leq \frac{|p(x)|}{3(1+2\lambda)(3-T_3)}$$

For $\lambda = 0$ and $t = 0$ in theorem (3.1), the following Fekete-Szego inequality is obtained

Corollary 3.3. *Let f given by (1.1) be in the class $\mathcal{C}(t, \lambda)$, then*

For $t = 0$

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|p(x)|}{6(1+2\lambda)}, & 0 \leq |h(\mu)| \leq \dfrac{1}{12(1+2\lambda)} \\ 2|p(x)||h(\mu)|, & |h(\mu)| \geq \dfrac{1}{12(1+2\lambda)} \end{cases}$$

For $\lambda = 0$

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|p(x)|}{3(3-T_3)}, & 0 \leq |h(\mu)| \leq \dfrac{1}{6(3-T_3)} \\ 2|p(x)||h(\mu)|, & |h(\mu)| \geq \dfrac{1}{6(3-T_3)} \end{cases}$$

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