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EXISTENCE, GLOBAL REGULARITY AND UNIQUENESS OF SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN SPACE DIMENSION 3 WHEN THE INITIAL DATA ARE REGULAR

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ABSTRACT. The existence, regularity, and uniqueness of global solutions of the Navier-Stokes equations in $\mathbb{R}^3$ are given for when the initial velocity $u_0 \in W^{q,1}(\mathbb{R}^3)^3$ for all integers $q \geq 0$ and div $u_0 = 0$.

Key words and phrases: BV functions; Divergence theorem; Finite perimeter; Global regular solution; Haussdorff measure; Hypersurface; Measure; Navier-Stokes equations; Regular set; Singular set.

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CONTENTS

1. INTRODUCTION

The Navier-Stokes equations correspond to the system

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = 0 \quad \text{in } \mathbb{R}^3 \times (0,T), \tag{1.1}$$

$$\text{div } u = 0 \quad \text{in } \mathbb{R}^3 \times (0,T) \tag{1.2}$$

complemented with the initial condition

$$u(\cdot,0) = u_0(\cdot) \ \text{ in } \mathbb{R}^3, \tag{1.3}$$

where u_0 is a divergence-free vector field, that is; div $u_0 = 0$ in $\mathbb{R}^3$. Above $u = u(x,t)$ denotes the velocity field at the point $x \in \mathbb{R}^3$ and at time $t \in (0,T)$ with $T > 0$ and $p = p(x,t)$ denotes the pressure while ν denotes the kinematic viscosity.

The only global solution known to exist for general initial data is the weak solution of Leray [7, 8]; Consult for instance [9, 10]. The existence and uniqueness of a smooth solution to the Navier-Stokes system on a short time interval is known. Such a result is obtained using estimates derived from techniques employed in the proof of existence of global weak solutions; Consult for instance [9, 10] where more information about existing results and references is given.

The existence, regularity, and uniqueness of global solutions of the Navier-Stokes equations in $\mathbb{R}^3$ are given for when the initial velocity $u_0 \in W^{q,1}(\mathbb{R}^3)^3$ for all integers $q \geq 0$ and div $u_0 = 0$.

In Sections 3-7, basic properties of the measures introduced in [12] are covered. In Section 8, this measure theory to the vorticity-stream formulation of the Navier-Stokes equations is applied. A brief description of how this measure theory is applied to Navier-Stokes equations is given below. For the definition of the functional spaces and other notations used throughout this paper consult the subsection below and Sections 3-4.

Let $u_0 \in W^{q,1}(\mathbb{R}^3)^3$ for all integers $q \geq 0$ with div $u_0 = 0$. Let u and T_r be the corresponding solution and time given by the short time existence and regularity theorem for Navier-Stokes equations (8.1)-(8.3); See Theorem 8.1. Let $\omega_0 =$ curl u_0 and $\omega =$ curl u. Then the primitive formulation of Navier-Stokes equations (8.1)-(8.3) is equivalent to the following vorticity-stream formulation

$$\partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u - \nu\Delta\omega = 0 \ \text{ in } \mathbb{R}^3 \times (0,T) \tag{1.4}$$

$$\omega|_{t=0} = \omega_0 \ \text{ in } \mathbb{R}^3, \tag{1.5}$$

with $u =$ curl Ψ and div $\Psi = 0$, with Ψ a potential vector satisfying $-\Delta\Psi = \omega$ and $0 < T < T_r$.

The application of the measure theory developed in Sections 3-7 to the vorticity formulation of the Navier-Stokes equations relies in a fundamental way on the structure of the nonlinearities in these equations. Let $t_1, t_2 \in [0, T_r)$ with $t_1 < t_2$. For any scalar function $w \in C^\infty(\mathbb{R}^3 \times (t_1,t_2); \mathbb{R})$ and any vector function $U \in C^\infty(\mathbb{R}^3 \times (t_1,t_2); \mathbb{R}^3)$, let $\mu_{\{w,U\}}$ be the measure introduced in [12]; See also Theorem 3.1 below. Let $l = (l_1, l_2, l_3)$ be any multi-index of nonnegative integers l_1, l_2, and l_3. Let $i \in \{1,2,3\}$. Let $v_i = D^l\omega_i$, where ω is the solution to the vorticity-stream formulation of the Navier-Stokes equations; See Eqs. (1.4)-(1.5). Let ν_{v_i} denote the measure associated with $v_i = D^l\omega_i$ and given by

$$\nu_{v_i} = \mu_{\{v_i, D^l(\omega_i u)\}} - \mu_{\{v_i, D^l(u_i\omega)\}} - \nu\mu_{\{v_i, \nabla v_i\}}. \tag{1.6}$$

Then using the vorticity-stream formulation of Navier-Stokes equations (1.4)-(1.5), one obtains

$$\nu_{v_i} = \partial_t |v_i| + \text{div}\,[\text{sg}(v_i)(D^l(\omega_i u) - D^l(u_i\omega) - \nu\nabla v_i)] \ \text{ in } \mathcal{M}(\mathbb{R}^3 \times (t_1, t_2)). \tag{1.7}$$

Let $\psi \in C_c^\infty(\mathbb{R}^3)$ be such that $\psi(x) = 1$ for $|x| \leq 1$, $\psi(x) = 0$ for $|x| \geq 2$, and $0 \leq \psi(x) \leq 1$ for all $x \in \mathbb{R}^3$. Define ψ_m, $m \geq 1$, by $\psi_m(x) = \psi(\frac{x}{m})$. Then using (1.7) and the regularity of ω, one obtains for any $\phi \in C_c^1(t_1, t_2)$, up to a subsequence,

$$\lim_{m\to\infty} \int \phi\psi_m d\nu_{v_i} = \int_{t_1}^{t_2} \phi \frac{d}{dt}[\int_{\mathbb{R}^3} |v_i|(x,t)dx]dt. \tag{1.8}$$

If one proves that for any nonnegative $\phi \in C_c^1(t_1, t_2)$,

$$\lim_{m\to\infty} \int \phi\psi_m d\nu_{v_i} \leq 0, \tag{1.9}$$

then using (1.8), one deduces

$$\frac{d}{dt}[\int_{\mathbb{R}^3} |v_i|(x,t)dx] \leq 0 \ \text{ in } \mathcal{M}(t_1, t_2). \tag{1.10}$$

The regularity of ω and (1.10) then yield

$$\frac{d}{dt}[\int_{\mathbb{R}^3} |v_i|(x,t)dx] \leq 0 \ \text{ a.e. on } (t_1, t_2). \tag{1.11}$$

Using (1.11) and the absolute continuity of $\int_{\mathbb{R}^3} |v_i|(x,t)dx$ in $[t_1, t_2]$, one then deduces for all $t_1 \leq s < t \leq t_2$,

$$\int_{\mathbb{R}^3} |D^l\omega_i|(x,t)dx \leq \int_{\mathbb{R}^3} |D^l\omega_i|(x,s)dx. \tag{1.12}$$

By a continuity argument, one then obtains for all $0 \leq s < t$,

$$\int_{\mathbb{R}^3} |D^l\omega_i|(x,t)dx \leq \int_{\mathbb{R}^3} |D^l\omega_i|(x,s)dx. \tag{1.13}$$

These estimates yield the global regularity for the Navier-Stokes equations.
It remains to show that for any nonnegative $\phi \in C_c^1(t_1, t_2)$,

$$\lim_{m\to\infty} \int \phi\psi_m d\nu_{v_i} \leq 0. \tag{1.14}$$

The measures associated with the convective terms and the stretching terms in the vorticity-stream formulation of the Navier-Stokes equations; Eqs. (1.4)-(1.5), satisfy, up to a subsequence,

$$\lim_{m\to\infty} \int \phi\psi_m d\mu_{\{D^l\omega_i, D^l(\omega_i u)\}} = 0 \tag{1.15}$$

and

$$\lim_{m\to\infty} \int \phi\psi_m d\mu_{\{D^l\omega_i, D^l(u_i\omega)\}} = 0. \tag{1.16}$$

Then using (1.15), (1.16), and (1.6) yields (1.14).

The proof of the convergence in (1.15) for the measures associated with the convective terms in the vorticity-stream formulation depends on the properties of the measures $\mu_{\{w,U\}}$ established in Sections 3 - 7 and the following crucial facts:

(1) div $u = 0$;

(2) the "regular part" of the space-time set $\{(x,t) \in \mathbb{R}^3 \times (t_1, t_2) |\ v_i(x,t) = 0\}$ enjoys some fine geometric properties;

(3) the pairs of functions w and U defining the measures are related in a precise manner, namely, $w = D^l\omega_i$ and $U = D^l(\omega_i u)$.

The proof of the convergence in (1.16) for the measures associated with the stretching terms in the vorticity-stream formulation depends on the properties of the measures $\mu_{\{w,U\}}$ established in Sections 3-7 and the following crucial facts:

(4) div $u = 0$;

(5) the "regular part" of the space-time set $\{(x,t) \in \mathbb{R}^3 \times (t_1, t_2) |\ v_i(x,t) = 0\}$ enjoys some fine geometric properties;

(6) the pairs of functions w and U defining the measures are related in a precise manner, namely, $w = D^l\omega_i$ and $U = D^l(u_i\omega)$;

The above shows that the proofs of the results that yield the solution to the Navier-Stokes global regularity rely in a fundamental way on the exact structures of the convective terms and the stretching terms in the vorticity-stream formulation of Navier-Stokes equations; See, in particular, Points (3) and (6) above.

In Section 2, the main results stating the existence, global regularity, and uniqueness of solutions of Navier-Stokes equations (1.1)-(1.3) are given.

In Sections 3 and 4, a family of measures $\mu^+_{\{v,U\}}$, $\mu^-_{\{v,U\}}$, and $\mu_{\{v,U\}}$ generated by pairs of regular functions v and U and a family of measures $\mu^+_{\{v,U,\mathcal{O}\}}$, $\mu^-_{\{v,U,\mathcal{O}\}}$, and $\mu_{\{v,U,\mathcal{O}\}}$ generated by triplets formed of pairs of regular functions v and U, and open sets $\mathcal{O}$ are constructed. These families of measures were introduced by the author in [12]. Basic properties and characterizations of these measures, their concentration sets, and the regularity properties of these concentration sets along with their actions on some particular manifolds are given.

In Section 5, the actions of the measures $\mu^+_{\{v,U\}}$, $\mu^-_{\{v,U\}}$, and $\mu_{\{v,U\}}$ on some particular functions when the vector function generator U satisfies: div $U = 0$ are studied.

In Sections 6-7, further properties of the measures of Sections 3-5 for several classes of generators are obtained. As particular cases of these applications, the convergence in (1.15) and (1.16) above is obtained.

In Section 8, the measure theory of Sections 3-7 to prove the existence, the regularity, and the uniqueness of global solutions of the Navier-Stokes equations in space dimension 3 is applied.

In [13], the existence, global regularity, and uniqueness of solutions of the Navier-Stokes equations in space dimension 3 with external forces are given for when the initial velocity $u_0 \in W^{q,1}(\mathbb{R}^3)^3$ for all integers $q \geq 0$, div $u_0 = 0$, and the external force is in $C^\infty([0,\infty); W^{q,1}(\mathbb{R}^3))^3$ for all integers $q \geq 0$. In [14], the existence, global regularity, and uniqueness of solutions of the Euler equations in space dimension 3 are given for when the initial velocity $u_0 \in W^{q,1}(\mathbb{R}^3)^3$ for all integers $q \geq 0$ and div $u_0 = 0$. In [15], the existence, global regularity, and uniqueness of solutions of the Euler equations in space dimension 3 with external forces are given for when the initial velocity $u_0 \in W^{q,1}(\mathbb{R}^3)^3$ for all integers $q \geq 0$, div $u_0 = 0$, and the external force is in $C^\infty([0,\infty); W^{q,1}(\mathbb{R}^3))^3$ for all integers $q \geq 0$.

Basic notations. Here, the main notations that will be used throughout this paper are given.

Let N be any nonnegative integer. e_i, $i = 1, \cdots, N$ denotes the canonical basis of $\mathbb{R}^N$. $\partial_k^r v$ or $\dfrac{\partial^r v}{\partial x_k^r}$ denotes the partial derivative of order r with respect to the variable x_k of the function v. For a vector field u in $\mathbb{R}^N$, the expression div $u = \sum_{i=1}^N \partial_i u_i$ denotes the divergence of u and $(u \cdot \nabla)$ denotes the operator $\sum_{j=1}^N u_j \partial_j$. For a scalar function v, $\Delta v = \sum_{i=1}^N \partial_i^2 v$ denotes the Laplacean of the function v.

Let k and l denote the multi-indices $k = (k_1, \cdots, k_n)$ and $l = (l_1, \cdots, l_N)$ of nonnegative integers k_i, l_i, $i = 1, \cdots, N$. Then define

$$
\begin{aligned}
&|l| = \sum_{i=1}^N l_i \ \text{ and } \ D^l v = \frac{\partial^{l_1 + \cdots + l_N}}{\partial x_1^{l_1} \partial x_2^{l_2} \cdots \partial x_N^{l_N}} v, \\
&k \leq l \ \text{ if and only if } \ k_i \leq l_i, \quad i = 1, \cdots, N.
\end{aligned} \tag{1.17}
$$

When there is no need to specify the components of the multi-index $k = (k_1, \cdots, k_N)$ of nonnegative integers $k_1, \cdots, k_N$, the multi-index k is simply referred to as an N-multi-index k of nonnegative integers or by $k \in \mathbb{N}^N$. Let m be any nonnegative integer and $q \geq 1$. Let Ω be an open subset of $\mathbb{R}^N$. The sets $H^m(\Omega)$ and $W^{q,m}(\Omega)$ denote the usual Sobolev spaces:

$$
\begin{aligned}
H^m(\Omega) &= \{v \in L^2(\Omega) |\ D^l v \in L^2(\Omega),\ l = (l_1, \cdots, l_N) \text{ and } 0 \leq |l| \leq m\} \\
W^{q,m}(\Omega) &= \{v \in L^m(\Omega) |\ D^l v \in L^m(\Omega),\ l = (l_1, \cdots, l_N) \text{ and } 0 \leq |l| \leq q\}
\end{aligned}
$$

where $L^m(\Omega)$ denotes the space of functions f such that $|f|^m$ is integrable over the domain Ω. Let s be a nonnegative integer. The set $C^s(E)$ denotes the space of functions v defined in E with $D^k v$, $0 \leq |k| \leq s$, continuous functions in E. The notation $\mathcal{L}^N$ refers to the Lebesgue measure on $\mathbb{R}^N$ and $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure on $\mathbb{R}^N$. $\mathcal{M}(\Omega)$ denotes the space of Radon measures on Ω. $C_c^m(\Omega)$, m nonnegative integer, denotes the space of real-valued functions in $C^m(\Omega)$ having compact supports contained in Ω. $\mathcal{D}(\Omega)$ is another notation for the space $C_c^\infty(\Omega)$. $\mathcal{D}'(\Omega)$ denotes the space of distributions over Ω.

The sets $\{w > \alpha\}$, $\{w \geq \alpha\}$, $\{w < \alpha\}$, $\{w \leq \alpha\}$, and $\{w = \alpha\}$, for a real valued function w defined in Ω, and α a real number, correspond to the following subsets of $\mathbb{R}^N$

$$
\begin{aligned}
&\{w > \alpha\} = \{y \in \Omega |\ w(y) > \alpha\}, \ \ \{w \geq \alpha\} = \{y \in \Omega |\ w(y) \geq \alpha\} \\
&\{w < \alpha\} = \{y \in \Omega |\ w(y) < \alpha\}, \ \ \{w \leq \alpha\} = \{y \in \Omega |\ w(y) \leq \alpha\} \\
&\{w = \alpha\} = \{y \in \Omega |\ w(y) = \alpha\}.
\end{aligned}
$$

For any subset E of $\mathbb{R}^N$ unless specified otherwise, $\bar{E}$ resp. E^o denotes the closure resp. the interior of E with respect to the canonical topology of $\mathbb{R}^N$, ∂E denotes its topological boundary and χ_E denotes its characteristic function; that is $\chi_E(y) = 1$ for $y \in E$ and $\chi_E(y) = 0$ otherwise.

The interior of the set $\{w = 0\}$, denoted by $\{w = 0\}^o$ is also denoted by $\mathcal{O}_w$.

For a given pair of subsets A and B of $\mathbb{R}^N$, $A \setminus B$ denotes the set of points that are in A but not in B. $A \cap B$ denotes the intersection of A and B, while $A \cup B$ denotes the union of A and B. For any $x \in \mathbb{R}^N$, $d(x, A) = \inf\{|x - y| |\ y \in A\}$ and $d(A, B) = \inf\{|x - y| |\ x \in A,\ y \in B\}$.

The function sg denotes the sign function; that is, sg$(y) = -1$ for $y < 0$, sg$(y) = 0$ for $y = 0$, and sg$(y) = 1$ for $y > 0$.

For a given function f, spt(f) denotes the support of f.

Here and below, ψ_m, $m \geq 1$, denote the sequence of functions defined in $\mathbb{R}^N$ (N integer ≥ 1) by $\psi_m(x) = \psi(\frac{x}{m})$, where $\psi \in C_c^\infty(\mathbb{R}^N)$ and is such that $\psi(x) = 1$ for $|x| \leq 1$, $\psi(x) = 0$ for $|x| \geq 2$, and $0 \leq \psi(x) \leq 1$ for all $x \in \mathbb{R}^N$.

Regarding the sequences and subsequences studied in the sequel. If necessary, one passes to a further subsequence so that all limits are taken on the same subsequence. Also, unless specified otherwise, one uses the same notation for a subsequence and the sequence it is derived from. Finally, throughout this paper, constants C in different occurrences are not necessarily the same.

2. EXISTENCE, GLOBAL REGULARITY, AND UNIQUENESS OF SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

The Navier-Stokes equations correspond to the system

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = 0 \ \text{ in } \mathbb{R}^3 \times (0, T), \tag{2.1}$$

$$\text{div } u = 0 \ \text{ in } \mathbb{R}^3 \times (0, T), \tag{2.2}$$

$$u|_{t=0} = u_0 \ \text{ in } \mathbb{R}^3. \tag{2.3}$$

Here, u denotes the velocity field, p denotes the pressure, ν denotes the kinematic viscosity, u_0 denotes the initial velocity field, and $T > 0$ is a time parameter.

Theorem 2.1. *($N = 3$) Let u_0 be such that div $u_0 = 0$ and $u_0 \in W^{q,1}(\mathbb{R}^3)^3$ for all integers $q \geq 0$. Then*

(1) There exists a global regular and unique solution $u \in C^\infty([0, T]; W^{q,1}(\mathbb{R}^3))^3$ and a unique, up to an additive constant, function p such that $\nabla p \in C^\infty([0, T]; W^{q,1}(\mathbb{R}^3))^3$ for all integers $q \geq 0$, to the Navier-Stokes equations (2.1)-(2.3) for all $T > 0$. Moreover, $u \in C^\infty(\mathbb{R}^3 \times [0, \infty))^3$ and $p \in C^\infty(\mathbb{R}^3 \times [0, \infty))$.

(2) The solution u satisfies the strong energy equality

$$\int_{\mathbb{R}^3} \frac{1}{2}|u|^2(x,t)dx + \nu \int_s^t \int_{\mathbb{R}^3} |\nabla u|^2 dx d\tau = \int_{\mathbb{R}^3} \frac{1}{2}|u|^2(x,s)dx,$$
$$\textit{for all } \ t > s \geq 0.$$

(3) The solution u satisfies the local energy equality

$$\frac{d}{dt}\int_{\mathbb{R}^3} \frac{1}{2}|u|^2(x,t)dx + \nu \int_{\mathbb{R}^3} |\nabla u|^2 dx = 0 \ \textit{ for all } \ t \in (0, \infty).$$

(4) u and p satisfy the local energy equality

$$\partial_t(\frac{1}{2}|u|^2) + \textit{div}(u(\frac{1}{2}|u|^2 + p)) - \nu\Delta(\frac{1}{2}|u|^2) + \nu|\nabla u|^2 = 0$$
$$\textit{for all } (x,t) \in \mathbb{R}^3 \times (0, \infty).$$

Let $\omega =$ curl u. Let $i \in \{1, 2, 3\}$. Then for all $t > s \geq 0$,

(5)

$$\int_{\{D^l\omega_i(\cdot,t)>0\}} D^l\omega_i(x,t)dx \leq \int_{\{D^l\omega_i(\cdot,s)>0\}} D^l\omega_i(x,s)dx.$$

(6)

$$\int_{\{D^l\omega_i(\cdot,t)<0\}} (-D^l\omega_i)(x,t)dx \leq \int_{\{D^l\omega_i(\cdot,s)<0\}} (-D^l\omega_i)(x,s)dx.$$

(7)

$$\int_{\mathbb{R}^3} |D^l\omega_i|(x,t)dx \leq \int_{\mathbb{R}^3} |D^l\omega_i|(x,s)dx.$$

The proof of Theorem 2.1 is given in Section 8.

3. THE FIRST CLASS OF MEASURES OF [12]

In this section, based on the results of [12], basic properties and characterizations of the family of measures $\mu^+_{\{v,U\}}$, $\mu^-_{\{v,U\}}$, and $\mu_{\{v,U\}}$ generated by the pairs (v, U) of scalar and vector functions v and U and introduced by the author in [12] are given. In particular, in Subsection 3.1, the precise definition of these measures and the fact that they are Radon measures are given. In Subsection 3.2, a characterization of these measures in terms of integrals over Ω is given. In Subsection 3.3, several basic properties of these measures are given. First, a characterization of these measures in terms of boundary integrals is given. Second, a characterization of concentration sets of these measures in terms of the singular sets of the boundary sets of v is given. Third, the actions of these measures on manifolds of lower dimensions are studied. Fourth, it is shown that the regular sets of the boundary sets of v are the union of two sets: one that is a countable unions of $C^\infty-$hypersurfaces and the other that is a countable unions of collections of C^∞-submanifolds of dimension less or equal to $N-2$ whose $(N-1)-$dimensional Hausdorff measures are 0. In Subsection 3.4, formula characterizing the measures in terms of distributional derivatives of characteristic functions is obtained. In Subsection 3.5, formula characterizing the measures as functionals on $BV(\mathbb{R}^N \times (t_1,t_2))$ is obtained.

The content of this section is either contained in the author's paper [12] or can be deduced from the results in [12]. The reader is referred to this paper for more on these topics and the proofs of the theorems listed in this section.

3.1. **The measures.** Throughout this section and the rest of this paper the following notations and definitions will be used. Let N be any integer ≥ 1. Let $t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R}^N)$, let $W \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R}^{N+1})$, and let $v \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R})$. Then for $(x,t) \in \mathbb{R}^N \times (t_1,t_2)$,

$$\begin{aligned}
&\text{div}\, U(x,t) = \partial_{x_1}U_1 + \cdots + \partial_{x_N}U_N, \quad \nabla v(x,t) = (\partial_{x_1}v, \cdots, \partial_{x_N}v)^t \\
&\text{div}_{x,t}\, W(x,t) = \partial_{x_1}W_1 + \cdots + \partial_{x_N}W_N + \partial_t W_{N+1}, \\
&\nabla_{x,t}v(x,t) = (\partial_{x_1}v, \cdots, \partial_{x_N}v, \partial_t v)^t.
\end{aligned}$$

Let $0 \leq t_1 < t_2$. Let $\Omega = \mathbb{R}^N \times (t_1,t_2)$. Let $v \in C^\infty(\Omega,\mathbb{R})$. Let $V_+ = \{v > 0\}$ $(= \{(x,t) \in \Omega |\ v(x,t) > 0\})$ and $V_- = \{v < 0\}$ $(= \{(x,t) \in \Omega |\ v(x,t) < 0\})$. Let S_+ and B_+ resp. S_- and B_- be the subsets of ∂V_+ resp. ∂V_- such that

$$\begin{aligned}
S_+ &= \partial\{v > 0\} \cap \Omega, \qquad S_- = \partial\{v < 0\} \cap \Omega, \\
B_+ &= \partial\{v > 0\} \cap \partial\Omega, \qquad B_- = \partial\{v < 0\} \cap \partial\Omega.
\end{aligned}$$

Then $\partial V_+ = S_+ \cup B_+$ and $\partial V_- = S_- \cup B_-$. Since B_+ and B_- are subsets of $\partial\Omega$ and $\partial\Omega$ is Lipschitz, it is clear that the Hausdorff measures of dimension N of these subsets of $\mathbb{R}^{N+1}$ are locally finite.

Let S^r_+, S^s_+, S^r_-, and S^s_- denote resp. the "regular" and "singular" subsets of S_+ and S_- defined by

$$\begin{aligned} S^r_+ &= \{y \in S_+ |\ D^\alpha v(y) \neq 0 \text{ for some } (N+1)\text{-multi-index } \alpha\} \\ S^r_- &= \{y \in S_- |\ D^\alpha v(y) \neq 0 \text{ for some } (N+1)\text{-multi-index } \alpha\} \\ S^s_+ &= \{y \in S_+ |\ D^\alpha v(y) = 0 \text{ for every } (N+1)\text{-multi-index } \alpha\} \\ S^s_- &= \{y \in S_- |\ D^\alpha v(y) = 0 \text{ for every } (N+1)\text{-multi-index } \alpha\}. \end{aligned}$$

Then $S_+ = S^r_+ \cup S^s_+$ and $S_- = S^r_- \cup S^s_-$. Define $\Gamma^s_v = S^s_+ \cup S^s_-$, $\Gamma^s_{v,+} = S^s_+$, $\Gamma^s_{v,-} = S^s_-$, $\Gamma^r_v = S^r_+ \cup S^r_-$, $\Gamma^r_{v,+} = S^r_+$, and $\Gamma^r_{v,-} = S^r_-$.

The following theorem is a corollary of a theorem obtained by the author in [12]; Consult Theorem 3.1 of [12]. It yields a family of measures generated by pairs of regular functions.

Theorem 3.1. *Let N be any integer ≥ 1. Let $0 \leq t_1 < t_2$. Let $v \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$.*
(1) Let γ be any real number. Then for any compact set K of $\mathbb{R}^N \times (t_1, t_2)$,

$$\int_{\{|v-\gamma|<\alpha\}\cap K} |U \cdot \nabla v| \frac{1}{\alpha} dx d\tau < C,$$

where C is a positive constant independent of $\alpha \in (0, 1)$.
(2) The estimates in (1) with $\gamma = 0$ show that, up to a subsequence, as α goes to 0, the following weak convergence in the sense of measures, holds

$$\begin{aligned} &U \cdot \nabla v \frac{1}{\alpha} \chi_{\{|v|<\alpha\}} \to \mu_{\{v,U\}}, \quad U \cdot \nabla v \frac{1}{\alpha} \chi_{\{0<v<\alpha\}} \to \mu^+_{\{v,U\}}, \\ &U \cdot \nabla v \frac{1}{\alpha} \chi_{\{-\alpha<v<0\}} \to \mu^-_{\{v,U\}}, \end{aligned}$$

where $\mu_{\{v,U\}}$, $\mu^+_{\{v,U\}}$, and $\mu^-_{\{v,U\}}$ are measures on $\mathbb{R}^N \times (t_1, t_2)$ concentrated resp. on $\partial\{v > 0\} \cup \partial\{v < 0\}$, $\partial\{v > 0\}$, and $\partial\{v < 0\}$, which are also Radon measures. Moreover,

$$\mu_{\{v,U\}} = \mu^+_{\{v,U\}} + \mu^-_{\{v,U\}}.$$

Remark 3.2. The notation $\mu^+_{\{v,U\}}$ and $\mu^-_{\{v,U\}}$ should not be confused with the nonnegative and nonpositive parts of a measure. The upper index $+$ resp. $-$ is merely used to refer to the fact that the measure $\mu^+_{\{v,U\}}$ with the upper index $+$ is concentrated on $\partial\{v > 0\}$ resp. the measure $\mu^-_{\{v,U\}}$ with the upper index $-$ is concentrated on $\partial\{v < 0\}$.

The following terminology is adopted. For the measures $\mu^+_{\{v,U\}}$, $\mu^-_{\{v,U\}}$, and $\mu_{\{v,U\}}$, the scalar function v resp. the vector function U is called the scalar function generator resp. the vector function generator of the measures $\mu^+_{\{v,U\}}$, $\mu^-_{\{v,U\}}$, and $\mu_{\{v,U\}}$.

For a discussion of Hausdorff measures and other notions of measures and related issues used here, the reader is referred to [2, 3, 4, 6] and the references therein.

Proof of Theorem 3.1. The proof of Theorem 3.1 is deduced from that of Theorem 3.1 of [12].

Let $V = (U_1, \cdots, U_N, 0)^t$. By the regularity of U, $V \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^{N+1})$. Then applying Parts *(1)-(2)* of Theorem 3.1 of [12] with (v, U) of Theorem 3.1 of [12] corresponding to (v, V) of this proof, one obtains the proof of Parts *(1)-(2)* of the theorem. Thus, the proof of Theorem 3.1 is completed. ∎

3.2. **Relations with integrals over Ω.**

Theorem 3.3. *Let N be any integer ≥ 1. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$. Let $v \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$. Let $\mu^+_{\{v,U\}}$, $\mu^-_{\{v,U\}}$, and $\mu_{\{v,U\}}$ be the measures corresponding to the pair (v, U) obtained in Theorem 3.1. Then for every Lipschitz function $\varphi \in C_c(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$,*

(1)

$$\int \varphi d\mu^+_{\{v,U\}} = \lim_{\alpha \to 0} \int_{\{0<v<\alpha\}} \frac{U \cdot \nabla v}{\alpha} \varphi dx d\tau = - \int_{\{v>0\}} \textit{div}\ (\varphi U) dx d\tau,$$

(2)

$$\int \varphi d\mu^-_{\{v,U\}} = \lim_{\alpha \to 0} \int_{\{-\alpha<v<0\}} \frac{U \cdot \nabla v}{\alpha} \varphi dx d\tau = \int_{\{v<0\}} \textit{div}\ (\varphi U) dx d\tau,$$

(3)

$$\begin{aligned} \int \varphi d\mu_{\{v,U\}} &= \lim_{\alpha \to 0} \int_{\{|v|<\alpha\}} \frac{U \cdot \nabla v}{\alpha} \varphi dx d\tau = - \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \textit{div}\ (\varphi U) \textit{sg}(v) dx d\tau \\ &= \int \varphi d\mu^+_{\{v,U\}} + \int \varphi d\mu^-_{\{v,U\}}. \end{aligned}$$

The basic properties of the measures stated in Parts *(1)-(3)* of Theorem 3.3 can be deduced directly from Parts *(1)-(3)* of Theorem 3.3 of [12]. Because the ideas and constructions in the proof will be used in the proof of other theorems in this paper, the complete proof of Part *(1)* of the theorem is given.

Proof of Theorem 3.3. Let $\alpha \in (0, 1)$ be fixed. Define the following sequences of Lipschitz continuous functions on $\mathbb{R}$ by

$$G^{(1)}_\alpha(y) = \begin{cases} -1 & \text{if } y < -\alpha, \\ y/\alpha & \text{if } |y| \leq \alpha, \\ 1 & \text{if } y > \alpha, \end{cases}$$

$$G^{(2)}_\alpha(y) = \begin{cases} 0 & \text{if } y < 0, \\ y/\alpha & \text{if } 0 \leq y \leq \alpha, \\ 1 & \text{if } y > \alpha, \end{cases}$$

$$G^{(3)}_\alpha(y) = \begin{cases} -1 & \text{if } y < -\alpha, \\ y/\alpha & \text{if } -\alpha \leq y \leq 0, \\ 0 & \text{if } y > 0, \end{cases}$$

Proof of (1)

Since φ is Lipschitz, using the regularity of U and v yields,

$$\begin{aligned} &\int_{t_1}^{t_2} \int_{\mathbb{R}^N} \mathrm{div}(\varphi U G^{(2)}_\alpha(v)) dx d\tau \\ = &\int_{t_1}^{t_2} \int_{\mathbb{R}^N} \mathrm{div}(\varphi U) G^{(2)}_\alpha(v) dx d\tau + \int_{\{0<v<\alpha\}} \frac{U \cdot \nabla v}{\alpha} \varphi dx d\tau. \end{aligned}$$

By the regularity of U and v, and the fact that φ is Lipschitz and of compact support in $\mathbb{R}^N \times (t_1, t_2)$, $\varphi U G_\alpha^{(2)}(v)$ is in $W^{1,1}(\mathbb{R}^N \times (t_1, t_2))$ and $\int_{t_1}^{t_2} \int_{\mathbb{R}^N} \text{div}(\varphi U G_\alpha^{(2)}(v)) dx d\tau = 0$. Therefore,

$$\int_{\{0<v<\alpha\}} \frac{U \cdot \nabla v}{\alpha} \varphi dx d\tau = -\int_{t_1}^{t_2} \int_{\mathbb{R}^N} \text{div}(\varphi U) G_\alpha^{(2)}(v) dx d\tau. \tag{3.1}$$

As $\alpha \to 0$, $G_\alpha^{(2)}(v)$ converges everywhere to $\chi_{\{v>0\}}$. Now $|\text{div}(\varphi U) G_\alpha^{(2)}(v)| \leq |\text{div}(\varphi U)|$. Since φ is Lipschitz and of compact support, one can use the regularity of U to conclude that $\text{div}(\varphi U)$ is in $L^1(\mathbb{R}^N \times (t_1, t_2))$. Therefore using convergence dominated theorem, up to a subsequence,

$$\lim_{\alpha \to 0} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \text{div}(\varphi U) G_\alpha^{(2)}(v) dx d\tau = \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \text{div}(\varphi U) \chi_{\{v>0\}} dx d\tau. \tag{3.2}$$

Hence using (3.1)-(3.2) and Part *(2)* of Theorem 3.1 yields, up to a subsequence, as $\alpha \to 0$,

$$\int \varphi d\mu^+_{\{v,U\}} = \lim_{\alpha \to 0} \int_{\{0<v<\alpha\}} \frac{U \cdot \nabla v}{\alpha} \varphi dx d\tau = -\int_{t_1}^{t_2} \int_{\mathbb{R}^N} \text{div}(\varphi U) \chi_{\{v>0\}} dx d\tau.$$

Proof of (2)-(3)

Taking $G_\alpha^{(3)}$ resp. $G_\alpha^{(1)}$ in place of $G_\alpha^{(2)}$ and proceeding as in the proof of Part *(1)* yields the proof of Part *(2)* resp. Part *(3)* of the theorem. ∎

3.3. **Relations with boundary integrals, concentration sets, and actions on manifolds of lower dimensions.**

Theorem 3.4. *Let N be any integer ≥ 1. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$. Let $v \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$. Let $\mu^+_{\{v,U\}}$, $\mu^-_{\{v,U\}}$, and $\mu_{\{v,U\}}$ be the measures corresponding to the pair (v, U) obtained in Theorem 3.1. Let $\Gamma^s_{v,+}$, $\Gamma^s_{v,-}$, and Γ^s_v be the singular sets corresponding to v introduced in Subsection 3.1. Then*
(1) For any $\varphi \in C_c(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$,

$$\int_{\mathbb{R}^N \times (t_1,t_2)} \varphi d\mu^+_{v,U} = -\int_{\partial\{v>0\} \setminus \Gamma^s_{v,+}} \varphi V \cdot n^+ d\mathcal{H}^N, \tag{3.3}$$

and

$$\int_{\mathbb{R}^N \times (t_1,t_2)} \varphi d\mu^-_{v,U} = \int_{\partial\{v<0\} \setminus \Gamma^s_{v,-}} \varphi V \cdot n^- d\mathcal{H}^N, \tag{3.4}$$

where $V = (U_1, \cdots, U_N, 0)^t$, and n^+ resp. n^- correspond to the unit exterior normal vector to $\partial\{v > 0\} \setminus \Gamma^s_{v,+}$ resp. $\partial\{v < 0\} \setminus \Gamma^s_{v,-}$.
(2) The measures $\mu^+_{v,U}$, $\mu^-_{v,U}$, and $\mu_{\{v,U\}}$ are concentrated resp. on $\Gamma^+_v = \partial\{v > 0\} \setminus \Gamma^s_{v,+}$, $\Gamma^-_v = \partial\{v < 0\} \setminus \Gamma^s_{v,-}$, and $\Gamma_v = (\partial\{v > 0\} \cup \partial\{v < 0\}) \setminus \Gamma^s_v$.
(3) For any subset Γ of any submanifold of $\mathbb{R}^N \times (t_1, t_2)$ of dimension $\leq N - 1$, one has: The restrictions of the measures $\mu^+_{v,U}$, $\mu^-_{v,U}$, and $\mu_{v,U}$ to Γ are all identically 0.
(4) Let $\Gamma^r_{v,+}$, $\Gamma^r_{v,-}$, and Γ^r_v be the regular sets corresponding to v introduced in Subsection 3.1. Then $\Gamma^r_{v,+}$, $\Gamma^r_{v,-}$, and Γ^r_v are countable unions of collections of C^∞-hypersurfaces (C^∞-manifolds of dimension N) of $\mathbb{R}^N \times (t_1, t_2)$ and collections of C^∞-submanifolds of dimension

less or equal to $N-1$ whose $N-$dimensional Hausdorff measures are 0. That is; $\Gamma^r_{v,+} = (\cup_{i\in I_1} H_i) \cup (\cup_{j\in I_2}\Gamma_j)$ and $\Gamma^r_{v,-} = (\cup_{i\in I_3} H'_i) \cup (\cup_{j\in I_4}\Gamma'_j)$, where H_i, $i \in I_1$ with I_1 a countable set and H'_i, $i \in I_3$ with I_3 a countable set, are C^∞-hypersurfaces, and Γ_j, $j \in I_2$ with I_2 a countable set and Γ'_j, $j \in I_4$ with I_4 a countable set, are C^∞-submanifolds of dimension less or equal to $N-1$ whose $N-$dimensional Hausdorff measures are 0. Moreover, $\Gamma^r_v = \Gamma^r_{v,+} \cup \Gamma^r_{v,-}$ and so Γ^r_v is the union of the collections above that form $\Gamma^r_{v,+}$ and $\Gamma^r_{v,-}$. Here, if $\Gamma^r_{v,+}$ resp. $\Gamma^r_{v,-}$ is a manifold of dimension less or equal to $N-1$, then I_1 resp. I_3 is empty.

Part *(1)* of Theorem 3.4 can be deduced directly from the proof of Theorem 3.1 of [12]. Part *(2)* can be deduced directly from Part *(4)* of Theorem 3.3 of [12]. Part *(4)* can be deduced from the proof of Theorem 2.1 of [12]. Part *(3)* is a consequence of Part *(1)* of Theorem 3.4. However, some of the ideas and constructions in the proof will be used in the proof of other theorems in this paper, and so, a rather complete proof of the theorem is given below.

Proof of Theorem 3.4. Let $V = (U_1, \cdots, U_N, 0)^t$. By the regularity of U, $V \in C^\infty(\mathbb{R}^N \times (t_1,t_2); \mathbb{R}^{N+1})$. Then proceeding as in the proof of Theorem 3.1 of [12] with $(UH_\zeta\varphi, U\cdot\nabla v, (a,b), \mathrm{div})$ of the proof of Theorem 3.1 of [12] replaced by $(V\varphi, v, (t_1,t_2), \mathrm{div}_{x,t})$ of the current theorem, one obtains

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \mathrm{div}_{x,t}\,(V\varphi)\chi_{\{v>0\}}dxd\tau = \int_{\partial\{v>0\}\setminus\Gamma^s_{v,+}} \varphi V\cdot n^+ d\mathcal{H}^N, \tag{3.5}$$

and

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \mathrm{div}_{x,t}\,(V\varphi)\chi_{\{v<0\}}dxd\tau = \int_{\partial\{v<0\}\setminus\Gamma^s_{v,-}} \varphi V\cdot n^- d\mathcal{H}^N, \tag{3.6}$$

for any Lipschitz function $\varphi \in C_c(\mathbb{R}^N \times (t_1,t_2); \mathbb{R})$. Above, n^+ resp. n^- correspond to the unit exterior normal vector to $\Gamma^+_v = \partial\{v>0\}\setminus\Gamma^s_{v,+}$ resp. $\Gamma^-_v = \partial\{v<0\}\setminus\Gamma^s_{v,-}$. By definition of V, one has

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \mathrm{div}\,(U\varphi)\chi_{\{v>0\}}dxd\tau = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \mathrm{div}_{x,t}\,(V\varphi)\chi_{\{v>0\}}dxd\tau, \tag{3.7}$$

and

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \mathrm{div}\,(U\varphi)\chi_{\{v<0\}}dxd\tau = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \mathrm{div}_{x,t}\,(V\varphi)\chi_{\{v<0\}}dxd\tau. \tag{3.8}$$

Using Parts *(1)-(2)* of Theorem 3.3 and (3.5)-(3.8), one obtains for any Lipschitz function $\varphi \in C_c(\mathbb{R}^N \times (t_1,t_2); \mathbb{R})$

$$\int \varphi d\mu^+_{v,U} = -\int_{\partial\{v>0\}\setminus\Gamma^s_{v,+}} \varphi V\cdot n^+ d\mathcal{H}^N, \tag{3.9}$$

and

$$\int \varphi d\mu^-_{v,U} = \int_{\partial\{v<0\}\setminus\Gamma^s_{v,-}} \varphi V\cdot n^- d\mathcal{H}^N. \tag{3.10}$$

Then using (3.9)-(3.10), one deduces that for any $\varphi \in C_c(\mathbb{R}^N \times (t_1,t_2); \mathbb{R})$

$$\int \varphi d\mu^+_{v,U} = -\int_{\partial\{v>0\}\setminus\Gamma^s_{v,+}} \varphi V \cdot n^+ d\mathcal{H}^N, \tag{3.11}$$

and

$$\int \varphi d\mu^-_{v,U} = \int_{\partial\{v<0\}\setminus\Gamma^s_{v,-}} \varphi V \cdot n^- d\mathcal{H}^N. \tag{3.12}$$

Now using (3.11)-(3.12) and Part *(3)* of Theorem 3.3, one concludes that the measures $\mu^+_{v,U}$, $\mu^-_{v,U}$, and $\mu_{\{v,U\}}$ are concentrated resp. on Γ^+_v, Γ^-_v, and $\Gamma_v = (\partial\{v>0\}\cup\partial\{v<0\})\setminus\Gamma^s_v$.

The proof of (3.9)-(3.10) is obtained in Steps 1-4 of the proof of Part *(1)* given below. The proof of (3.9)-(3.10) is obtained in Step 5 of the proof of Part *(1)* given below. Finally, the proof of (3.11)-(3.12) is obtained in Step 6 of the proof of Part *(1)* given below.

The following construction will be used in the proof of this theorem and several others in this paper. Let h_m be a sequence of functions in $C^\infty_c(\mathbb{R};\mathbb{R})$ such that $h_m(\tau) = 0$ if τ is not in $[t_1 + 1/m, t_2 - 1/m]$ and $h_m(\tau) = 1$ if $\tau \in [t_1 + 2/m, t_2 - 2/m]$ and $0 \le h_m \le 1$, and $m > \frac{4}{t_2 - t_1}$.

Let $\rho_{1,\eta_1} = \frac{1}{\eta_1^N}\rho_1(\frac{\cdot}{\eta_1})$, $\eta_1 \in (0,1]$, be a mollifier with $\rho_1 \in C^\infty_c(\mathbb{R}^N)$, $\rho_1 \ge 0$, spt$(\rho_1) \subset B(0,1)$ with $B(0,1)$ the unit ball of $\mathbb{R}^N$, and $\int_{\mathbb{R}^N} \rho_1 dy = 1$. Let $\rho_{2,\eta_2} = \frac{1}{\eta_2}\rho_2(\frac{\cdot}{\eta_2})$, $\eta_2 \in (0,1]$, be a mollifier with $\rho_2 \in C^\infty_c(\mathbb{R})$, $\rho_2 \ge 0$, spt$(\rho_2) \subset [-1,0]$, and $\int_{\mathbb{R}} \rho_2 d\tau = 1$. Set $\eta = (\eta_1, \eta_2)$ and $\rho_\eta = \rho_{1,\eta_1}\rho_{2,\eta_2}$. For every $\varphi \in C_c(\mathbb{R}^N \times (t_1,t_2);\mathbb{R})$ and for every nonnegative integer $m > \frac{4}{t_2-t_1}$, define

$$\varphi_\eta = \rho_\eta \star \varphi, \quad \varphi_{\eta,m} = h_m \psi_m \varphi_\eta. \tag{3.13}$$

Then, $\varphi_{\eta,m} \in C^\infty_c(\mathbb{R}^N \times (t_1 + 1/m, t_2 - 1/m))$. Above, ψ_m, $m \ge 1$, denote the sequence of functions defined in $\mathbb{R}^N$ by $\psi_m(x) = \psi(\frac{x}{m})$, where $\psi \in C^\infty_c(\mathbb{R}^N)$ and is such that $\psi(x) = 1$ for $|x| \le 1$, $\psi(x) = 0$ for $|x| \ge 2$, and $0 \le \psi(x) \le 1$ for all $x \in \mathbb{R}^N$. Also above, $\star$ denotes the standard convolution product and spt(f) denotes the support of the function f.

Proof of (1)

Let $\varphi \in C^1_c(\mathbb{R}^N \times (t_1,t_2);\mathbb{R})$. Let $V = (U_1, \cdots, U_N, 0)^t$. By the regularity of U, $V \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R}^{N+1})$. One proceeds as in the proof of Theorem 3.1 of [12] with $(UH_\zeta\varphi, U\cdot\nabla v, (a,b), \text{div})$ of the proof of Theorem 3.1 of [12] replaced by $(V\varphi, v, (t_1,t_2), \text{div}_{x,t})$ of the current theorem.

1. Let $\Gamma = \partial\{v>0\}\cup\partial\{v<0\}$. Using the regularity of v and U, one can take $(v, 0, \mathbb{R}^{N+1}, \mathbb{R}^N \times (t_1,t_2))$ in place of $(w, \gamma, \mathbb{R}^N, \Omega)$ in Theorem 2.1 of [12]. Then one concludes that $\mathcal{H}^N((\Gamma\setminus\Gamma^s_v)\cap K) < \infty$ for any compact set K of $\mathbb{R}^{N+1}$.

Using the regularity of v, the set Γ^s_v is a Borel subset of $\mathbb{R}^{N+1}$. Hence there exists a sequence of compact subsets, A_j, $j = 1,2,3,\cdots$ of Γ^s_v such that $A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_n \cdots$ and $\mathcal{H}^N(A_j) < \infty$ and $\mathcal{H}^N(\Gamma^s_v) = \lim_{j\to\infty}\mathcal{H}^N(A_j)$, and $\mathcal{H}^N(\Gamma^s_v \setminus \cup_{j=1}^\infty A_j) = 0$; Consult for example Theorem 2.10.48 of [4].

Let B_j, $j = 1,2,\cdots$ be a sequence of sets such that:

(1) For $j = 1,2,\cdots$, B_j is an open neighborhood of A_j in the sense that $A_j \subset B_j$ and B_j is an open subset of $\mathbb{R}^{N+1}$; and

(2) For $j = 1,2,\cdots$, $B_j \cap A_{j+1}$ is a strict subset of A_{j+1} if A_j is a strict subset of A_{j+1}.

Let

$$\tilde{B}_j = \{y \in B_j |\ d(y, B_{j+1} \setminus B_j) < \gamma\}$$

with γ a sufficiently small positive number.

Let Ω_1 denote the interior of B_1 and Ω_{j+1} denote the interior of $\tilde{B}_j \cup (B_{j+1} \setminus B_j)$, $j = 1, 2, \cdots$.

Let Ω_0 denote an open set of $\mathbb{R}^{N+1}$ such that:

(1) $\Omega_0 \cap V(\Gamma_v^s) = \emptyset$ with $V(\Gamma_v^s)$ an open neighborhood of Γ_v^s in the sense that $\Gamma_v^s \subset V(\Gamma_v^s)$ and $V(\Gamma_v^s)$ is an open subset of $\mathbb{R}^{N+1}$; and

(2) $\cup_{j=0}^{\infty} \Omega_j$ is an open cover of $\mathbb{R}^{N+1}$.

Then one can construct a sequence β_j, $j = 0, 1, 2, \cdots$ of infinitely differentiable functions with compact supports such that for $j = 0, 1, 2, \cdots$:

(1) $0 \le \beta_j \le 1$;

(2) The support of β_j is included in Ω_j;

(3) For each $y \in \mathbb{R}^{N+1}$, there is an open neighborhood V_y of y such that only a finite number of functions β_i is not identically 0 on V_y; and

(4) $\sum_{j=0}^{\infty} \beta_j$ is identically 1 in $\mathbb{R}^{N+1}$.

2. Let $y \in \Gamma \cap \mathbb{R}^N \times (t_1, t_2)$. Step 1 above shows that there is an open neighborhood V_y such that only a finite number of functions β_i is not identically 0 on V_y. Let φ be a Lipschitz function in $C_c(V_y \cap \mathbb{R}^N \times (t_1, t_2); \mathbb{R})$. Let $\Phi \in C_c^1(V_y; \mathbb{R})$ be such that $\Phi \equiv 1$ on spt(φ), where spt(φ) denotes the support of the function φ. Let I denote the finite set of indices corresponding to those functions β_i which are not identically 0 on V_y. Now these β_i have their supports included in a finite number of Ω_i. By construction; See Step 1 above, the intersection of $\cup_{i \in I} \Omega_i$ with Γ is included in $A_k \cup (\Gamma \setminus \Gamma_v^s)$ for some k. Therefore, for each $i \in I$, the intersection of the support of β_i with Γ is included in $A_k \cup (\Gamma \setminus \Gamma_v^s)$. Here a use of the fact that if $i = 0 \in I$, then by construction the support of β_0 does not intersect Γ_v^s, has been made. Then one obtains that $\mathcal{H}^N(\partial\{v > 0\} \cap V_y)$ is finite. Hence, since $\Phi\chi_{\{v>0\}}$ and $\Phi\chi_{\{v<0\}}$ are in $L^1(\mathbb{R}^N \times (t_1, t_2))$, one deduces that $\Phi\chi_{\{v>0\}}$ and $\Phi\chi_{\{v<0\}}$ are in $BV(\mathbb{R}^N \times (t_1, t_2))$. Therefore, one can use the chain rule and integrate by parts

$$\begin{aligned}
\int_{t_1}^{t_2} \int_{\mathbb{R}^N} \mathrm{div}_{x,t}\ (V\varphi)\Phi\chi_{\{v>0\}} dx d\tau &= \sum_{j \in I} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \mathrm{div}_{x,t}\ (V\beta_j\varphi)\Phi\chi_{\{v>0\}} dx d\tau \\
&= -\sum_{j \in I} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \varphi\beta_j V \cdot \nabla_{x,t} v \delta_{\{\partial\{v>0\}\}} dx d\tau \\
&= -\sum_{j \in I} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \varphi\beta_j U \cdot \nabla v \delta_{\{\partial\{v>0\}\}} dx d\tau. \qquad (3.14)
\end{aligned}$$

Above, δ is the Dirac measure. On the other hand, one can use the divergence theorem

$$\sum_{j \in I} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \mathrm{div}_{x,t}\ (V\beta_j\varphi)\Phi\chi_{\{v>0\}} dx d\tau = \sum_{j \in I} \int_{\partial\{v>0\}} \varphi\beta_j V \cdot n^+ d\mathcal{H}^N. \qquad (3.15)$$

Here, n^+ denotes the unit exterior normal vector to $\partial\{v > 0\} \cap V_y$. Since $\Phi \equiv 1$ on spt(φ), the relations (3.14) and (3.15) show that

$$
\begin{aligned}
\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \operatorname{div}_{x,t}\,(V\varphi)\chi_{\{v>0\}}dxd\tau &= \sum_{j\in I}\int_{\partial\{v>0\}}\varphi\beta_j V\cdot n^+ d\mathcal{H}^N\\
&= -\sum_{j\in I}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\varphi\beta_j U\cdot\nabla v\delta_{\{\partial\{v>0\}\}}dxd\tau. \qquad (3.16)
\end{aligned}
$$

Now using (3.16) and the properties of β_j and φ, one obtains

$$
\begin{aligned}
&\sum_{j\in I}\int_{\partial\{v>0\}}\varphi\beta_j V\cdot n^+ d\mathcal{H}^N\\
&= -\sum_{j\in I}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\varphi|_{\{A_k\cup(\Gamma\setminus\Gamma_v^s)\}}\beta_j U\cdot\nabla v\delta_{\{\partial\{v>0\}\}}dxd\tau\\
&= -\sum_{j\in I}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\varphi|_{\{\Gamma\setminus\Gamma_v^s\}}\beta_j U\cdot\nabla v\delta_{\{\partial\{v>0\}\}}dxd\tau. \qquad (3.17)
\end{aligned}
$$

The last equality above is obtained by using the definition of Γ_v^s; Consult Subsection 3.1 above, which shows that on Γ_v^s one has: $\nabla v\equiv 0$. Since $A_k\subset\Gamma_v^s$, one has on A_k: $\nabla v\equiv 0$.

By (3.16) and (3.17), one obtains

$$
\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\operatorname{div}_{x,t}(V\varphi)\chi_{\{v>0\}}dxd\tau = \int_{\partial\{v>0\}\cap\{\Gamma\setminus\Gamma_v^s\}}\varphi V\cdot n^+ d\mathcal{H}^N.
$$

3. Since y was arbitrary on $\Gamma\cap\mathbb{R}^N\times(t_1,t_2)$, one concludes that

$$
\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\operatorname{div}_{x,t}(V\varphi)\chi_{\{v>0\}}dxd\tau = \int_{\partial\{v>0\}\cap\{\Gamma\setminus\Gamma_v^s\}}\varphi V\cdot n^+ d\mathcal{H}^N, \qquad (3.18)
$$

for any Lipschitz function $\varphi\in C_c(\mathbb{R}^N\times(t_1,t_2);\mathbb{R})$.

4. Proceeding as in Steps 1-3 above, one obtains

$$
\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\operatorname{div}_{x,t}(V\varphi)\chi_{\{v<0\}}dxd\tau = \int_{\partial\{v<0\}\cap\{\Gamma\setminus\Gamma_v^s\}}\varphi V\cdot n^+ d\mathcal{H}^N, \qquad (3.19)
$$

for any Lipschitz function $\varphi\in C_c(\mathbb{R}^N\times(t_1,t_2);\mathbb{R})$.

5. By construction of V, one has

$$
\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\operatorname{div}(U\varphi)\chi_{\{v>0\}}dxd\tau = \int_{t_1}^{t_2}\int_{\mathbb{R}^N}\operatorname{div}_{x,t}(V\varphi)\chi_{\{v>0\}}dxd\tau, \qquad (3.20)
$$

and

$$
\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\operatorname{div}(U\varphi)\chi_{\{v<0\}}dxd\tau = \int_{t_1}^{t_2}\int_{\mathbb{R}^N}\operatorname{div}_{x,t}(V\varphi)\chi_{\{v<0\}}dxd\tau. \qquad (3.21)
$$

Then using Parts *(1)*-*(2)* of Theorem 3.3 and (3.18)-(3.21), one obtains

$$\int \varphi d\mu_{v,U}^{+} = -\int_{\partial\{v>0\}\setminus\Gamma_{v,+}^{s}} \varphi V\cdot n^{+} d\mathcal{H}^{N}, \tag{3.22}$$

and

$$\int \varphi d\mu_{v,U}^{-} = \int_{\partial\{v<0\}\setminus\Gamma_{v,-}^{s}} \varphi V\cdot n^{-} d\mathcal{H}^{N}, \tag{3.23}$$

for any Lipschitz function $\varphi \in C_c(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$.

6. Let φ be any function in $C_c(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$. Let φ_η and $\varphi_{\eta,m}$ be the sequences of functions associated to φ by (3.13); Consult the beginning of the proof of the theorem. Then for every η and every nonnegative integer $m > \frac{4}{t_2-t_1}$, $\varphi_{\eta,m} = h_m\psi_m\varphi_\eta \in C_c^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$. Using this function in (3.22)-(3.23), one obtains

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} h_m\psi_m\varphi_\eta d\mu_{v,U}^{+} = -\int_{\partial\{v>0\}\setminus\Gamma_{v,+}^{s}} h_m\psi_m\varphi_\eta V\cdot n^{+} d\mathcal{H}^{N}, \tag{3.24}$$

and

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} h_m\psi_m\varphi_\eta d\mu_{v,U}^{-} = \int_{\partial\{v<0\}\setminus\Gamma_{v,-}^{s}} h_m\psi_m\varphi_\eta V\cdot n^{-} d\mathcal{H}^{N}. \tag{3.25}$$

By the properties of the mollifiers and the continuity of φ, as η goes to 0, $h_m\psi_m\varphi_\eta$ converges everywhere to $h_m\psi_m\varphi$. Moreover, by the regularity of U and the properties of the mollifiers, one has

$$\begin{aligned}
&|h_m\psi_m\varphi_\eta| \le Ch_m\psi_m,\\
&|h_m\psi_m\varphi_\eta V\cdot n^{+}\chi_{\{\partial\{v>0\}\setminus\Gamma_{v,+}^{s}\}}| \le Ch_m\psi_m\chi_{\{\partial\{v>0\}\setminus\Gamma_{v,+}^{s}\}},\\
&|h_m\psi_m\varphi_\eta V\cdot n^{-}\chi_{\{\partial\{v<0\}\setminus\Gamma_{v,-}^{s}\}}| \le Ch_m\psi_m\chi_{\{\partial\{v<0\}\setminus\Gamma_{v,-}^{s}\}},
\end{aligned}$$

where C is a positive constant independent of η. Also, since $h_m\psi_m \in C_c(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$, $h_m\psi_m$ is integrable with respect to the measures $\mu_{\{v,U\}}^{+}$ and $\mu_{\{v,U\}}^{-}$. Moreover, using Theorem 2.1 of [12] with $(w, \gamma, \mathbb{R}^N, \Omega)$ in Theorem 2.1 of [12] replaced by $(v, 0, \mathbb{R}^{N+1}, \mathbb{R}^N\times(t_1, t_2))$ of this proof, shows that $\mathcal{H}^N((\partial\{v>0\}\setminus\Gamma_{v,+}^{s})\cap K) < \infty$ and $\mathcal{H}^N((\partial\{v<0\}\setminus\Gamma_{v,-}^{s})\cap K) < \infty$ for any compact set K of $\mathbb{R}^{N+1}$. Hence, $h_m\psi_m\chi_{\{\partial\{v>0\}\setminus\Gamma_{v,+}^{s}\}}$ and $h_m\psi_m\chi_{\{\partial\{v<0\}\setminus\Gamma_{v,-}^{s}\}}$ are integrable with respect to the Hausdorff measure $\mathcal{H}^N$. Therefore, one can apply dominated convergence theorem for the measures $\mu_{\{v,U\}}^{+}$, $\mu_{\{v,U\}}^{-}$, and $\mathcal{H}^N$ and obtain, up to a subsequence, as η goes to 0,

$$\int h_m\psi_m\varphi d\mu_{\{v,U\}}^{+} = \lim_{\eta\to 0}\int h_m\psi_m\varphi_\eta d\mu_{\{v,U\}}^{+}, \tag{3.26}$$

$$-\int_{\partial\{v>0\}\setminus\Gamma_{v,+}^{s}} h_m\psi_m\varphi V\cdot n^{+} d\mathcal{H}^{N} = -\lim_{\eta\to 0}\int_{\partial\{v>0\}\setminus\Gamma_{v,+}^{s}} h_m\psi_m\varphi_\eta V\cdot n^{+} d\mathcal{H}^{N}, \tag{3.27}$$

and

$$\int h_m\psi_m\varphi d\mu^-_{\{v,U\}} = \lim_{\eta\to 0}\int h_m\psi_m\varphi_\eta d\mu^-_{\{v,U\}}, \tag{3.28}$$

$$\int_{\partial\{v<0\}\setminus\Gamma^s_{v,-}} h_m\psi_m\varphi V\cdot n^- d\mathcal{H}^N = \lim_{\eta\to 0}\int_{\partial\{v<0\}\setminus\Gamma^s_{v,-}} h_m\psi_m\varphi_\eta V\cdot n^- d\mathcal{H}^N. \tag{3.29}$$

Using (3.24) and (3.25), the right sides of (3.26) and (3.27) are equal and the right sides of (3.28) and (3.29) are equal. Hence,

$$\int h_m\psi_m\varphi d\mu^+_{\{v,U\}} = -\int_{\partial\{v>0\}\setminus\Gamma^s_{v,+}} h_m\psi_m\varphi V\cdot n^+ d\mathcal{H}^N, \tag{3.30}$$

and

$$\int h_m\psi_m\varphi d\mu^-_{\{v,U\}} = \int_{\partial\{v<0\}\setminus\Gamma^s_{v,-}} h_m\psi_m\varphi V\cdot n^- d\mathcal{H}^N. \tag{3.31}$$

By construction of the functions h_m and ψ_m, as m goes to ∞, $h_m\psi_m\varphi$ converges everywhere to φ. Moreover, $|h_m\psi_m\varphi|\leq|\varphi|$ and $|h_m\psi_m\varphi V\cdot n^+\chi_{\{\partial\{v>0\}\setminus\Gamma^s_{v,+}\}}|\leq C|\varphi|\chi_{\{\partial\{v>0\}\setminus\Gamma^s_{v,+}\}}$, and $|h_m\psi_m\varphi V\cdot n^-\chi_{\{\partial\{v<0\}\setminus\Gamma^s_{v,-}\}}|\leq C|\varphi|\chi_{\{\partial\{v<0\}\setminus\Gamma^s_{v,-}\}}$, where C is a positive constant independent of m. Also, since $\varphi\in C_c(\mathbb{R}^N\times(t_1,t_2);\mathbb{R})$, φ is integrable with respect to the measures $\mu^+_{\{v,U\}}$ and $\mu^-_{\{v,U\}}$. Moreover, using Theorem 2.1 of [12] with $(w,\gamma,\mathbb{R}^N,\Omega)$ in Theorem 2.1 of [12] replaced by $(v,0,\mathbb{R}^{N+1},\mathbb{R}^N\times(t_1,t_2))$ of this proof, shows that $\mathcal{H}^N((\partial\{v>0\}\setminus\Gamma^s_{v,+})\cap K)<\infty$ and $\mathcal{H}^N((\partial\{v<0\}\setminus\Gamma^s_{v,-})\cap K)<\infty$ for any compact set K of $\mathbb{R}^{N+1}$. Hence, $\varphi\chi_{\{\partial\{v>0\}\setminus\Gamma^s_{v,+}\}}$ and $\varphi\chi_{\{\partial\{v<0\}\setminus\Gamma^s_{v,-}\}}$ are integrable with respect to the Hausdorff measure $\mathcal{H}^N$. Therefore, one can apply dominated convergence theorem for the measures $\mu^+_{\{v,U\}}$, $\mu^-_{\{v,U\}}$, and $\mathcal{H}^N$ and obtain, up to a subsequence, as m goes to ∞,

$$\int\varphi d\mu^+_{\{v,U\}} = \lim_{m\to\infty}\int h_m\psi_m\varphi d\mu^+_{\{v,U\}} \tag{3.32}$$

$$-\int_{\partial\{v>0\}\setminus\Gamma^s_{v,+}}\varphi V\cdot n^+ d\mathcal{H}^N = -\lim_{m\to\infty}\int_{\partial\{v>0\}\setminus\Gamma^s_{v,+}} h_m\psi_m\varphi V\cdot n^+ d\mathcal{H}^N, \tag{3.33}$$

and

$$\int\varphi d\mu^-_{\{v,U\}} = \lim_{m\to\infty}\int h_m\psi_m\varphi d\mu^-_{\{v,U\}} \tag{3.34}$$

$$\int_{\partial\{v<0\}\setminus\Gamma^s_{v,-}}\varphi V\cdot n^- d\mathcal{H}^N = \lim_{m\to\infty}\int_{\partial\{v<0\}\setminus\Gamma^s_{v,-}} h_m\psi_m\varphi V\cdot n^- d\mathcal{H}^N. \tag{3.35}$$

Using (3.30) and (3.31), the right sides of (3.32) and (3.33) are equal and the right sides of (3.34) and (3.35) are equal. Hence,

$$\int\varphi d\mu^+_{\{v,U\}} = -\int_{\partial\{v>0\}\setminus\Gamma^s_{v,+}}\varphi V\cdot n^+ d\mathcal{H}^N, \tag{3.36}$$

and

$$\int \varphi d\mu^-_{\{v,U\}} = \int_{\partial\{v<0\}\setminus\Gamma^s_{v,-}} \varphi V\cdot n^- d\mathcal{H}^N. \tag{3.37}$$

(3.36) and (3.37) yield (3.3) and (3.4), and conclude the proof of Part *(1)* of the theorem.

Proof of (2)

Using (3.3)-(3.4) of Part *(1)* of the theorem and Part *(3)* of Theorem 3.3, one concludes that the measures $\mu^+_{v,U}$, $\mu^-_{v,U}$, and $\mu_{\{v,U\}}$ are concentrated resp. on $\Gamma^+_v = \partial\{v>0\}\setminus\Gamma^s_{v,+}$, $\Gamma^-_v = \partial\{v<0\}\setminus\Gamma^s_{v,-}$, and $\Gamma_v = (\partial\{v>0\}\cup\partial\{v<0\})\setminus\Gamma^s_v$.

Proof of (3)

Using (3.3)-(3.4) of Part *(1)* of the theorem, one deduces that for any Γ that is a subset of a submanifold of $\mathbb{R}^N\times(t_1,t_2)$ of dimension $\leq N-1$, one has: the restrictions of the measures $\mu^+_{v,U}$ and $\mu^-_{v,U}$ to Γ are identically 0. Then using Part *(3)* of Theorem 3.3, one has: the restriction of the measure $\mu_{v,U}$ to Γ is identically 0.

Proof of (4)

One deduces from Theorem 2.1 of [12] with $(w,\gamma,\mathbb{R}^N,\Omega)$ in Theorem 2.1 of [12] replaced by $(v,0,\mathbb{R}^{N+1},\mathbb{R}^N\times(t_1,t_2))$ of this proof that $\Gamma^r_{v,+}$, $\Gamma^r_{v,-}$, and Γ^r_v are countable unions of collections of C^∞-hypersurfaces (C^∞-manifolds of dimension N) of $\mathbb{R}^N\times(t_1,t_2)$ and collections of C^∞-submanifolds of dimension less or equal to $N-1$ whose $N-$dimensional Hausdorff measures are 0, as stated in Part *(4)* of the theorem. Moreover, by construction; Consult Subsection 3.1, $\Gamma^r_v = \Gamma^r_{v,+}\cup\Gamma^r_{v,-}$ and so Γ^r_v is the union of the collections above that form $\Gamma^r_{v,+}$ and $\Gamma^r_{v,-}$. The proof of Theorem 3.4 is now completed. ∎

3.4. **Characterization of the measures in terms of distributional derivatives of characteristic functions.** Let $E,\Omega\subset\mathbb{R}^N\times(t_1,t_2)$ with E $\mathcal{L}^{N+1}$-measurable. Denote by $P(E,\Omega)$ the perimeter of E in Ω. For any function f defined in $\mathbb{R}^N\times(t_1,t_2)$, denote by Df its distributional derivative with respect to (x,t).

Theorem 3.5. *Let N be any integer ≥ 2. Let $0\leq t_1<t_2$. Let $U\in C^\infty(\mathbb{R}^N\times(t_1,t_2);\mathbb{R}^N)$. Let $w\in C^\infty_c(\mathbb{R}^N\times(t_1,t_2);\mathbb{R})$. Then,*

(1) Let a be any real number in the range of w. Then $P(\{w-a>0\},\mathbb{R}^N\times(t_1,t_2))$ resp. $P(\{w-a<0\},\mathbb{R}^N\times(t_1,t_2))$ is finite and $D\chi_{\{w-a>0\}}$ resp. $D\chi_{\{w-a<0\}}$ is a finite Radon measure in $\mathbb{R}^N\times(t_1,t_2)$ and

$$P(\{w-a>0\},\mathbb{R}^N\times(t_1,t_2)) = |D\chi_{\{w-a>0\}}|(\mathbb{R}^N\times(t_1,t_2)) \leq \mathcal{H}^N(\Gamma^r_{w-a,+}) < C,$$
$$P(\{w-a<0\},\mathbb{R}^N\times(t_1,t_2)) = |D\chi_{\{w-a<0\}}|(\mathbb{R}^N\times(t_1,t_2)) \leq \mathcal{H}^N(\Gamma^r_{w-a,-}) < C,$$

where C is a positive constant.

(2) Let a be any real number in the range of w. Let $\mu^+_{\{w-a,U\}}$ and $\mu^-_{\{w-a,U\}}$ be the measures corresponding to the pair $(w-a,U)$ obtained in Theorem 3.1. Then

$$\mu^+_{w-a,U} = \sum_{i=1}^N U_i D_i\chi_{\{w-a>0\}}, \quad \mu^-_{w-a,U} = -\sum_{i=1}^N U_i D_i\chi_{\{w-a<0\}}.$$

Proof of Theorem 3.5.

1. Let $V\in C^1_c(\mathbb{R}^N\times(t_1,t_2);\mathbb{R}^{N+1})$. Using (3.18) of Step 3 of the proof of Part (1) of Theorem 3.4 with $V\varphi$ resp. v and $\partial\{v>0\}\cap\{\Gamma\setminus\Gamma^s_v\}$ of (3.18) replaced by V resp. $w-a$ and $\partial\{w-a>0\}\setminus\Gamma^s_{w-a}$ of this step, one obtains

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \mathrm{div}_{x,t} V\chi_{\{w-a>0\}} dx d\tau = \int_{\partial\{w-a>0\}\setminus\Gamma^s_{w-a}} V\cdot n^+ d\mathcal{H}^N. \tag{3.38}$$

Here, the set Γ^s_{w-a} corresponding to w was introduced in Subsection 3.1. Taking the supremum over V with $\|V\|_\infty \le 1$, one concludes that $P(\{w-a>0\}, \mathbb{R}^N\times(t_1,t_2))$ (or simply the perimeter of $\{w-a>0\}$) satisfies

$$P(\{w-a>0\}, \mathbb{R}^N\times(t_1,t_2)) \le \mathcal{H}^N(\Gamma^r_{w-a,+}), \tag{3.39}$$

where $\Gamma^r_{w-a,+} = \partial\{w-a>0\}\setminus\Gamma^s_{w-a}$. Using the fact that w is of compact support and Part *(4)* of Theorem 3.4, one concludes that $\Gamma^r_{w-a,+}$ is a union of a finite number of connected C^∞-hypersurfaces of $\mathbb{R}^N\times(t_1,t_2)$. Hence, $\mathcal{H}^N(\Gamma^r_{w-a,+}) < C$, where C is a positive constant. Then using (3.39), one obtains

$$P(\{w-a>0\}, \mathbb{R}^N\times(t_1,t_2)) \le \mathcal{H}^N(\Gamma^r_{w-a,+}) < C.$$

Hence, $P(\{w-a>0\}, \mathbb{R}^N\times(t_1,t_2))$ is finite. Then one deduces that the distributional derivative with respect to (x,t), $D\chi_{\{w-a>0\}}$ is a finite Radon measure on $\mathbb{R}^N\times(t_1,t_2)$. Moreover,

$$|D\chi_{\{w-a>0\}}|(\mathbb{R}^N\times(t_1,t_2)) = P(\{w-a>0\}, \mathbb{R}^N\times(t_1,t_2)) \le \mathcal{H}^N(\Gamma^r_{w-a,+}). \tag{3.40}$$

By the fact that $\chi_{\{w-a>0\}} \in L^1_{loc}(\mathbb{R}^N\times(t_1,t_2))$, one has: $\chi_{\{w-a>0\}} \in BV_{loc}(\mathbb{R}^N\times(t_1,t_2))$.

Proceeding as above with appropriate adaptations, one obtains the proof of the statements in Part *(1)* corresponding to $\{w-a<0\}$.

2. Using (3.18) of Step 3 of the proof of Part (1) of Theorem 3.4 with V resp. v and $\partial\{v>0\}\cap\{\Gamma\setminus\Gamma^s_v\}$ of (3.18) replaced by $V=(U_1,\cdots,U_N,0)^t$ resp. $w-a$ and $\partial\{w-a>0\}\setminus\Gamma^s_{w-a}$ of this proof, one obtains for any Lipschitz function $\varphi\in C_c(\mathbb{R}^N\times(t_1,t_2);\mathbb{R})$,

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \mathrm{div}_{x,t}(\varphi V)\chi_{\{w-a>0\}} dx d\tau = \int_{\partial\{w-a>0\}\setminus\Gamma^s_{w-a}} \varphi V\cdot n^+ d\mathcal{H}^N. \tag{3.41}$$

Using the fact that by Part *(1)* of the theorem, $D\chi_{\{w-a>0\}}$ is a finite Radon measure, one can integrate by part in (3.41) and obtain

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \mathrm{div}_{x,t}(\varphi V)\chi_{\{w-a>0\}} dx d\tau = -\sum_{i=1}^N \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \varphi U_i d[D_i\chi_{\{w-a>0\}}]. \tag{3.42}$$

Using Part *(1)* of Theorem 3.4 with $(v, U, \partial\{v>0\}\setminus\Gamma^s_{v,+})$ replaced by $(w-a, U, \partial\{w-a>0\}\setminus\Gamma^s_{w-a,+})$ of this step with $\Gamma^s_{w-a,+}$ the set corresponding to $w-a$ introduced in Subsection 3.1, one obtains for any $\varphi\in C_c(\mathbb{R}^N\times(t_1,t_2);\mathbb{R})$,

$$\int_{\mathbb{R}^N\times(t_1,t_2)} \varphi d\mu^+_{w-a,U} = -\int_{\partial\{w-a>0\}\setminus\Gamma^s_{w-a,+}} \varphi V\cdot n^+ d\mathcal{H}^N. \tag{3.43}$$

Combining (3.41)-(3.43) one obtains for any Lipschitz function $\varphi\in C_c(\mathbb{R}^N\times(t_1,t_2);\mathbb{R})$,

$$\int_{\mathbb{R}^N\times(t_1,t_2)} \varphi d\mu^+_{w-a,U} = \sum_{i=1}^N \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \varphi U_i d[D_i\chi_{\{w-a>0\}}]. \tag{3.44}$$

3. Let φ be any function in $C_c(\Omega;\mathbb{R})$. Let K be the compact support of φ. Let $\widetilde{\varphi}$ be any function in $C_c^\infty(\Omega;\mathbb{R})$ such that $\widetilde{\varphi}\equiv 1$ on K. Let φ_η and $\varphi_{\eta,m}$ be the sequences of functions associated to φ by (3.13); Consult the beginning of the proof of Theorem 3.4. Then for every η and every nonnegative integer $m>\frac{4}{t_2-t_1}$, $\tilde{\varphi}\varphi_{\eta,m}=\tilde{\varphi}h_m\psi_m\varphi_\eta\in C_c^\infty(\Omega;\mathbb{R})$. Plugging this function in (3.44) and proceeding as in Step 6 of the proof of Part *(1)* of Theorem 3.4, one concludes that (3.44) holds for $\varphi\in C_c(\Omega;\mathbb{R})$; that is,

$$\int_{\mathbb{R}^N\times(t_1,t_2)}\varphi d\mu^+_{w-a,U}=\sum_{i=1}^N\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\varphi U_i d[D_i\chi_{\{w-a>0\}}]. \tag{3.45}$$

This proves the first statement in Part *(2)*. Proceeding as above with appropriate adaptations, one obtains the second statement in Part *(2)*. The proof of Theorem 3.5 is completed. ∎

3.5. Characterization of the measures as functionals on the space BV.

Theorem 3.6. *Let N be any integer ≥ 1. Let $0\leq t_1<t_2$. Let $U\in C^\infty(\mathbb{R}^N\times(t_1,t_2);\mathbb{R}^N)$. Let $v\in C_c^\infty(\mathbb{R}^N\times(t_1,t_2);\mathbb{R})$. Let $\mu^+_{\{v,U\}}$, $\mu^-_{\{v,U\}}$, and $\mu_{\{v,U\}}$ be the measures corresponding to the pair (v,U) obtained in Theorem 3.1.*

(1) Set $K=$ spt(v). Then spt$(\mu^+_{\{v,U\}})\subset K$.

(2) For any $\varphi\in BV_{loc}(\mathbb{R}^N\times(t_1,t_2))$ such that the distributional derivatives with respect to x, $D_i\varphi$, $i=1,\cdots,N$ are finite Radon measures in $\mathbb{R}^N\times(t_1,t_2)$, one has

(3.46)
$$\int\varphi d\mu^+_{\{v,U\}}=-\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\varphi \boldsymbol{div}(U)\chi_{\{v>0\}}dxd\tau-\sum_{i=1}^N\int_{t_1}^{t_2}\int_{\mathbb{R}^N}U_i\chi_{\{v>0\}\cup\partial\{v>0\}}d[D_i\varphi],$$

and

$$\int\varphi d\mu^-_{\{v,U\}}=\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\varphi \boldsymbol{div}(U)\chi_{\{v<0\}}dxd\tau+\sum_{i=1}^N\int_{t_1}^{t_2}\int_{\mathbb{R}^N}U_i\chi_{\{v<0\}\cup\partial\{v<0\}}d[D_i\varphi], \tag{3.47}$$

Moreover, $\mu^+_{\{v,U\}}$ and $\mu^-_{\{v,U\}}$ are in the dual space of $BV(\mathbb{R}^N\times(t_1,t_2))$ denoted here by $(BV(\mathbb{R}^N\times(t_1,t_2))^\star$ and

$$\|\mu^+_{\{v,U\}}\|_{(BV(\mathbb{R}^N\times(t_1,t_2)))^\star}\leq\|\boldsymbol{div}(U)\chi_{\{v>0\}}\|_{\infty,K}+\|U\chi_{\{v>0\}\cup\partial\{v>0\}}\|_{\infty,K}, \tag{3.48}$$

and

$$\|\mu^-_{\{v,U\}}\|_{(BV(\mathbb{R}^N\times(t_1,t_2)))^\star}\leq\|\boldsymbol{div}(U)\chi_{\{v<0\}}\|_{\infty,K}+\|U\chi_{\{v<0\}\cup\partial\{v<0\}}\|_{\infty,K}, \tag{3.49}$$

(3) $\mu^+_{\{v,U\}}$ and $\mu^-_{\{v,U\}}$ are in the dual space of $W^{1,1}(\mathbb{R}^N\times(t_1,t_2))$ denoted here by $(W^{1,1}(\mathbb{R}^N\times(t_1,t_2)))^\star$.

Proof of Theorem 3.6.

Proof of (1)

Let $K=\mathrm{spt}(v)$. By assumption, K is compact. By Part *(2)* of Theorem 3.4, the measure $\mu^+_{\{v,U\}}$ is concentrated on $\Gamma^+_v=\partial\{v>0\}\setminus\Gamma^s_{v,+}$, where $\Gamma^s_{v,+}$ is the set corresponding to v introduced in Subsection 3.1. Hence, spt$(\mu^+_{\{v,U\}})\subset K$.

Proof of (2)

1. Let $\varphi \in BV(\mathbb{R}^N \times (t_1, t_2))$. Let h_m be a sequence of functions in $C_c^\infty(\mathbb{R}; \mathbb{R})$ such that $h_m(\tau) = 0$ if τ is not in $[t_1 + 1/m, t_2 - 1/m]$ and $h_m(\tau) = 1$ if $\tau \in [t_1 + 2/m, t_2 - 2/m]$ and $0 \leq h_m \leq 1$, and $m > \frac{4}{t_2 - t_1}$. Let $\rho_{1,\eta_1} = \frac{1}{\eta_1^N}\rho_1(\frac{\cdot}{\eta_1})$, $\eta_1 \in (0, 1]$, be a mollifier with $\rho_1 \in C_c^\infty(\mathbb{R}^N)$, $\rho_1 \geq 0$, spt$(\rho_1) \subset B(0, 1)$ with $B(0, 1)$ the unit ball of $\mathbb{R}^N$, and $\int_{\mathbb{R}^N} \rho_1 dy = 1$. Let $\rho_{2,\eta_2} = \frac{1}{\eta_2}\rho_2(\frac{\cdot}{\eta_2})$, $\eta_2 \in (0, 1]$, be a mollifier with $\rho_2 \in C_c^\infty(\mathbb{R})$, $\rho_2 \geq 0$, spt$(\rho_2) \subset [-1, 1]$, and $\int_{\mathbb{R}} \rho_2 d\tau = 1$. Then set $\eta = (\eta_1, \eta_2)$ and $\rho_\eta = \rho_{1,\eta_1}\rho_{2,\eta_2}$.

Set $\varphi_\eta = \varphi \star \rho_\eta$. Using Part *(1)* of Theorem 3.3 yields

$$\int h_m \varphi_\eta d\mu^+_{\{v,U\}} = -\int_{\{v>0\}} \text{div}(h_m \varphi_\eta U) dx d\tau. \tag{3.50}$$

Now one has

$$\int_{\{v>0\}} \text{div}(h_m\varphi_\eta U) dx d\tau = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \varphi_\eta h_m \text{div}(U)\chi_{\{v>0\}} dx d\tau + \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \nabla\varphi_\eta h_m \cdot U\chi_{\{v>0\}} dx d\tau. \tag{3.51}$$

Using the properties of the convolution, the symmetry of ρ_η, Fubini Theorem, the fact that $h_m U\chi_{\{v>0\}} \in L^1(\mathbb{R}^N \times (t_1, t_2))$, and the definition of h_m, one obtains

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \nabla\varphi_\eta h_m \cdot U\chi_{\{v>0\}} dx d\tau = \sum_{i=1}^N \int_{t_1}^{t_2}\int_{\mathbb{R}^N} (h_m U_i \chi_{\{v>0\}}) \star \rho_\eta d[D_i\varphi]. \tag{3.52}$$

Using the fact that $v,\ h_m$, and U are all continuous, as η goes to 0, $(h_m U\chi_{\{v>0\}}) \star \rho_\eta \to h_m U\chi_{\{v>0\}\cup\partial\{v>0\}}$ everywhere. Moreover, by the properties of the regularization, $|(h_m U\chi_{\{v>0\}}) \star \rho_\eta| \leq C\|h_m U\|_{\infty,K}$ for some positive constant independent of η. Now since $\varphi \in BV(\mathbb{R}^N \times (t_1, t_2))$, one can use dominated convergence theorem and conclude that

$$\lim_{\eta\to 0}\sum_{i=1}^N \int_{t_1}^{t_2}\int_{\mathbb{R}^N} (h_m U_i\chi_{\{v>0\}})\star\rho_\eta d[D_i\varphi] = \sum_{i=1}^N \int_{t_1}^{t_2}\int_{\mathbb{R}^N} h_m U_i \chi_{\{v>0\}\cup\partial\{v>0\}} d[D_i\varphi]. \tag{3.53}$$

Now assume that $\varphi \in BV(\mathbb{R}^N \times (t_1, t_2)) \cap C^1(\mathbb{R}^N \times (t_1, t_2))$. Then by the properties of the regularization, the continuity of φ and the fact that $\varphi \in L^1_{loc}(\mathbb{R}^N \times (t_1, t_2))$, spt$(v)$ is compact, and $h_m$div$(U)\chi_{\{v>0\}}$ is bounded,

$$\lim_{\eta\to 0}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \varphi_\eta h_m \text{div}(U)\chi_{\{v>0\}} dx d\tau = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \varphi h_m \text{div}(U)\chi_{\{v>0\}} dx d\tau. \tag{3.54}$$

Using again the properties of the regularization and the continuity of φ, one obtains: $\lim_{\eta\to 0}\varphi_\eta = \varphi$. Moreover, by the continuity of φ, φ is bounded in K. Also, h_m is continuous and by Part *(1)* of the theorem spt$(\mu^+_{v,U}) \subset K$. Hence, by dominated convergence theorem for the measure $\mu^+_{v,U}$, one obtains

$$\lim_{\eta\to 0}\int h_m\varphi_\eta d\mu^+_{\{v,U\}} = \int h_m\varphi d\mu^+_{\{v,U\}}. \tag{3.55}$$

Combining (3.50)-(3.55) yields

(3.56)
$$\int h_m\varphi d\mu^+_{\{v,U\}} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\varphi h_m \mathrm{div}(U)\chi_{\{v>0\}}dxd\tau - \sum_{i=1}^{N}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} h_m U_i\chi_{\{v>0\}\cup\partial\{v>0\}}d[D_i\varphi].$$

Now as m goes to ∞, $h_m \to \chi_{(t_1,t_2)}$. Moreover, $|h_m| \leq \chi_{(t_1,t_2)}$. Then proceeding as above, one can let m go to ∞ in (3.56) and obtain

(3.57)
$$\int \varphi d\mu^+_{\{v,U\}} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\varphi \mathrm{div}(U)\chi_{\{v>0\}}dxd\tau - \sum_{i=1}^{N}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} U_i\chi_{\{v>0\}\cup\partial\{v>0\}}d[D_i\varphi].$$

2. Using the density of $BV(\mathbb{R}^N\times(t_1,t_2))\cap C^1(\mathbb{R}^N\times(t_1,t_2))$ in $BV(\mathbb{R}^N\times(t_1,t_2))$, the fact that $\mu^+_{\{v,U\}}$ is a finite Radon measure, and (3.57), one obtains for any $\varphi\in BV(\mathbb{R}^N\times(t_1,t_2))$,

(3.58)
$$\int \varphi d\mu^+_{\{v,U\}} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\varphi \mathrm{div}(U)\chi_{\{v>0\}}dxd\tau - \sum_{i=1}^{N}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} U_i\chi_{\{v>0\}\cup\partial\{v>0\}}d[D_i\varphi].$$

One then deduces that $\mu^+_{\{v,U\}}$ is in the dual space of $BV(\mathbb{R}^N\times(t_1,t_2))$ denoted here by $(BV(\mathbb{R}^N\times(t_1,t_2)))^\star$ and

$$\|\mu^+_{\{v,U\}}\|_{(BV(\mathbb{R}^N\times(t_1,t_2)))^\star} \leq \|\mathrm{div}(U)\chi_{\{v>0\}}\|_{\infty,K} + \|U\chi_{\{v>0\}\cup\partial\{v>0\}}\|_{\infty,K}. \tag{3.59}$$

This yields (3.48).

3. Proceeding as in Steps 1-2 above with appropriate adaptations, one obtains (3.58) with φ any function in $BV_{loc}(\mathbb{R}^N\times(t_1,t_2))$ such that the distributional derivatives with respect to x, $D_i\varphi$, $i=1,\cdots,N$ are finite Radon measures in $\mathbb{R}^N\times(t_1,t_2)$. This yields (3.46).

4. Proceeding as in Steps 1-3 above for $\mu^-_{v,U}$ with appropriate adaptations, one obtains (3.47) and (3.49) and this completes the proof of Part*(2)* of the theorem.

5. Since $W^{1,1}(\mathbb{R}^N\times(t_1,t_2))\subset BV(\mathbb{R}^N\times(t_1,t_2))$, using Part *(2)* of the theorem one deduces by transposition that $\mu^+_{\{v,U\}}$ and $\mu^-_{\{v,U\}}$ are in the dual space of $W^{1,1}(\mathbb{R}^N\times(t_1,t_2))$ denoted here by $(W^{1,1}(\mathbb{R}^N\times(t_1,t_2)))^\star$. This completes the proof of Theorem 3.6. ∎

4. THE SECOND CLASS OF MEASURES OF [12]

In this section, based on the results of [12], basic properties and characterizations of the family of measures $\mu^+_{\{v,U,\mathcal{O}\}}$, $\mu^-_{\{v,U,\mathcal{O}\}}$, and $\mu_{\{v,U,\mathcal{O}\}}$ generated by the triplets $(v,U,\mathcal{O})$ of scalar and vector functions v and U, and open sets $\mathcal{O}$ and introduced by the author in [12] are given. In particular, in Section 4.1, the precise definition of these measures and the fact that they are Radon measures are given. In Section 4.2, relationships between these measures and the measures $\mu^+_{\{v,U\}}$ and $\mu^-_{\{v,U\}}$ for general open sets $\mathcal{O}$ are established. In Section 4.3, relationships between these measures for particular open set generators are established.

The results of this section are partly contained in the author's paper [12].

Let $a<b$. Let E,F be subsets of $\mathbb{R}^N\times(a,b)$. The set E^c denotes the complementary of E in $\mathbb{R}^N\times(a,b)$. That is; $E^c=\{y\in\mathbb{R}^N\times(a,b)|\ y \text{ is not in } E\}$. The set $E\setminus F$ denotes the set of points that are in E but not in F. That is; $E\setminus F = E\cap F^c$.

4.1. **The measures.** Let N be any integer ≥ 1. Let $0 \leq t_1 < t_2$. The following theorem is a corollary of a theorem obtained by the author in [12]; Consult Theorem 3.1 of [12]. It yields a family of measures, $\mu^+_{\{v,U,\mathcal{O}\}}$, $\mu^-_{\{v,U,\mathcal{O}\}}$, and $\mu_{\{v,U,\mathcal{O}\}}$ generated by $v \in C^\infty(\mathbb{R}^N \times (t_1,t_2),\mathbb{R})$, $U \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R}^N)$, and open sets $\mathcal{O}$ of $\mathbb{R}^N \times (t_1,t_2)$, which are also Radon measures.

Theorem 4.1. *Let N be any integer ≥ 1. Let $0 \leq t_1 < t_2$. Let $v \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R})$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R}^N)$. Let $\mathcal{O}$ be any open set of $\mathbb{R}^N \times (t_1,t_2)$. Then the estimates in Part (1) of Theorem 3.1 with $\gamma = 0$ show that, up to a subsequence, as α goes to 0, the following weak convergence in the sense of measures, holds*

$$
\begin{aligned}
&U \cdot \nabla v \frac{1}{\alpha}\chi_{\{0<v<\alpha\}}\chi_{\mathcal{O}} \rightarrow \mu^+_{\{v,U,\mathcal{O}\}}, \;\; U \cdot \nabla v \frac{1}{\alpha}\chi_{\{-\alpha<v<0\}}\chi_{\mathcal{O}} \rightarrow \mu^-_{\{v,U,\mathcal{O}\}},\\
&U \cdot \nabla v \frac{1}{\alpha}\chi_{\{|v|<\alpha\}}\chi_{\mathcal{O}} \rightarrow \mu_{\{v,U,\mathcal{O}\}},
\end{aligned}
$$

where $\mu^+_{\{v,U,\mathcal{O}\}}$, $\mu^-_{\{v,U,\mathcal{O}\}}$, and $\mu_{\{v,U,\mathcal{O}\}}$ are measures on $\mathbb{R}^N \times (t_1,t_2)$ concentrated resp. on $\partial\{v > 0\} \cap \partial\{\{v > 0\} \cap \mathcal{O}\}$, $\partial\{v < 0\} \cap \partial\{\{v < 0\} \cap \mathcal{O}\}$, and $(\partial\{v > 0\} \cup \partial\{v < 0\}) \cap \partial\{(\{v > 0\} \cup \{v < 0\}) \cap \mathcal{O}\}$, which are also Radon measures.

Remark 4.2. The notation $\mu^+_{\{v,U,\mathcal{O}\}}$ and $\mu^-_{\{v,U,\mathcal{O}\}}$ should not be confused with the nonnegative and nonpositive parts of a measure. The upper index $+$ resp. $-$ is merely used to refer to the fact that the measure $\mu^+_{\{v,U,\mathcal{O}\}}$ with the upper index $+$ is concentrated on $\partial\{v > 0\}$ resp. the measure $\mu^-_{\{v,U,\mathcal{O}\}}$ with the upper index $-$ is concentrated on $\partial\{v < 0\}$.

The following terminology is adopted. For the measures $\mu^+_{\{v,U,\mathcal{O}\}}$, $\mu^-_{\{v,U,\mathcal{O}\}}$, and $\mu_{\{v,U,\mathcal{O}\}}$, the scalar function v resp. the vector function U and the open set $\mathcal{O}$ is called the scalar function generator resp. the vector function generator and the open generator of the measures $\mu^+_{\{v,U,\mathcal{O}\}}$, $\mu^-_{\{v,U,\mathcal{O}\}}$, and $\mu_{\{v,U,\mathcal{O}\}}$.

Proof of Theorem 4.1. The proof of Theorem 4.1 is deduced from that of Theorem 3.1 above; See also the proof of Theorem 3.1 of [12]. Let $V = (U_1, \cdots, U_N, 0)^t$. By the regularity of U, $V \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R}^{N+1})$. Let $\mathcal{O}$ be any open set of $\mathbb{R}^N \times (t_1,t_2)$. Then the estimates in Part *(1)* of Theorem 3.1 with $\gamma = 0$, show that

$$
\int_{\{|v|<\alpha\}\cap K} |U \cdot \nabla v \chi_{\mathcal{O}}| \frac{1}{\alpha} dx d\tau \leq \int_{\{|v|<\alpha\}\cap K} |U \cdot \nabla v| \frac{1}{\alpha} dx d\tau < C, \tag{4.1}
$$

and then

$$
\int_{\{0<v<\alpha\}\cap K} |U \cdot \nabla v \chi_{\mathcal{O}}| \frac{1}{\alpha} dx d\tau < C, \quad \int_{\{-\alpha<v<0\}\cap K} |U \cdot \nabla v \chi_{\mathcal{O}}| \frac{1}{\alpha} dx d\tau < C, \tag{4.2}
$$

where C is a positive constant independent of $\alpha \in (0,1)$ and K is an arbitrary compact subset of $\mathbb{R}^N \times (t_1,t_2)$. Then (4.1)-(4.2) show that $U \cdot \nabla v \frac{1}{\alpha}\chi_{\{|v|<\alpha\}}\chi_{\mathcal{O}}$, $U \cdot \nabla v \frac{1}{\alpha}\chi_{\{0<v<\alpha\}}\chi_{\mathcal{O}}$, and $U \cdot \nabla v \frac{1}{\alpha}\chi_{\{-\alpha<v<0\}}\chi_{\mathcal{O}}$ are uniformly in α bounded in L^1_{loc}. Therefore, up to a subsequence, as α goes to 0, the following weak convergence in the sense of measures holds,

$$
\begin{aligned}
&U \cdot \nabla v \frac{1}{\alpha}\chi_{\{0<v<\alpha\}}\chi_{\mathcal{O}} \rightarrow \mu^+_{\{v,U,\mathcal{O}\}}, \;\; U \cdot \nabla v \frac{1}{\alpha}\chi_{\{-\alpha<v<0\}}\chi_{\mathcal{O}} \rightarrow \mu^-_{\{v,U,\mathcal{O}\}},\\
&U \cdot \nabla v \frac{1}{\alpha}\chi_{\{|v|<\alpha\}}\chi_{\mathcal{O}} \rightarrow \mu_{\{v,U,\mathcal{O}\}},
\end{aligned}
$$

where $\mu^+_{\{v,U,\mathcal{O}\}}$, $\mu^-_{\{v,U,\mathcal{O}\}}$, and $\mu_{\{v,U,\mathcal{O}\}}$ are Radon measures on $\mathbb{R}^N \times (t_1,t_2)$ concentrated resp. on $\partial\{v>0\}\cap\partial\{\{v>0\}\cap\mathcal{O}\}$, $\partial\{v<0\}\cap\partial\{\{v<0\}\cap\mathcal{O}\}$, and $(\partial\{v>0\}\cup\partial\{v<0\})\cap\partial\{(\{v>0\}\cup\{v<0\})\cap\mathcal{O}\}$. This completes the proof of Theorem 4.1. ∎

4.2. Relations with the first class of measures of [12].

Theorem 4.3. *Let N be any integer ≥ 1. Let $0\leq t_1<t_2$. Let $U\in C^\infty(\mathbb{R}^N\times(t_1,t_2);\mathbb{R}^N)$. Let $v\in C^\infty(\mathbb{R}^N\times(t_1,t_2);\mathbb{R})$. Let $\mu^+_{\{v,U\}}$ and $\mu^-_{\{v,U\}}$ be the measures corresponding to the pair (v,U) obtained in Theorem 3.1. Let $\mathcal{O}$ be any open set of $\mathbb{R}^N\times(t_1,t_2)$. Let $\mu^+_{\{v,U,\mathcal{O}\}}$ and $\mu^-_{\{v,U,\mathcal{O}\}}$ be the measures corresponding to the triplets $(v,U,\mathcal{O})$ obtained in Theorem 4.1.*

Set $\gamma_v^+=\partial\{v>0\}$ and $\gamma_v^-=\partial\{v<0\}$. Then for every $\varphi\in C_c(\mathbb{R}^N\times(t_1,t_2);\mathbb{R})$,

(1)

$$\int \chi_{\mathcal{O}}\varphi d\mu^+_{\{v,U,\mathcal{O}\}} = \int \chi_{\mathcal{O}}\varphi d\mu^+_{\{v,U\}}.$$

(2)

$$\begin{aligned}
&\int \varphi d\mu^+_{\{v,U,\mathcal{O}\}}\\
=&\int \varphi\chi_{\{\gamma_v^+\cap\partial\{\{v>0\}\cap\mathcal{O}\}\cap\mathcal{O}\}}d\mu^+_{\{v,U,\mathcal{O}\}}+\int \varphi\chi_{\{\gamma_v^+\cap\partial\{\{v>0\}\cap\mathcal{O}\}\cap\partial\mathcal{O}\}}d\mu^+_{\{v,U,\mathcal{O}\}}\\
=&\int \varphi\chi_{\mathcal{O}}d\mu^+_{\{v,U\}}+\int \varphi\chi_{\{\gamma_v^+\cap\partial\{\{v>0\}\cap\mathcal{O}\}\cap\partial\mathcal{O}\}}d\mu^+_{\{v,U,\mathcal{O}\}}.
\end{aligned}$$

(3)

$$\int \chi_{\mathcal{O}}\varphi d\mu^-_{\{v,U,\mathcal{O}\}} = \int \chi_{\mathcal{O}}\varphi d\mu^-_{\{v,U\}}.$$

(4)

$$\begin{aligned}
&\int \varphi d\mu^-_{\{v,U,\mathcal{O}\}}\\
=&\int \varphi\chi_{\{\gamma_v^-\cap\partial\{\{v<0\}\cap\mathcal{O}\}\cap\mathcal{O}\}}d\mu^-_{\{v,U,\mathcal{O}\}}+\int \varphi\chi_{\{\gamma_v^-\cap\partial\{\{v<0\}\cap\mathcal{O}\}\cap\partial\mathcal{O}\}}d\mu^-_{\{v,U,\mathcal{O}\}}\\
=&\int \varphi\chi_{\mathcal{O}}d\mu^-_{\{v,U\}}+\int \varphi\chi_{\{\gamma_v^-\cap\partial\{\{v<0\}\cap\mathcal{O}\}\cap\partial\mathcal{O}\}}d\mu^-_{\{v,U,\mathcal{O}\}}.
\end{aligned}$$

Proof of Theorem 4.3. If $\mathcal{O}\cap\{0<v<\alpha\}$ is empty for α sufficiently small, then all terms in Parts *(1)-(2)* of the theorem are 0. Therefore, it is assumed that, for all α sufficiently small, $\mathcal{O}\cap\{0<v<\alpha\}$ is not empty.

1. Theorem 4.1 yields, up to a subsequence, as α goes to 0,

$$\int \varphi d\mu^+_{\{v,U,\mathcal{O}\}} = \lim_{\alpha\to 0}\int_{\{0<v<\alpha\}} \frac{U\cdot\nabla v}{\alpha}\chi_{\mathcal{O}}\varphi dx d\tau, \tag{4.3}$$

for any $\varphi\in C_c(\mathbb{R}^N\times(t_1,t_2);\mathbb{R})$, where $\mu^+_{\{v,U,\mathcal{O}\}}$ is a Radon measure on $\mathbb{R}^N\times(t_1,t_2)$ concentrated on $\gamma_v^+\cap\partial\{\{v>0\}\cap\mathcal{O}\}$.

2. Let $y \in \gamma_v^+ \cap \partial\{\{v > 0\} \cap \mathcal{O}\} \cap \mathcal{O}$. Let $B(y, \epsilon)$ be a ball of radius ϵ centered at y with ϵ sufficiently small that $B(y, \epsilon)$ is a subset of $\mathcal{O}$. Let $\varphi \in C_c(B(y, \epsilon); \mathbb{R})$. Then using (4.3) shows that, up to a subsequence,

$$\int \varphi d\mu^+_{\{v,U,\mathcal{O}\}} = \lim_{\alpha \to 0} \int_{\{0<v<\alpha\}} \frac{U \cdot \nabla v}{\alpha} \chi_{\mathcal{O}} \varphi dx d\tau = \lim_{\alpha \to 0} \int_{\{0<v<\alpha\}} \frac{U \cdot \nabla v}{\alpha} \varphi dx d\tau. \tag{4.4}$$

On the other hand, using Part *(2)* of Theorem 3.1 yields, up to a subsequence,

$$\int \varphi d\mu^+_{\{v,U\}} = \lim_{\alpha \to 0} \int_{\{0<v<\alpha\}} \frac{U \cdot \nabla v}{\alpha} \varphi dx d\tau. \tag{4.5}$$

(4.4)-(4.5) then yield

$$\int \varphi d\mu^+_{\{v,U\}} = \int \varphi d\mu^+_{\{v,U,\mathcal{O}\}}.$$

Since y was arbitrary on $\gamma_v^+ \cap \partial\{\{v > 0\} \cap \mathcal{O}\} \cap \mathcal{O}$, one deduces that for any $\varphi \in C_c(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$,

$$\int \varphi \chi_{\mathcal{O}} d\mu^+_{\{v,U,\mathcal{O}\}} = \int \varphi \chi_{\{\gamma_v^+ \cap \partial\{\{v>0\} \cap \mathcal{O}\} \cap \mathcal{O}\}} d\mu^+_{\{v,U\}}. \tag{4.6}$$

This yields Part *(1)* of the theorem.

3. Using Theorem 4.1 yields for every function $\varphi \in C_c(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$,

$$\int \varphi d\mu^+_{\{v,U,\mathcal{O}\}} = \int \varphi \chi_{\{\gamma_v^+ \cap \partial\{\{v>0\} \cap \mathcal{O}\} \cap \mathcal{O}\}} d\mu^+_{\{v,U,\mathcal{O}\}} + \int \varphi \chi_{\{\gamma_v^+ \cap \partial\{\{v>0\} \cap \mathcal{O}\} \cap \partial\mathcal{O}\}} d\mu^+_{\{v,U,\mathcal{O}\}}.$$

Then using (4.6) of Step 2, one obtains for every function $\varphi \in C_c(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$,

$$\int \varphi d\mu^+_{\{v,U,\mathcal{O}\}} = \int \varphi \chi_{\mathcal{O}} d\mu^+_{\{v,U\}} + \int \varphi \chi_{\{\gamma_v^+ \cap \partial\{\{v>0\} \cap \mathcal{O}\} \cap \partial\mathcal{O}\}} d\mu^+_{\{v,U,\mathcal{O}\}}.$$

This yields Part *(2)* of the theorem.

5. Let $\mathcal{O}$ be an open set of $\mathbb{R}^N \times (t_1, t_2)$. If $\mathcal{O} \cap \{-\alpha < v < 0\}$ is empty for α sufficiently small, then all terms in Parts *(3)-(4)* of the theorem are 0. Therefore, it is assumed that, for all α sufficiently small, $\mathcal{O} \cap \{-\alpha < v < 0\}$ is not empty. By the regularity of v, $\{v < 0\}$ is an open set. Proceeding as in Steps 1-3, one obtains Parts *(3)-(4)* of the theorem. This completes the proof of Theorem 4.3. ∎

4.3. **The measures for particular open set generators.**

Theorem 4.4. *Let N be any integer ≥ 1. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$. Let $w, v \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$. Let $\mu^+_{\{v,U\}}$ and $\mu^-_{\{v,U\}}$ be the measures corresponding to the pair (v, U) obtained in Theorem 3.1. Let $\mu^+_{\{w,U,\{v>0\}\}}$, $\mu^+_{\{w,U,\{v<0\}\}}$, $\mu^-_{\{w,U,\{v>0\}\}}$, and $\mu^-_{\{w,U,\{v<0\}\}}$ be the measures corresponding to w, v, U, and the open sets $\{v > 0\}$ and $\{v < 0\}$ obtained in Theorem 4.1. Then for every Lipschitz function $\varphi \in C_c(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$, the following identities hold.*

(1)

$$\int \chi_{\{w>0\}}\varphi d\mu^+_{\{v,U\}} + \int \varphi d\mu^+_{\{w,U,\{v>0\}\}} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^N} div(\varphi U)\chi_{\{v>0\}}\chi_{\{w>0\}}dxd\tau.$$

(2)

$$\int \chi_{\{w<0\}}\varphi d\mu^+_{\{v,U\}} - \int \varphi d\mu^-_{\{w,U,\{v>0\}\}} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^N} div(\varphi U)\chi_{\{v>0\}}\chi_{\{w<0\}}dxd\tau.$$

(3)

$$\int \chi_{\{w>0\}}\varphi d\mu^-_{\{v,U\}} - \int \varphi d\mu^+_{\{w,U,\{v<0\}\}} = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} div(\varphi U)\chi_{\{v<0\}}\chi_{\{w>0\}}dxd\tau.$$

(4)

$$\int \chi_{\{w<0\}}\varphi d\mu^-_{\{v,U\}} + \int \varphi d\mu^-_{\{w,U,\{v<0\}\}} = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} div(\varphi U)\chi_{\{v<0\}}\chi_{\{w<0\}}dxd\tau.$$

Proof of Theorem 4.4. Let $\alpha,\ \epsilon \in (0,1)$. Then for $j = 1,2,3$, let $G_\epsilon^{(j)}$ and $G_\alpha^{(j)}$ be the sequence of functions introduced at the beginning of the proof of Theorem 3.3.

Proof of (1). Let φ be any Lipschitz function in $C_c(\mathbb{R}^N \times (t_1,t_2);\mathbb{R})$. Using the regularity of U, v, and w, yields

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \text{div}(\varphi U G_\alpha^{(2)}(v)G_\epsilon^{(2)}(w))dy = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \text{div}(\varphi U)G_\alpha^{(2)}(v)G_\epsilon^{(2)}(w)dy + \int_{\{0<v<\alpha\}} \frac{U\cdot\nabla v}{\alpha}\varphi G_\epsilon^{(2)}(w)dy + \int_{\{0<w<\epsilon\}} \frac{U\cdot\nabla w}{\epsilon}\varphi G_\alpha^{(2)}(v)dy.$$

By the regularity of U, v, and w, and the fact that φ is of compact support, $\varphi U G_\alpha^{(2)}(v)G_\epsilon^{(2)}(w)$ is in $W^{1,1}(\mathbb{R}^N \times (t_1,t_2))$ and

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \text{div}(\varphi U G_\alpha^{(2)}(v)G_\epsilon^{(2)}(w))dy = 0.$$

Therefore

$$\begin{aligned} &\int_{\{0<v<\alpha\}} \frac{U\cdot\nabla v}{\alpha}G_\epsilon^{(2)}(w)\varphi dy + \int_{\{0<w<\epsilon\}} \frac{U\cdot\nabla w}{\epsilon}G_\alpha^{(2)}(v)\varphi dy \\ (4.7)\qquad &= -\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \text{div}(\varphi U)G_\alpha^{(2)}(v)G_\epsilon^{(2)}(w)dy. \end{aligned}$$

Here α is let go to 0 first, then ϵ is let go to 0 second. As α goes to 0, $G_\alpha^{(2)}(v)$ converges everywhere to $\chi_{\{v>0\}}$. Now $|\text{div}(\varphi U)G_\alpha^{(2)}(v)G_\epsilon^{(2)}(w)| \leq |\text{div}(\varphi U)|$. Since φ is Lipschitz and of compact support one can use the regularity of U to conclude that $\text{div}(\varphi U)$ is in $L^1(\mathbb{R}^N \times (t_1,t_2))$. Therefore, using dominated convergence theorem, up to a subsequence,

$$(4.8)\qquad \lim_{\alpha\to 0}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \text{div}(\varphi U)G_\alpha^{(2)}(v)G_\epsilon^{(2)}(w)dy = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \text{div}(\varphi U)\chi_{\{v>0\}}G_\epsilon^{(2)}(w)dy.$$

The fact that $G_\epsilon^{(2)}(w)\varphi$ is continuous with compact support in $\mathbb{R}^N \times (t_1,t_2)$ and Part *(2)* of Theorem 3.1 yield, up to a subsequence,

$$\lim_{\alpha\to 0}\int_{\{0<v<\alpha\}} \frac{U\cdot\nabla v}{\alpha} G_\epsilon^{(2)}(w)\varphi dy = \int G_\epsilon^{(2)}(w)\varphi d\mu^+_{\{v,U\}}. \tag{4.9}$$

Now $|\frac{U\cdot\nabla w}{\epsilon}G_\alpha^{(2)}(v)\varphi| \le |\frac{U\cdot\nabla w}{\epsilon}\varphi|$ and thanks to the regularity of U and w, $\frac{U\cdot\nabla w}{\epsilon}\varphi \in L^1(\mathbb{R}^N \times (t_1,t_2))$. Therefore, one can use dominated convergence theorem and obtain, up to a subsequence,

$$\lim_{\alpha\to 0}\int_{\{0<w<\epsilon\}} \frac{U\cdot\nabla w}{\epsilon} G_\alpha^{(2)}(v)\varphi dy = \int_{\{0<w<\epsilon\}} \frac{U\cdot\nabla w}{\epsilon}\chi_{\{v>0\}}\varphi dy. \tag{4.10}$$

Combining (4.7)-(4.10) yields

(4.11)
$$\int G_\epsilon^{(2)}(w)\varphi d\mu^+_{\{v,U\}} + \int_{\{0<w<\epsilon\}} \frac{U\cdot\nabla w}{\epsilon}\chi_{\{v>0\}}\varphi dy = -\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \text{div}(\varphi U)\chi_{\{v>0\}}G_\epsilon^{(2)}(w)dy.$$

As ϵ goes to 0, $G_\epsilon^{(2)}(w)$ converges everywhere to $\chi_{\{w>0\}}$. Moreover $|\text{div}(\varphi U)G_\epsilon^{(2)}(w)\chi_{\{v>0\}}| \le |\text{div}(\varphi U)|$. The regularity of U and the fact that φ is Lipschitz and of compact support show that $\text{div}(\varphi U) \in L^1(\mathbb{R}^N \times (t_1,t_2))$. Therefore, using dominated convergence theorem, up to a subsequence,

$$\lim_{\epsilon\to 0}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \text{div}(\varphi U)G_\epsilon^{(2)}(w)\chi_{\{v>0\}}dy = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \text{div}(\varphi U)\chi_{\{v>0\}}\chi_{\{w>0\}}dy. \tag{4.12}$$

One also has $|G_\epsilon^{(2)}(w)\varphi| \le |\varphi|$ and φ is integrable with respect to $\mu^+_{\{v,U\}}$; Consult Part *(2)* of Theorem 3.1. Therefore, using dominated convergence theorem for the measure $\mu^+_{\{v,U\}}$, yields, up to a subsequence,

$$\lim_{\epsilon\to 0}\int G_\epsilon^{(2)}(w)\varphi d\mu^+_{\{v,U\}} = \int \chi_{\{w>0\}}\varphi d\mu^+_{\{v,U\}}. \tag{4.13}$$

The fact that φ is continuous with compact support in $\mathbb{R}^N \times (t_1,t_2)$ and Theorem 4.1 with $\mathcal{O} = \{v>0\}$ show that, up to a subsequence,

$$\lim_{\epsilon\to 0}\int_{\{0<w<\epsilon\}} \frac{U\cdot\nabla w}{\epsilon}\chi_{\{v>0\}}\varphi dy = \int \varphi d\mu^+_{\{w,U,\mathcal{O}\}}. \tag{4.14}$$

(4.11)-(4.14) yield

$$\int \chi_{\{w>0\}}\varphi d\mu^+_{\{v,U\}} + \int \varphi d\mu^+_{\{w,U,\{v>0\}\}} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \text{div}(\varphi U)\chi_{\{v>0\}}\chi_{\{w>0\}}dy. \tag{4.15}$$

This concludes the proof of *(1)*.

Proof of (2). Taking $G_\alpha^{(2)}(v)G_\epsilon^{(3)}(w)$ in place of $G_\alpha^{(2)}(v)G_\epsilon^{(2)}(w)$ and proceeding as in the proof of Part *(1)* of the theorem, yields the proof of Part (2) of the theorem.

Proof of (3). Taking $G_\alpha^{(3)}(v)G_\epsilon^{(2)}(w)$ in place of $G_\alpha^{(2)}(v)G_\epsilon^{(2)}(w)$ and proceeding as in the proof of Part *(1)* of the theorem, yields the proof of Part (3) of the theorem.

Proof of (4). Taking $G_\alpha^{(3)}(v)G_\epsilon^{(3)}(w)$ in place of $G_\alpha^{(2)}(v)G_\epsilon^{(2)}(w)$ and proceeding as in the proof of Part *(1)* of the theorem, yields the proof of Part (4) of the theorem. The proof of Theorem 4.4 is now completed. ∎

5. ACTIONS OF THE MEASURES OF SECTIONS 3-4 ON SOME PARTICULAR FUNCTIONS WHEN THE VECTOR FUNCTION GENERATOR U SATISFIES: DIV $U = 0$

The main theorems of this section are Theorem 5.19. Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let k and l be any $N-$multi-indices of nonnegative integers such that $0 \leq k \leq l$. Set $\tilde{v} = D^l\varpi$ and $\tilde{w} = D^k\varpi$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$ be such that div $U = 0$. Assume that $\text{spt}_x(\tilde{w}) \setminus \text{spt}_x(U)$ is either empty or a subset of the projection of $\{\tilde{v} = 0\}^o$ into $\mathbb{R}^N$. Let $\mu^+_{\{\tilde{v},U\}}$, $\mu^-_{\{\tilde{v},U\}}$, and $\mu_{\{\tilde{v},U\}}$ be the measures corresponding to the pair $(\tilde{v}, U)$ obtained in Theorem 3.1. Let $\phi \in C_c(t_1, t_2)$. Then Theorem 5.19 says that

$$\int \phi\tilde{w}d\mu^+_{\{\tilde{v},U\}} = 0, \quad \int \phi\tilde{w}d\mu^-_{\{\tilde{v},U\}} = 0, \quad \int \phi\tilde{w}d\mu_{\{\tilde{v},U\}} = 0. \tag{5.1}$$

Let $\tilde{\phi} \in C_c^\infty(t_1, t_2)$ be such that $\tilde{\phi} \equiv 1$ on spt(ϕ) and $0 \leq \tilde{\phi} \leq 1$. In Subsections 5.1-5.8, (5.1) is proved when $\tilde{\phi}\tilde{w}$ satisfies: $0 \leq \tilde{\phi}\tilde{w} \leq 1$. In Subsection 5.8, (5.1) is proved in the general case. In Subsection 5.1, a brief description of the key elements of the proof of Theorem 5.19 when $\tilde{\phi}\tilde{w}$ satisfies: $0 \leq \tilde{\phi}\tilde{w} \leq 1$ is given. In Subsection 5.2, basic properties of the measures involved in the proof of Theorem 5.19 are given. In Subsections 5.3-5.7, several results that will play a key role in the proof of Theorem 5.19 are proved. In Subsection 5.8, the main results corresponding to Theorem 5.19 are stated and their proofs are given. Finally, in Subsection 5.9, an example showing that the conclusions of Theorem 5.19 are not true if $\text{spt}_x(\tilde{w}) \setminus \text{spt}_x(U)$ is neither empty nor a subset of the projection of $\{\tilde{v} = 0\}^o$ into $\mathbb{R}^N$.

Throughout this section, $\mu^+_{\{v,U\}}$, $\mu^-_{\{v,U\}}$, and $\mu_{\{v,U\}}$ denote the measures corresponding to the pair (v, U) obtained in Theorem 3.1. While $\mu^+_{\{v,U,\mathcal{O}\}}$ and $\mu^-_{\{v,U,\mathcal{O}\}}$ denote the measures corresponding to the triplet $(v, U, \mathcal{O})$ obtained in Theorem 4.1. Here, $\mathcal{O}$ is an open set of $\mathbb{R}^N \times (t_1, t_2)$. For any function f defined in $\mathbb{R}^N \times (t_1, t_2)$, $\text{spt}_x(f)$ denotes the projection of its support, spt(f), into $\mathbb{R}^N$. Also, $\mu\lfloor E$ denotes the restriction of the measure μ to the set E.

5.1. **Description of the key elements of the proof of Theorem 5.19.** Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ such that $0 \leq k \leq l$. Set $\tilde{v} = D^l\varpi$ and $\tilde{w} = D^k\varpi$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$ be such that div $U = 0$. Let $\mu^+_{\{\tilde{v},U\}}$, $\mu^-_{\{\tilde{v},U\}}$, and $\mu_{\{\tilde{v},U\}}$ be the measures corresponding to the pair $(\tilde{v}, U)$ obtained in Theorem 3.1. Let $\phi \in C_c(t_1, t_2)$. Set $K = \text{spt}_x(\tilde{w}) \times \text{spt}(\phi)$.

If $\tilde{w} \equiv 0$ there is nothing to prove. Therefore, it is assumed that $\tilde{w}$ is not identically 0. If $k = l$, then $\tilde{v} = \tilde{w}$. By Theorem 3.1, $\mu_{\{\tilde{v},U\}}$, $\mu^+_{\{\tilde{v},U\}}$, and $\mu^-_{\{\tilde{v},U\}}$ are concentrated resp. on $\partial\{\tilde{v} > 0\} \cup \partial\{\tilde{v} < 0\}$, $\partial\{\tilde{v} > 0\}$, and $\partial\{\tilde{v} < 0\}$. Hence, (5.1) is clearly satisfied. Therefore, it is assumed that

$$k \neq l. \tag{5.2}$$

Since $\mathrm{spt}_x(\tilde{w})$ is compact, $\mathrm{spt}_x(\tilde{v})$ is compact. Let $\tilde{\phi} \in C_c^\infty(t_1, t_2)$ be such that $\tilde{\phi} \equiv 1$ on $\mathrm{spt}(\phi)$ and $0 \leq \tilde{\phi} \leq 1$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Then $w, v \in C_c^\infty(\mathbb{R}^N \times (t_1, t_2))$. Moreover, $w \equiv \tilde{w}$ on K and $v \equiv \tilde{v}$ on K. Also, $v = D^{l-k}w$. Assume that w satisfies: $0 \leq w \leq 1$.

5.1.1. *Construction of the sets $A_{n,q}$, $A^+_{n,q}$, $A^-_{n,q}$, $A_{n,q,i}$, $A^+_{n,q,i}$, and $A^-_{n,q,i}$ and their properties.* Since the support of w, $\mathrm{spt}(w)$, is compact the set of numbers c such that $\{w = c\}^o \neq \emptyset$, is either empty or finite. It is assumed throughout this section that it is finite. The proofs for when this set is empty can be obtained by an adaptation of the proofs for when this set is finite. Let $\mathcal{W}_1 = \cup_{i=1}^{m_0}\{w = c_i\}^o$ with $0 < c_1 < \cdots < c_{m_0} \leq 1$ for some nonnegative integer m_0. Let $\mathcal{W}_2$ be the open set such that $\mathcal{W}_2 \cap \mathcal{W}_1 = \emptyset$ and $\mathrm{spt}(w) = \overline{\mathcal{W}}_2 \cup \overline{\mathcal{W}}_1$. Also, since $\mathrm{spt}(w)$ is compact, the set E_w^m resp. E_w^M of points in the range of w corresponding to local minima resp. local maxima is finite:

$$E_w^m = \{\bar{w}_0, \cdots, \bar{w}_{m+1}\}, \quad E_w^M = \{\tilde{w}_1, \cdots, \tilde{w}_M\} \tag{5.3}$$

for some nonnegative integers m and M. Two elements in E_w^m resp. E_w^M are not necessarily distinct. Let $m_w = \min\{|\tilde{w}_{i''} - \bar{w}_{i'}|; 0 \leq i' \leq m+1,\ 1 \leq i'' \leq M,\ \tilde{w}_{i''} \neq \bar{w}_{i'}\}$. Let $i \in J$ with J a subset of $\{0, \cdots, m+1\}$, which may be empty such that $w_i = m_{p_i}/2^{p_i}$ with p_i a nonnegative integer and $m_{p_i} \in I_{p_i}$, where $w_i \in E_w^m$. Here, for any nonnegative integer p, $I_p = \{0, 1, 2, \cdots, 2^p - 1\}$. Let n be any nonnegative integer such that $2^n > 1/m_w$ and $n > \max\{p_i|\ i \in J\}$. Let $q \in I_n$. Let $A_{n,q}$, $A^-_{n,q}$, and $A^+_{n,q}$ be the sets defined by

$$A^-_{n,q} = \{y \in \mathbb{R}^N \times (t_1, t_2)|\ q/2^n - (1/2^n)^2 \leq w(y) \leq q/2^n\} \tag{5.4}$$

$$A_{n,q} = \{y \in \mathbb{R}^N \times (t_1, t_2)|\ q/2^n < w(y) \leq (q+1)/2^n\} \tag{5.5}$$

$$A^+_{n,q} = \{y \in \mathbb{R}^N \times (t_1, t_2)|\ (q+1)/2^n < w(y) \leq (q+1)/2^n + (1/2^n)^2\}. \tag{5.6}$$

Assume that $A_{n,q}$ is not empty. Then $A_{n,q} = \cup_{i \in I_{n,q}} A_{n,q,i}$, where $A_{n,q,i}$ are the nonempty connected components of $A_{n,q}$. Since $\mathrm{spt}(w)$ is compact the set $I_{n,q}$ is finite. Since $2^n > 1/m_w$, one deduces that for all $0 \leq i' \leq m+1$ and $1 \leq i'' \leq M$ such that $\tilde{w}_{i''} \neq \bar{w}_{i'}$, the set $\{\bar{w}_{i'} \leq w \leq \tilde{w}_{i''}\}$ if $\bar{w}_{i'} < \tilde{w}_{i''}$ or $\{\tilde{w}_{i''} \leq w \leq \bar{w}_{i'}\}$ if $\tilde{w}_{i''} < \bar{w}_{i'}$ cannot be a subset of $A_{n,q,i}$ for any $i \in I_{n,q}$.

For any $i \in I_{n,q}$, denote by $A^-_{n,q,i}$ resp. $A^+_{n,q,i}$ the connected components of $A^-_{n,q}$ resp. $A^+_{n,q}$ that are adjacent to $A_{n,q,i}$. If $A^-_{n,q,i} = \emptyset$, then $A_{n,q,i} = \{w = (q+1)/2^n\} \cap A_{n,q,i}$ or $A_{n,q,i} = \{w_{j_i} \leq w \leq (q+1)/2^n\} \cap A_{n,q,i}$ with $\{w = w_{j_i}\} \cap A_{n,q,i} \neq \emptyset$ and $\{w = (q+1)/2^n\} \cap A_{n,q,i} \neq \emptyset$ and $w_{j_i} \in E_w^m$ for some $j_i \in \{0, \cdots, m+1\}$. If $A^+_{n,q,i} = \emptyset$, then using the fact that $2^n > 1/m_w$; See the observation above, $A_{n,q,i} = \{q/2^n < w \leq \tilde{w}_{j_i}\} \cap A_{n,q,i}$ with $\{w = q/2^n\} \cap \overline{A_{n,q,i}} \neq \emptyset$ and $\{w = \tilde{w}_{j_i}\} \cap A_{n,q,i} \neq \emptyset$ and $\tilde{w}_{j_i} \in E_w^M$ for some $j_i \in \{1, \cdots, M\}$.

Let $I^1_{n,q}$ denote the subset of $I_{n,q}$ such that for any $i \in I^1_{n,q}$, $A^-_{n,q,i} = \emptyset$ and $A_{n,q,i} = \{w_{j_i} \leq w \leq (q+1)/2^n\} \cap A_{n,q,i}$ with $\{w = w_{j_i}\} \cap A_{n,q,i} \neq \emptyset$ and $\{w = (q+1)/2^n\} \cap A_{n,q,i} \neq \emptyset$ and $w_{j_i} \in E_w^m$ for some $j_i \in \{0, \cdots, m+1\}$ and w_{j_i} is not of the form $m_p/2^p$ for any nonnegative integer p and any $m_p \in I_p$. Let $I^2_{n,q}$ denote the subset of $I_{n,q}$ such that for any $i \in I^2_{n,q}$, $A^-_{n,q,i}$ is a nonempty connected component of $\{w = q/2^n\}$ with $q/2^n = m_{p_j}/2^{p_j}$ for some $j \in J$ and $m_{p_j} \in I_{p_j}$. Let $I^3_{n,q}$ denote the subset of $I_{n,q}$ such that for any $i \in I^3_{n,q}$, $A^-_{n,q,i} \cap \{w = q/2^n - (1/2^n)^2\} \neq \emptyset$ and $A^+_{n,q,i} \neq \emptyset$. Let $I^4_{n,q}$ denote the subset of $I_{n,q}$ such that for any $i \in I^4_{n,q}$, $A^-_{n,q,i} \cap \{w = q/2^n - (1/2^n)^2\} \neq \emptyset$ and $A^+_{n,q,i} = \emptyset$.

By definition of w, the sets $I^2_{n,q}$, $I^3_{n,q}$, and $I^4_{n,q}$ are not empty. It is assumed throughout this section that $I^1_{n,q}$ is not empty. The case of $I^1_{n,q} = \emptyset$ can be obtained by an adaptation of the proofs for the case $I^1_{n,q} \neq \emptyset$.

5.1.2. *Construction of the sets* $\Omega^1_{n,q,i}$, $\Omega^{2,+}_{n,q,i}$, $\Omega^{2,-}_{n,q,i}$, $\tilde{A}_{n,q}$, $\tilde{A}^+_{n,q}$, *and* $\tilde{A}^-_{n,q}$ *and their properties.* Set $D^+_{n,q,i} = A^+_{n,q,i} \setminus \{w = (q+1)/2^n + (1/2^n)^2\}$ and $D^-_{n,q,i} = A^-_{n,q,i} \setminus \{w = q/2^n - (1/2^n)^2\}$.

(a) Let $i \in I^2_{n,q}$. Then $A^-_{n,q,i}$ is a nonempty connected component of $\{w = q/2^n\}$. In this case, w reaches a local minimum on $\{w = q/2^n\}$ with $q/2^n = m_{p_j}/2^{p_j}$ for some $j \in J$ and $m_{p_j} \in I_{p_j}$. Since $2^n > 1/m_w$, one deduces that $A^+_{n,q,i}$ cannot be empty; See the observation following the introduction of $A_{n,q,i}$ above.

(a.1) If $q = 0$ and $A^-_{n,q,i}$ is unbounded, set

$$\tilde{A}_{n,q,i} = \overline{A_{n,q,i}} \text{ and } D = \tilde{A}_{n,q,i}$$

if $D^+_{n,q,i} \cap \mathcal{W}_1 = \emptyset$ and

$$\tilde{A}_{n,q,i} = \overline{A_{n,q,i}} \text{ and } D = \tilde{A}_{n,q,i} \cup (A^+_{n,q,i} \cap \{(q+1)/2^n < w \leq c_{i'}\})$$

if $\{w = c_{i'}\}^o \subset D^+_{n,q,i}$ for some $i' \in \{1, \cdots, m_0\}$. If $q = 0$ and $A^-_{n,q,i}$ is bounded or if $q \neq 0$ set

$$\tilde{A}_{n,q,i} = A_{n,q,i} \cup A^-_{n,q,i} \text{ and } D = \tilde{A}_{n,q,i}$$

if $D^+_{n,q,i} \cap \mathcal{W}_1 = \emptyset$ and

$$\tilde{A}_{n,q,i} = A_{n,q,i} \cup A^-_{n,q,i} \text{ and } D = \tilde{A}_{n,q,i} \cup (A^+_{n,q,i} \cap \{(q+1)/2^n < w \leq c_{i'}\})$$

if $\{w = c_{i'}\}^o \subset D^+_{n,q,i}$ for some $i' \in \{1, \cdots, m_0\}$.

Since $2^n > 1/m_w$, there exists at most one $j \in \{0, \cdots, m_0\}$ such that $\{w = c_j\}^o \subset D^+_{n,q,i}$. This fact has been used above. Set

$$d = d(D, A^+_{n,q,i} \cap \{w = (q+1)/2^n + (1/2^n)^2\}).$$

Then set $\epsilon_{n,q,i} = \epsilon_i d$, where ϵ_i is a sufficiently small positive number; See below.

Let $B_{n,q}$ denote the interior of the set

$$\{y|\ d(y, D) \leq \epsilon_{n,q,i}\}.$$

Let

$$C_{n,q} = B_{n,q} \setminus \overline{D}.$$

Then $B_{n,q}$ and $C_{n,q}$ are nonempty open sets of $\mathbb{R}^{N+1}$. Since the support of w is compact, the open sets D^o, $B_{n,q}$, and $C_{n,q}$ are all bounded.

Let $\Gamma_{n,q} = \{y \in C_{n,q}|\ d(y, \partial C_{n,q}) = \epsilon_{n,q,i}/2\}$. Let Q_j, $j = 1, \cdots, J_{n,q}$ be a sequence of closed cubes such that: (1) the interiors of Q_j are mutually disjoint; (2) if two closed cubes touch at a point y, they touch at a whole side containing y; (3) for each $j \in \{1, \cdots, J_{n,q}\}$, diam$(Q_j) < \epsilon_{n,q,i}/8$; and (4) $\Gamma_{n,q} \subset (\cup_j Q_j)^o$. Here, a use of the fact that $C_{n,q}$ is a bounded set, and hence, $J_{n,q}$ is finite has been made.

By construction $\cup_j Q_j \subset C_{n,q}$ and $d(\cup_j Q_j, \partial C_{n,q}) > \epsilon_{n,q,i}/4$. Let $\widetilde{\Gamma}_{n,q}$ denote the union of the sides of Q_j, $j \in \{1, \cdots, J_{n,q}\}$ that are contained in $\{y \in C_{n,q}|\ d(y, D) < \epsilon_{n,q,i}/2\}$. Then by construction $\widetilde{\Gamma}_{n,q}$ is the graph of a locally Lipschitz continuous function. Let $\Omega^1_{n,q,i}$ be the open set containing $\overline{D}$ and bounded by $\widetilde{\Gamma}_{n,q}$. Then by the above, $\Omega^1_{n,q,i}$ satisfies the strong local Lipschitz condition. Moreover, $\Omega^1_{n,q,i}$ and its boundary satisfy:

$$\overline{D} \subset \Omega^1_{n,q,i}, \quad \frac{\epsilon_{n,q,i}}{4} < d(\partial\Omega^1_{n,q,i}, \partial D) < \epsilon_{n,q,i}.$$

(a.2) Let $B_{n,q}$ denote the interior of the set

$$\{y|\ d(y, D) \leq \epsilon_{n,q,i}\},$$

where

$$D = A^+_{n,q,i} \cap \{w = (q+1)/2^n + (1/2^n)^2\}.$$

Let

$$C_{n,q} = B_{n,q} \setminus D.$$

Then proceeding as above, one can build an open bounded set $\Omega^{2,+}_{n,q,i}$ such that $\Omega^{2,+}_{n,q,i}$ satisfies the strong local Lipschitz condition and together with its boundary satisfy:

$$D \subset \Omega^{2,+}_{n,q,i}, \quad \frac{\epsilon_{n,q,i}}{4} < d(\partial\Omega^{2,+}_{n,q,i}, D) < \epsilon_{n,q,i}.$$

(b) Let $i \in I^1_{n,q}$. Then $A^-_{n,q,i} = \emptyset$ and $A_{n,q,i} = \{w_{j_i} \leq w \leq (q+1)/2^n\} \cap A_{n,q,i}$ with $\{w = w_{j_i}\} \cap A_{n,q,i} \neq \emptyset$ and $\{w = (q+1)/2^n\} \cap A_{n,q,i} \neq \emptyset$ and $w_{j_i} \in E^m_w$ for some $j_i \in \{0, \cdots, m+1\}$ and w_{j_i} is not of the form $m_p/2^p$ for any nonnegative integer p and any $m_p \in I_p$. In this case, w reaches a local minimum on $\{w = w_{j_i}\}$. Moreover, since $2^n > 1/m_w$, one deduces that $q \neq 0$ and $A^+_{n,q,i}$ cannot be empty; See the observation following the introduction of $A_{n,q,i}$ above.

(b.1) If $D^+_{n,q,i} \cap \mathcal{W}_1 = \emptyset$, set

$$D = A_{n,q,i}.$$

If $\{w = c_{i'}\}^o \subset D^+_{n,q,i}$ for some $i' \in \{1, \cdots, m_0\}$, set

$$D = A_{n,q,i} \cup (A^+_{n,q,i} \cap \{(q+1)/2^n < w \leq c_{i'}\}).$$

Then set

$$d = d(D, A^+_{n,q,i} \cap \{w = (q+1)/2^n + (1/2^n)^2\}).$$

Since $2^n > 1/m_w$, there exists at most one $j \in \{0, \cdots, m_0\}$ such that $\{w = c_j\}^o \subset D^+_{n,q,i}$. This fact has been used above. Then set $\epsilon_{n,q,i} = \epsilon_i d$, where ϵ_i is a sufficiently small positive number; See below.

Let $B_{n,q}$ denote the interior of the set

$$\{y|\ d(y, D) \leq \epsilon_{n,q,i}\}.$$

Let

$$C_{n,q} = B_{n,q} \setminus \overline{D}.$$

Then proceeding as in *(a)* above, one can build an open bounded set $\Omega^1_{n,q,i}$ that satisfies the strong local Lipschitz condition and the following:

$$\overline{D} \subset \Omega^1_{n,q,i}, \quad \frac{\epsilon_{n,q,i}}{4} < d(\partial\Omega^1_{n,q,i}, \partial D) < \epsilon_{n,q,i}.$$

(b.2) Let $B_{n,q}$ denote the interior of the set $\{y|\ d(y, D) \leq \epsilon_{n,q,i}\}$, where

$$D = A^+_{n,q,i} \cap \{w = (q+1)/2^n + (1/2^n)^2\}.$$

Let $C_{n,q} = B_{n,q} \setminus D$. Then proceeding as above, one can build an open bounded set $\Omega^{2,+}_{n,q,i}$ such that $\Omega^{2,+}_{n,q,i}$ satisfies the strong local Lipschitz condition and together with its boundary satisfy:

$$D \subset \Omega^{2,+}_{n,q,i}, \quad \frac{\epsilon_{n,q,i}}{4} < d(\partial\Omega^{2,+}_{n,q,i}, D) < \epsilon_{n,q,i}.$$

(c) Let $i \in I^3_{n,q}$. Then $A^-_{n,q,i} \cap \{w = q/2^n - (1/2^n)^2\} \neq \emptyset$ and $A^+_{n,q,i} \neq \emptyset$. Hence, $A_{n,q,i} = \{q/2^n < w \leq (q+1)/2^n\} \cap A_{n,q,i}$ with $\{w = q/2^n\} \cap \overline{A_{n,q,i}} \neq \emptyset$ and $\{w = (q+1)/2^n\} \cap A_{n,q,i} \neq \emptyset$. In this case $q \neq 0$.

(c.1) If $D^-_{n,q,i} \cap \mathcal{W}_1 = \emptyset$ and $D^+_{n,q,i} \cap \mathcal{W}_1 = \emptyset$, set

$$D = A_{n,q,i} \cup (A^-_{n,q,i} \cap \{w = q/2^n\}).$$

Then set

$$d_1 = d(D, A^-_{n,q,i} \cap \{w = q/2^n - (1/2^n)^2\}),$$
$$d_2 = d(D, A^+_{n,q,i} \cap \{w = (q+1)/2^n + (1/2^n)^2\}).$$

If $D^-_{n,q,i} \cap \mathcal{W}_1 = \emptyset$ and $\{w = c_{i'}\}^o \subset D^+_{n,q,i}$ for some $i' \in \{1, \cdots, m_0\}$, set

$$D = A_{n,q,i} \cup (A^-_{n,q,i} \cap \{w = q/2^n\}) \cup (A^+_{n,q,i} \cap \{(q+1)/2^n < w \leq c_{i'}\}).$$

Then set

$$d_1 = d(D, A^-_{n,q,i} \cap \{w = q/2^n - (1/2^n)^2\}),$$
$$d_2 = d(D, A^+_{n,q,i} \cap \{w = (q+1)/2^n + (1/2^n)^2\}).$$

If $\{w = c_{i'}\}^o \subset D^-_{n,q,i}$ for some $i' \in \{1, \cdots, m_0\}$ and $D^+_{n,q,i} \cap \mathcal{W}_1 = \emptyset$, set

$$D = A_{n,q,i} \cup (A^-_{n,q,i} \cap \{w = q/2^n\}) \cup (A^-_{n,q,i} \cap \{c_{i'} \leq w < q/2^n\}).$$

Then set

$$d_1 = d(D, A^-_{n,q,i} \cap \{w = q/2^n - (1/2^n)^2\}),$$
$$d_2 = d(D, A^+_{n,q,i} \cap \{w = (q+1)/2^n + (1/2^n)^2\})$$

If $\{w = c_{i'}\}^o \subset D^-_{n,q,i}$ and $\{w = c_{i''}\}^o \subset D^+_{n,q,i}$ for some $i', i'' \in \{1, \cdots, m_0\}$, set

$$D = A_{n,q,i} \cup (A^-_{n,q,i} \cap \{w = q/2^n\}) \cup (A^-_{n,q,i} \cap \{c_{i'} \leq w < q/2^n\}) \cup$$
$$\cup (A^+_{n,q,i} \cap \{(q+1)/2^n < w \leq c_{i''}\}).$$

Then set

$$d_1 = d(D, A^-_{n,q,i} \cap \{w = q/2^n - (1/2^n)^2\}),$$
$$d_2 = d(D, A^+_{n,q,i} \cap \{w = (q+1)/2^n + (1/2^n)^2\}).$$

Since $2^n > 1/m_w$, there exists at most one $j \in \{0, \cdots, m_0\}$ such that $\{w = c_j\}^o \subset D^+_{n,q,i}$ and there exists at most one $j \in \{0, \cdots, m_0\}$ such that $\{w = c_j\}^o \subset D^-_{n,q,i}$. This fact has been used above.

Now set $d = \min(d_1, d_2)$. Then set $\epsilon_{n,q,i} = \epsilon_i d$, where ϵ_i is a sufficiently small positive number; See Below.

Let $B_{n,q}$ denote the interior of the set

$$\{y|\ d(y,D) \leq \epsilon_{n,q,i}\}$$

Then proceeding as in *(a)* above, one can build an open bounded set $\Omega^1_{n,q,i}$ that satisfies the strong local Lipschitz condition and the following:

$$\overline{D} \subset \Omega^1_{n,q,i}, \quad \frac{\epsilon_{n,q,i}}{4} < d(\partial\Omega^1_{n,q,i}, \partial D) < \epsilon_{n,q,i}.$$

(c.2) Let $B_{n,q}$ denote the interior of the set

$$\{y|\ d(y,D) \leq \epsilon_{n,q,i}\},$$

where

$$D = A^+_{n,q,i} \cap \{w = (q+1)/2^n + (1/2^n)^2\}.$$

Let

$$C_{n,q} = B_{n,q} \setminus D.$$

Then proceeding as above, one can build an open bounded set $\Omega^{2,+}_{n,q,i}$ such that $\Omega^{2,+}_{n,q,i}$ satisfies the strong local Lipschitz condition and together with its boundary satisfy:

$$D \subset \Omega^{2,+}_{n,q,i}, \quad \frac{\epsilon_{n,q,i}}{4} < d(\partial\Omega^{2,+}_{n,q,i}, D) < \epsilon_{n,q,i}.$$

(c.3) Let $B_{n,q}$ denote the interior of the set

$$\{y|\ d(y,D) \leq \epsilon_{n,q,i}\},$$

where

$$D = A^-_{n,q,i} \cap \{w = q/2^n - (1/2^n)^2\}.$$

Let

$$C_{n,q} = B_{n,q} \setminus D.$$

Then proceeding as above, one can build an open bounded set $\Omega^{2,-}_{n,q,i}$ such that $\Omega^{2,-}_{n,q,i}$ satisfies the strong local Lipschitz condition and together with its boundary satisfy:

$$D \subset \Omega^{2,-}_{n,q,i}, \quad \frac{\epsilon_{n,q,i}}{4} < d(\partial\Omega^{2,-}_{n,q,i}, D) < \epsilon_{n,q,i}.$$

(d) Let $i \in I^4_{n,q}$. Then $A^-_{n,q,i} \cap \{w = q/2^n - (1/2^n)^2\} \neq \emptyset$ and $A^+_{n,q,i} = \emptyset$. Hence, $A_{n,q,i} = \{q/2^n < w \leq \tilde{w}_{j_i}\} \cap A_{n,q,i}$ with $\{w = q/2^n\} \cap \overline{A_{n,q,i}} \neq \emptyset$ and $\{w = \tilde{w}_{j_i}\} \cap A_{n,q,i} \neq \emptyset$ and $\tilde{w}_{j_i} \in E^M_w$ for some $j_i \in \{1, \cdots, M\}$. In this case w reaches a local maximum on $\{w = \tilde{w}_{j_i}\}$ and since $2^n > 1/m_w$, $q \neq 0$.

(d.1) If $D^-_{n,q,i} \cap \mathcal{W}_1 = \emptyset$, set

$$D = A_{n,q,i} \cup (A^-_{n,q,i} \cap \{w = q/2^n\}).$$

If $\{w = c_{i'}\}^o \subset D^-_{n,q,i}$ for some $i' \in \{1, \cdots, m_0\}$, set

$$D = A_{n,q,i} \cup (A^-_{n,q,i} \cap \{w = q/2^n\}) \cup (A^-_{n,q,i} \cap \{c_{i'} \leq w < q/2^n\}).$$

Then set

$$d = d(D, A^-_{n,q,i} \cap \{w = q/2^n - (1/2^n)^2\}).$$

Since $2^n > 1/m_w$, there exists at most one $j \in \{0, \cdots, m_0\}$ such that $\{w = c_j\}^o \subset D^-_{n,q,i}$. This fact has been used above. Then set $\epsilon_{n,q,i} = \epsilon_i d$, where ϵ_i is a sufficiently small positive number; See below.

Let $B_{n,q}$ denote the interior of the set

$$\{y|\ d(y, D) \le \epsilon_{n,q,i}\}.$$

Then proceeding as in *(a)* above, one can build an open bounded set $\Omega^1_{n,q,i}$ that satisfies the strong local Lipschitz condition and the following:

$$\overline{D} \subset \Omega^1_{n,q,i}, \quad \frac{\epsilon_{n,q,i}}{4} < d(\Omega^1_{n,q,i}, D) < \epsilon_{n,q,i}.$$

(d.2) Let $B_{n,q}$ denote the interior of the set

$$\{y|\ d(y, A^-_{n,q,i} \cap \{w = q/2^n - (1/2^n)^2\}) \le \epsilon_{n,q,i}\}.$$

Let

$$C_{n,q} = B_{n,q} \setminus A^-_{n,q,i} \cap \{w = q/2^n - (1/2^n)^2\}.$$

Then proceeding as above, one can build an open bounded set $\Omega^{2,-}_{n,q,i}$ such that $\Omega^{2,-}_{n,q,i}$ satisfies the strong local Lipschitz condition and together with its boundary satisfy:

$$\begin{aligned} &A^-_{n,q,i} \cap \{w = q/2^n - (1/2^n)^2\} \subset \Omega^{2,-}_{n,q,i}, \\ &\frac{\epsilon_{n,q,i}}{4} < d(\partial\Omega^{2,-}_{n,q,i}, A^-_{n,q,i} \cap \{w = q/2^n - (1/2^n)^2\}) < \epsilon_{n,q,i}. \end{aligned}$$

Now set $\tilde{I}_{n,q} = I^1_{n,q} \cup I^2_{n,q} \cup I^3_{n,q} \cup I^4_{n,q}$ and

(5.7)
$$\tilde{A}_{n,q} = (\cup_{i \in I^2_{n,q}} \tilde{A}_{n,q,i}) \cup (\cup_{i \in \tilde{I}_{n,q} \setminus I^2_{n,q}} A_{n,q,i}), \quad \tilde{A}^-_{n,q} = \cup_{i \in I^3_{n,q} \cup I^4_{n,q}} A^-_{n,q,i}, \quad \tilde{A}^+_{n,q} = \cup_{i \in \tilde{I}_{n,q} \setminus I^4_{n,q}} A^+_{n,q,i}.$$

By construction, since one includes the connected components $A^-_{n,q,i}$ that are connected components of $\{w = q/2^n\}$ in $\tilde{A}_{n,q}$, the sets $A_{n,q}$ and $\tilde{A}_{n,q}$ are not equal. Likewise, since one excludes the connected components $A^-_{n,q,i}$ that are connected components of $\{w = q/2^n\}$ in $\tilde{A}^-_{n,q}$, the sets $A^-_{n,q}$ and $\tilde{A}^-_{n,q}$ are not equal. However,

$$A_{n,q} \cup A^-_{n,q} \cup A^+_{n,q} = \tilde{A}_{n,q} \cup \tilde{A}^-_{n,q} \cup \tilde{A}^+_{n,q}. \tag{5.8}$$

Moreover,

$$\sum_{q \in I_n} w \chi_{\tilde{A}_{n,q}} = w. \tag{5.9}$$

By the construction above there exist open neighborhoods

$$V^2_{n,q,i}(\partial\{w > q/2^n - (1/2^n)^2\} \cap A^-_{n,q,i}) \subset \Omega^{2,-}_{n,q,i}, \tag{5.10}$$
$$V^4_{n,q,i}(\partial\{w > (q+1)/2^n + (1/2^n)^2\} \cap A^+_{n,q,i}) \subset \Omega^{2,+}_{n,q,i}, \tag{5.11}$$
$$V^1_{n,q,i}(\partial\{w > q/2^n\} \cap \overline{A_{n,q,i}}) \subset \Omega^1_{n,q,i}, \tag{5.12}$$
$$V^3_{n,q,i}(\partial\{w > (q+1)/2^n\} \cap \overline{A_{n,q,i}}) \subset \Omega^1_{n,q,i}, \tag{5.13}$$

where in (5.10), $i \in I^3_{n,q} \cup I^4_{n,q}$ and in (5.12), $i \in I^2_{n,q} \cup I^3_{n,q} \cup I^4_{n,q}$, and in (5.11) and (5.13), $i \in I^1_{n,q} \cup I^2_{n,q} \cup I^3_{n,q}$. Here, open neighborhood $V_{n,q,i}(\Gamma)$ is meant in the sense that Γ is strictly included in $V_{n,q,i}$ and $V_{n,q,i}$ is an open set of $\mathbb{R}^N \times (t_1, t_2)$.

5.1.3. *Construction of extension functions* $w_{n,q,i}$ *and* $v_{n,q,i}$ *and their properties.* For $i \in I^1_{n,q}$ set

$$E_{n,q,i} = \Omega^{2,+}_{n,q,i} \cup (\overline{A_{n,q,i} \cup A^+_{n,q,i}})^c.$$

For $i \in I^2_{n,q}$ set

$$E_{n,q,i} = \Omega^{2,+}_{n,q,i} \cup ((\overline{\tilde{A}_{n,q,i} \cup A^+_{n,q,i}})^c \setminus (\Omega^1_{n,q,i} \cap \{w = q/2^n\}))$$

if $q = 0$ and $A^-_{n,q,i}$ is unbounded and set

$$E_{n,q,i} = \Omega^{2,+}_{n,q,i} \cup (\overline{\tilde{A}_{n,q,i} \cup A^+_{n,q,i}})^c$$

if $q = 0$ and $A^-_{n,q,i}$ is bounded or if $q \neq 0$. For $i \in I^3_{n,q}$ set

$$E_{n,q,i} = \Omega^{2,+}_{n,q,i} \cup \Omega^{2,-}_{n,q,i} \cup (\overline{A_{n,q,i} \cup A^+_{n,q,i} \cup A^-_{n,q,i}})^c.$$

And for $i \in I^4_{n,q}$ set

$$E_{n,q,i} = \Omega^{2,-}_{n,q,i} \cup (\overline{A_{n,q,i} \cup A^-_{n,q,i}})^c.$$

Let $i \in \tilde{I}_{n,q}$. Taking $\epsilon_i > 0$ sufficiently small in the definition of $\epsilon_{n,q,i}$ in ***(a)-(d)*** above, one has $E_{n,q,i} \cap \Omega^1_{n,q,i} = \emptyset$. Moreover, by construction; See above, $\Omega_{n,q,i} = E_{n,q,i} \cup \Omega^1_{n,q,i}$ satisfies the strong local Lipschitz condition. Hence, by classical extension theorems; See [1, 11], there is a total extension operator, L, for $\Omega_{n,q,i}$. By definition, L is an p-extension operator for $\Omega_{n,q,i}$ for every integer p. In particular, L extends functions in $C^p(\overline{\Omega_{n,q,i}})$ to functions in $C^p(\mathbb{R}^{N+1})$ for every p.

Let $f_{n,q,i}$ be the function defined by

$$f_{n,q,i}(y) = w(y) \text{ if } y \in \overline{E_{n,q,i}}, \tag{5.14}$$
$$f_{n,q,i}(y) = q/2^n \text{ if } y \in \overline{\Omega^1_{n,q,i}}. \tag{5.15}$$

Set $w_{n,q,i} = L(f_{n,q,i})$. Then by the extension results recalled above, $w_{n,q,i} \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$. Moreover, using the fact that $w \in C^\infty_c(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$ and (5.14)-(5.15), one concludes that $w_{n,q,i} \in C^\infty_c(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$. By construction, $w_{n,q,i}$ satisfies

$$w_{n,q,i}(y) = w(y) \quad \text{if } y \in \overline{E_{n,q,i}} \text{ and either } w(y) \le q/2^n - (1/2^n)^2 \text{ or } w(y) \ge (q+1)/2^n + (1/2^n)^2, \tag{5.16}$$
$$w_{n,q,i}(y) = q/2^n \quad \text{if } y \in \overline{\Omega^1_{n,q,i}} \text{ and } q/2^n \le w(y) \le (q+1)/2^n. \tag{5.17}$$

Now define $v_{n,q,i}$ by

$$v_{n,q,i} = D^{l-k} w_{n,q,i}. \tag{5.18}$$

Then using (5.10)-(5.13) one obtains

$$V^1_{n,q,i}(\partial\{w > q/2^n\} \cap \overline{A_{n,q,i}}) \subset \Omega^1_{n,q,i} \subset \{w_{n,q,i} = q/2^n\}^o \subset \{v_{n,q,i} = 0\}^o, \tag{5.19}$$

$$V^3_{n,q,i}(\partial\{w > (q+1)/2^n\} \cap A_{n,q,i}) \subset \Omega^1_{n,q,i} \subset \{w_{n,q,i} = q/2^n\}^o$$
$$\subset \{v_{n,q,i} = 0\}^o, \tag{5.20}$$

$$V^2_{n,q,i}(\partial\{w > q/2^n - (1/2^n)^2\} \cap A^-_{n,q,i}) \subset \Omega^{2,-}_{n,q,i} \subset \{w_{n,q,i} = w\}^o$$
$$\subset \{v_{n,q,i} = v\}^o, \tag{5.21}$$

$$V^4_{n,q,i}(\partial\{w > (q+1)/2^n + (1/2^n)^2\} \cap A^+_{n,q,i}) \subset \Omega^{2,+}_{n,q,i} \subset \{w_{n,q,i} = w\}^o$$
$$\subset \{v_{n,q,i} = v\}^o. \tag{5.22}$$

where in (5.19), $i \in I^2_{n,q} \cup I^3_{n,q} \cup I^4_{n,q}$, and in (5.21), $i \in I^3_{n,q} \cup I^4_{n,q}$, and in (5.20) and (5.22), $i \in I^1_{n,q} \cup I^2_{n,q} \cup I^3_{n,q}$. The third inclusions in (5.19)-(5.20) are obtained thanks to Assumption (5.2).

5.1.4. *Construction of the sets $\mathcal{A}_{n,j}$ and $J_{n,j}$, $j = 0, \cdots, m+1$.* Assume that the local minima are ordered in a descending order: $\bar{w}_{m+1} \geq \cdots \geq \bar{w}_0$. The sets $\mathcal{A}_{n,j}$ and $J_{n,j}$, $j = 0, \cdots, m+1$ are built in a descending order. The local minimum $\bar{w}_{m+1}$ is either a number that is not of the form $m_p/2^p$ for any integer p and any $m_p \in I_p$ or $\bar{w}_{m+1} = m_p/2^p$ for some integer p and $m_p \in I_p$. Let $q \in I_n$ be such that $q/2^n < \bar{w}_{m+1} \leq (q+1)/2^n$ in the first case or $\bar{w}_{m+1} = q/2^n = m_p/2^p$ in the second case. Denote this q by q_{m+1}. Let $A_{n,q_{m+1},i^0_{m+1}}$ denote the connected component of $A_{n,q_{m+1}}$ such that $\{w = \bar{w}_{m+1}\} \subset A_{n,q_{m+1},i^0_{m+1}}$ with $i^0_{m+1} \in I^1_{n,q_{m+1}}$ in the first case and $\{w = \bar{w}_{m+1}\} \cap \overline{A_{n,q_{m+1},i^0_{m+1}}}$ is not empty with $i^0_{m+1} \in I^2_{n,q_{m+1}}$ in the second case. Here, a use of the fact that $\bar{w}_i$, $i = 0, \cdots, m+1$ are not necessarily distinct has been made. Set $\tilde{A}_{n,q_{m+1},i^0_{m+1}} = A_{n,q_{m+1},i^0_{m+1}}$ in the first case and $\tilde{A}_{n,q_{m+1},i^0_{m+1}} = A_{n,q_{m+1},i^0_{m+1}} \cup A^-_{n,q_{m+1},i^0_{m+1}}$ in the second case. Repeating this process with $\bar{w}_m, \cdots, \bar{w}_0$, one obtains the definition of q_m and the set $\tilde{A}_{n,q_m,i^0_m}$ resp. q_{m-1} and the set $\tilde{A}_{n,q_{m-1},i^0_{m-1}}$, $\cdots$, $q_0 = 0$ and the set $\tilde{A}_{n,q_0,i^0_0}$.

In the following construction of $\mathcal{A}_{n,j}$, $j = m+1, \cdots, 0$, it is assumed that for each fixed j and s, $\{(q_j+s)/2^n < w \leq (q_j+s+1)/2^n\} \cap \mathcal{A}_{n,j}$ consists only of one connected component.

Let $\mathcal{A}_{n,m+1}$ denote the set formed of the following sets: $\tilde{A}_{n,q_{m+1},i^0_{m+1}}, A_{n,q_{m+1}+1,i^1_{m+1}}, \cdots,$ $A_{n,q_{m+1}+r_{m+1},i^{r_{m+1}}_{m+1}}$, where $A_{n,q_{m+1}+1,i^1_{m+1}}$ is a connected component of $A_{n,q_{m+1}+1}$ with $i^1_{m+1} \in I^3_{n,q_{m+1}+1}$ such that $\tilde{A}_{n,q_{m+1},i^0_{m+1}}$ and $A_{n,q_{m+1}+1,i^1_{m+1}}$ are adjacent, $\cdots$, $A_{n,q_{m+1}+r_{m+1}-1,i^{r_{m+1}-1}_{m+1}}$ is a connected component of $A_{n,q_{m+1}+r_{m+1}-1}$ with $i^{r_{m+1}-1}_{m+1} \in I^3_{n,q_{m+1}+r_{m+1}-1}$ such that $A_{n,q_{m+1}+r_{m+1}-2,i^{r_{m+1}-2}_{m+1}}$ and $A_{n,q_{m+1}+r_{m+1}-1,i^{r_{m+1}-1}_{m+1}}$ are adjacent, and $A_{n,q_{m+1}+r_{m+1},i^{r_{m+1}}_{m+1}}$ is a connected component of $A_{n,q_{m+1}+r_{m+1}}$ with $i^{r_{m+1}}_{m+1} \in I^4_{n,q_{m+1}+r_{m+1}}$ such that $A_{n,q_{m+1}+r_{m+1}-1,i^{r_{m+1}-1}_{m+1}}$ and $A_{n,q_{m+1}+r_{m+1},i^{r_{m+1}}_{m+1}}$ are adjacent.

Let $J_{n,m+1}$ denote the set consisting of (q_{m+1}, i^0_{m+1}) and all pairs (q, i) such that there exists $s \in \{1, \cdots, r_{m+1}\}$ satisfying $q = q_{m+1} + s$, $i = i^s_{m+1}$, and $A_{n,q_{m+1}+s,i^s_{m+1}} \in \mathcal{A}_{n,m+1}$.

Let $\mathcal{B}_{n,m}$ denote the set formed of the following sets: $\tilde{A}_{n,q_m,i^0_m}, A_{n,q_m+1,i^1_m}, \cdots, A_{n,q_m+r_m,i^{r_m}_m}$, where A_{n,q_m+1,i^1_m} is a connected component of A_{n,q_m+1} with $i^1_m \in I^3_{n,q_m+1}$ such that $\tilde{A}_{n,q_m,i^0_m}$ and A_{n,q_m+1,i^1_m} are adjacent, $\cdots$, $A_{n,q_m+r_m-1,i^{r_m-1}_m}$ is a connected component of A_{n,q_m+r_m-1} with $i^{r_m-1}_m \in I^3_{n,q_m+r_m-1}$ such that $A_{n,q_m+r_m-2,i^{r_m-2}_m}$ and $A_{n,q_m+r_m-1,i^{r_m-1}_m}$ are adjacent, and

$A_{n,q_m+r_m,i_m^{r_m}}$ is a connected component of A_{n,q_m+r_m} with $i_m^{r_m} \in I^4_{n,q_m+r_m}$ such that $A_{n,q_m+r_m-1,i_m^{r_m-1}}$ and $A_{n,q_m+r_m,i_m^{r_m}}$ are adjacent.

Let $J'_{n,m}$ denote the set consisting of (q_m, i_m^0) and all pairs (q,i) such that there exists $s \in \{1, \cdots, r_m\}$ satisfying $q = q_m + s$, $i = i_m^s$, and $A_{n,q_m+s,i_m^s} \in \mathcal{B}_{n,m}$. Let $\mathcal{A}_{n,m}$ denote the set formed of the sets $\tilde{A}_{n,q_m,i_m^0}$ and $A_{n,q,i}$ such that $(q,i) \in J'_{n,m} \setminus J_{n,m+1}$. Set $J_{n,m} = J'_{n,m} \setminus J_{n,m+1}$. By construction if $s \in \{2, \cdots, r_m\}$ and $A_{n,q_m+s,i_m^s} \in \mathcal{A}_{n,m}$, then $A_{n,q_m+s-1,i_m^{s-1}} \in \mathcal{A}_{n,m}$ and A_{n,q_m+s,i_m^s} and $A_{n,q_m+s-1,i_m^{s-1}}$ are adjacent.

Let $\mathcal{B}_{n,m-1}$ denote the set formed of the following sets: $\tilde{A}_{n,q_{m-1},i_{m-1}^0}, A_{n,q_{m-1}+1,i_{m-1}^1}, \cdots,$ $A_{n,q_{m-1}+r_{m-1},i_{m-1}^{r_{m-1}}}$, where $A_{n,q_{m-1}+1,i_{m-1}^1}$ is a connected component of $A_{n,q_{m-1}+1}$ with $i_{m-1}^1 \in I^3_{n,q_{m-1}+1}$ such that $\tilde{A}_{n,q_{m-1},i_{m-1}^0}$ and $A_{n,q_{m-1}+1,i_{m-1}^1}$ are adjacent, $\cdots$, $A_{n,q_{m-1}+r_{m-1}-1,i_{m-1}^{r_{m-1}-1}}$ is a connected component of $A_{n,q_{m-1}+r_{m-1}-1}$ with $i_{m-1}^{r_{m-1}-1} \in I^3_{n,q_{m-1}+r_{m-1}-1}$ such that $A_{n,q_{m-1}+r_{m-1}-2,i_{m-1}^{r_{m-1}-2}}$ and $A_{n,q_{m-1}+r_{m-1}-1,i_{m-1}^{r_{m-1}-1}}$ are adjacent, and $A_{n,q_{m-1}+r_{m-1},i_{m-1}^{r_{m-1}}}$ is a connected component of $A_{n,q_{m-1}+r_{m-1}}$ with $i_{m-1}^{r_{m-1}} \in I^4_{n,q_{m-1}+r_{m-1}}$ such that $A_{n,q_{m-1}+r_{m-1}-1,i_{m-1}^{r_{m-1}-1}}$ and $A_{n,q_{m-1}+r_{m-1},i_{m-1}^{r_{m-1}}}$ are adjacent.

Let $J'_{n,m-1}$ denote the set consisting of (q_{m-1}, i_{m-1}^0) and all pairs (q,i) such that there exists $s \in \{1, \cdots, r_{m-1}\}$ satisfying $q = q_{m-1} + s$, $i = i_{m-1}^s$, and $A_{n,q_{m-1}+s,i_{m-1}^s} \in \mathcal{B}_{n,m-1}$. Let $\mathcal{A}_{n,m-1}$ denote the set formed of the sets $\tilde{A}_{n,q_{m-1},i_{m-1}^0}$ and $A_{n,q,i}$ such that $(q,i) \in J'_{n,m-1} \setminus (J_{n,m+1} \cup J_{n,m})$. Set $J_{n,m-1} = J'_{n,m-1} \setminus (J_{n,m+1} \cup J_{n,m})$. By construction if $s \in \{2, \cdots, r_{m-1}\}$ and $A_{n,q_{m-1}+s,i_{m-1}^s} \in \mathcal{A}_{n,m-1}$, then $A_{n,q_{m-1}+s-1,i_{m-1}^{s-1}} \in \mathcal{A}_{n,m-1}$ and $A_{n,q_{m-1}+s,i_{m-1}^s}$ and $A_{n,q_{m-1}+s-1,i_{m-1}^{s-1}}$ are adjacent.

By repeating the process above, one builds $\mathcal{B}_{n,m-2}$, $J'_{n,m-2}$, $\mathcal{A}_{n,m-2}$, and $J_{n,m-2}$, $\cdots$, $\mathcal{B}_{n,0}$, $J'_{n,0}$, $\mathcal{A}_{n,0}$, and $J_{n,0}$.

To build the sets $\mathcal{A}_{n,j}$ and $J_{n,j}$ in the case where there are j and s such that $\{(q_j+s)/2^n < w \le (q_j+s+1)/2^n\}$ cut $\mathcal{A}_{n,j}$ into several (finite number of) connected components, one proceeds as above with appropriate adaptations.

By the construction above, one has

$$\sum_{j=0}^{m+1} (w\chi_{\tilde{A}_{n,q_j,i_j^0}} + w \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \chi_{A_{n,q_j+s,i_j^s}}) = w. \tag{5.23}$$

5.1.5. *Brief description of the proof of Theorem 5.19.* By the properties of $w, v, \tilde{w}$, and $\tilde{v}$, one has

$$\int \phi \tilde{w} d\mu^+_{\{\tilde{v},U\}} = \int \phi w d\mu^+_{\{v,U\}}, \quad \int \phi \tilde{w} d\mu^-_{\{\tilde{v},U\}} = \int \phi w d\mu^-_{\{v,U\}}, \tag{5.24}$$

$$\int \phi \tilde{w} d\mu_{\{\tilde{v},U\}} = \int \phi w d\mu_{\{v,U\}}. \tag{5.25}$$

Let $\mu^+_{\{v_{n,q,i},U\}}$, $\mu^-_{\{v_{n,q,i},U\}}$, and $\mu_{\{v_{n,q,i},U\}}$ be the measures corresponding to the pair $(v_{n,q,i}, U)$ obtained in Theorem 3.1. Define the measures $\mu^+_{n,q,i} = \mu^+_{\{v,U\}} - \mu^+_{\{v_{n,q,i},U\}}$, $\mu^-_{n,q,i} = \mu^-_{\{v,U\}} - \mu^-_{\{v_{n,q,i},U\}}$, and $\mu_{n,q,i} = \mu_{\{v,U\}} - \mu_{\{v_{n,q,i},U\}}$. Using the results obtained in Subsections 5.2-5.7, it is proved that, up to a subsequence, as n goes to ∞,

$$\sum_{q\in I_n, i\in \tilde{I}_{n,q}} \int w\phi d\mu^+_{n,q,i} \to 0, \qquad \sum_{q\in I_n, i\in \tilde{I}_{n,q}} \int w\phi \chi_{\tilde{A}^-_{n,q}\cup \tilde{A}^+_{n,q}} d\mu^+_{n,q,i} \to 0, \tag{5.26}$$

where $\tilde{A}^-_{n,q}$ and $\tilde{A}^+_{n,q}$ were introduced in (5.7) above. Then using the basic properties of these measures, it is deduced that

$$\int \phi w d\mu^+_{\{v,U\}} = 0. \tag{5.27}$$

Proceeding similarly for the measure $\mu^-_{\{v,U\}}$ with appropriate adaptations, one obtains

$$\int \phi w d\mu^-_{\{v,U\}} = 0. \tag{5.28}$$

Now using the last statement in Part *(2)* of Theorem 3.1 and the above, one obtains

$$\int \phi w d\mu_{\{v,U\}} = 0. \tag{5.29}$$

Then using (5.27)-(5.29) and (5.24)-(5.25), one obtains

$$\int \phi\tilde{w} d\mu^+_{\{\tilde{v},U\}} = 0, \quad \int \phi\tilde{w} d\mu^-_{\{\tilde{v},U\}} = 0, \quad \int \phi\tilde{w} d\mu_{\{\tilde{v},U\}} = 0.$$

This corresponds to Theorem 5.19 when $\tilde{\phi}\tilde{w}$ satisfies: $0 \le \tilde{\phi}\tilde{w} \le 1$. The proof of Theorem 5.19 in the general case is obtained by an appropriate adaptation of the proof for this particular case.

The fact that one can select $w_{n,q,i}$ and $v_{n,q,i}$ such that $v_{n,q,i} = D^{l-k} w_{n,q,i}$ and $w_{n,q,i}$ and $v_{n,q,i}$ satisfy the basic properties shown below and that enabled the proof, is possible thanks to the assumption that $v = D^{l-k}w$, $\mathrm{spt}_x(\tilde{w})$ is compact, and $\mathrm{spt}_x(\tilde{w}) \subset \mathrm{spt}_x(U)$.

5.2. **Basic properties of the measures introduced in 5.1.5.** By the regularity of w, the sets $A_{n,q}$, $A^-_{n,q}$, $A^+_{n,q}$, $\tilde{A}_{n,q}$, $\tilde{A}^-_{n,q}$, $\tilde{A}^+_{n,q}$, $A_{n,q,i}$, $A^-_{n,q,i}$, $A^+_{n,q,i}$, and $\tilde{A}_{n,q,i}$ are all Boreleans and so, they are all measurable with respect to the Radon measures $\mu^+_{\{v_{n,q,i},U\}}$, $\mu^+_{\{v,U\}}$, $\mu^+_{n,q,i}$, $\mu^-_{\{v_{n,q,i},U\}}$, $\mu^-_{\{v,U\}}$, and $\mu^-_{n,q,i}$.

Theorem 5.1. *Let N be any integer ≥ 2. Let $0 \le t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1,t_2); \mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1,t_2); \mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ be such that $0 \le k \le l$. Set $\tilde{v} = D^l\varpi$ and $\tilde{w} = D^k\varpi$. Let $\phi \in C_c(t_1,t_2)$. Set $K = \mathrm{spt}_x(\tilde{w}) \times \mathrm{spt}(\phi)$. Let $\tilde{\phi} \in C^\infty_c(t_1,t_2)$ be such that $\tilde{\phi} \equiv 1$ on spt(ϕ) and $0 \le \tilde{\phi} \le 1$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Then $w, v \in C^\infty_c(\mathbb{R}^N \times (t_1,t_2))$ and $v = D^{l-k}w$. Moreover, $w \equiv \tilde{w}$ on K and $v \equiv \tilde{v}$ on K. Assume that w satisfies: $0 \le w \le 1$.*

Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0,1,2,\cdots,2^n-1\}$. Let $q \in I_n$. Let $\tilde{I}_{n,q}$ be the set introduced in §5.1.2. Let $i \in \tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k}w_{n,q,i}$. Define the measures $\mu^+_{n,q,i} = \mu^+_{\{v,U\}} - \mu^+_{\{v_{n,q,i},U\}}$ and $\mu^-_{n,q,i} = \mu^-_{\{v,U\}} - \mu^-_{\{v_{n,q,i},U\}}$.

Then the measures $\mu^+_{n,q,i}$ and $\mu^-_{n,q,i}$ are concentrated on the set $\tilde{A}_{n,q,i} \cup A^+_{n,q,i}$ if $i \in I^2_{n,q}$ and on $A^-_{n,q,i} \cup A_{n,q,i} \cup A^+_{n,q,i}$ if $i \in \tilde{I}_{n,q} \setminus I^2_{n,q}$, where $I^2_{n,q}$ was introduced in §5.1.1.

Proof of Theorem 5.1 Since the projection of the support of $\tilde{w}$ into $\mathbb{R}^N$ is compact and spt$(\tilde{\phi})$ is compact, the support of $w = \tilde{\phi}\tilde{w}$ is compact and the support of $\tilde{v} = D^{l-k}\tilde{w}$ is compact. By definition $v = \tilde{\phi}\tilde{v} = \tilde{\phi}D^{l-k}\tilde{w}$. Then by the above, spt$(v)$ is compact. Since $\varpi \in C^\infty(\mathbb{R}^N \times (t_1, t_2))$, $\tilde{w}, \tilde{v} \in C^\infty(\mathbb{R}^N \times (t_1, t_2))$. Moreover, $\tilde{\phi} \in C_c^\infty(t_1, t_2)$. One then deduces that $w.v \in C_c^\infty(\mathbb{R}^N \times (t_1, t_2))$ and $v = D^{l-k}w$. Since $\tilde{\phi} \equiv 1$ on spt(ϕ), $w = \tilde{w}$ on K and $v = \tilde{v}$ on K. This proves the first statements in the theorem.

Let $q \in I_n$ and $i \in \tilde{I}_{n,q}$. Set $\tilde{B}_{n,q,i} = \tilde{A}_{n,q,i} \cup A^+_{n,q,i}$ if $i \in I^2_{n,q}$ and $\tilde{B}_{n,q,i} = A^-_{n,q,i} \cup A_{n,q,i} \cup A^+_{n,q,i}$ if $i \in \tilde{I}_{n,q} \setminus I^2_{n,q}$. By the regularity of w, $\tilde{B}_{n,q,i}$ is a closed subset of $\mathbb{R}^{N+1}$. Let B be any ball such that $B \subset \tilde{B}^c_{n,q,i}$. Let $y \in B$. Then for $q = 0$ either $w(y) = 0$ or $w(y) > 1/2^n + (1/2^n)^2$ and for all other q, $w(y) < q/2^n - (1/2^n)^2$ or $w(y) > (q+1)/2^n + (1/2^n)^2$. Hence, by definition of $w_{n,q,i}$, $w_{n,q,i}$ is identically equal to w on B. Then by the assumption of the theorem that: $v = D^{l-k}w$, and by the definition of $v_{n,q,i}$: $v_{n,q,i} = D^{l-k}w_{n,q,i}$, one has $v_{n,q,i}$ is identically equal to v on B. Let φ be any Lipschitz function in $C_c(B; \mathbb{R})$. By Theorem 3.3, one has

$$\text{(5.30)} \quad \int \varphi d\mu^+_{\{v,U\}} = -\int_{\{v>0\}} \text{div}(\varphi U)dy, \quad \int \varphi d\mu^+_{\{v_{n,q,i},U\}} = -\int_{\{v_{n,q,i}>0\}} \text{div}(\varphi U)dy.$$

(5.30) and the fact shown above that $v_{n,q,i}$ is identically equal to v on B yield

$$\text{(5.31)} \qquad \int \varphi d\mu^+_{n,q,i} = \int_B (\chi_{\{v_{n,q,i}>0\}} - \chi_{\{v>0\}})\text{div}(\varphi U)dy = 0.$$

Let φ be any function in $C_c(B; \mathbb{R})$. Let $\widetilde{K}$ be the compact support of φ. Let $\widetilde{\varphi}$ be any function in $C_c^\infty(B; \mathbb{R})$ such that $\widetilde{\varphi} \equiv 1$ on $\widetilde{K}$. Let φ_η and $\varphi_{\eta,m}$ be the sequences of functions associated to φ by (3.13); Consult the beginning of the proof of Theorem 3.4. Then for every η and every nonnegative integer $m > \frac{4}{t_2 - t_1}$, $\tilde{\varphi}\varphi_{\eta,m} = \tilde{\varphi}h_m\psi_m\varphi_\eta \in C_c^\infty(B; \mathbb{R})$. Plugging this function in (5.31) and proceeding as in Step 6 of the proof of Part *(1)* of Theorem 3.4, one concludes that for every $\varphi \in C_c(B; \mathbb{R})$,

$$\int \varphi d\mu^+_{n,q,i} = 0.$$

This completes the proof that the support of the measure $\mu^+_{n,q,i}$ is concentrated on the set $\tilde{B}_{n,q,i}$. Proceeding as above for the measure $\mu^-_{n,q,i}$ one obtains the conclusions of the theorem for this measure. This completes the proof of Theorem 5.1. ∎

Theorem 5.2. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ be such that $0 \leq k \leq l$. Set $\tilde{v} = D^l\varpi$ and $\tilde{w} = D^k\varpi$. Let $\phi \in C_c(t_1, t_2)$. Set $K = spt_x(\tilde{w}) \times spt(\phi)$. Let $\tilde{\phi} \in C_c^\infty(t_1, t_2)$ be such that $\tilde{\phi} \equiv 1$ on spt(ϕ) and $0 \leq \tilde{\phi} \leq 1$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Assume that w satisfies: $0 \leq w \leq 1$.*

Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0, 1, 2, \cdots, 2^n - 1\}$. Let $q \in I_n$. Let $\tilde{I}_{n,q}$ resp. $I^2_{n,q}$ be the set introduced in §5.1.2 resp. 5.1.1. Let $i \in \tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k}w_{n,q,i}$. Then

(1) For any $\varphi \in C_c(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$,

$$\int \chi_{\tilde{A}^o_{n,q,i}} \varphi d\mu^+_{\{v_{n,q,i},U\}} = 0, \quad \int \chi_{\tilde{A}^o_{n,q,i}} \varphi d\mu^-_{\{v_{n,q,i},U\}} = 0 \textit{ if } i \in I^2_{n,q},$$
$$\int \chi_{A^o_{n,q,i}} \varphi d\mu^+_{\{v_{n,q,i},U\}} = 0, \quad \int \chi_{A^o_{n,q,i}} \varphi d\mu^-_{\{v_{n,q,i},U\}} = 0 \textit{ if } i \in \tilde{I}_{n,q} \setminus I^2_{n,q}.$$

(2)

$$\int \chi_{\partial \tilde{A}_{n,q,i} \cap \{w=(q+1)/2^n\}} \phi d\mu^+_{\{v_{n,q,i},U\}} = 0, \int \chi_{\partial \tilde{A}_{n,q,i} \cap \{w=(q+1)/2^n\}} \phi d\mu^-_{\{v_{n,q,i},U\}} = 0,$$
$$\int \chi_{\partial \tilde{A}_{n,q,i} \cap \{w=q/2^n\}} \phi d\mu^+_{\{v_{n,q,i},U\}} = 0, \int \chi_{\partial \tilde{A}_{n,q,i} \cap \{w=q/2^n\}} \phi d\mu^-_{\{v_{n,q,i},U\}} = 0$$
$$\textit{if } i \in I^2_{n,q},$$
$$\int \chi_{\partial A_{n,q,i} \cap \{w=(q+1)/2^n\}} \phi d\mu^+_{\{v_{n,q,i},U\}} = 0, \int \chi_{\partial A_{n,q,i} \cap \{w=(q+1)/2^n\}} \phi d\mu^-_{\{v_{n,q,i},U\}} = 0$$
$$\textit{if } i \in \tilde{I} \setminus I^2_{n,q}.$$

(3) For $i \in I^2_{n,q}$,

$$\int w \chi_{\tilde{A}_{n,q,i}} \phi d\mu^+_{\{v_{n,q,i},U\}} = 0, \quad \int \chi_{\tilde{A}_{n,q,i}} \phi d\mu^+_{\{v_{n,q,i},U\}} = 0,$$
$$\int w \chi_{\tilde{A}_{n,q,i}} \phi d\mu^-_{\{v_{n,q,i},U\}} = 0, \quad \int \chi_{\tilde{A}_{n,q,i}} \phi d\mu^-_{\{v_{n,q,i},U\}} = 0,$$

and for $i \in \tilde{I}_{n,q} \setminus I^2_{n,q}$,

$$\int w \chi_{A_{n,q,i}} \phi d\mu^+_{\{v_{n,q,i},U\}} = 0, \quad \int \chi_{A_{n,q,i}} \phi d\mu^+_{\{v_{n,q,i},U\}} = 0,$$
$$\int w \chi_{A_{n,q,i}} \phi d\mu^-_{\{v_{n,q,i},U\}} = 0, \quad \int \chi_{A_{n,q,i}} \phi d\mu^-_{\{v_{n,q,i},U\}} = 0.$$

Proof of Theorem 5.2
If $k = l$, the conclusions of the theorem are obvious. Therefore, it is assumed that $k \neq l$.

Proof of Part (1)

1. Let $q \in I_n$. Let $i \in I^2_{n,q}$. By construction, $\tilde{A}^o_{n,q,i} \subset \{w_{n,q,i} = q/2^n\}^o$. On $\{w_{n,q,i} = q/2^n\}^o$, $\nabla v_{n,q,i} = \nabla D^{l-k} w_{n,q.i}$ is identically 0. Hence, $\tilde{A}^o_{n,q,i} \subset \{w_{n,q,i} = q/2^n\}^o \subset \{\nabla v_{n,q,i} = 0\}^o$.

2. Let $\varphi \in C_c(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$. Using Step 1 and Theorem 4.1 with $\mathcal{O} = \{w_{n,q,i} = q/2^n\}^o$ yields, up to a subsequence,

$$\int \varphi d\mu^+_{\{v_{n,q,i},U,\{w_{n,q,i}=q/2^n\}^o\}} = \lim_{\alpha \to 0} \int_{\{0<v_{n,q,i}<\alpha\}} \frac{U \cdot \nabla v_{n,q,i}}{\alpha} \chi_{\{w_{n,q,i}=q/2^n\}^o} \varphi dy = 0. \tag{5.32}$$

3. By Part *(1)* of Theorem 4.3, one has for any $\varphi \in C_c(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$,

$$\int \chi_{\{w_{n,q,i}=q/2^n\}^o} \varphi d\mu^+_{\{v_{n,q,i},U,\{w_{n,q,i}=q/2^n\}^o\}} = \int \chi_{\{w_{n,q,i}=q/2^n\}^o} \varphi d\mu^+_{\{v_{n,q,i},U\}}. \tag{5.33}$$

Then using (5.33), (5.32), and the fact that by Step 1 above, $\tilde{A}^o_{n,q,i}$ is a subset of $\{w_{n,q,i} = q/2^n\}^o$, one obtains

$$\int \chi_{\{w_{n,q,i}=q/2^n\}^o}\varphi d\mu^+_{\{v_{n,q,i},U\}} = 0, \quad \int \chi_{\tilde{A}^o_{n,q,i}}\varphi d\mu^+_{\{v_{n,q,i},U\}} = 0.$$

This yields the first statement in the theorem.

4. Proceeding as in Steps 1-3 for the measures $\mu^-_{\{v_{n,q,i},U\}}$ with appropriate adaptation yields the second statement in *(1)*.

5. For $i \in \tilde{I}_{n,q} \setminus I^2_{n,q}$ proceeding as in Steps 1-4 for $\mu^+_{\{v_{n,q,i},U\}}$ and $\mu^-_{\{v_{n,q,i},U\}}$ with appropriate adaptation yields the third and fourth statement in *(1)*.

Proof of Part (2)

Let $i \in I^2_{n,q}$. Since $k \neq l$, one can use (5.19) and (5.20) to conclude that $\partial\tilde{A}_{n,q,i} \cap \{w = (q+1)/2^n\}$ and $\partial\tilde{A}_{n,q,i} \cap \{w = q/2^n\}$ are subsets of $\{v_{n,q,i} = 0\}^o$. By Theorem 3.1, the measure $\mu^+_{\{v_{n,q,i},U\}}$ is concentrated on $\partial\{v_{n,q,i} > 0\}$. Hence, $\int \chi_{\partial\tilde{A}_{n,q,i}\cap\{w=(q+1)/2^n\}}\phi d\mu^+_{\{v_{n,q,i},U\}} = 0$ and $\int \chi_{\partial\tilde{A}_{n,q,i}\cap\{w=q/2^n\}}\phi d\mu^+_{\{v_{n,q,i},U\}} = 0$. Proceeding similarly for $\mu^-_{\{v_{n,q,i},U\}}$, one obtains $\int \chi_{\partial\tilde{A}_{n,q,i}\cap\{w=(q+1)/2^n\}}\phi d\mu^-_{\{v_{n,q,i},U\}} = 0$ and $\int \chi_{\partial\tilde{A}_{n,q,i}\cap\{w=q/2^n\}}\phi d\mu^-_{\{v_{n,q,i},U\}} = 0$. This yields the first part of *(2)*. Proceeding as above with appropriate adaptations for $i \in \tilde{I}_{n,q} \setminus I^2_{n,q}$, one obtains the second part of *(2)*.

Proof of Part (3)

1. Using Part *(1)* of the theorem with $\varphi = \phi w$, one obtains

$$\begin{aligned} &\int w\chi_{\tilde{A}_{n,q,i}}\phi d\mu^+_{\{v_{n,q,i},U\}} = \int w\chi_{\partial\tilde{A}_{n,q,i}\cap\{w=(q+1)/2^n\}}\phi d\mu^+_{\{v_{n,q,i},U\}} + \\ &\int w\chi_{\partial\tilde{A}_{n,q,i}\cap\{w=q/2^n\}}\phi d\mu^+_{\{v_{n,q,i},U\}} + \int w\chi_{\tilde{A}^o_{n,q,i}}\phi d\mu^+_{\{v_{n,q,i},U\}} \\ = &\int w\chi_{\partial\tilde{A}_{n,q,i}\cap\{w=(q+1)/2^n\}}\phi d\mu^+_{\{v_{n,q,i},U\}} + \int w\chi_{\partial\tilde{A}_{n,q,i}\cap\{w=q/2^n\}}\phi d\mu^+_{\{v_{n,q,i},U\}}. \end{aligned}$$

Then using Part *(2)* of the theorem yields the first statement in Part *(3)*.

2. Using Part *(1)* of the theorem with $\varphi = \phi\psi_m$ with m so large that $\psi_m \equiv 1$ on $\mathrm{spt}_x(\tilde{w})$, one obtains

$$\begin{aligned} &\int \chi_{\tilde{A}_{n,q,i}}\phi d\mu^+_{\{v_{n,q,i},U\}} = \int \chi_{\partial\tilde{A}_{n,q,i}\cap\{w=(q+1)/2^n\}}\phi d\mu^+_{\{v_{n,q,i},U\}} + \\ &\int \chi_{\partial\tilde{A}_{n,q,i}\cap\{w=q/2^n\}}\phi d\mu^+_{\{v_{n,q,i},U\}} + \int \chi_{\tilde{A}^o_{n,q,i}}\phi d\mu^+_{\{v_{n,q,i},U\}} \\ = &\int \chi_{\partial\tilde{A}_{n,q,i}\cap\{w=(q+1)/2^n\}}\phi d\mu^+_{\{v_{n,q,i},U\}} + \int \chi_{\partial\tilde{A}_{n,q,i}\cap\{w=q/2^n\}}\phi d\mu^+_{\{v_{n,q,i},U\}}. \end{aligned}$$

Then using Part *(2)* of the theorem yields the second statement in Part *(3)*.

3. Proceeding as in Steps 1-2 above with appropriate adaptations for the measure $\mu^-_{\{v_{n,q,i},U\}}$ yields the third and fourth statement in Part *(3)*. This completes the proof of Theorem 5.2. ∎

Theorem 5.3. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ be such that $0 \leq k < l$. Set $\tilde{v} = D^l\varpi$ and $\tilde{w} = D^k\varpi$. Let*

$\phi \in C_c(t_1, t_2)$. Set $K = \text{spt}_x(\tilde{w}) \times \text{spt}(\phi)$. Let $\tilde{\phi} \in C_c^\infty(t_1, t_2)$ be such that $\tilde{\phi} \equiv 1$ on spt(ϕ) and $0 \le \tilde{\phi} \le 1$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Assume that w satisfies: $0 \le w \le 1$.

Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0, 1, 2, \cdots, 2^n - 1\}$. Let $q \in I_n$. Let $\tilde{I}_{n,q}$ be the set introduced in §5.1.2. Let $i \in \tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k}w_{n,q,i}$. Let $\mu^+_{n,q,i} = \mu^+_{\{v,U\}} - \mu^+_{\{v_{n,q,i},U\}}$ and $\mu^-_{n,q,i} = \mu^-_{\{v,U\}} - \mu^-_{\{v_{n,q,i},U\}}$. Then

(1) The measures $\phi\mu^+_{\{v,U\}}$, $\phi\mu^-_{\{v,U\}}$, $\phi\mu^+_{\{v_{n,q,i},U\}}$, and $\phi\mu^-_{\{v_{n,q,i},U\}}$ are concentrated on the compact set $K = \text{spt}_x(\tilde{w}) \times \text{spt}(\phi)$.

(2)

$$\int \phi d\mu^+_{\{v,U\}} = 0, \quad \int \phi d\mu^-_{\{v,U\}} = 0, \quad \int \phi d\mu^+_{\{v_{n,q,i},U\}} = 0, \quad \int \phi d\mu^-_{\{v_{n,q,i},U\}} = 0.$$

(3)

$$\int \phi d\mu^+_{n,q,i} = 0, \quad \int \phi d\mu^-_{n,q,i} = 0.$$

(4) As n goes to ∞,

$$\sum_{j=0}^{m+1} \Big(\int (w - q_j/2^n) \chi_{\tilde{A}_{n,q_j,i^0_j}} \phi d\mu^+_{n,q_j,i^0_j} + \sum_{s=1,(q_j+s,i^s_j)\in J_{n,j}}^{r_j} \int (w - (q_j+s)/2^n) \chi_{A_{n,q_j+s,i^s_j}} \phi d\mu^+_{n,q_j+s,i^s_j} \Big) \to 0,$$

$$\sum_{j=0}^{m+1} \Big(\int (w - q_j/2^n) \chi_{\tilde{A}_{n,q_j,i^0_j}} \phi d\mu^-_{n,q_j,i^0_j} + \sum_{s=1,(q_j+s,i^s_j)\in J_{n,j}}^{r_j} \int (w - (q_j+s)/2^n) \chi_{A_{n,q_j+s,i^s_j}} \phi d\mu^-_{n,q_j+s,i^s_j} \Big) \to 0.$$

Here, m was introduced in §5.1.1, q_j, r_j, and $J_{n,j}$ were introduced in §5.1.4.

Proof of Theorem 5.3

Proof of (1). By construction, spt$(v_{n,q,i}) \subset$ spt$(w_{n,q,i}) \subset$ spt(w). Hence, the projection of spt$(v_{n,q,i})$ into $\mathbb{R}^N$ is a subset of spt$_x(\tilde{w})$. Moreover, spt(ϕ) is compact. Then by construction, the measures $\phi\mu^+_{\{v,U\}}$, $\phi\mu^-_{\{v,U\}}$, $\phi\mu^+_{\{v_{n,q,i},U\}}$, and $\phi\mu^-_{\{v_{n,q,i},U\}}$ are concentrated on the compact set $K = \text{spt}_x(\tilde{w}) \times \text{spt}(\phi)$.

Proof of (2). Let m be so large that $\psi_m \equiv 1$ on spt$_x(\tilde{w})$. Using Part *(1)* of Theorem 3.3 with $\varphi = \phi\psi_m$, one obtains

$$\int \phi\psi_m d\mu^+_{\{v,U\}} = -\int_{\{v>0\}} \text{div}(\phi\psi_m U) dy, \tag{5.34}$$

$$\int \phi\psi_m d\mu^+_{\{v_{n,q,i},U\}} = -\int_{\{v_{n,q,i}>0\}} \text{div}(\phi\psi_m U) dy. \tag{5.35}$$

By hypothesis, div $U = 0$. Moreover, $\psi_m \equiv 1$ on spt$_x(\tilde{w})$. Then one has

$$\int_{\{v>0\}} \mathrm{div}(\phi\psi_m U)dy = \int_{\{v>0\}\cap K} \mathrm{div}(\phi\psi_m U)dy = \int_{\{v>0\}\cap K} \mathrm{div}(\phi U)dy$$
$$= \int_{\{v>0\}\cap K} \phi \mathrm{div}(U)dy = 0, \tag{5.36}$$

$$\int_{\{v_{n,q,i}>0\}} \mathrm{div}(\phi\psi_m U)dy = \int_{\{v_{n,q,i}>0\}\cap K} \phi \mathrm{div}(\psi_m U)dy$$
$$= \int_{\{v_{n,q,i}>0\}\cap K} \phi \mathrm{div}(U)dy = 0. \tag{5.37}$$

Combining (5.34)-(5.37), one obtains

$$\int \phi d\mu^+_{\{v,U\}} = \int \phi\psi_m d\mu^+_{\{v,U\}} = 0,$$
$$\int \phi d\mu^+_{\{v_{n,q,i},U\}} = \int \phi\psi_m d\mu^+_{\{v_{n,q,i},U\}} = 0.$$

Proceeding as above for $\mu^-_{\{v,U\}}$ and $\mu^-_{\{v_{n,q,i},U\}}$, one obtains the remaining results in Part *(2)* of the theorem.

Proof of (3). Using the definition of the measures $\mu^+_{n,q,i}$ and Part *(2)* of the theorem yields

$$\int \phi d\mu^+_{n,q,i} = \int \phi d\mu^+_{v,U} - \int \phi d\mu^+_{v_{n,q,i},U} = 0.$$

Similarly, one obtains

$$\int \phi d\mu^-_{n,q,i} = 0.$$

Proof of (4). By definition of $\mu^+_{n,q,i}$,

$$\begin{aligned}
&\sum_{j=0}^{m+1}\Big(\int (w - q_j/2^n)\chi_{\tilde{A}_{n,q_j,i_j^0}} \phi d\mu^+_{n,q_j,i_j^0} + \\
&\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int (w-(q_j+s)/2^n)\chi_{A_{n,q_j+s,i_j^s}} \phi d\mu^+_{n,q_j+s,i_j^s}\Big) \\
=\;& \sum_{j=0}^{m+1}\Big(\int (w - q_j/2^n)\chi_{\tilde{A}_{n,q_j,i_j^0}} \phi d\mu^+_{v,U} + \\
&\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int (w-(q_j+s)/2^n)\chi_{A_{n,q_j+s,i_j^s}} \phi d\mu^+_{v,U}\Big) \\
&-\sum_{j=0}^{m+1}\Big(\int (w - q_j/2^n)\chi_{\tilde{A}_{n,q_j,i_j^0}} \phi d\mu^+_{v_{n,q_j,i_j^0},U} \\
&+ \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int (w-(q_j+s)/2^n)\chi_{A_{n,q_j+s,i_j^s}} \phi d\mu^+_{v_{n,q_j+s,i_j^s},U}\Big).
\end{aligned} \tag{5.38}$$

1. Set $w_n = \sum_{j=0}^{m+1}(q_j/2^n\chi_{\tilde{A}_{n,q_j,i_j^0}} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j}(q_j+s)/2^n\chi_{A_{n,q_j+s,i_j^s}})$. It will be proved that

$$0 \leq |w - w_n| \leq 1/2^n. \tag{5.39}$$

Let $y \in \mathbb{R}^N \times (t_1, t_2)$ be such that y does not belong to any of the sets involved in the definition of w_n above. Then using (5.23) one has necessarily $w(y) = 0$. One also has $w_n(y) = 0$. Then $0 \leq w(y) - w_n(y) \leq 1/2^n$. Now let $j \in \{0, \cdots, m+1\}$. Let $y \in \tilde{A}_{n,q_j,i_j^0}$. Then by definition of w_n: $w_n(y) = q_j/2^n$, and by definition of $\tilde{A}_{n,q_j,i_j^0}$: $|w(y) - q_j/2^n| \leq 1/2^n$. Hence, $0 \leq |w(y) - w_n(y)| \leq 1/2^n$. Repeating this process with the other sets, one obtains $0 \leq |w(y) - w_n(y)| \leq 1/2^n$ for any $y \in \mathbb{R}^N \times (t_1, t_2)$. This concludes the proof of (5.39).

Using (5.23) and the definition of w_n, the first term in the right side of (5.38) is equal to $\int (w_n - w)\phi d\mu^+_{\{v,U\}}$. Hence, by (5.39), one has

$$|\int (w - w_n)\phi d\mu^+_{\{v,U\}}| \leq (1/2^n)|(\phi\mu^+_{\{v,U\}})(K)|. \tag{5.40}$$

By Part *(1)* of the theorem, $\phi\mu^+_{\{v,U\}}$ is concentrated on the compact set $K = \text{spt}_x(\tilde{w}) \times \text{spt}(\phi)$, and by the fact that $\phi\mu^+_{\{v,U\}}$ is a Radon measure, one has: $|(\phi\mu^+_{\{v,U\}})(K)|$ is finite. Hence, the left side of (5.40) goes to 0 as n goes to ∞. Therefore, the first term in the right side of (5.38) goes to 0 as n goes to ∞.

2. By Part *(3)* of Theorem 5.2,

$$\int \chi_{\tilde{A}_{n,q_j,i_j^0}}\phi d\mu^+_{v_{n,q_j,i_j^0},U} = 0, \quad \int w\chi_{\tilde{A}_{n,q_j,i_j^0}}\phi d\mu^+_{v_{n,q_j,i_j^0},U} = 0, \tag{5.41}$$

and for any $s \in \{1, \cdots, r_j\}$,

$$\int \chi_{A_{n,q_j+s,i_j^s}}\phi d\mu^+_{v_{n,q_j+s,i_j^s},U} = 0, \quad \int w\chi_{A_{n,q_j+s,i_j^s}}\phi d\mu^+_{v_{n,q_j+s,i_j^s},U} = 0, \tag{5.42}$$

Then using (5.41)-(5.42) shows that the second term in the right side of (5.38) is 0.

3. Combining Steps 1 and 2 above, one concludes that the left side of (5.38) converges to 0 as n goes to ∞. This proves the first convergence in Part *(4)* of the theorem.

4. Proceeding as in Steps 1-3 above for $\mu^-_{n,q,i}$, one obtains the second convergence in Part *(4)* of the theorem. This completes the proof of Theorem 5.3. ∎

5.3. **Actions of the measures introduced in 5.1.5 on some particular functions: Part 1.** Here, basic properties of $\int \phi w\chi_{A^-_{n,q,i}}d\mu^+_{v_{n,q,i},U}$, $\int \phi(w - q/2^n)\chi_{A^-_{n,q,i}}d\mu^+_{v_{n,q,i},U}$, $\int \phi w\chi_{A^-_{n,q,i}}d\mu^+_{v,U}$, $\int \phi(w - q/2^n)\chi_{A^-_{n,q,i}}d\mu^+_{v,U}$, $\int \phi w\chi_{A^-_{n,q,i}}d\mu^-_{v_{n,q,i},U}$, $\int \phi(w - q/2^n)\chi_{A^-_{n,q,i}}d\mu^-_{v_{n,q,i},U}$, $\int \phi w\chi_{A^-_{n,q,i}}d\mu^-_{v,U}$, and $\int \phi(w - q/2^n)\chi_{A^-_{n,q,i}}d\mu^-_{v,U}$ are obtained.

5.3.1. *Basic properties of* $\int \phi w\chi_{A^-_{n,q,i}}d\mu^+_{v_{n,q,i},U}$ *and* $\int \phi w\chi_{A^-_{n,q,i}}d\mu^-_{v_{n,q,i},U}$, $i \in I^3_{n,q} \cup I^4_{n,q}$.

Theorem 5.4. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ be such that $0 \leq k < l$. Set $\tilde{v} = D^l\varpi$ and $\tilde{w} = D^k\varpi$. Let*

$\phi \in C_c(t_1, t_2)$. Set $K = spt_x(\tilde{w}) \times spt(\phi)$. Let $\tilde{\phi} \in C_c^\infty(t_1, t_2)$ be such that $\tilde{\phi} \equiv 1$ on $spt(\phi)$ and $0 \leq \tilde{\phi} \leq 1$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Assume that w satisfies: $0 \leq w \leq 1$.

Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0, 1, 2, \cdots, 2^n - 1\}$. Let $q \in I_n$. Let $\tilde{I}_{n,q}$, $I^3_{n,q}$, and $I^4_{n,q}$ be the sets introduced in §§5.1.2,5.1.1. Let $i \in \tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k}w_{n,q,i}$. Let $i \in I^3_{n,q} \cup I^4_{n,q}$. Then

(1)

$$\int \phi w \chi_{A^-_{n,q,i}} d\mu^+_{v_{n,q,i},U} = \int \phi w \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U} \lfloor A^-_{n,q,i}] - \int \phi w d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v>0\}\}} \lfloor A^-_{n,q,i}] - \int_{t_1}^{t_2} \int_{\mathbb{R}^n} div(\phi w U) \chi_{\{v_{n,q,i}>0\}} \chi_{B^-_{n,q,i}} dx d\tau.$$

(2)

$$\int \phi w \chi_{A^-_{n,q,i}} d\mu^-_{v_{n,q,i},U} = \int \phi w \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^-_{v,U} \lfloor A^-_{n,q,i}] + \int \phi w d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v<0\}\}} \lfloor A^-_{n,q,i}] + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} div(\phi w U) \chi_{\{v_{n,q,i}<0\}} \chi_{B^-_{n,q,i}} dx d\tau.$$

Here, $B^-_{n,q,i} = A^-_{n,q,i} \cap \{q/2^n - (1/2^n)^2 < w \leq q/2^n\}$.

Proof of Theorem 5.4.

The method of proof of this theorem given below will be used with appropriate adaptations for the proof of each of the theorems 5.5-5.11.

1. Let n be any integer such that $2^n > 1/m_w$. Let $q \in I_n$. Let $i \in I^3_{n,q} \cup I^4_{n,q}$. Then $A^-_{n,q,i} \neq \emptyset$. Let $V(A^-_{n,q,i})$ denote an open neighborhood of $A^-_{n,q,i}$ such that

$$V(A^-_{n,q,i}) \cap \cup_{j \in (I^3_{n,q} \cup I^4_{n,q}) \setminus \{i\}} A^-_{n,q,j} = \emptyset.$$

Let φ be any Lipschitz function in $C_c(V(A^-_{n,q,i}))$. By definition of $A^-_{n,q,i}$ and φ, one has

$$\int \varphi \chi_{A^-_{n,q,i}} d\mu^+_{v_{n,q,i},U} = \int \varphi \chi_{\{w-q/2^n+(1/2^n)^2 \geq 0\}} d\mu^+_{v_{n,q,i},U} - \int \varphi \chi_{\{w-q/2^n>0\}} d\mu^+_{v_{n,q,i},U}. \tag{5.43}$$

Taking $(\varphi, v_{n,q,i}, w-q/2^n+(1/2^n)^2, U)$ of this step in place of (φ, v, w, U) in the first equality in Part *(1)* of Theorem 4.4 yields

$$\int \chi_{\{w-q/2^n+(1/2^n)^2>0\}} \varphi d\mu^+_{\{v_{n,q,i},U\}} = -\int \varphi d\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v_{n,q,i}>0\}\}} - \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathrm{div}(\varphi U) \chi_{\{v_{n,q,i}>0\}} \chi_{\{w-q/2^n+(1/2^n)^2>0\}} dx d\tau. \tag{5.44}$$

Taking $(\varphi, v_{n,q,i}, w - q/2^n, U)$ of this step in place of (φ, v, w, U) in the first equality in Part *(1)* of Theorem 4.4 yields

$$\begin{aligned} \int \chi_{\{w-q/2^n>0\}} \varphi d\mu^+_{\{v_{n,q,i},U\}} &= -\int \varphi d\mu^+_{\{w-q/2^n,U,\{v_{n,q,i}>0\}\}} \\ &- \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\varphi U)\chi_{\{v_{n,q,i}>0\}}\chi_{\{w-q/2^n>0\}} dx d\tau. \end{aligned} \tag{5.45}$$

Combining (5.44) and (5.45) and using (5.43) one obtains

$$\begin{aligned} \int \varphi \chi_{A^-_{n,q,i}} d\mu^+_{v_{n,q,i},U} &= \int \varphi \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d\mu^+_{v_{n,q,i},U} \\ &- \int \varphi d\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v_{n,q,i}>0\}\}} + \int \varphi d\mu^+_{\{w-q/2^n,U,\{v_{n,q,i}>0\}\}} \\ &- \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\varphi U)\chi_{\{v_{n,q,i}>0\}}\chi_{B^-_{n,q,i}} dx d\tau, \end{aligned} \tag{5.46}$$

where $B^-_{n,q,i} = \{q/2^n - (1/2^n)^2 < w \le q/2^n\} \cap A^-_{n,q,i}$. By definition of $A^-_{n,q,i}$ and φ and classical properties of measures, one obtains

$$\begin{aligned} \int \varphi \chi_{A^-_{n,q,i}} d\mu^+_{v_{n,q,i},U} &= \int \varphi \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v_{n,q,i},U}\lfloor A^-_{n,q,i}] \\ &- \int \varphi d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v_{n,q,i}>0\}\}}\lfloor A^-_{n,q,i}] + \int \varphi d[\mu^+_{\{w-q/2^n,U,\{v_{n,q,i}>0\}\}}\lfloor A^-_{n,q,i}] \\ &- \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\varphi U)\chi_{\{v_{n,q,i}>0\}}\chi_{B^-_{n,q,i}} dx d\tau, \end{aligned} \tag{5.47}$$

where $\mu\lfloor E$ denote the restriction of the measure μ to the set E; See the beginning of this section.

2. By Theorem 4.1, $\mu^+_{\{w-q/2^n,U,\{v_{n,q,i}>0\}\}}$ is concentrated on γ, where

$$\begin{aligned} &\gamma = \gamma_1 \cap (\partial\{v_{n,q,i}>0\} \cup \{v_{n,q,i}>0\}), \\ &\gamma_1 = \partial\{w-q/2^n>0\} \cap \partial\{\{w-q/2^n>0\}\cap\{v_{n,q,i}>0\}\}. \end{aligned}$$

By assumption, $k \neq l$. Hence, one can use (5.19) and conclude that

$$\partial\{w-q/2^n>0\} \cap A^-_{n,q,i} = \partial\{w-q/2^n>0\} \cap \overline{A_{n,q,i}}$$

is a subset of $\{v_{n,q,i}=0\}^o$. Thus, $\gamma = \emptyset$. And so,

$$\int \varphi d\mu^+_{\{w-q/2^n,U,\{v_{n,q,i}>0\}\}} = 0.$$

3. By Theorem 3.1, $\mu^+_{v_{n,q,i},U}$ is concentrated on $\partial\{v_{n,q,i}>0\}$. By construction of $\Omega^{2,-}_{n,q,i}$; See §5.1.2, $\{w-q/2^n+(1/2^n)^2=0\}\cap\partial\{v_{n,q,i}>0\}\cap A^-_{n,q,i} \subset \Omega^{2,-}_{n,q,i}$. Then by (5.16) and (5.18), $\{w-q/2^n+(1/2^n)^2=0\}\cap\partial\{v_{n,q,i}>0\}\cap A^-_{n,q,i} \subset \{w_{n,q,i}=w\}^o \subset \{v_{n,q,i}=v\}^o$. Then one obtains

$$\int \varphi \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v_{n,q,i},U}\lfloor A^-_{n,q,i}] = \int \varphi \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U}\lfloor A^-_{n,q,i}].$$

4. By Theorem 4.1, $\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v_{n,q,i}>0\}\}}$ is concentrated on γ, where

$$\gamma = \gamma_1 \cap (\partial\{v_{n,q,i} > 0\} \cup \{v_{n,q,i} > 0\}),$$
$$\gamma_1 = \partial\{w - q/2^n + (1/2^n)^2 > 0\} \cap \partial\{\{w - q/2^n + (1/2^n)^2 > 0\} \cap \{v_{n,q,i} > 0\}\}.$$

By (5.16) and (5.18), $\gamma \subset \{w_{n,q,i} = w\}^o \subset \{v_{n,q,i} = v\}^o$. Then one obtains

$$\int \varphi d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v_{n,q,i}>0\}\}} \lfloor A^-_{n,q,i}] = \int \varphi d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v>0\}\}} \lfloor A^-_{n,q,i}].$$

5. Using (5.47) of Step 1 and Steps 2-4, one obtains

$$\int \varphi \chi_{A^-_{n,q,i}} d\mu^+_{v_{n,q,i},U} = \int \varphi \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U} \lfloor A^-_{n,q,i}]$$
$$- \int \varphi d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v>0\}\}} \lfloor A^-_{n,q,i}] - \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \operatorname{div}(\varphi U) \chi_{\{v_{n,q,i}>0\}} \chi_{B^-_{n,q,i}} dx d\tau.$$

6. Since spt(w) is compact and $q \neq 0$, $A^-_{n,q,i}$ is compact. Then it is clear that one can select φ a Lipschitz function in $C_c(V(A^-_{n,q,i}))$ such that $\varphi \equiv \phi w$ on $A^-_{n,q,i}$. Then using Step 5, one obtains

$$\int \phi w \chi_{A^-_{n,q,i}} d\mu^+_{v_{n,q,i},U} = \int \phi w \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U} \lfloor A^-_{n,q,i}]$$
$$- \int \phi w d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v>0\}\}} \lfloor A^-_{n,q,i}] - \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \operatorname{div}(\phi w U) \chi_{\{v_{n,q,i}>0\}} \chi_{B^-_{n,q,i}} dx d\tau.$$

This completes the proof of Part *(1)* of Theorem 5.4.

7. Proceeding as in Steps 1-6 for $\mu^-_{v_{n,q,i},U}$ with appropriate adaptations, one obtains Part *(2)* of Theorem 5.4. ∎

5.3.2. *Basic properties of* $\int \phi(w - q/2^n)\chi_{A^-_{n,q,i}} d\mu^+_{v_{n,q,i},U}$ *and* $\int \phi(w - q/2^n)\chi_{A^-_{n,q,i}} d\mu^-_{v_{n,q,i},U}$.

Theorem 5.5. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ be such that $0 \leq k < l$. Set $\tilde{v} = D^l \varpi$ and $\tilde{w} = D^k \varpi$. Let $\phi \in C_c(t_1, t_2)$. Set $K =$ spt$_x(\tilde{w}) \times$ spt(ϕ). Let $\tilde{\phi} \in C_c^\infty(t_1, t_2)$ be such that $\tilde{\phi} \equiv 1$ on spt(ϕ) and $0 \leq \tilde{\phi} \leq 1$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Assume that w satisfies: $0 \leq w \leq 1$.*

Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0, 1, 2, \cdots, 2^n - 1\}$. Let $q \in I_n$. Let $\tilde{I}_{n,q}$, $I^3_{n,q}$, and $I^4_{n,q}$ be the sets introduced in §§5.1.2,5.1.1. Let $i \in \tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k} w_{n,q,i}$. Then for $i \in I^3_{n,q} \cup I^4_{n,q}$,

(1)

$$\int \phi(w-q/2^n)\chi_{A^-_{n,q,i}} d\mu^+_{v_{n,q,i},U} = \int \phi(w-q/2^n)\chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U} \lfloor A^-_{n,q,i}]$$
$$- \int \phi(w-q/2^n) d[\mu^+_{\{w-q/2^n+(1/2^n)^2, U, \{v>0\}\}} \lfloor A^-_{n,q,i}]$$
$$- \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathit{div}(\phi w U)\chi_{\{v_{n,q,i}>0\}}\chi_{B^-_{n,q,i}} dx d\tau.$$

(2)

$$\int \phi(w-q/2^n)\chi_{A^-_{n,q,i}} d\mu^-_{v_{n,q,i},U} = \int \phi w \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^-_{v,U} \lfloor A^-_{n,q,i}]$$
$$+ \int \phi(w-q/2^n) d[\mu^+_{\{w-q/2^n+(1/2^n)^2, U, \{v<0\}\}} \lfloor A^-_{n,q,i}]$$
$$+ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathit{div}(\phi w U)\chi_{\{v_{n,q,i}<0\}}\chi_{B^-_{n,q,i}} dx d\tau.$$

Here, $B^-_{n,q,i} = A^-_{n,q,i} \cap \{q/2^n - (1/2^n)^2 < w \leq q/2^n\}$.

Proof of Theorem 5.5.

Let n be any integer such that $2^n > 1/m_w$. Let $q \in I_n$. Let $i \in I^3_{n,q} \cup I^4_{n,q}$. Let $V(A^-_{n,q,i})$ denote an open neighborhood of $A^-_{n,q,i}$ such that $V(A^-_{n,q,i}) \cap \cup_{j \in (I^3_{n,q} \cup I^4_{n,q}) \setminus \{i\}} A^-_{n,q,j} = \emptyset$. Since spt$(w)$ is compact and $q \neq 0$, $A^-_{n,q,i}$ is compact. Then it is clear that one can select φ a Lipschitz function in $C_c(V(A^-_{n,q,i}))$ such that $\varphi \equiv \phi(w-q/2^n)$ on $A^-_{n,q,i}$. Then using Step 5 of the proof of Theorem 5.4, one obtains

$$\int \phi(w-q/2^n)\chi_{A^-_{n,q,i}} d\mu^+_{v_{n,q,i},U} = \int \phi(w-q/2^n)\chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U} \lfloor A^-_{n,q,i}]$$
$$- \int \phi(w-q/2^n) d[\mu^+_{\{w-q/2^n+(1/2^n)^2, U, \{v>0\}\}} \lfloor A^-_{n,q,i}]$$
$$- \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathrm{div}(\phi w U)\chi_{\{v_{n,q,i}>0\}}\chi_{B^-_{n,q,i}} dx d\tau.$$

Here a use of the fact that div $U = 0$, has been made. This completes the proof of Part *(1)* of Theorem 5.5. Proceeding as above for $\mu^-_{v_{n,q,i},U}$ with appropriate adaptations, one obtains Part *(2)* of Theorem 5.5. ∎

5.3.3. *Basic properties of* $\int \phi w \chi_{A^-_{n,q,i}} d\mu^+_{v,U}$ *and* $\int \phi w \chi_{A^-_{n,q,i}} d\mu^-_{v,U}$.

Theorem 5.6. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1,t_2); \mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1,t_2); \mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ be such that $0 \leq k < l$. Set $\tilde{v} = D^l \varpi$ and $\tilde{w} = D^k \varpi$. Let $\phi \in C_c(t_1,t_2)$. Set $K = spt_x(\tilde{w}) \times spt(\phi)$. Let $\tilde{\phi} \in C^\infty_c(t_1,t_2)$ be such that $\tilde{\phi} \equiv 1$ on spt(ϕ) and $0 \leq \tilde{\phi} \leq 1$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Assume that w satisfies: $0 \leq w \leq 1$.*

Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0, 1, 2, \cdots, 2^n - 1\}$. Let $q \in I_n$. Let $\tilde{I}_{n,q}$, $I^3_{n,q}$, and $I^4_{n,q}$ be the sets introduced in §§5.1.2,5.1.1. Let $i \in \tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for

$\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k}w_{n,q,i}$. Then for $i \in I^3_{n,q} \cup I^4_{n,q}$,

(1)

$$\begin{aligned}\int \phi w \chi_{A^-_{n,q,i}} d\mu^+_{v,U} &= \int \phi w \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U}\lfloor A^-_{n,q,i}] \\ &- \int \phi w d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v>0\}\}}\lfloor A^-_{n,q,i}] + \int \phi w d[\mu^+_{\{w-q/2^n,U,\{v>0\}\}}\lfloor A^-_{n,q,i}] \\ &- \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \textit{div}(\phi w U)\chi_{\{v>0\}}\chi_{B^-_{n,q,i}} dx d\tau.\end{aligned}$$

(2)

$$\begin{aligned}\int \phi w \chi_{A^-_{n,q,i}} d\mu^-_{v,U} &= \int \phi w \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^-_{v,U}\lfloor A^-_{n,q,i}] \\ &+ \int \phi w d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v<0\}\}}\lfloor A^-_{n,q,i}] - \int \phi w d[\mu^+_{\{w-q/2^n,U,\{v<0\}\}}\lfloor A^-_{n,q,i}] \\ &+ \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \textit{div}(\phi w U)\chi_{\{v<0\}}\chi_{B^-_{n,q,i}} dx d\tau.\end{aligned}$$

Here, $B^-_{n,q,i} = A^-_{n,q,i} \cap \{q/2^n - (1/2^n)^2 < w \le q/2^n\}$.

Proof of Theorem 5.6.

The proof is obtained by an appropriate adaptation of the proof of Theorem 5.4.

Proof of (1)

1. Let n be any integer such that $2^n > 1/m_w$. Let $q \in I_n$. Let $i \in I^3_{n,q} \cup I^4_{n,q}$. Let $V(A^-_{n,q,i})$ denote an open neighborhood of $A^-_{n,q,i}$ such that $V(A^-_{n,q,i}) \cap \cup_{j\in(I^3_{n,q}\cup I^4_{n,q})\setminus\{i\}} A^-_{n,q,j} = \emptyset$. Let φ be any Lipschitz function in $C_c(V(A^-_{n,q,i}))$. By definition of $A^-_{n,q,i}$ and φ, one obtains

$$\int \varphi \chi_{A^-_{n,q,i}} d\mu^+_{v,U} = \int \varphi \chi_{\{w-q/2^n+(1/2^n)^2\ge 0\}} d\mu^+_{v,U} - \int \varphi \chi_{\{w-q/2^n>0\}} d\mu^+_{v,U}. \tag{5.48}$$

Then proceeding as in Step 1 of the proof of Theorem 5.4, one obtains

$$\begin{aligned}\int \varphi \chi_{A^-_{n,q,i}} d\mu^+_{v,U} &= \int \varphi \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U}\lfloor A^-_{n,q,i}] \\ &- \int \varphi d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v>0\}\}}\lfloor A^-_{n,q,i}] + \int \varphi d[\mu^+_{\{w-q/2^n,U,\{v>0\}\}}\lfloor A^-_{n,q,i}] \\ &- \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \mathrm{div}(\varphi U)\chi_{\{v>0\}}\chi_{B^-_{n,q,i}} dx d\tau.\end{aligned} \tag{5.49}$$

2. Since spt(w) is compact and $q \neq 0$, $A^-_{n,q,i}$ is compact. Then it is clear that one can select φ a Lipschitz function in $C_c(V(A^-_{n,q,i}))$ such that $\varphi \equiv \phi w$ in $A^-_{n,q,i}$. Then plugging this function in (5.49) above, one obtains

$$\int \phi w \chi_{A^-_{n,q,i}} d\mu^+_{v,U} = \int \phi w \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U} \lfloor A^-_{n,q,i}]$$
$$- \int \phi w d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v>0\}\}} \lfloor A^-_{n,q,i}] + \int \phi w d[\mu^+_{\{w-q/2^n,U,\{v>0\}\}} \lfloor A^-_{n,q,i}]$$
$$- \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathrm{div}(\phi w U) \chi_{\{v>0\}} \chi_{B^-_{n,q,i}} dx d\tau.$$

This completes the proof of Part *(1)* of theorem 5.6.

3. Proceeding as in Steps 1-2 for $\mu^-_{v,U}$ with appropriate adaptations, one obtains Part *(2)* of Theorem 5.6. ∎

5.3.4. *Basic properties of* $\int \phi(w - q/2^n) \chi_{A^-_{n,q,i}} d\mu^+_{v,U}$ *and* $\int \phi(w - q/2^n) \chi_{A^-_{n,q,i}} d\mu^-_{v,U}$.

Theorem 5.7. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ be such that $0 \leq k < l$. Set $\tilde{v} = D^l \varpi$ and $\tilde{w} = D^k \varpi$. Let $\phi \in C_c(t_1, t_2)$. Set $K = spt_x(\tilde{w}) \times spt(\phi)$. Let $\tilde{\phi} \in C_c^\infty(t_1, t_2)$ be such that $\tilde{\phi} \equiv 1$ on spt(ϕ) and $0 \leq \tilde{\phi} \leq 1$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Assume that w satisfies: $0 \leq w \leq 1$.*

Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0, 1, 2, \cdots, 2^n - 1\}$. Let $q \in I_n$. Let $\tilde{I}_{n,q}$, $I^3_{n,q}$, and $I^4_{n,q}$ be the sets introduced in §§5.1.2,5.1.1. Let $i \in \tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k} w_{n,q,i}$. Then for $i \in I^3_{n,q} \cup I^4_{n,q}$,

(1)

$$\int \phi(w - q/2^n) \chi_{A^-_{n,q,i}} d\mu^+_{v,U} = \int \phi(w - q/2^n) \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U} \lfloor A^-_{n,q,i}]$$
$$- \int \phi(w - q/2^n) d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v>0\}\}} \lfloor A^-_{n,q,i}]$$
$$- \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathit{div}(\phi w U) \chi_{\{v>0\}} \chi_{B^-_{n,q,i}} dx d\tau.$$

(2)

$$\int \phi(w - q/2^n) \chi_{A^-_{n,q,i}} d\mu^-_{v,U} = \int \phi(w - q/2^n) \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^-_{v,U} \lfloor A^-_{n,q,i}]$$
$$+ \int \phi(w - q/2^n) d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v<0\}\}} \lfloor A^-_{n,q,i}]$$
$$+ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathit{div}(\phi w U) \chi_{\{v<0\}} \chi_{B^-_{n,q,i}} dx d\tau.$$

Here, $B^-_{n,q,i} = A^-_{n,q,i} \cap \{q/2^n - (1/2^n)^2 < w \leq q/2^n\}$.

Proof of Theorem 5.7.
The proof is obtained by an appropriate adaptation of the proof of Theorem 5.4.

1. Let n be any integer such that $2^n > 1/m_w$. Let $q \in I_n$. Let $i \in I^3_{n,q} \cup I^4_{n,q}$. Let $V(A^-_{n,q,i})$ denote an open neighborhood of $A^-_{n,q,i}$ such that $V(A^-_{n,q,i}) \cap \cup_{j\in(I^3_{n,q}\cup I^4_{n,q})\setminus\{i\}} A^-_{n,q,j} = \emptyset$. Let φ be any Lipschitz function in $C_c(V(A^-_{n,q,i}))$. Step 1 of the proof of Theorem 5.6 yields

$$\int \varphi\chi_{A^-_{n,q,i}} d\mu^+_{v,U} = \int \varphi\chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U}\lfloor A^-_{n,q,i}]$$
$$-\int \varphi d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v>0\}\}}\lfloor A^-_{n,q,i}] + \int \varphi d[\mu^+_{\{w-q/2^n,U,\{v>0\}\}}\lfloor A^-_{n,q,i}]$$
$$-\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \text{div}(\varphi U)\chi_{\{v>0\}}\chi_{B^-_{n,q,i}} dx d\tau.$$

2. By Part *(2)* of Theorem 5.3, the measure $\phi\mu^+_{\{v,U\}}$ is concentrated on the compact set $K = \text{spt}_x(\tilde{w}) \times \text{spt}(\phi)$. Let $\tilde{\varphi}$ be a function in $C^1_c(\mathbb{R}^n \times (t_1, t_2)$ such that $\tilde{\varphi} \equiv 1$ on K. Then $\tilde{\varphi}(w - q/2^n) \in C^1_c(\mathbb{R}^n \times (t_1, t_2))$ and $\tilde{\varphi}(w - q/2^n) \equiv (w - q/2^n)$ on K. By Theorem 4.1, $\mu^+_{\{w-q/2^n,U,\{v>0\}\}}$ is concentrated on $\partial\{w - q/2^n > 0\}$. Hence,

$$\int \tilde{\varphi}\phi(w - q/2^n) d\mu^+_{\{w-q/2^n,U,\{v>0\}\}} = \int \phi(w - q/2^n) d\mu^+_{\{w-q/2^n,U,\{v>0\}\}} = 0.$$

3. Since spt(w) is compact and $q \neq 0$, $A^-_{n,q,i}$ is compact. Then it is clear that one can select φ a Lipschitz function in $C_c(V(A^-_{n,q,i}))$ such that $\varphi \equiv \phi(w - q/2^n)$ on $A^-_{n,q,i}$. Plugging this φ in Step 1 above and using Step 2 above, one obtains

$$\int \phi(w - q/2^n)\chi_{A^-_{n,q,i}} d\mu^+_{v,U} = \int \phi(w - q/2^n)\chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U}\lfloor A^-_{n,q,i}]$$
$$-\int \phi(w - q/2^n) d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v>0\}\}}\lfloor A^-_{n,q,i}] - \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \text{div}(\phi w U)\chi_{\{v>0\}}\chi_{B^-_{n,q,i}} dx d\tau.$$

Here a use of the fact that div $U = 0$, has been made. This completes the proof of Part *(1)* of Theorem 5.7.

4. Proceeding as in Steps 1-3 for $\mu^-_{v,U}$ with appropriate adaptations, one obtains Part *(2)* of Theorem 5.7. ∎

5.4. **Actions of the measures introduced in 5.1.5 on some particular functions: Part 2.** Here, basic properties of $\int \phi w\chi_{A^+_{n,q,i}} d\mu^+_{v_{n,q,i},U}$, $\int \phi(w-q/2^n)\chi_{A^+_{n,q,i}} d\mu^+_{v_{n,q,i},U}$, $\int \phi w\chi_{A^+_{n,q,i}} d\mu^+_{v,U}$, $\int \phi(w - q/2^n)\chi_{A^+_{n,q,i}} d\mu^+_{v,U}$, $\int \phi w\chi_{A^+_{n,q,i}} d\mu^-_{v_{n,q,i},U}$, $\int \phi(w - q/2^n)\chi_{A^+_{n,q,i}} d\mu^-_{v_{n,q,i},U}$, $\int \phi w\chi_{A^+_{n,q,i}} d\mu^-_{v,U}$, and $\int \phi(w - q/2^n)\chi_{A^+_{n,q,i}} d\mu^-_{v,U}$ are obtained.

5.4.1. *Basic properties of* $\int \phi w\chi_{A^+_{n,q,i}} d\mu^+_{v_{n,q,i},U}$ *and* $\int \phi w\chi_{A^+_{n,q,i}} d\mu^-_{v_{n,q,i},U}$.

Theorem 5.8. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ be such that $0 \leq k < l$. Set $\tilde{v} = D^l\varpi$ and $\tilde{w} = D^k\varpi$. Let $\phi \in C_c(t_1, t_2)$. Set $K = spt_x(\tilde{w}) \times spt(\phi)$. Let $\tilde{\phi} \in C^\infty_c(t_1, t_2)$ be such that $\tilde{\phi} \equiv 1$ on spt(ϕ) and $0 \leq \tilde{\phi} \leq 1$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Assume that w satisfies: $0 \leq w \leq 1$.*

Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0, 1, 2, \cdots, 2^n - 1\}$. Let $q \in I_n$. Let $\tilde{I}_{n,q}$, $I^1_{n,q}$, $I^2_{n,q}$, and $I^3_{n,q}$ be the sets introduced in §§5.1.2,5.1.1. Let $i \in \tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets *corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k}w_{n,q,i}$. Then for $i \in I^1_{n,q} \cup I^2_{n,q} \cup I^3_{n,q}$,*

(1)

$$\int \phi w \chi_{A^+_{n,q,i}} d\mu^+_{v_{n,q,i},U} = \int \phi w d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v>0\}\}} \lfloor \overline{A^+_{n,q,i}}]$$
$$- \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathit{div}(\phi w U) \chi_{\{v_{n,q,i}>0\}} \chi_{A^+_{n,q,i}} dx d\tau.$$

(2)

$$\int \phi w \chi_{A^+_{n,q,i}} d\mu^-_{v_{n,q,i},U} = - \int \phi w d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v<0\}\}} \lfloor \overline{A^+_{n,q,i}}]$$
$$+ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathit{div}(\phi w U) \chi_{\{v_{n,q,i}<0\}} \chi_{A^+_{n,q,i}} dx d\tau.$$

Proof of Theorem 5.8.

The proof is obtained by an appropriate adaptation of the proof of Theorem 5.4.

1. Let n be any integer such that $2^n > 1/m_w$. Let $q \in I_n$. Let $i \in I^1_{n,q} \cup I^2_{n,q} \cup I^3_{n,q}$. Let $V(A^+_{n,q,i})$ denote an open neighborhood of $A^+_{n,q,i}$ such that $V(A^+_{n,q,i}) \cap \cup_{j \in (I^1_{n,q} \cup I^2_{n,q} \cup I^3_{n,q}) \setminus \{i\}} A^+_{n,q,j} = \emptyset$. Let φ be any Lipschitz function in $C_c(V(A^+_{n,q,i}))$. By definition of $A^+_{n,q,i}$ and φ, one obtains

$$\int \varphi \chi_{A^+_{n,q,i}} d\mu^+_{v_{n,q,i},U} = \int \varphi \chi_{\{w-(q+1)/2^n>0\}} d\mu^+_{v_{n,q,i},U} - \int \varphi \chi_{\{w-(q+1)/2^n-(1/2^n)^2>0\}} d\mu^+_{v_{n,q,i},U}. \tag{5.50}$$

Taking $(\varphi, v_{n,q,i}, w-(q+1)/2^n, U)$ of this step in place of (φ, v, w, U) in the first equality in Part *(1)* of Theorem 4.4 yields

$$\begin{aligned} \int \chi_{\{w-(q+1)/2^n>0\}} \varphi d\mu^+_{\{v_{n,q,i},U\}} &= - \int \varphi d\mu^+_{\{w-(q+1)/2^n,U,\{v_{n,q,i}>0\}\}} \\ &- \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathrm{div}(\varphi U) \chi_{\{v_{n,q,i}>0\}} \chi_{\{w-(q+1)/2^n>0\}} dx d\tau. \end{aligned} \tag{5.51}$$

Taking $(\varphi, v_{n,q,i}, w-(q+1)/2^n-(1/2^n)^2, U)$ of this step in place of (φ, v, w, U) in the first equality in Part *(1)* of Theorem 4.4 yields

$$\begin{aligned} \int \chi_{\{w-(q+1)/2^n-(1/2^n)^2>0\}} \varphi d\mu^+_{\{v_{n,q,i},U\}} &= - \int \varphi d\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v_{n,q,i}>0\}\}} \\ &- \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathrm{div}(\varphi U) \chi_{\{v_{n,q,i}>0\}} \chi_{\{w-(q+1)/2^n-(1/2^n)^2>0\}} dx d\tau. \end{aligned} \tag{5.52}$$

Combining (5.50)-(5.52), one obtains

$$\int \varphi \chi_{A^+_{n,q,i}} d\mu^+_{v_{n,q,i},U} = -\int \varphi d\mu^+_{\{w-(q+1)/2^n,U,\{v_{n,q,i}>0\}\}}$$
$$+\int \varphi d\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v_{n,q,i}>0\}\}}$$
$$-\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \mathrm{div}(\varphi U)\chi_{\{v_{n,q,i}>0\}}\chi_{\{0<w-(q+1)/2^n\leq(1/2^n)^2\}}dxd\tau. \tag{5.53}$$

2. By Theorem 4.1, $\mu^+_{\{w-(q+1)/2^n,U,\{v_{n,q,i}>0\}\}}$ is concentrated on γ, where

$$\gamma = \gamma_1 \cap (\partial\{v_{n,q,i}>0\}\cup\{v_{n,q,i}>0\}),$$
$$\gamma_1 = \partial\{w-(q+1)/2^n>0\}\cap\partial\{\{w-(q+1)/2^n>0\}\cap\{v_{n,q,i}>0\}\}.$$

Since by assumption $k\neq l$, one can use (5.20) and conclude that $\partial\{w-(q+1)/2^n>0\}\cap A_{n,q,i}$ is a subset of $\{v_{n,q,i}=0\}^o$. Hence, $\gamma=\emptyset$. And so, one has

$$\int \varphi d\mu^+_{\{w-(q+1)/2^n,U,\{v_{n,q,i}>0\}\}} = 0. \tag{5.54}$$

3. By Theorem 4.1, $\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v_{n,q,i}>0\}\}}$ is concentrated on γ, where

$$\gamma = \gamma_1 \cap (\partial\{v_{n,q,i}>0\}\cup\{v_{n,q,i}>0\}),$$
$$\gamma_1 = \partial\{w-(q+1)/2^n-(1/2^n)^2>0\}\cap$$
$$\partial\{\{w-(q+1)/2^n-(1/2^n)^2>0\}\cap\{v_{n,q,i}>0\}\}.$$

Then by (5.22), $\gamma\subset\{w_{n,q,i}=w\}^o\subset\{v_{n,q,i}=v\}^o$. Then one obtains

$$\int \varphi d\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v_{n,q,i}>0\}\}} = \int \varphi d\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v>0\}\}}.$$

4. Using (5.53) of Step 1 and Steps 2-3, one obtains

$$\int \varphi \chi_{A^+_{n,q,i}} d\mu^+_{v_{n,q,i},U} = \int \varphi d\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v>0\}\}}$$
$$-\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \mathrm{div}(\varphi U)\chi_{\{v_{n,q,i}>0\}}\chi_{A^+_{n,q,i}}dxd\tau.$$

5. Since spt(w) is compact, $\overline{A^+_{n,q,i}}$ is compact. Then it is clear that one can select φ a Lipschitz function in $C_c(V(A^+_{n,q,i}))$ such that $\varphi\equiv\phi w$ on $\overline{A^+_{n,q,i}}$. Then using Step 4, one obtains

$$\int \phi w \chi_{A^+_{n,q,i}} d\mu^+_{v_{n,q,i},U} = \int \phi w d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v>0\}\}}\lfloor\overline{A^+_{n,q,i}}]$$
$$-\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \mathrm{div}(\phi w U)\chi_{\{v_{n,q,i}>0\}}\chi_{A^+_{n,q,i}}dxd\tau.$$

This completes the proof of Part *(1)* of Theorem 5.8.

6. Proceeding as in Steps 1-5 for $\mu^-_{v_{n,q,i},U}$ with appropriate adaptations, one obtains Part *(2)* of Theorem 5.8. ∎

5.4.2. *Basic properties of* $\int \phi(w-(q+1)/2^n)\chi_{A^+_{n,q,i}} d\mu^+_{v_{n,q},U}$ *and* $\int \phi(w-(q+1)/2^n)\chi_{A^+_{n,q,i}} d\mu^-_{v_{n,q},U}$.

Theorem 5.9. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ be such that $0 \leq k < l$. Set $\tilde{v} = D^l\varpi$ and $\tilde{w} = D^k\varpi$. Let $\phi \in C_c(t_1,t_2)$. Set $K = spt_x(\tilde{w}) \times spt(\phi)$. Let $\tilde{\phi} \in C_c^\infty(t_1,t_2)$ be such that $\tilde{\phi} \equiv 1$ on spt(ϕ) and $0 \leq \tilde{\phi} \leq 1$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Assume that w satisfies: $0 \leq w \leq 1$.*

Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0,1,2,\cdots,2^n-1\}$. Let $q \in I_n$. Let $\tilde{I}_{n,q}$, $I^1_{n,q}$, $I^2_{n,q}$, and $I^3_{n,q}$ be the sets introduced in §§5.1.2,5.1.1. Let $i \in \tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k}w_{n,q,i}$. Then for $i \in I^1_{n,q} \cup I^2_{n,q} \cup I^3_{n,q}$,

(1)

$$\int \phi(w-(q+1)/2^n)\chi_{A^+_{n,q,i}} d\mu^+_{v_{n,q,i},U} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \text{div}(\phi w U)\chi_{\{v_{n,q,i}>0\}}\chi_{A^+_{n,q,i}} dx d\tau$$
$$+\int \phi(w-(q+1)/2^n) d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v>0\}\}} \lfloor \overline{A^+_{n,q,i}}].$$

(2)

$$\int \phi(w-(q+1)/2^n)\chi_{A^+_{n,q,i}} d\mu^-_{v_{n,q,i},U} = \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \text{div}(\phi w U)\chi_{\{v_{n,q,i}<0\}}\chi_{A^+_{n,q,i}} dx d\tau$$
$$-\int \phi(w-(q+1)/2^n) d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v<0\}\}} \lfloor \overline{A^+_{n,q,i}}].$$

Proof of Theorem 5.9.

Let n be any integer such that $2^n > 1/m_w$. Let $q \in I_n$. let $i \in I^1_{n,q} \cup I^2_{n,q} \cup I^3_{n,q}$. Let $V(A^+_{n,q,i})$ denote an open neighborhood of $A^+_{n,q,i}$ such that $V(A^+_{n,q,i}) \cap \cup_{j\in(I^1_{n,q}\cup I^2_{n,q}\cup I^3_{n,q})\setminus\{i\}} A^+_{n,q,j} = \emptyset$. Since spt$(w)$ is compact, $\overline{A^+_{n,q,i}}$ is compact. Then it is clear that one can select φ a Lipschitz function in $C_c(V(A^+_{n,q,i}))$ such that $\varphi \equiv \phi(w-(q+1)/2^n)$ on $\overline{A^+_{n,q,i}}$. Then proceeding as in the proof of Theorem 5.8, one obtains

$$\int \phi(w-(q+1)/2^n)\chi_{A^+_{n,q,i}} d\mu^+_{v_{n,q,i},U} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \text{div}(\phi w U)\chi_{\{v_{n,q,i}>0\}}\chi_{A^+_{n,q,i}} dx d\tau$$
$$+\int \phi(w-(q+1)/2^n) d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v>0\}\}} \lfloor \overline{A^+_{n,q,i}}].$$

Here, a use of the fact that div $U = 0$, has been made. This completes the proof of Part *(1)* of Theorem 5.9.

Proceeding as above for $\mu^-_{v_{n,q},U}$ with appropriate adaptations, one obtains Part *(2)* of Theorem 5.9. ∎

5.4.3. *Basic properties of* $\int \phi w \chi_{A^+_{n,q,i}} d\mu^+_{v,U}$ *and* $\int \phi w \chi_{A^+_{n,q,i}} d\mu^-_{v,U}$.

Theorem 5.10. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R})$ be such that the projection of its support* *into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ be such that $0 \leq k < l$. Set $\tilde{v} = D^l \varpi$ and $\tilde{w} = D^k \varpi$. Let $\phi \in C_c(t_1, t_2)$. Set $K = spt_x(\tilde{w}) \times spt(\phi)$. Let $\tilde{\phi} \in C_c^\infty(t_1, t_2)$ be such that $\tilde{\phi} \equiv 1$ on spt(ϕ) and $0 \leq \tilde{\phi} \leq 1$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Assume that w satisfies: $0 \leq w \leq 1$.*

Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0, 1, 2, \cdots, 2^n - 1\}$. Let $q \in I_n$. Let $\tilde{I}_{n,q}$, $I^1_{n,q}$, $I^2_{n,q}$, and $I^3_{n,q}$ be the sets introduced in §§5.1.2,5.1.1. Let $i \in \tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k} w_{n,q,i}$. Then for $i \in I^1_{n,q} \cup I^2_{n,q} \cup I^3_{n,q}$,

(1)

$$\int \phi w \chi_{A^+_{n,q,i}} d\mu^+_{v,U} = -\int \phi w d[\mu^+_{\{w-(q+1)/2^n, U, \{v>0\}\}} \lfloor \overline{A^+_{n,q,i}}]$$
$$+ \int \phi w d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2, U, \{v>0\}\}} \lfloor \overline{A^+_{n,q,i}}] - \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathit{div}(\phi w U) \chi_{\{v>0\}} \chi_{A^+_{n,q,i}} dx d\tau.$$

(2)

$$\int \phi w \chi_{A^+_{n,q,i}} d\mu^-_{v,U} = \int \phi w d[\mu^+_{\{w-(q+1)/2^n, U, \{v<0\}\}} \lfloor \overline{A^+_{n,q,i}}]$$
$$- \int \phi w d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2, U, \{v<0\}\}} \lfloor \overline{A^+_{n,q,i}}] + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathit{div}(\phi w U) \chi_{\{v<0\}} \chi_{A^+_{n,q,i}} dx d\tau.$$

Proof of Theorem 5.10.

The proof is obtained by an appropriate adaptation of the proof of Theorem 5.4.

1. Let n be any integer such that $2^n > 1/m_w$. Let $q \in I_n$. Let $i \in I^1_{n,q} \cup I^2_{n,q} \cup I^3_{n,q}$. Let $V(A^+_{n,q,i})$ denote an open neighborhood of $A^+_{n,q,i}$ such that $V(A^+_{n,q,i}) \cap \cup_{j \in (I^1_{n,q} \cup I^2_{n,q} \cup I^3_{n,q}) \setminus \{i\}} A^+_{n,q,j} = \emptyset$. Let φ be any Lipschitz function in $C_c(V(A^+_{n,q,i}))$. By definition of $A^+_{n,q,i}$ and φ, one obtains

$$\int \varphi \chi_{A^+_{n,q,i}} d\mu^+_{v,U} = \int \varphi \chi_{\{w-(q+1)/2^n>0\}} d\mu^+_{v,U} - \int \varphi \chi_{\{w-(q+1)/2^n-(1/2^n)^2>0\}} d\mu^+_{v,U}.$$

Then proceeding as in Step 1 of the proof of Theorem 5.8, one obtains

$$\int \varphi \chi_{A^+_{n,q,i}} d\mu^+_{v,U} = -\int \varphi d\mu^+_{\{w-(q+1)/2^n, U, \{v>0\}\}} + \int \varphi d\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2, U, \{v>0\}\}}$$
$$- \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathrm{div}(\varphi U) \chi_{\{v>0\}} \chi_{A^+_{n,q,i}} dx d\tau. \tag{5.55}$$

2. Since spt(w) is compact, $\overline{A^+_{n,q,i}}$ is compact. Then it is clear that one can select φ a Lipschitz function in $C_c(V(A^+_{n,q,i}))$ such that $\varphi \equiv \phi w$ on $\overline{A^+_{n,q,i}}$. Then plugging this function in (5.55) above, one obtains

$$\int \phi w \chi_{A^+_{n,q,i}} d\mu^+_{v,U} = -\int \phi w d[\mu^+_{\{w-(q+1)/2^n, U, \{v>0\}\}} \lfloor \overline{A^+_{n,q,i}}]$$
$$+ \int \phi w d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2, U, \{v>0\}\}} \lfloor \overline{A^+_{n,q,i}}] - \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathrm{div}(\phi w U) \chi_{\{v>0\}} \chi_{A^+_{n,q,i}} dx d\tau.$$

This completes the proof of Part *(1)* of Theorem 5.10.

3. Proceeding as in Steps 1-2 for $\mu^-_{v,U}$ with appropriate adaptations, one obtains Part *(2)* of Theorem 5.10. ∎

5.4.4. *Basic properties of* $\int \phi(w-(q+1)/2^n)\chi_{A^+_{n,q,i}} d\mu^+_{v,U}$ *and* $\int \phi(w-(q+1)/2^n)\chi_{A^+_{n,q,i}} d\mu^-_{v,U}$.

Theorem 5.11. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1,t_2); \mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1,t_2); \mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ be such that $0 \leq k < l$. Set $\tilde v = D^l \varpi$ and $\tilde w = D^k \varpi$. Let $\phi \in C_c(t_1,t_2)$. Set $K =$ spt$_x(\tilde w) \times$ spt(ϕ). Let $\tilde\phi \in C^\infty_c(t_1,t_2)$ be such that $\tilde\phi \equiv 1$ on spt(ϕ) and $0 \leq \tilde\phi \leq 1$. Set $w = \tilde\phi \tilde w$ and $v = \tilde\phi \tilde v$. Assume that w satisfies: $0 \leq w \leq 1$.*

Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0,1,2,\cdots,2^n-1\}$. Let $q \in I_n$. Let $\tilde I_{n,q}$, $I^1_{n,q}$, $I^2_{n,q}$, and $I^3_{n,q}$ be the sets introduced in §§5.1.2,5.1.1. Let $i \in \tilde I_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde A_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k} w_{n,q,i}$. Then for $i \in I^1_{n,q} \cup I^2_{n,q} \cup I^3_{n,q}$,

(1)

$$\int \phi(w-(q+1)/2^n)\chi_{A^+_{n,q,i}} d\mu^+_{v,U} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \textit{div}(\phi w U)\chi_{\{v>0\}}\chi_{A^+_{n,q,i}} dx d\tau$$
$$+\int \phi(w-(q+1)/2^n) d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v>0\}\}} \lfloor \overline{A^+_{n,q,i}}].$$

(2)

$$\int \phi(w-(q+1)/2^n)\chi_{A^+_{n,q,i}} d\mu^-_{v,U} = \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \textit{div}(\phi w U)\chi_{\{v<0\}}\chi_{A^+_{n,q,i}} dx d\tau$$
$$-\int \phi(w-(q+1)/2^n) d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v<0\}\}} \lfloor \overline{A^+_{n,q,i}}].$$

Proof of Theorem 5.11.

Let n be any integer such that $2^n > 1/m_w$. Let $q \in I_n$. let $i \in I^1_{n,q} \cup I^2_{n,q} \cup I^3_{n,q}$. Let $V(A^+_{n,q,i})$ denote an open neighborhood of $A^+_{n,q,i}$ such that $V(A^+_{n,q,i}) \cap \cup_{j\in(I^1_{n,q}\cup I^2_{n,q}\cup I^3_{n,q})\setminus\{i\}} A^+_{n,q,j} = \emptyset$. Since spt$(w)$ is compact, $\overline{A^+_{n,q,i}}$ is compact. Then it is clear that one can select φ a Lipschitz function in $C_c(V(A^+_{n,q,i}))$ such that $\varphi \equiv \phi(w-(q+1)/2^n)$ on $\overline{A^+_{n,q,i}}$. Then proceeding as in the proof of Theorem 5.10 with this φ, one obtains

$$\int \phi(w-(q+1)/2^n)\chi_{A^+_{n,q,i}} d\mu^+_{v,U} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \phi \mathrm{div}(wU)\chi_{\{v>0\}}\chi_{A^+_{n,q,i}} dx d\tau +$$
$$-\int \phi(w-(q+1)/2^n) d[\mu^+_{\{w-(q+1)/2^n,U,\{v>0\}\}} \lfloor \overline{A^+_{n,q,i}}]$$
$$+\int \phi(w-(q+1)/2^n) d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v>0\}\}} \lfloor \overline{A^+_{n,q,i}}].$$

Here a use of the fact that div $U = 0$, has been made. By Theorem 4.1, $\mu^+_{\{w-(q+1)/2^n,U,\{v>0\}\}}$ is concentrated on $\partial\{w-(q+1)/2^n > 0\}$. Hence, the second term in the right side of the above equality is 0. This completes the proof of Part *(1)* of Theorem 5.11.

3. Proceeding as in Steps 1-2 for $\mu^-_{v,U}$ with appropriate adaptations, one obtains Part *(2)* of Theorem 5.11. ∎

5.5. **Basic properties of $R_n^{(1)}$ and $R_n^{(2)}$.** Here,

$$R_n^{(1)} = \sum_{(q,i)\in I_n\times I_{n,q}} \int \phi w \chi_{A^-_{n,q,i}\cup A^+_{n,q,i}} d\mu^+_{n,q,i} \text{ and } R_n^{(2)} = \sum_{(q,i)\in I_n\times I_{n,q}} \int \phi w \chi_{A^-_{n,q,i}\cup A^+_{n,q,i}} d\mu^-_{n,q,i}.$$

The main objective of this subsection is to express the above sums in terms of sums whose limits up to a subsequence as n goes to ∞ are 0. For such an objective, the assumption that $\text{spt}_x(\tilde{w}) \setminus \text{spt}_x(U)$ is either empty or a subset of the projection of $\{\tilde{v} = 0\}^o$ into $\mathbb{R}^N$ plays a key role. In its turn, this convergence plays a key role in the proof of Theorem 5.19. In Subsection 5.9, it is proven that if $\text{spt}_x(\tilde{w}) \setminus \text{spt}_x(U)$ is neither empty nor a subset of the projection of $\{\tilde{v} = 0\}^o$ into $\mathbb{R}^N$, then the cancellations in Steps 3.1-3.2 of the proof of Theorem 5.12 do not hold. Then as a first consequence, the convergence in Theorem 5.17 does not hold and, as a second consequence, the conclusions in Theorem 5.19 do not hold.

Theorem 5.12. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1,t_2); \mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1,t_2); \mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ be such that $0 \leq k < l$. Set $\tilde{v} = D^l\varpi$ and $\tilde{w} = D^k\varpi$. Assume that spt$_x(\tilde{w}) \setminus$ spt$_x(U)$ is either empty or a subset of the projection of $\{\tilde{v} = 0\}^o$ into $\mathbb{R}^N$. Let $\phi \in C_c(t_1,t_2)$. Set $K =$ spt$_x(\tilde{w}) \times$ spt(ϕ). Let $\tilde{\phi} \in C_c^\infty(t_1,t_2)$ be such that $\tilde{\phi} \equiv 1$ on spt(ϕ) and $0 \leq \tilde{\phi} \leq 1$. Set $\mathcal{U} =$ spt$_x(U) \times$ spt$(\tilde{\phi})$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Assume that w satisfies: $0 \leq w \leq 1$. Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0,1,2,\cdots,2^n-1\}$. Let $q \in I_n$. Let $\tilde{I}_{n,q}$, $I^1_{n,q}$, $I^2_{n,q}$, $I^3_{n,q}$, and $I^4_{n,q}$ be the sets introduced in §§5.1.2,5.1.1. Let $i \in \tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k}w_{n,q,i}$. Then*

(1)

$$\sum_{j=0}^{m+1}\Big(\int \phi w \chi_{A^+_{n,q_j,i_j^0}} d\mu^+_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w \chi_{A^-_{n,q_j+s,i_j^s}\cup A^+_{n,q_j+s,i_j^s}} d\mu^+_{n,q_j+s,i_j^s}\Big) =$$

$$-\sum_{j=0}^{m+1}\Big(\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \textit{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q_j,i_j^0}>0\}})\chi_{A^+_{n,q_j,i_j^0}} dx d\tau +$$

$$\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \textit{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q_j+s,i_j^s}>0\}})(\chi_{B^-_{n,q_j+s,i_j^s}} + \chi_{A^+_{n,q_j+s,i_j^s}}) dx d\tau\Big).$$

(2)

$$\sum_{j=0}^{m+1}\Big(\int \phi w \chi_{A^+_{n,q_j,i_j^0}} d\mu^-_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w \chi_{A^-_{n,q_j+s,i_j^s}\cup A^+_{n,q_j+s,i_j^s}} d\mu^-_{n,q_j+s,i_j^s}\Big) =$$

$$\sum_{j=0}^{m+1}\Big(\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \textit{div}(\phi w U)(\chi_{\{v<0\}} - \chi_{\{v_{n,q_j,i_j^0}<0\}})\chi_{A^+_{n,q_j,i_j^0}} dx d\tau +$$

$$\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \textit{div}(\phi w U)(\chi_{\{v<0\}} - \chi_{\{v_{n,q_j+s,i_j^s}<0\}})(\chi_{B^-_{n,q_j+s,i_j^s}} + \chi_{A^+_{n,q_j+s,i_j^s}}) dx d\tau\Big).$$

Here, m was introduced in §5.1.1, q_j, r_j, and $J_{n,j}$ were introduced in §5.1.4.

Proof of Theorem 5.12. It is assumed that $\mathrm{spt}(\tilde{w}) \setminus \mathrm{spt}(U)$ is empty. The proof for the case corresponding to $\mathrm{spt}(\tilde{w}) \setminus \mathrm{spt}(U)$ being a subset of the projection of $\{\tilde{v} = 0\}^o$ into $\mathbb{R}^N$ is obtained by an appropriate adaptation of the proof given below.

1. Part *(1)* of Theorem 5.4 yields for any $i \in I^3_{n,q} \cup I^4_{n,q}$,

$$\int \phi w \chi_{A^-_{n,q,i}} d\mu^+_{v_{n,q,i},U} = \int \phi w \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U} \lfloor A^-_{n,q,i}] - \int \phi w d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v>0\}\}} \lfloor A^-_{n,q,i}] - \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathrm{div}(\phi w U) \chi_{\{v_{n,q,i}>0\}} \chi_{B^-_{n,q,i}} dx d\tau. \tag{5.56}$$

Part *(1)* of Theorem 5.6 yields for any $i \in I^3_{n,q} \cup I^4_{n,q}$,

$$\begin{aligned} \int \phi w \chi_{A^-_{n,q,i}} d\mu^+_{v,U} &= \int \phi w \chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U} \lfloor A^-_{n,q,i}] \\ &- \int \phi w d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v>0\}\}} \lfloor A^-_{n,q,i}] + \int \phi w d[\mu^+_{\{w-q/2^n,U,\{v>0\}\}} \lfloor A^-_{n,q,i}] \\ &- \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathrm{div}(\phi w U) \chi_{\{v>0\}} \chi_{B^-_{n,q,i}} dx d\tau. \end{aligned} \tag{5.57}$$

By definition of $\mu^+_{n,q,i}$ and (5.56)-(5.57), one has

$$\int \phi w \chi_{A^-_{n,q,i}} d\mu^+_{n,q,i} = \int \phi w \chi_{A^-_{n,q,i}} d\mu^+_{v,U} - \int \phi w \chi_{A^-_{n,q,i}} d\mu^+_{v_{n,q,i},U} = \int \phi w d[\mu^+_{\{w-q/2^n,U,\{v>0\}\}} \lfloor A^-_{n,q,i}] - \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathrm{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q,i}>0\}}) \chi_{B^-_{n,q,i}} dx d\tau. \tag{5.58}$$

2. Part *(1)* of Theorem 5.8 yields for any $i \in I^1_{n,q} \cup I^2_{n,q} \cup I^3_{n,q}$,

$$\begin{aligned} \int \phi w \chi_{A^+_{n,q,i}} d\mu^+_{v_{n,q,i},U} &= \int \phi w d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v>0\}\}} \lfloor \overline{A^+_{n,q,i}}] \\ &- \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathrm{div}(\phi w U) \chi_{\{v_{n,q,i}>0\}} \chi_{A^+_{n,q,i}} dx d\tau. \end{aligned} \tag{5.59}$$

Part *(1)* of Theorem 5.10 yields for any $i \in I^1_{n,q} \cup I^2_{n,q} \cup I^3_{n,q}$,

$$\int \phi w \chi_{A^+_{n,q,i}} d\mu^+_{v,U} = - \int \phi w d[\mu^+_{\{w-(q+1)/2^n,U,\{v>0\}\}} \lfloor \overline{A^+_{n,q,i}}] + \int \phi w d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v>0\}\}} \lfloor \overline{A^+_{n,q,i}}] - \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathrm{div}(\phi w U) \chi_{\{v>0\}} \chi_{A^+_{n,q,i}} dx d\tau. \tag{5.60}$$

By definition of $\mu^+_{n,q,i}$ and (5.59)-(5.60), one has

$$\begin{aligned} \int \phi w \chi_{A^+_{n,q,i}} d\mu^+_{n,q,i} &= - \int \phi w d[\mu^+_{\{w-(q+1)/2^n,U,\{v>0\}\}} \lfloor \overline{A^+_{n,q,i}}] \\ &- \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \mathrm{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q,i}>0\}}) \chi_{A^+_{n,q,i}} dx d\tau. \end{aligned} \tag{5.61}$$

3. Let $J_{n,j}$ and $\mathcal{A}_{n,j}$, $j \in \{0, \cdots, m+1\}$ be the sets introduced in §5.1.4. The following sum will be evaluated

$$\sum_{j=0}^{m+1}\Big(\int \phi w \chi_{A^+_{n,q_j,i^0_j}} d\mu^+_{n,q_j,i^0_j} + \sum_{s=1,(q_j+s,i^s_j)\in J_{n,j}}^{r_j} \int \phi w \chi_{A^-_{n,q_j+s,i^s_j}\cup A^+_{n,q_j+s,i^s_j}} d\mu^+_{n,q_j+s,i^s_j}\Big). \tag{5.62}$$

3.1 Let $j \in \{0, \cdots, m+1\}$. By construction $A_{n,q_j+1,i^1_j} \in \mathcal{A}_{n,j}$. Using (5.61) and (5.58), one obtains

$$\begin{aligned}
&\int \phi w \chi_{A^+_{n,q_j,i^0_j}} d\mu^+_{n,q_j,i^0_j} + \int \phi w \chi_{A^-_{n,q_j+1,i^1_j}} d\mu^+_{n,q_j+1,i^1_j} = \\
&-\int \phi w d[\mu^+_{\{w-(q_j+1)/2^n,U,\{v>0\}\}} \lfloor \overline{A^+_{n,q_j,i^0_j}}] \\
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \mathrm{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q_j,i^0_j}>0\}})\chi_{A^+_{n,q_j,i^0_j}} dx d\tau \\
&+\int \phi w d[\mu^+_{\{w-(q_j+1)/2^n,U,\{v>0\}\}} \lfloor A^-_{n,q_j+1,i^1_j}] \\
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \mathrm{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q_j+1,i^1_j}>0\}})\chi_{B^-_{n,q_j+1,i^1_j}} dx d\tau.
\end{aligned} \tag{5.63}$$

The measures $\mu^+_{\{w-(q_j+1)/2^n,U,\{v>0\}\}} \lfloor \overline{A^+_{n,q_j,i^0_j}}$ and $\mu^+_{\{w-(q_j+1)/2^n,U,\{v>0\}\}} \lfloor A^-_{n,q_j+1,i^1_j}$ are identically equal on $\overline{A^+_{n,q_j,i^0_j}} \cap A^-_{n,q_j+1,i^1_j}$. Since this intersection is a subset of $\partial\{w > (q_j+1)/2^n\}$ and since by Theorem 4.1, $\mu^+_{\{w-(q_j+1)/2^n,U,\{v>0\}\}}$ is concentrated on $\partial\{w > (q_j+1)/2^n\}$, one concludes that the sum of the first and third term in the right side of (5.63) is 0. Here, a use of the fact that spt$(w) \subset \mathcal{U}$ and hence, $\overline{A^+_{n,q_j,i^0_j}} \cup A^-_{n,q_j+1,i^1_j} \subset \mathcal{U}$ has been made; See §5.9.

3.2 Let $j \in \{0, \cdots, m+1\}$. Let $s \in \{2, \cdots, r_j\}$ be such that $(q_j+s, i^s_j) \in J_{n,j}$. Then there exist $j' \in \{0, \cdots, m+1\}$ and $s' \in \{1, \cdots, r_{j'}\}$ such that $(q_{j'}+s', i^{s'}_{j'}) \in J_{n,j'}$ and A_{n,q_j+s,i^s_j} and $A_{n,q_{j'}+s',i^{s'}_{j'}}$ are adjacent. For $s \in \{2, \cdots, r_j - 1\}$, this happens for $q_{j'}+s' = q_j+s-1$ and for $q_{j'}+s' = q_j+s+1$. For $s = r_j$, this happens only for $q_{j'}+s' = q_j+r_j-1$. In the first and third case, $A^+_{n,q_{j'}+s',i^{s'}_{j'}} \subset \overline{A_{n,q_j+s,i^s_j}}$. In the second case, $A^+_{n,q_j+s,i^s_j} \subset \overline{A_{n,q_{j'}+s',i^{s'}_{j'}}}$.

By the assumption on the construction of the sets $\mathcal{A}_{n,r}$; See §5.1.4, for fixed j and s as above, the corresponding pair (j', s') is unique. In the first and third case, using (5.61) and (5.58) one calculates

$$\begin{aligned}
&\int \phi w \chi_{A^-_{n,q_j+s,i^s_j}} d\mu^+_{n,q_j+s,i^s_j} + \int \phi w \chi_{A^+_{n,q_{j'}+s',i^{s'}_{j'}}} d\mu^+_{n,q_{j'}+s',i^{s'}_{j'}} = \\
&\int \phi w d[\mu^+_{\{w-(q_j+s)/2^n,U,\{v>0\}\}} \lfloor A^-_{n,q_j+s,i^s_j}] \\
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \mathrm{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q_j+s,i^s_j}>0\}})\chi_{B^-_{n,q_j+s,i^s_j}} dx d\tau \\
&-\int \phi w d[\mu^+_{\{w-(q_j+s)/2^n,U,\{v>0\}\}} \lfloor \overline{A^+_{n,q_{j'}+s',i^{s'}_{j'}}}] +
\end{aligned}$$

$$-\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q_{j'}+s',i^{s'}_{j'}}>0\}})\chi_{A^+_{n,q_{j'}+s',i^{s'}_{j'}}} dx d\tau. \tag{5.64}$$

The measures $\mu^+_{\{w-(q_j+s)/2^n,U,\{v>0\}\}}\lfloor A^-_{n,q_j+s,i^s_j}$ and $\mu^+_{\{w-(q_j+s)/2^n,U,\{v>0\}\}}\lfloor \overline{A^+_{n,q_{j'}+s',i^{s'}_{j'}}}$ are identically equal on $A^-_{n,q_j+s,i^s_j} \cap \overline{A^+_{n,q_{j'}+s',i^{s'}_{j'}}}$. Since by the above, this intersection is included in $A^-_{n,q_j+s,i^s_j} \cap \overline{A_{n,q_j+s,i^s_j}}$, this intersection is a subset of $\partial\{w > (q_j + s)/2^n\}$. Since by Theorem 4.1, $\mu^+_{\{w-(q_j+s)/2^n,U,\{v>0\}\}}$ is concentrated on $\partial\{w > (q_j + s)/2^n\}$, one concludes that the sum of the first and third term in the right side of (5.64) is 0. Here, a use of the fact that spt$(w) \subset \mathcal{U}$ and hence, $A^-_{n,q_j+s,i^s_j} \cup \overline{A^+_{n,q_{j'}+s',i^{s'}_{j'}}} \subset \mathcal{U}$ has been made; See §5.9.

In the second case, using (5.61) and (5.58) one calculates

$$\begin{aligned}
&\int \phi w \chi_{A^+_{n,q_j+s,i^s_j}} d\mu^+_{n,q_j+s,i^s_j} + \int \phi w \chi_{A^-_{n,q_{j'}+s',i^{s'}_{j'}}} d\mu^+_{n,q_{j'}+s',i^{s'}_{j'}} \\
= \ & -\int \phi w d[\mu^+_{\{w-(q_j+s+1)/2^n,U,\{v>0\}\}}\lfloor \overline{A^+_{n,q_j+s,i^s_j}}] \\
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q_j+s,i^s_j}>0\}})\chi_{A^+_{n,q_j+s,i^s_j}} dx d\tau. \\
&+\int \phi w d[\mu^+_{\{w-(q_j+s+1)/2^n,U,\{v>0\}\}}\lfloor A^-_{n,q_{j'}+s',i^{s'}_{j'}}] \\
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q_{j'}+s',i^{s'}_{j'}}>0\}})\chi_{B^-_{n,q_{j'}+s',i^{s'}_{j'}}} dx d\tau.
\end{aligned} \tag{5.65}$$

The measures $\mu^+_{\{w-(q_j+s+1)/2^n,U,\{v>0\}\}}\lfloor \overline{A^+_{n,q_j+s,i^s_j}}$ and $\mu^+_{\{w-(q_j+s+1)/2^n,U,\{v>0\}\}}\lfloor A^-_{n,q_{j'}+s',i^{s'}_{j'}}$ are identically equal on $\overline{A^+_{n,q_j+s,i^s_j}} \cap A^-_{n,q_{j'}+s',i^{s'}_{j'}}$. Since by the above, this intersection is included in $\overline{A_{n,q_{j'}+s',i^{s'}_{j'}}} \cap A^-_{n,q_{j'}+s',i^{s'}_{j'}}$, this intersection is a subset of $\partial\{w > (q_j + s + 1)/2^n\}$. Since by Theorem 4.1, $\mu^+_{\{w-(q_j+s+1)/2^n,U,\{v>0\}\}}$ is concentrated on $\partial\{w > (q_j+s+1)/2^n\}$, one concludes that the sum of the first and third term in the right side of (5.65) is 0. Here, a use of the fact that spt$(w) \subset \mathcal{U}$ and hence, $\overline{A^+_{n,q_j+s,i^s_j}} \cup A^-_{n,q_{j'}+s',i^{s'}_{j'}} \subset \mathcal{U}$ has been made; See §5.9.

3.3 Steps 3.1-3.2 show that the sum of the terms in (5.62) involving the measures with three parameters is 0. Thus, one has:

$$\begin{aligned}
&\sum_{j=0}^{m+1}\Big(\int \phi w \chi_{A^+_{n,q_j,i^0_j}} d\mu^+_{n,q_j,i^0_j} + \sum_{s=1,(q_j+s,i^s_j)\in J_{n,j}}^{r_j} \int \phi w \chi_{A^-_{n,q_j+s,i^s_j} \cup A^+_{n,q_j+s,i^s_j}} d\mu^+_{n,q_j+s,i^s_j}\Big) \\
= \ & -\sum_{j=0}^{m+1}\Big(\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q_j,i^0_j}>0\}})\chi_{A^+_{n,q_j,i^0_j}} dx d\tau \\
&+ \sum_{s=1,(q_j+s,i^s_j)\in J_{n,j}}^{r_j} \Big(\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q_j+s,i^s_j}>0\}})\chi_{B^-_{n,q_j+s,i^s_j}} dx d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v>0\}}-\chi_{\{v_{n,q_j+s,i_j^s}>0\}})\chi_{A^+_{n,q_j+s,i_j^s}}dxd\tau))\\
= \;& -\sum_{j=0}^{m+1}(\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v>0\}}-\chi_{\{v_{n,q_j,i_j^0}>0\}})\chi_{A^+_{n,q_j,i_j^0}}dxd\tau+\\
&\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j}\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v>0\}}-\chi_{\{v_{n,q_j+s,i_j^s}>0\}})(\chi_{B^-_{n,q_j+s,i_j^s}}+\chi_{A^+_{n,q_j+s,i_j^s}})\\
&\qquad dxd\tau).
\end{aligned}
$$

4. Proceeding as in Steps 1-3 for the terms involving μ^-_{n,q_j+s,i_j^s} with appropriate adaptations, one obtains Part *(2)* of Theorem 5.12. This completes the proof of Theorem 5.12. ∎

5.6. **Basic properties of $R_n^{(3)}$ and $R_n^{(4)}$.** Here,

$$
R_n^{(3)}=\sum_{(q,i)\in I_n\times I_{n,q}}\int \phi(w-q/2^n)\chi_{A^-_{n,q,i}}d\mu^+_{n,q,i},\quad R_n^{(4)}=\sum_{(q,i)\in I_n\times I_{n,q}}\int \phi(w-q/2^n)\chi_{A^-_{n,q,i}}d\mu^-_{n,q,i}.
$$

Theorem 5.13. *Let N be any integer ≥ 2. Let $0\leq t_1<t_2$. Let $U\in C^\infty(\mathbb{R}^N\times(t_1,t_2);\mathbb{R}^N)$ be such that div $U=0$. Let $\varpi\in C^\infty(\mathbb{R}^N\times(t_1,t_2);\mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k,l\in\mathbb{N}^N$ be such that $0\leq k<l$. Set $\tilde v=D^l\varpi$ and $\tilde w=D^k\varpi$. Let $\phi\in C_c(t_1,t_2)$. Set $K=spt_x(\tilde w)\times spt(\phi)$. Let $\tilde\phi\in C_c^\infty(t_1,t_2)$ be such that $\tilde\phi\equiv 1$ on $spt(\phi)$ and $0\leq\tilde\phi\leq 1$. Set $w=\tilde\phi\tilde w$ and $v=\tilde\phi\tilde v$. Assume that w satisfies: $0\leq w\leq 1$.*

Let n be any integer such that $2^n>1/m_w$, where m_w was introduced in §5.1.1. Let $I_n=\{0,1,2,\cdots,2^n-1\}$. Let $q\in I_n$. Let $\tilde I_{n,q}$, $I^1_{n,q}$, $I^2_{n,q}$, $I^3_{n,q}$, and $I^4_{n,q}$ be the sets introduced in §§5.1.2,5.1.1. Let $i\in\tilde I_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde A_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i}=L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i}=D^{l-k}w_{n,q,i}$. Then

(1)

$$
\begin{aligned}
&\sum_{j=0}^{m+1}\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j}\int \phi(w-(q_j+s)/2^n)\chi_{A^-_{n,q_j+s,i_j^s}}d\mu^+_{n,q_j+s,i_j^s}\\
=\;&\sum_{j=0}^{m+1}\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j}\int_{t_1}^{t_2}\int_{\mathbb{R}^n} div(\phi w U)(\chi_{\{v_{n,q_j+s,i_j^s}>0\}}-\chi_{\{v>0\}})\chi_{B^-_{n,q_j+s,i_j^s}}dxd\tau.
\end{aligned}
$$

(2)

$$
\begin{aligned}
&\sum_{j=0}^{m+1}(\int \phi(w-(q_j+1)/2^n)\chi_{A^+_{n,q_j,i_j^0}}d\mu^+_{n,q_j,i_j^0}+\\
&\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j}\int \phi(w-(q_j+s+1)/2^n)\chi_{A^+_{n,q_j+s,i_j^s}}d\mu^+_{n,q_j+s,i_j^s})
\end{aligned}
$$

$$= \sum_{j=0}^{m+1}\Big(\int_{t_1}^{t_2}\int_{\mathbb{R}^n} div(\phi w U)(\chi_{\{v_{n,q_j,i_j^0}>0\}} - \chi_{\{v>0\}})\chi_{A^+_{n,q_j,i_j^0}} dx d\tau$$
$$+ \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int_{t_1}^{t_2}\int_{\mathbb{R}^n} div(\phi w U)(\chi_{\{v_{n,q_j+s,i_j^s}>0\}} - \chi_{\{v>0\}})\chi_{A^+_{n,q_j+s,i_j^s}} dx d\tau\Big).$$

(3)

$$\sum_{j=0}^{m+1}\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi(w-(q_j+s)/2^n)\chi_{A^-_{n,q_j+s,i_j^s}} d\mu^-_{n,q_j+s,i_j^s}$$
$$= \sum_{j=0}^{m+1}\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int_{t_1}^{t_2}\int_{\mathbb{R}^n} div(\phi w U)(\chi_{\{v<0\}} - \chi_{\{v_{n,q_j+s,i_j^s}<0\}})\chi_{B^-_{n,q_j+s,i_j^s}} dx d\tau.$$

(4)

$$\sum_{j=0}^{m+1}\Big(\int \phi(w-(q_j+1)/2^n)\chi_{A^+_{n,q_j,i_j^0}} d\mu^-_{n,q_j,i_j^0} +$$
$$\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi(w-(q_j+s+1)/2^n)\chi_{A^+_{n,q_j+s,i_j^s}} d\mu^-_{n,q_j+s,i_j^s}\Big)$$

$$= \sum_{j=0}^{m+1}\Big(\int_{t_1}^{t_2}\int_{\mathbb{R}^n} div(\phi w U)(\chi_{\{v<0\}} - \chi_{\{v_{n,q_j,i_j^0}<0\}})\chi_{A^+_{n,q_j,i_j^0}} dx d\tau$$
$$+ \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int_{t_1}^{t_2}\int_{\mathbb{R}^n} div(\phi w U)(\chi_{\{v<0\}} - \chi_{\{v_{n,q_j+s,i_j^s}<0\}})\chi_{A^+_{n,q_j+s,i_j^s}} dx d\tau\Big).$$

Here, m was introduced in §5.1.1, q_j, r_j, and $J_{n,j}$ were introduced in §5.1.4.

Proof of Theorem 5.13. The proof of the theorem is obtained by following the proof of Theorem 5.12.

1. Theorem 5.5 yields for any $i \in I^3_{n,q} \cup I^4_{n,q}$,

$$\int \phi(w-q/2^n)\chi_{A^-_{n,q,i}} d\mu^+_{v_{n,q,i},U} = \int \phi(w-q/2^n)\chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U}\lfloor A^-_{n,q,i}]$$
$$-\int \phi(w-q/2^n) d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v>0\}\}}\lfloor A^-_{n,q,i}]$$
$$-\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \mathrm{div}(\phi w U)\chi_{\{v_{n,q,i}>0\}}\chi_{B^-_{n,q,i}} dx d\tau. \tag{5.66}$$

Theorem 5.7 yields for any $i \in I^3_{n,q} \cup I^4_{n,q}$,

$$\int \phi(w-q/2^n)\chi_{A^-_{n,q,i}} d\mu^+_{v,U} = \int \phi(w-q/2^n)\chi_{\{w-q/2^n+(1/2^n)^2=0\}} d[\mu^+_{v,U}\lfloor A^-_{n,q,i}]$$
$$-\int \phi(w-q/2^n) d[\mu^+_{\{w-q/2^n+(1/2^n)^2,U,\{v>0\}\}}\lfloor A^-_{n,q,i}] +$$

$$-\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \text{div}(\phi w U)\chi_{\{v>0\}}\chi_{B^-_{n,q,i}} dx d\tau. \tag{5.67}$$

By definition of $\mu^+_{n,q,i}$ and (5.66)-(5.67), one has

$$\begin{aligned}
&\int \phi(w-q/2^n)\chi_{A^-_{n,q,i}} d\mu^+_{n,q,i}\\
&= \int \phi(w-q/2^n)\chi_{A^-_{n,q,i}} d\mu^+_{v,U} - \int \phi(w-q/2^n)\chi_{A^-_{n,q,i}} d\mu^+_{v_{n,q,i},U}\\
&= \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \text{div}(\phi w U)(\chi_{\{v_{n,q,i}>0\}} - \chi_{\{v>0\}})\chi_{B^-_{n,q,i}} dx d\tau.
\end{aligned} \tag{5.68}$$

2. Using (5.68), one obtains

$$\begin{aligned}
&\sum_{j=0}^{m+1} \sum_{s=1,(q_j+s,i^s_j)\in J_{n,j}}^{r_j} \int \phi(w-(q_j+s)/2^n)\chi_{A^-_{n,q_j+s,i^s_j}} d\mu^+_{n,q_j+s,i^s_j}\\
= &\sum_{j=0}^{m+1} \sum_{s=1,(q_j+s,i^s_j)\in J_{n,j}}^{r_j} \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \text{div}(\phi w U)(\chi_{\{v_{n,q_j+s,i^s_j}>0\}} - \chi_{\{v>0\}})\chi_{B^-_{n,q_j+s,i^s_j}} dx d\tau.
\end{aligned}$$

This yields Part *(1)* of the theorem.

3. Part *(1)* of Theorem 5.9 for any $i \in I^1_{n,q}\cup I^2_{n,q}\cup I^3_{n,q}$,

$$\begin{aligned}
\int \phi(w-(q+1)/2^n)\chi_{A^+_{n,q,i}} d\mu^+_{v_{n,q,i},U} &= -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \text{div}(\phi w U)\chi_{\{v_{n,q,i}>0\}}\chi_{A^+_{n,q,i}} dx d\tau\\
+\int \phi(w-(q+1)/2^n) d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v>0\}\}}&\lfloor \overline{A^+_{n,q,i}}].
\end{aligned} \tag{5.69}$$

Part *(1)* of Theorem 5.11 yields for any $i \in I^1_{n,q}\cup I^2_{n,q}\cup I^3_{n,q}$,

$$\begin{aligned}
\int \phi(w-(q+1)/2^n)\chi_{A^+_{n,q,i}} d\mu^+_{v,U} &= -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \text{div}(\phi w U)\chi_{\{v>0\}}\chi_{A^+_{n,q,i}} dx d\tau\\
\int \phi(w-(q+1)/2^n) d[\mu^+_{\{w-(q+1)/2^n-(1/2^n)^2,U,\{v>0\}\}}&\lfloor \overline{A^+_{n,q,i}}].
\end{aligned} \tag{5.70}$$

By definition of $\mu^+_{n,q,i}$ and (5.69)-(5.70), one obtains

$$\begin{aligned}
&\int \phi(w-(q+1)/2^n)\chi_{A^+_{n,q,i}} d\mu^+_{n,q,i}\\
&= \int \phi(w-(q+1)/2^n)\chi_{A^+_{n,q,i}} d\mu^+_{v,U} - \int \phi(w-(q+1)/2^n)\chi_{A^+_{n,q,i}} d\mu^+_{v_{n,q,i},U}\\
&= \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \text{div}(\phi w U)(\chi_{\{v_{n,q,i}>0\}} - \chi_{\{v>0\}})\chi_{A^+_{n,q,i}} dx d\tau.
\end{aligned} \tag{5.71}$$

4. Using (5.71), one obtains

$$
\begin{aligned}
&\sum_{j=0}^{m+1}\Big(\int \phi(w-(q_j+1)/2^n)\chi_{A^+_{n,q_j,i_j^0}}d\mu^+_{n,q_j,i_j^0} + \\
&\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j}\int \phi(w-(q_j+s+1)/2^n)\chi_{A^+_{n,q_j+s,i_j^s}}d\mu^+_{n,q_j+s,i_j^s}\Big). \\
= \ &\sum_{j=0}^{m+1}\Big(\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\text{div}(\phi w U)(\chi_{\{v_{n,q_j,i_j^0}>0\}}-\chi_{\{v>0\}})\chi_{A^+_{n,q_j,i_j^0}}dxd\tau \\
&+\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j}\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\text{div}(\phi w U)(\chi_{\{v_{n,q_j+s,i_j^s}>0\}}-\chi_{\{v>0\}})\chi_{A^+_{n,q_j+s,i_j^s}}dxd\tau\Big).
\end{aligned}
$$

This yields Part *(2)* of the theorem.

5. Proceeding as in Steps 1-2 resp. Steps 3-4 for the terms involving μ^-_{n,q_j+s,i_j^s} with appropriate adaptations, one obtains Part *(3)* resp. Part *(4)* of Theorem 5.12. This completes the proof of Theorem 5.13. ∎

5.7. **Convergence properties of $R_n^{(1)}$, $R_n^{(2)}$, $R_n^{(5)}$, and $R_n^{(6)}$.** Here,

$$
\begin{aligned}
R_n^{(1)} &= \sum_{(q,i)\in I_n\times I_{n,q}}\int \phi w\chi_{A^-_{n,q,i}\cup A^+_{n,q,i}}d\mu^+_{n,q,i}, \ R_n^{(2)} = \sum_{(q,i)\in I_n\times I_{n,q}}\int \phi w\chi_{A^-_{n,q,i}\cup A^+_{n,q,i}}d\mu^-_{n,q,i}, \\
R_n^{(5)} &= \sum_{(q,i)\in I_n\times I_{n,q}}\int \phi(w-q/2^n)\chi_{A^-_{n,q,i}\cup A^+_{n,q,i}}d\mu^+_{n,q,i}, \\
R_n^{(6)} &= \sum_{(q,i)\in I_n\times I_{n,q}}\int \phi(w-q/2^n)\chi_{A^-_{n,q,i}\cup A^+_{n,q,i}}d\mu^-_{n,q,i}.
\end{aligned}
$$

The main theorem of this section is Theorem 5.17.

5.7.1. *Convergence in measure of some sequences of functions involving the sets $A^-_{n,q,i}$ and $A^+_{n,q,i}$.* The results of Theorem 5.14 will be used to prove the convergence in Theorem 5.17 below.

Theorem 5.14. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N\times(t_1,t_2);\mathbb{R}^N)$ be such that div $U=0$. Let $\varpi \in C^\infty(\mathbb{R}^N\times(t_1,t_2);\mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k,l\in\mathbb{N}^N$ be such that $0\leq k<l$. Set $\tilde{v}=D^l\varpi$ and $\tilde{w}=D^k\varpi$. Let $\phi\in C_c(t_1,t_2)$. Set $K=$ spt$_x(\tilde{w})\times$ spt(ϕ). Let $\tilde{\phi}\in C_c^\infty(t_1,t_2)$ be such that $\tilde{\phi}\equiv 1$ on spt(ϕ) and $0\leq\tilde{\phi}\leq 1$. Set $w=\tilde{\phi}\tilde{w}$ and $v=\tilde{\phi}\tilde{v}$. Assume that w satisfies: $0\leq w\leq 1$.*

Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n=\{0,1,2,\cdots,2^n-1\}$. Let $q\in I_n$. Let $\tilde{I}_{n,q}$, $I^1_{n,q}$, $I^2_{n,q}$, $I^3_{n,q}$, and $I^4_{n,q}$ be the sets introduced in §§5.1.2,5.1.1. Let $i\in\tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i}=L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i}=D^{l-k}w_{n,q,i}$. Set

$$
B^-_n=\cup_{j=0}^{m+1}\cup_{s=1}^{r_j}B^-_{n,q_j+s,i_j^s} \quad \textit{and} \quad A^+_n=\cup_{j=0}^{m+1}\cup_{s=0}^{r_j-1}A^+_{n,q_j+s,i_j^s}.
$$

Here, m was introduced in §5.1.1, q_j, r_j, and $J_{n,j}$ were introduced in §5.1.4. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be the open sets associated with w introduced in §5.1.1.

(1) Let φ be any Lipschitz continuous function such that $\nabla\varphi$ is a.e. 0 on $\mathcal{W}_2^c$. Then, up to a subsequence, as n goes to ∞, ϕdiv$(\varphi U)\chi_{B_n^-}$ converges to 0 in measure in $\mathcal{W}_2$.

(2) Let φ be any Lipschitz continuous function such that $\nabla\varphi$ is a.e. 0 on $\mathcal{W}_2^c$. Then, up to a subsequence, as n goes to ∞, ϕdiv$(\varphi U)\chi_{A_n^+}$ converges to 0 in measure in $\mathcal{W}_2$.

Remark 5.15. Let $Q = \{q/2^p : \ q \in I_p, \ p \in \mathbb{N}\}$. Let $c_{i'}$ be any number in $\{c_j : j = 1, \cdots, m_0\} \cap Q$, where m_0 was introduced in 5.1.1. Assume that $\{w = c_{i'}\}^o$ is not empty and that ϕU is not identically 0 on $\{w = c_{i'}\}^o$. Let $y_0 \in \{w = c_{i'}\}^o$. Let $B(y_0, \eta)$ be a ball centered at y_0 and of radius $\eta > 0$ so small that $B(y_0, \eta) \subset \{w = c_{i'}\}^o$. Let φ be any Lipschitz function in $C_c(B(y_0, \eta))$. Since $c_{i'} \in Q$, $c_{i'} = r'_p/2^p$ for some integer $p \geq 1$ and $r'_p \in I_p$. Let n be any integer $> p$. Set $r'_n = 2^n r'_p/2^p$. Then using the fact that $c_{i'} > 0$, $r'_n \in I_n$. Now

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{\mathbb{R}^N} |\phi \text{div}(\varphi U)|\chi_{B_n^-} dy = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} |\phi \text{div}(\varphi U)|\chi_{A^-_{n,r'_n,i^s_j}} dy\\ &= \int_{t_1}^{t_2}\int_{\mathbb{R}^N} |\phi \text{div}(\varphi U)| dy,\end{aligned}$$

for some $j \in \{0, \cdots, m_w\}$ and $i^s_j \in \{1, \cdots, r_j\}$. Then one can select φ such that $|\phi\text{div}(\varphi U)| > \delta > 0$ on some open subset of $B(y_0, \eta)$ for some $\delta > 0$. This shows that ϕdiv$(\varphi U)\chi_{B_n^-}$ does not converge to 0 in measure.

Proof of Theorem 5.14.

Let φ be any Lipschitz continuous function such that $\nabla\varphi$ is a.e. 0 on $\mathcal{W}_2^c$. If $\varphi = 0$ the conclusions of the theorem are clearly true. Therefore, it is assumed that φ is not identically 0 on $\mathcal{W}_2$. It will be proved that, up to a subsequence, as n goes to ∞, ϕdiv$(\varphi U)\chi_{B_n^-}$ converges to 0 in measure in $\mathcal{W}_2$. Set $h = \phi$div(φU).

1. Here, it is assumed that $\mathcal{W}_1 = \emptyset$. Recall that r_j depends on n. By definition $B^-_{n,q,i} = A^-_{n,q,i} \cap \{q/2^n - (1/2^n)^2 < w \leq q/2^n\}$; See Theorem 5.4. Hence, $B_n^- = \cup_{j=0}^{m+1} \cup_{s=1}^{r_j} A^-_{n,q_j+s,i^s_j} \cap \{(q_j+s)/2^n - (1/2^n)^2 < w \leq (q_j+s)/2^n\}$. By the properties of the reciprocal function w^{-1}, one has:

$$\begin{aligned}B_n^- &\subset \cup_{j=0}^{m+1} \cup_{s=1}^{r_j} \{(q_j+s)/2^n - (1/2^n)^2 < w \leq (q_j+s)/2^n\}\\ &= w^{-1}(\cup_{j=0}^{m+1} \cup_{s=1}^{r_j} ((q_j+s)/2^n - (1/2^n)^2, (q_j+s)/2^n]).\end{aligned} \tag{5.72}$$

Now, one has:

$$\begin{aligned}&\mathcal{L}^1(\cup_{j=0}^{m+1} \cup_{s=1}^{r_j} ((q_j+s)/2^n - (1/2^n)^2, (q_j+s)/2^n])\\ &\leq \sum_{j=0}^{m+1}\sum_{s=1}^{r_j}(1/2^n)^2 \leq (m+2)2^n(1/2^n)^2 = (m+1)(1/2^n).\end{aligned} \tag{5.73}$$

Thus, this term converges to 0 as n goes to ∞. Then using the regularity of w, the fact that $\mathcal{W}_1 = \emptyset$, and the properties of $\mathcal{L}^{N+1}$, one deduces that

$$\mathcal{L}^{N+1}(\cup_{j=0}^{m+1} \cup_{s=1}^{r_j} \{(q_j+s)/2^n - (1/2^n)^2 < w \leq (q_j+s)/2^n\}) \to 0$$

as n goes to ∞. Therefore, using the fact that div $U = 0$ and $\nabla\varphi$ is a.e. 0 on $\mathcal{W}_2^c$, one obtains for any $\delta > 0$,

$$\lim_{n\to\infty} \mathcal{L}^{N+1}(\{y \in \mathcal{W}_2 : \ (|h|\chi_{B_n^-})(y) \geq \delta\}) = 0.$$

That is; as n goes to ∞, ϕdiv$(\varphi U)\chi_{B_n^-}$ converges to 0 in measure in $\mathcal{W}_2$.

2. Here, it is assumed that $\mathcal{W}_1 \neq \emptyset$. Let N_w^1 denote the subset of $\{1, \cdots, m_0\}$ such for $i' \in N_w^1$, $c_{i'} = m_{p_{i'}}/2^{p_{i'}}$ for some fixed nonnegative integer $p_{i'}$ and $m_{p_{i'}} \in I_{p_{i'}}$. Let $N_w^2 = \{1, \cdots, m_0\} \setminus N_w^1$. For n large, let $L_{n,j}^1$ denote the subset of $\{1, \cdots, r_j\}$ such that for $s \in L_{n,j}^1$, $\{(q_j + s)/2^n - (1/2^n)^2 < w \leq (q_j + s)/2^n\} \cap \mathcal{W}_1 = \emptyset$. Let $L_{n,j}^2$ denote the subset of $\{1, \cdots, r_j\}$ such that for $s \in L_{n,j}^2$, there corresponds $i' \in N_w^1$ such that $(q_j + s)/2^n = c_{i'}$. Let $L_{n,j}^3$ denote the subset of $\{1, \cdots, r_j\}$ such that for $s \in L_{n,j}^3$, there corresponds $i' \in N_w^2$ such that $\{(q_j + s)/2^n - (1/2^n)^2 < w < (q_j + s)/2^n\} \cap \mathcal{W}_1 = \{w = c_{i'}\}$. In this case, $(q_j + s)/2^n$ converges to $c_{i'}$.

2.1 Proceeding as in Step 1 with appropriate adaptations, one obtains as n goes to ∞,

$$\mathcal{L}^{N+1}(\cup_{j=0}^{m+1} \cup_{s\in L_{n,j}^1} \{(q_j + s)/2^n - (1/2^n)^2 < w \leq (q_j + s)/2^n\}) \to 0$$

2.2 One has

$$\begin{aligned}
&\mathcal{L}^{N+1}(\cup_{j=0}^{m+1} \cup_{s\in L_{n,j}^2} \{(q_j + s)/2^n - (1/2^n)^2 < w \leq (q_j + s)/2^n\}) \\
&= \mathcal{L}^{N+1}(\cup_{j=0}^{m+1} \cup_{i'\in N_w^1} \{c_{i'} - (1/2^n)^2 < w \leq c_{i'}\}).
\end{aligned}$$

Then by dominated convergence theorem, one obtains as n goes to ∞,

$$\begin{aligned}
&\mathcal{L}^{N+1}(\cup_{j=0}^{m+1} \cup_{s\in L_{n,j}^2} \{(q_j + s)/2^n - (1/2^n)^2 < w \leq (q_j + s)/2^n\}) \to \\
&\mathcal{L}^{N+1}(\cup_{j=0}^{m+1} \cup_{i'\in N_w^1} \{w = c_{i'}\}).
\end{aligned}$$

2.3 Using the fact that the number of $c_{i'}$ is finite and dominated convergence theorem, one obtains as n goes to ∞,

$$\begin{aligned}
&\mathcal{L}^{N+1}(\cup_{j=0}^{m+1} \cup_{s\in L_{n,j}^3} \{(q_j + s)/2^n - (1/2^n)^2 < w \leq (q_j + s)/2^n\}) \\
&\to \mathcal{L}^{N+1}(\cup_{j=0}^{m+1} \cup_{i'\in N_w^2} \{w = c_{i'}\}).
\end{aligned}$$

2.4 Combining Steps 2.1-2.3, one obtains as n goes to ∞,

$$\begin{aligned}
&\mathcal{L}^{N+1}(\cup_{j=0}^{m+1} \cup_{s\in \{1,\cdots,r_j\}} \{(q_j + s)/2^n - (1/2^n)^2 < w \leq (q_j + s)/2^n\}) \to \\
&\mathcal{L}^{N+1}(\cup_{j=0}^{m+1} \cup_{i'\in \{1,\cdots,m_0\}} \{w = c_{i'}\}).
\end{aligned}$$

Therefore, using the fact that div $U = 0$ and $\nabla\varphi$ is a.e. 0 on $\mathcal{W}_2^c$, one obtains for any $\delta > 0$,

$$\lim_{n\to\infty} \mathcal{L}^{N+1}(\{y \in \mathcal{W}_2 : \ (|h|\chi_{B_n^-})(y) \geq \delta\}) = 0.$$

That is; as n goes to ∞, ϕdiv$(\varphi U)\chi_{B_n^-}$ converges to 0 in measure in $\mathcal{W}_2$.

3. Proceeding as in Steps 1-2 above with appropriate adaptations, one obtains the proof of Part *(2)* of the theorem. This completes the proof of Theorem 5.14. ∎

5.7.2. *Convergence of terms involving* $\chi_{A^+_{n,q,i}}$.

Theorem 5.16. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ be such that $0 \leq k < l$. Set $\tilde{v} = D^l\varpi$ and $\tilde{w} = D^k\varpi$. Let $\phi \in C_c(t_1,t_2)$. Set $K = spt_x(\tilde{w}) \times spt(\phi)$. Let $\tilde{\phi} \in C_c^\infty(t_1,t_2)$ be such that $\tilde{\phi} \equiv 1$ on spt(ϕ) and $0 \leq \tilde{\phi} \leq 1$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Assume that w satisfies: $0 \leq w \leq 1$.*

Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0,1,2,\cdots,2^n-1\}$. Let $q \in I_n$. Let $\tilde{I}_{n,q}$, $I^1_{n,q}$, $I^2_{n,q}$, and $I^3_{n,q}$ be the sets introduced in §§5.1.2,5.1.1. Let $i \in \tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k}w_{n,q,i}$. Then as n goes to ∞,

$$\frac{1}{2^n}\sum_{j=0}^{m+1}\Big(\int \phi\chi_{A^+_{n,q_j,i^0_j}}d\mu^+_{n,q_j,i^0_j} + \sum_{s=1,(q_j+s,i^s_j)\in J_{n,j}}^{r_j}\int \phi\chi_{A^+_{n,q_j+s,i^s_j}}d\mu^+_{n,q_j+s,i^s_j}\Big) \to 0,$$

$$\frac{1}{2^n}\sum_{j=0}^{m+1}\Big(\int \phi\chi_{A^+_{n,q_j,i^0_j}}d\mu^-_{n,q_j,i^0_j} + \sum_{s=1,(q_j+s,i^s_j)\in J_{n,j}}^{r_j}\int \phi\chi_{A^+_{n,q_j+s,i^s_j}}d\mu^-_{n,q_j+s,i^s_j}\Big) \to 0.$$

Here, m was introduced in §5.1.1, q_j, r_j, and $J_{n,j}$ were introduced in §5.1.4.

Proof of Theorem 5.16. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be the open sets associated with w introduced in §5.1.1. In Steps 1-4, it is assumed that $\mathcal{W}_1 = \emptyset$. The proof for the case $\mathcal{W}_1 \neq \emptyset$ is obtained in Step 5.

1. Let $L_{n,j}$ denote the set of $s \in \{1,\cdots,r_j\}$ such that $(q_j+s, i^s_j) \in J_{n,j}$. The following will be evaluated

$$\sum_{j=0}^{m+1}\Big(\int \phi\chi_{A^+_{n,q_j,i^0_j}}d\mu^+_{n,q_j,i^0_j} + \sum_{s\in L_{n,j}}\int \phi\chi_{A^+_{n,q_j+s,i^s_j}}d\mu^+_{n,q_j+s,i^s_j}\Big). \tag{5.74}$$

Let $j \in \{0,\cdots,m+1\}$. Let $s \in L_{n,j}\cup\{0\}$. If A^+_{n,q_j+s,i^s_j} is empty its contribution to the integrals above will be 0. Assume now that A^+_{n,q_j+s,i^s_j} is not empty. By definition of A^+_{n,q_j+s,i^s_j} and the fact that by Theorem 5.1, the measure μ^+_{n,q_j+s,i^s_j} is concentrated on the set $\tilde{A}_{n,q_j+s,i^s_j} \cup A^+_{n,q_j+s,i^s_j}$ if $i^s_j \in I^2_{n,q_j+s}$ and $A^-_{n,q_j+s,i^s_j} \cup A_{n,q_j+s,i^s_j} \cup A^+_{n,q_j+s,i^s_j}$ if $i^s_j \in \tilde{I}_{n,q_j+s}\setminus I^2_{n,q_j+s}$, one has

$$\int \phi\chi_{A^+_{n,q_j+s,i^s_j}}d\mu^+_{n,q_j+s,i^s_j} = \int \phi(\chi_{\{w>(q_j+s+1)/2^n\}} - \chi_{\{w>(q_j+s+1)/2^n+(1/2^n)^2\}})d\mu^+_{n,q_j+s,i^s_j}. \tag{5.75}$$

By the regularity of w and the fact that spt(w) is compact, $\chi_{\{w>(q_j+s+1)/2^n\}}$ and $\chi_{\{w>(q_j+s+1)/2^n+(1/2^n)^2\}}$ are in $L^1(\mathbb{R}^N\times(t_1,t_2))$. By Part *(1)* of Theorem 3.5, $D\chi_{\{w>(q_j+s+1)/2^n\}}$ and $D\chi_{\{w>(q_j+s+1)/2^n+(1/2^n)^2\}}$ are finite Radon measures in $\mathbb{R}^N\times(t_1,t_2)$. Hence, one deduces that $\chi_{\{w>(q_j+s+1)/2^n\}}$ and $\chi_{\{w>(q_j+s+1)/2^n+(1/2^n)^2\}}$ are in $BV(\mathbb{R}^N\times(t_1,t_2))$. Therefore, one can apply Part *(2)* of Theorem 3.6 with (v,U,φ) replaced with $(v,U,\phi\chi_{\{w>(q_j+s+1)/2^n\}})$ resp.

$(v_{n,q_j+s,i_j^s}, U, \phi\chi_{\{w>(q_j+s+1)/2^n\}})$, $(v, U, \phi\chi_{\{w>(q_j+s+1)/2^n+(1/2^n)^2\}})$, and $(v_{n,q_j+s,i_j^s}, U, \phi\chi_{\{w>(q_j+s+1)/2^n+(1/2^n)^2\}})$ and obtain resp.

$$\int \phi\chi_{\{w>(q_j+s+1)/2^n\}} d(\mu^+_{\{v,U\}} - \mu^+_{v_{n,q_j+s,i_j^s},U}) = -\sum_{i=1}^{N}\int \phi U_i(\chi_{\{v>0\}\cup\partial\{v>0\}} - \chi_{\{v_{n,q_j+s,i_j^s}>0\}\cup\partial\{v_{n,q_j+s,i_j^s}>0\}})d[D_i\chi_{\{w>(q_j+s+1)/2^n\}}], \tag{5.76}$$

and

$$\int \phi\chi_{\{w>(q_j+s+1)/2^n+(1/2^n)^2\}} d(\mu^+_{\{v,U\}} - \mu^+_{v_{n,q_j+s,i_j^s},U}) = -\sum_{i=1}^{N}\int \phi U_i(\chi_{\{v>0\}\cup\partial\{v>0\}} - \chi_{\{v_{n,q_j+s,i_j^s}>0\}\cup\partial\{v_{n,q_j+s,i_j^s}>0\}})d[D_i\chi_{\{w>(q_j+s+1)/2^n+(1/2^n)^2\}}]. \tag{5.77}$$

Here, a use of the fact that div $U = 0$ has been made. (5.75)-(5.77) yield

$$\int \phi\chi_{A^+_{n,q_j+s,i_j^s}} d\mu^+_{n,q_j+s,i_j^s} = -\sum_{i=1}^{N}\int \phi U_i(\chi_{\{v>0\}\cup\partial\{v>0\}} - \chi_{\{v_{n,q_j+s,i_j^s}>0\}\cup\partial\{v_{n,q_j+s,i_j^s}>0\}})d[D_i\chi_{A^+_{n,q_j+s,i_j^s}}]. \tag{5.78}$$

Using (5.78), one obtains

$$\begin{aligned}
&\sum_{j=0}^{m+1}\Big(\int \phi\chi_{A^+_{n,q_j,i_j^0}} d\mu^+_{n,q_j,i_j^0} + \sum_{s\in L_{n,j}}\int \phi\chi_{A^+_{n,q_j+s,i_j^s}} d\mu^+_{n,q_j+s,i_j^s}\Big)\\
= &-\sum_{j=0}^{m+1}\sum_{i=1}^{N}\Big(\int \phi U_i(\chi_{\{v>0\}\cup\partial\{v>0\}} - \chi_{\{v_{n,q_j,i_j^0}>0\}\cup\partial\{v_{n,q_j,i_j^0}>0\}})d[D_i\chi_{A^+_{n,q_j,i_j^0}}] +\\
&\sum_{s\in L_{n,j}}\int \phi U_i(\chi_{\{v>0\}\cup\partial\{v>0\}} - \chi_{\{v_{n,q_j+s,i_j^s}>0\}\cup\partial\{v_{n,q_j+s,i_j^s}>0\}})d[D_i\chi_{A^+_{n,q_j+s,i_j^s}}]\Big).
\end{aligned}$$

Then one has

$$\begin{aligned}
&\Big|\sum_{j=0}^{m+1}\Big(\int \phi\chi_{A^+_{n,q_j,i_j^0}} d\mu^+_{n,q_j,i_j^0} + \sum_{s\in L_{n,j}}\int \phi\chi_{A^+_{n,q_j+s,i_j^s}} d\mu^+_{n,q_j+s,i_j^s}\Big)\Big|\\
\leq\ &\sum_{i=1}^{N}\sum_{j=0}^{m+1} 2\|\phi U_i\|_{\infty,K}\Big(|D_i\chi_{A^+_{n,q_j,i_j^0}}| + \sum_{s\in L_{n,j}}|D_i\chi_{A^+_{n,q_j+s,i_j^s}}|\Big)\\
\leq\ &\Big(4\sum_{i=1}^{N}\|\phi U_i\|_{\infty,K}\Big)(m+2)2^n|D_{i'}\chi_{A^+_{n,q_{j'}+s',i_{j'}^{s'}}}|, \qquad (5.79)
\end{aligned}$$

where $|D_{i'}\chi_{A^+_{n,q_{j'}+s',i_{j'}^{s'}}}| = \max_{(i,j,s)\in\{1,\cdots,N+1\}\times\{0,\cdots,m+1\}\times(L_{n,j}\cup\{0\})}|D_i\chi_{A^+_{n,q_j+s,i_j^s}}|$.

2. By definition of A^+_{n,q_j+s,i_j^s},

$$D_i\chi_{A^+_{n,q_j+s,i^s_j}} = (D_i\chi_{\{w>(q_j+s+1)/2^n\}} - D_i\chi_{\{w>(q_j+s+1)/2^n+(1/2^n)^2\}})\lfloor\overline{A^+_{n,q_j+s,i^s_j}}. \tag{5.80}$$

By Part *(2)* of Theorem 3.5,

$$D_i\chi_{\{w>(q_j+s+1)/2^n\}} = \mu^+_{\{w-((q_j+s+1)/2^n),e_i\}}\lfloor\overline{A^+_{n,q_j+s,i^s_j}}, \tag{5.81}$$

$$D_i\chi_{\{w>(q_j+s+1)/2^n+(1/2^n)^2\}} = \mu^+_{\{w-((q_j+s+1)/2^n+(1/2^n)^2),e_i\}}\lfloor\overline{A^+_{n,q_j+s,i^s_j}}. \tag{5.82}$$

Then by Part *(2)* and *(4)* of Theorem 3.4 and (5.80)-(5.82), $D_iA^+_{n,q_j+s,i^s_j}$ is concentrated on Γ_{n,q_j+s,i^s_j} with

$$\Gamma_{n,q_j+s,i^s_j} = (\Gamma^+_{\{w-((q_j+s+1)/2^n)\}} \cup \Gamma^+_{\{w-((q_j+s+1)/2^n+(1/2^n)^2)\}}) \cap \overline{A^+_{n,q_j+s,i^s_j}},$$

where $\Gamma^+_{\{w-((q_j+s+1)/2^n)\}}$ and $\Gamma^+_{\{w-((q_j+s+1)/2^n+(1/2^n)^2)\}}$ are C^∞-hypersurfaces of $\mathbb{R}^N \times (t_1,t_2)$. Since spt$(w)$ is compact, these hypersurfaces are compact. One then deduces that $\mathcal{H}^N(\Gamma^+_{\{w-((q_j+s+1)/2^n)\}})$ and $\mathcal{H}^N(\Gamma^+_{\{w-((q_j+s+1)/2^n+(1/2^n)^2)\}})$ are finite. Moreover, $\chi_{A^+_{n,q_j+s,i^s_j}} \in L^1(\mathbb{R}^N \times (t_1,t_2))$. Therefore, $D\chi_{A^+_{n,q_j+s,i^s_j}}$ is a finite Radon measure and A^+_{n,q_j+s,i^s_j} is of finite perimeter in $\mathbb{R}^N \times (t_1,t_2)$ with $P(A^+_{n,q_j+s,i^s_j}, \mathbb{R}^N \times (t_1,t_2)) = |D\chi_{A^+_{n,q_j+s,i^s_j}}|$. Moreover, the following divergence theorem holds

$$\int_{A^+_{n,q_j+s,i^s_j}} \text{div}(\varphi)dy = \int_{\partial_\star A^+_{n,q_j+s,i^s_j}} \varphi\cdot n^+ d\mathcal{H}^N = \int_{\Gamma_{n,q_j+s,i^s_j}} \varphi\cdot n^+ d\mathcal{H}^N, \tag{5.83}$$

for all $\varphi \in C^1_c(\mathbb{R}^N \times (t_1,t_2); \mathbb{R}^{N+1})$. Here, $\partial_\star A^+_{n,q_j+s,i^s_j}$ is the essential boundary of A^+_{n,q_j+s,i^s_j}. Then using the fact that Γ_{n,q_j+s,i^s_j} is a C^∞-hypersurface of $\mathbb{R}^N \times (t_1,t_2)$, one deduces that

$$P(A^+_{n,q_j+s,i^s_j}, \mathbb{R}^N \times (t_1,t_2)) = |D\chi_{A^+_{n,q_j+s,i^s_j}}| = \mathcal{H}^N(\Gamma_{n,q_j+s,i^s_j}). \tag{5.84}$$

3. Let $i \in \{1,\cdots,N+1\}$. Let $j \in \{0,\cdots,m+1\}$. It will be proved that $|D_i\chi_{A^+_{n,q_j+s,i^s_j}}|$ goes to 0 as n goes to ∞.

By an adaptation of Step 1 of the proof of Part *(1)* of Theorem 5.14, one obtains using the fact that $\mathcal{W}_1 = \emptyset$: $\chi_{A^+_{n,q_j+s,i^s_j}}$ converges to 0 in measure. By classical measure theory, for any sequence converging in measure to some function, one can extract a subsequence converging a.e. to the same function. Thus, a subsequence of $\chi_{A^+_{n,q_j+s,i^s_j}}$ still denoted $\chi_{A^+_{n,q_j+s,i^s_j}}$ converges to 0 a.e. Then by dominated convergence theorem, one concludes that $\mathcal{L}^{N+1}(A^+_{n,q_j+s,i^s_j})$ converges to 0. Using the regularity of Γ_{n,q_j+s,i^s_j}, one deduces that the classical $N-$dimensional area of Γ_{n,q_j+s,i^s_j} converges to 0. Since by regularity, the classical $N-$dimensional area of Γ_{n,q_j+s,i^s_j} is identical to $\mathcal{H}^N(\Gamma_{n,q_j+s,i^s_j})$, one deduces that $\mathcal{H}^N(\Gamma_{n,q_j+s,i^s_j})$ converges to 0. Then using (5.84), one deduces that $P(A^+_{n,q_j+s,i^s_j}, \mathbb{R}^N \times (t_1,t_2)) = |D\chi_{A^+_{n,q_j+s,i^s_j}}|$ converges to 0.

4. Using (5.79) and Steps 2-3, one obtains as n goes to ∞,

$$\frac{1}{2^n}\sum_{j=0}^{m+1}\Big(\int \phi\chi_{A^+_{n,q_j,i^0_j}} d\mu^+_{n,q_j,i^0_j} + \sum_{s\in L_{n,j}} \int \phi\chi_{A^+_{n,q_j+s,i^s_j}} d\mu^+_{n,q_j+s,i^s_j}\Big) \to 0.$$

This completes the proof of Part *(1)* of the theorem when $\mathcal{W}_1 = \emptyset$.

5. Here, it is assumed that $\mathcal{W}_1 \neq \emptyset$. Let $L^1_{n,j}$ denote the subset of $L_{n,j} \cup \{0\}$ such that for $s \in L^1_{n,j}$, $A^+_{n,q_j+s,i^s_j} \cap \mathcal{W}_1 = \emptyset$. Let $L^2_{n,j}$ denote the subset of $L_{n,j} \cup \{0\}$ such that for $s \in L^2_{n,j}$, there corresponds a $c_{i'}$ for some $i' \in \{1, \cdots, m_0\}$ such that $A^+_{n,q_j+s,i^s_j} \cap \mathcal{W}_1 = \{c_{i'}\}$.

5.1 Proceeding as in Steps 1-4 with appropriate adaptations, one obtains as n goes to ∞,

$$\frac{1}{2^n} \sum_{j=0}^{m+1} \sum_{s \in L^1_{n,j}} \int \phi \chi_{A^+_{n,q_j+s,i^s_j}} d\mu^+_{n,q_j+s,i^s_j} \to 0.$$

5.2 Let $s \in L^2_{n,j}$. Then

$$\int \phi \chi_{A^+_{n,q_j+s,i^s_j}} d\mu^+_{n,q_j+s,i^s_j} = \int \phi \chi_{A^+_{n,q_j+s,i^s_j} \cap \{(q_j+s+1)/2^n < w < c_{i'}\}} d\mu^+_{n,q_j+s,i^s_j}$$
$$+ \int \phi \chi_{A^+_{n,q_j+s,i^s_j} \cap \{c_{i'} < w < (q_j+s+1)/2^n + (1/2^n)^2\}} d\mu^+_{n,q_j+s,i^s_j}.$$

Here, a use of the fact that on $\overline{\{w = c_{i'}\}^o}$, $\mu^+_{v,U} \equiv 0$ and on $\overline{\{w = c_{i'}\}^o} \subset \Omega^1_{n,q_j+s,i^s_j}$, $v_{n,q_j+s,i^s_j} \equiv 0$ and hence, $\mu^+_{v_{n,q_j+s,i^s_j},U} \equiv 0$; See §5.1. Then proceeding as in Steps 1-4 with appropriate adaptations, one obtains as n goes to ∞,

$$\frac{1}{2^n} \sum_{j=0}^{m+1} \sum_{s \in L^2_{n,j}} \int \phi \chi_{A^+_{n,q_j+s,i^s_j} \cap \{(q_j+s+1)/2^n < w < c_{i'}\}} d\mu^+_{n,q_j+s,i^s_j} \to 0$$

and

$$\frac{1}{2^n} \sum_{j=0}^{m+1} \sum_{s \in L^2_{n,j}} \int \phi \chi_{A^+_{n,q_j+s,i^s_j} \cap \{c_{i'} < w < (q_j+s+1)/2^n + (1/2^n)^2\}} d\mu^+_{n,q_j+s,i^s_j} \to 0.$$

5.3 Combining Steps 5.1-5.2, one obtains as n goes to ∞

$$\frac{1}{2^n} \sum_{j=0}^{m+1} \sum_{s \in L_{n,j} \cup \{0\}} \int \phi \chi_{A^+_{n,q_j+s,i^s_j}} d\mu^+_{n,q_j+s,i^s_j} \to 0.$$

This completes the proof of Part *(1)* of the theorem when $\mathcal{W}_1 \neq \emptyset$.

6. Proceeding as in Steps 1-5 with appropriate adaptations, one obtains the proof of Part *(2)* of Theorem 5.16. ∎

5.7.3. *The first convergence theorem.*

Theorem 5.17. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ be such that $0 \leq k < l$. Set $\tilde{v} = D^l \varpi$ and $\tilde{w} = D^k \varpi$. Assume that spt$_x(\tilde{w}) \setminus$ spt$_x(U)$ is either empty or a subset of the projection of $\{\tilde{v} = 0\}^o$ into $\mathbb{R}^N$. Let $\phi \in C_c(t_1, t_2)$. Set $K =$ spt$_x(\tilde{w}) \times$ spt(ϕ). Let $\tilde{\phi} \in C^\infty_c(t_1, t_2)$ be such that $\tilde{\phi} \equiv 1$ on spt(ϕ) and $0 \leq \tilde{\phi} \leq 1$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Assume that w satisfies: $0 \leq w \leq 1$.*

Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0, 1, 2, \cdots, 2^n - 1\}$. Let $q \in I_n$. Let $\tilde{I}_{n,q}$ be the set introduced in §5.1.2. Let $i \in \tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q

introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k} w_{n,q,i}$. Then, up to a subsequence, as n goes to ∞,

$$\sum_{j=0}^{m+1}\Big(\int \phi w \chi_{A^+_{n,q_j,i_j^0}} d\mu^+_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w \chi_{A^-_{n,q_j+s,i_j^s} \cup A^+_{n,q_j+s,i_j^s}} d\mu^+_{n,q_j+s,i_j^s}\Big) \to 0.$$

$$\sum_{j=0}^{m+1}\Big(\int \phi w \chi_{A^+_{n,q_j,i_j^0}} d\mu^-_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w \chi_{A^-_{n,q_j+s,i_j^s} \cup A^+_{n,q_j+s,i_j^s}} d\mu^-_{n,q_j+s,i_j^s}\Big) \to 0.$$

Here, m was introduced in §5.1.1, q_j, r_j, and $J_{n,j}$ were introduced in §5.1.4.

Proof of Theorem 5.17. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be the open sets associated with w introduced in §5.1.1.

1. Theorem 5.12 shows that

$$\begin{aligned}
&\sum_{j=0}^{m+1}\Big(\int \phi w \chi_{A^+_{n,q_j,i_j^0}} d\mu^+_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w \chi_{A^-_{n,q_j+s,i_j^s} \cup A^+_{n,q_j+s,i_j^s}} d\mu^+_{n,q_j+s,i_j^s}\Big) = \\
&-\sum_{j=0}^{m+1}\Big(\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \mathrm{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q_j,i_j^0}>0\}})\chi_{A^+_{n,q_j,i_j^0}} dx d\tau + \\
&\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \mathrm{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q_j+s,i_j^s}>0\}})(\chi_{B^-_{n,q_j+s,i_j^s}} + \chi_{A^+_{n,q_j+s,i_j^s}}) dx d\tau\Big).
\end{aligned}$$

Hence one has,

$$\begin{aligned}
&\Big|\sum_{j=0}^{m+1}\Big(\int \phi w \chi_{A^+_{n,q_j,i_j^0}} d\mu^+_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w \chi_{A^-_{n,q_j+s,i_j^s} \cup A^+_{n,q_j+s,i_j^s}} d\mu^+_{n,q_j+s,i_j^s}\Big)\Big| \\
&\le 2\int_{t_1}^{t_2}\int_{\mathbb{R}^N} |\mathrm{div}(\phi w U)|(\chi_{B_n^-} + \chi_{A_n^+}) dx d\tau \\
&\le 2\int_{t_1}^{t_2}\int_{\mathbb{R}^N} |\phi \nabla w \cdot U|(\chi_{B_n^-} + \chi_{A_n^+}) dx d\tau, \qquad (5.85)
\end{aligned}$$

where $B_n^- = \cup_{j=0}^{m+1} \cup_{s=1}^{r_j} B^-_{n,q_j+s,i_j^s}$ and $A_n^+ = \cup_{j=0}^{m+1} \cup_{s=0}^{r_j-1} A^+_{n,q_j+s,i_j^s}$. Here, a use of the fact that div $U = 0$, and so: div$(\phi w U) = \phi \nabla w \cdot U$ has been made.

2. Since $\nabla w \equiv 0$ on $\overline{\mathcal{W}_1}$, the function $\phi \nabla w \cdot U$ is identically 0 on $\mathcal{W}_2^c$. Recall that spt$(w) = \overline{\mathcal{W}_1} \cup \overline{\mathcal{W}_2}$. Therefore, one can take (ϕ, w, U) of this proof in place of (ϕ, φ, U) in Parts *(1)-(2)* of Theorem 5.14, and obtain, up to a subsequence, as n goes to ∞, div$(\phi w U)(\chi_{B_n^-} + \chi_{A_n^+}) = \phi U \cdot \nabla w(\chi_{B_n^-} + \chi_{A_n^+})$ converges to 0 in measure.

By classical measure theory, for any sequence converging in measure to some function, one can extract a subsequence converging a.e. to the same function. Therefore, using the above convergence in measure, one can extract a subsequence of div$(\phi w U)(\chi_{B_n^-} + \chi_{A_n^+})$ still denoted div$(\phi w U)(\chi_{B_n^-} + \chi_{A_n^+})$ that converges to 0 $\mathcal{L}^{N+1}$ a.e.

Now, one has, $|\phi\nabla w \cdot U|(\chi_{B_n^-} + \chi_{A_n^+})| \leq 2|\phi \text{div}(wU)|$. By the regularity of w, U and ϕ, $\phi\text{div}(wU) \in L^1(\mathbb{R}^N \times (t_1, t_2))$. Hence, by dominated convergence theorem, up to a subsequence, as n goes to ∞,

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^N} |\phi\nabla w \cdot U|(\chi_{B_n^-} + \chi_{A_n^+}) dx d\tau \to 0.$$

Then using (5.85), one deduces that, up to a subsequence, as n goes to ∞,

$$\sum_{j=0}^{m+1} (\int \phi w \chi_{A^+_{n,q_j,i_j^0}} d\mu^+_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w \chi_{A^-_{n,q_j+s,i_j^s} \cup A^+_{n,q_j+s,i_j^s}} d\mu^+_{n,q_j+s,i_j^s}) \to 0.$$

This yields the first convergence of the theorem.

3. Proceeding as in Steps 1-2 for the terms involving $\mu^-_{n,q,i}$ with appropriate adaptations, one obtains the second convergence of Theorem 5.17. ∎

5.7.4. *The second convergence theorem.*

Theorem 5.18. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k, l \in \mathbb{N}^N$ be such that $0 \leq k < l$. Set $\tilde{v} = D^l \varpi$ and $\tilde{w} = D^k \varpi$. Let $\phi \in C_c(t_1, t_2)$. Set $K = spt_x(\tilde{w}) \times spt(\phi)$. Let $\tilde{\phi} \in C_c^\infty(t_1, t_2)$ be such that $\tilde{\phi} \equiv 1$ on spt(ϕ) and $0 \leq \tilde{\phi} \leq 1$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Assume that w satisfies: $0 \leq w \leq 1$.*

Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0, 1, 2, \cdots, 2^n - 1\}$. Let $q \in I_n$. Let $\tilde{I}_{n,q}$ be the set introduced in §5.1.2. Let $i \in \tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k} w_{n,q,i}$. Then, up to a subsequence, as n goes to ∞,

$$\sum_{j=0}^{m+1} (\int \phi(w - q_j/2^n) \chi_{A^+_{n,q_j,i_j^0}} d\mu^+_{n,q_j,i_j^0} +$$
$$\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi(w - (q_j+s)/2^n) \chi_{A^-_{n,q_j+s,i_j^s} \cup A^+_{n,q_j+s,i_j^s}} d\mu^+_{n,q_j+s,i_j^s}) \to 0,$$

$$\sum_{j=0}^{m+1} (\int \phi(w - q_j/2^n) \chi_{A^+_{n,q_j,i_j^0}} d\mu^-_{n,q_j,i_j^0} +$$
$$\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi(w - (q_j+s)/2^n) \chi_{A^-_{n,q_j+s,i_j^s} \cup A^+_{n,q_j+s,i_j^s}} d\mu^-_{n,q_j+s,i_j^s}) \to 0.$$

Here, m was introduced in §5.1.1, q_j, r_j, and $J_{n,j}$ were introduced in §5.1.4.

Proof of Theorem 5.18.

1. Part *(1)* of Theorem 5.13 yields

$$
(5.86) \quad \sum_{j=0}^{m+1} \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi(w-(q_j+s)/2^n)\chi_{A^-_{n,q_j+s,i_j^s}} d\mu^+_{n,q_j+s,i_j^s}
$$

$$
= \sum_{j=0}^{m+1} \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v_{n,q_j+s,i_j^s}>0\}} - \chi_{\{v>0\}})\chi_{B^-_{n,q_j+s,i_j^s}} dx d\tau.
$$

Proceeding as in the proof of Part *(1)* of Theorem 5.17 shows that, up to a subsequence, as n goes to ∞, the right side of (5.86) converge to 0 and so does the left side.

2. Part *(2)* of Theorem 5.13 yields

$$
\sum_{j=0}^{m+1}(\int \phi(w-(q_j+1)/2^n)\chi_{A^+_{n,q_j,i_j^0}} d\mu^+_{n,q_j,i_j^0} +
$$

$$
(5.87) \quad \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi(w-(q_j+s+1)/2^n)\chi_{A^+_{n,q_j+s,i_j^s}} d\mu^+_{n,q_j+s,i_j^s})
$$

$$
= \sum_{j=0}^{m+1}(\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v_{n,q_j,i_j^0}>0\}} - \chi_{\{v>0\}})\chi_{A^+_{n,q_j,i_j^0}} dx d\tau
$$

$$
+ \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v_{n,q_j+s,i_j^s}>0\}} - \chi_{\{v>0\}})\chi_{A^+_{n,q_j+s,i_j^s}} dx d\tau).
$$

Proceeding as in the proof of Part *(1)* of Theorem 5.17 shows that, up to a subsequence, as n goes to ∞, the right side of (5.87) converge to 0 and so does the left side.

3. Combining the convergence in Step 2 above and the first convergence in Theorem 5.16, one concludes that, up to a subsequence, as n goes to ∞,

$$
\sum_{j=0}^{m+1}(\int \phi(w-q_j/2^n)\chi_{A^+_{n,q_j,i_j^0}} d\mu^+_{n,q_j,i_j^0} +
$$

$$
\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi(w-(q_j+s)/2^n)\chi_{A^+_{n,q_j+s,i_j^s}} d\mu^+_{n,q_j+s,i_j^s}) \to 0.
$$

4. Combining the convergence in Step 1 and the convergence in Step 3 yields the first convergence in the theorem.

5. Proceeding as in Steps 1-4 for the terms involving $\mu^-_{n,q,i}$ with appropriate adaptations, one obtains the second convergence in Theorem 5.18. ∎

5.8. **The main theorem.** For any function $f \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R}^N)$ denote the projection of its support into $\mathbb{R}^N$ by $\operatorname{spt}_x(f)$.

Theorem 5.19. *Let N be any integer ≥ 2. Let $0 \leq t_1 < t_2$. Let $U \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R}^N)$ be such that div $U = 0$. Let $\varpi \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R})$ be such that the projection of its support into $\mathbb{R}^N$ is compact. Let $k,l \in \mathbb{N}^N$ be such that $0 \leq k \leq l$. Set $\tilde{v} = D^l\varpi$ and $\tilde{w} = D^k\varpi$.*

Assume that $spt_x(\tilde{w}) \setminus spt_x(U)$ *is either empty or a subset of the projection of* $\{\tilde{v} = 0\}^o$ *into* $\mathbb{R}^N$. *Let* $\phi \in C_c(t_1, t_2)$. *Then*

$$\int \phi \tilde{w} d\mu^+_{\{\tilde{v},U\}} = 0, \quad \int \phi \tilde{w} d\mu^-_{\{\tilde{v},U\}} = 0, \quad \int \phi \tilde{w} d\mu_{\{\tilde{v},U\}} = 0.$$

Proof of Theorem 5.19

1. Assume that $k = l$, then $\tilde{v} = \tilde{w}$. By Theorem 3.1, $\mu_{\{\tilde{v},U\}}$, $\mu^+_{\{\tilde{v},U\}}$, and $\mu^-_{\{\tilde{v},U\}}$ are concentrated resp. on $\partial\{\tilde{v} > 0\} \cup \partial\{\tilde{v} < 0\}$, $\partial\{\tilde{v} > 0\}$, and $\partial\{\tilde{v} < 0\}$. Hence,

$$\int \phi \tilde{w} d\mu^+_{\{\tilde{v},U\}} = 0, \quad \int \phi \tilde{w} d\mu^-_{\{\tilde{v},U\}} = 0, \quad \int \phi \tilde{w} d\mu_{\{\tilde{v},U\}} = 0.$$

Thus, Theorem 5.19 is proved for the case $k = l$. Throughout the rest of the proof, it is assumed that $0 \le k < l$.

2. Let $0 \le k < l$. Set $K = \text{spt}_x(\tilde{w}) \times \text{spt}(\phi)$. Let $\tilde{\phi} \in C_c^\infty(t_1, t_2)$ be such that $\tilde{\phi} \equiv 1$ on spt(ϕ) and $0 \le \tilde{\phi} \le 1$. Set $\mathcal{U} = \text{spt}_x(U) \times \text{spt}(\tilde{\phi})$. Set $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. Then $w, v \in C_c^\infty(\mathbb{R}^N \times (t_1, t_2))$ and $v = D^{l-k}w$. Moreover, spt$(w) \subset \mathcal{U}$ and $w \equiv \tilde{w}$ on K and $v \equiv \tilde{v}$ on K.

Let φ be any Lipschitz function in $C_c(\mathbb{R}^N \times (t_1, t_2))$. By definition of $\mu^+_{\{\tilde{v},U\}}$, one has

$$\int \phi\varphi d\mu^+_{\{\tilde{v},U\}} = \lim_{\alpha \to 0} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \chi_{\{0<\tilde{v}<\alpha\}} \frac{1}{\alpha} U \cdot \nabla\tilde{v}\phi\varphi dy$$
$$= \lim_{\alpha \to 0} \int_{\{0<v<\alpha\}} \frac{1}{\alpha} U \cdot \nabla v \phi\varphi dy = \int \phi\varphi d\mu^+_{\{v,U\}}.$$

Here, a use of the fact that on K, $v \equiv \tilde{v}$ and $\tilde{\phi} \equiv 1$ on spt(ϕ) has been made. Taking φ such that $\varphi = \tilde{w}$ on K and plugging it above and then using the fact that $w \equiv \tilde{w}$ on K, one has

$$\int \phi \tilde{w} d\mu^+_{\{\tilde{v},U\}} = \int \phi w d\mu^+_{\{v,U\}}. \tag{5.88}$$

Similarly, one has

$$\int \phi \tilde{w} d\mu^-_{\{\tilde{v},U\}} = \int \phi w d\mu^-_{\{v,U\}}, \quad \int \phi \tilde{w} d\mu_{\{\tilde{v},U\}} = \int \phi w d\mu_{\{v,U\}}. \tag{5.89}$$

3. In this step and Steps 4-7, it is assumed that w satisfies: $0 \le w \le 1$. Let n be any integer such that $2^n > 1/m_w$, where m_w was introduced in §5.1.1. Let $I_n = \{0, 1, 2, \cdots, 2^n - 1\}$. Let $q \in I_n$. Let $\tilde{I}_{n,q}$, $I^1_{n,q}$, $I^2_{n,q}$, and $I^3_{n,q}$ be the sets introduced in §§5.1.2,5.1.1. Let $i \in \tilde{I}_{n,q}$. Let $A_{n,q,i}$, $A^-_{n,q,i}$, and $A^+_{n,q,i}$ resp. $\tilde{A}_{n,q,i}$ and $\Omega_{n,q,i}$ be the sets corresponding to w, n, and q introduced in §5.1.1 resp. 5.1.4 and 5.1.3. Let L be a total extension operator for $\Omega_{n,q,i}$. Let $w_{n,q,i} = L(f_{n,q,i})$, where $f_{n,q,i}$ is defined by (5.14)-(5.15). Let $v_{n,q,i} = D^{l-k}w_{n,q,i}$.

4. Here, it will be proved that, up to a subsequence, as n goes to ∞,

$$\sum_{j=0}^{m+1} \Big(\int \phi w d\mu^+_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w d\mu^+_{n,q_j+s,i_j^s} \Big) \to 0, \tag{5.90}$$

where m was introduced in §5.1.1, q_j, r_j, and $J_{n,j}$ were introduced in §5.1.4. First, one has

$$\begin{aligned}
&\sum_{j=0}^{m+1}\Big(\int \phi w d\mu^{+}_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w d\mu^{+}_{n,q_j+s,i_j^s}\Big)\\
= &\sum_{j=0}^{m+1}\Big(\int \phi(w-q_j/2^n) d\mu^{+}_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi(w-(q_j+s)/2^n) d\mu^{+}_{n,q_j+s,i_j^s}\Big) +\\
&\sum_{j=0}^{m+1}\Big(\int \phi(q_j/2^n) d\mu^{+}_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi((q_j+s)/2^n) d\mu^{+}_{n,q_j+s,i_j^s}\Big). \qquad (5.91)
\end{aligned}$$

4.1 By Part *(3)* of Theorem 5.3,

$$\sum_{j=0}^{m+1}\Big(\int \phi(q_j/2^n) d\mu^{+}_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi((q_j+s)/2^n) d\mu^{+}_{n,q_j+s,i_j^s}\Big) = 0. \qquad (5.92)$$

4.2 Using the fact that by Theorem 5.1, the measure $\mu^{+}_{n,q,i}$ is concentrated on $\tilde{A}_{n,q,i}\cup A^{+}_{n,q,i}$ if $i\in I^2_{n,q}$ and $A^{-}_{n,q,i}\cup A_{n,q,i}\cup A^{+}_{n,q,i}$ if $i\in \tilde{I}_{n,q}\setminus I^2_{n,q}$, one obtains

$$\begin{aligned}
&\sum_{j=0}^{m+1}\Big(\int \phi(w-q_j/2^n) d\mu^{+}_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi(w-(q_j+s)/2^n) d\mu^{+}_{n,q_j+s,i_j^s}\Big) =\\
&\sum_{j=0}^{m+1}\Big(\int \phi(w-q_j/2^n)\chi_{\tilde{A}_{n,q_j,i_j^0}} d\mu^{+}_{n,q_j,i_j^0} +\\
&\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi(w-(q_j+s)/2^n)\chi_{A_{n,q_j+s,i_j^s}} d\mu^{+}_{n,q_j+s,i_j^s}\Big) +\\
&\sum_{j=0}^{m+1}\Big(\int \phi(w-q_j/2^n)\chi_{A^{+}_{n,q_j,i_j^0}} d\mu^{+}_{n,q_j,i_j^0} +\\
&\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi(w-(q_j+s)/2^n)\chi_{A^{-}_{n,q_j+s,i_j^s}\cup A^{+}_{n,q_j+s,i_j^s}} d\mu^{+}_{n,q_j+s,i_j^s}\Big). \qquad (5.93)
\end{aligned}$$

By Part *(4)* of Theorem 5.3, as n goes to ∞,

$$\begin{aligned}
&\sum_{j=0}^{m+1}\Big(\int \phi(w-q_j/2^n)\chi_{\tilde{A}_{n,q_j,i_j^0}} d\mu^{+}_{n,q_j,i_j^0} +\\
&\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi(w-(q_j+s)/2^n)\chi_{A_{n,q_j+s,i_j^s}} d\mu^{+}_{n,q_j+s,i_j^s}\Big) \to 0. \qquad (5.94)
\end{aligned}$$

By Part *(1)* of Theorem 5.18, up to a subsequence, as n goes to ∞,

$$(5.95)\quad \sum_{j=0}^{m+1}\Big(\int \phi(w-q_j/2^n)\chi_{A^+_{n,q_j,i_j^0}}d\mu^+_{n,q_j,i_j^0}+ \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j}\int \phi(w-(q_j+s)/2^n)\chi_{A^-_{n,q_j+s,i_j^s}\cup A^+_{n,q_j+s,i_j^s}}d\mu^+_{n,q_j+s,i_j^s}\Big)\to 0.$$

Combining (5.94) and (5.95), and using (5.93), one obtains, up to a subsequence, as n goes to ∞,

$$(5.96)\quad \sum_{j=0}^{m+1}\Big(\int \phi(w-q_j/2^n)d\mu^+_{n,q_j,i_j^0}+\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j}\int \phi(w-(q_j+s)/2^n)d\mu^+_{n,q_j+s,i_j^s}\Big)\to 0.$$

4.3 Combining the convergence in (5.92) of Step 4.1 and (5.96) of Step 4.2, and using (5.91), one obtains, up to a subsequence, as n goes to ∞,

$$(5.97)\quad \sum_{j=0}^{m+1}\Big(\int \phi w d\mu^+_{n,q_j,i_j^0}+\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j}\int \phi w d\mu^+_{n,q_j+s,i_j^s}\Big)\to 0.$$

This corresponds to the convergence in (5.90).

5. Here, it will be proved that, up to a subsequence, as n goes to ∞,

$$(5.98)\quad \sum_{j=0}^{m+1}\Big(\int \phi w\chi_{\tilde{A}_{n,q_j,i_j^0}}d\mu^+_{n,q_j,i_j^0}+\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j}\int \phi w\chi_{A_{n,q_j+s,i_j^s}}d\mu^+_{n,q_j+s,i_j^s}\Big)\to 0.$$

Using the fact that by Theorem 5.1, the measure $\mu^+_{n,q,i}$ is concentrated on $\tilde{A}_{n,q,i}\cup A^+_{n,q,i}$ if $i\in I^2_{n,q}$ and $A^-_{n,q,i}\cup A_{n,q,i}\cup A^+_{n,q,i}$ if $i\in \tilde{I}_{n,q}\setminus I^2_{n,q}$, one obtains

$$\begin{aligned}(5.99)\quad &\sum_{j=0}^{m+1}\Big(\int \phi w d\mu^+_{n,q_j,i_j^0}+\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j}\int \phi w d\mu^+_{n,q_j+s,i_j^s}\Big)=\\ &\sum_{j=0}^{m+1}\Big(\int \phi w\chi_{\tilde{A}_{n,q_j,i_j^0}}d\mu^+_{n,q_j,i_j^0}+\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j}\int \phi w\chi_{A_{n,q_j+s,i_j^s}}d\mu^+_{n,q_j+s,i_j^s}\Big)+\\ &\sum_{j=0}^{m+1}\Big(\int \phi w\chi_{A^+_{n,q_j,i_j^0}}d\mu^+_{n,q_j,i_j^0}+\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j}\int \phi w\chi_{A^-_{n,q_j+s,i_j^s}\cup A^+_{n,q_j+s,i_j^s}}d\mu^+_{n,q_j+s,i_j^s}\Big).\end{aligned}$$

By Part *(1)* of Theorem 5.17, up to a subsequence, as n goes to ∞,

$$(5.100)\quad \sum_{j=0}^{m+1}\Big(\int \phi w\chi_{A^+_{n,q_j,i_j^0}}d\mu^+_{n,q_j,i_j^0}+\sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j}\int \phi w\chi_{A^-_{n,q_j+s,i_j^s}\cup A^+_{n,q_j+s,i_j^s}}d\mu^+_{n,q_j+s,i_j^s}\Big)\to 0.$$

Combining the convergence in (5.97) of Step 4 and the convergence in (5.100), and using (5.99), one obtains, up to a subsequence, as n goes to ∞,

$$\sum_{j=0}^{m+1}\Big(\int \phi w \chi_{\tilde{A}_{n,q_j,i_j^0}} d\mu^+_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w \chi_{A_{n,q_j+s,i_j^s}} d\mu^+_{n,q_j+s,i_j^s}\Big) \to 0. \tag{5.101}$$

This corresponds to the convergence in (5.98).

6. Using Part *(3)* of Theorem 5.2, one obtains

$$\sum_{j=0}^{m+1}\Big(\int \phi w \chi_{\tilde{A}_{n,q_j,i_j^0}} d\mu^+_{\{v_{n,q_j,i_j^0},U\}} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w \chi_{A_{n,q_j+s,i_j^s}} d\mu^+_{\{v_{n,q_j+s,i_j^s},U\}}\Big) = 0. \tag{5.102}$$

The definition of $\mu^+_{n,q,i}$ and the definition of the sets $\tilde{A}_{n,q,i}$ and $A_{n,q,i}$ and (5.102) show that

$$\begin{aligned}
&\sum_{j=0}^{m+1}\Big(\int \phi w \chi_{\tilde{A}_{n,q_j,i_j^0}} d\mu^+_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w \chi_{A_{n,q_j+s,i_j^s}} d\mu^+_{n,q_j+s,i_j^s}\Big)\\
&= \sum_{j=0}^{m+1}\Big(\int \phi w \chi_{\tilde{A}_{n,q_j,i_j^0}} d\mu^+_{v,U} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w \chi_{A_{n,q_j+s,i_j^s}} d\mu^+_{v,U}\Big)\\
&\quad - \sum_{j=0}^{m+1}\Big(\int \phi w \chi_{\tilde{A}_{n,q_j,i_j^0}} d\mu^+_{\{v_{n,q_j,i_j^0},U\}} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w \chi_{A_{n,q_j+s,i_j^s}} d\mu^+_{\{v_{n,q_j+s,i_j^s},U\}}\Big)\\
&= \sum_{j=0}^{m+1}\Big(\int \phi w \chi_{\tilde{A}_{n,q_j,i_j^0}} d\mu^+_{v,U} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w \chi_{A_{n,q_j+s,i_j^s}} d\mu^+_{v,U}\Big)\\
&= \int \phi w d\mu^+_{\{v,U\}}.
\end{aligned} \tag{5.103}$$

where the last equality was obtained thanks to (5.23) of Subsection 5.1.4.

Now by (5.103) and the convergence in (5.101) of Step 5 above, one obtains,

$$\int \phi w d\mu^+_{\{v,U\}} = 0. \tag{5.104}$$

7. Proceeding as in Steps 4-6 for the measure $\mu^-_{\{v,U\}}$ with appropriate adaptations, one obtains

$$\int w\phi d\mu^-_{\{v,U\}} = 0. \tag{5.105}$$

Then using Part *(3)* of Theorem 3.3, one obtains

$$\int w\phi d\mu_{\{v,U\}} = 0. \tag{5.106}$$

Then combining (5.104)-(5.106) and using (5.88)-(5.89), one obtains

$$\int \phi \tilde{w} d\mu^+_{\{\tilde{v},U\}} = 0, \quad \int \phi \tilde{w} d\mu^-_{\{\tilde{v},U\}} = 0, \quad \int \phi \tilde{w} d\mu_{\{\tilde{v},U\}} = 0. \tag{5.107}$$

This completes the proof of Theorem 5.19 in the case where $0 \leq k < l$ and w satisfies: $0 \leq w \leq 1$.

8. In this step and Step 9, Theorem 5.19 will be proved in the case where $0 \leq k < l$ and w such that w does not satisfy: $0 \leq w \leq 1$.

If w is nonnegative, by the regularity of w and the compactness of its support, $w(y) \leq B$ for any $y \in \mathbb{R}^N \times (t_1, t_2)$, where B is the absolute maximum. The function w/B satisfies: $0 \leq w/B \leq 1$. Then proceeding as in Steps 2-7 above with w replaced by w/B with appropriates adaptations, one obtains the proof of Theorem 5.19 in this case. Below the proof in the case where w changes sign will be given.

8.1 Recall that by Step 2 above, $w = \tilde{\phi}\tilde{w}$ and $v = \tilde{\phi}\tilde{v}$. By the regularity of w and the compactness of its support, $|w(y)| \leq B$ for any $y \in \mathbb{R}^N \times (t_1, t_2)$, where B is an absolute extremum. Then define the function $\hat{w}$ by $\hat{w} = \frac{1}{B}w$ and $\hat{v}$ by $\hat{v} = D^{l-k}\hat{w}$. This function satisfies: $-1 \leq \hat{w} \leq 1$.

Let n be any nonnegative integer. Set $I_n^- = \{-(2^n+1), -2^n, \cdots, -2, -1\}$ and $I_n^+ = \{0, 1, 2, \cdots, 2^n - 1\}$. Set $I_n = I_n^- \cup I_n^+$. Let $q \in I_n$. Let $A_{n,q}$, $A_{n,q}^-$, and $A_{n,q}^+$ be the sets defined by

$$
\begin{aligned}
A_{n,q}^- &= \{y \in \mathbb{R}^N \times (t_1, t_2) |\ q/2^n - (1/2^n)^2 \leq w(y) \leq q/2^n\} \\
A_{n,q} &= \{y \in \mathbb{R}^N \times (t_1, t_2) |\ q/2^n < w(y) \leq (q+1)/2^n\} \\
A_{n,q}^+ &= \{y \in \mathbb{R}^N \times (t_1, t_2) |\ (q+1)/2^n < w(y) \leq (q+1)/2^n + (1/2^n)^2\}
\end{aligned}
$$

By an appropriate adaptation of the constructions and methods of Subsections 5.1-5.7 and those of Steps 2-6 above with (w, v) replaced by $(\hat{w}, \hat{v})$ of this step, one obtains

$$
\int \hat{w}\phi d\mu^+_{\{\hat{v},U\}} = 0. \tag{5.108}
$$

8.2 Let φ be any Lipschitz function in $C_c(\mathbb{R}^N \times (t_1, t_2))$. By Part *(1)* of Theorem 3.3, one has

$$
\int \varphi d\mu^+_{\{\hat{v},U\}} = -\int_{\{\hat{v}>0\}} \mathrm{div}(\varphi U)dy, \quad \int \varphi d\mu^+_{\{v,U\}} = -\int_{\{v>0\}} \mathrm{div}(\varphi U)dy,
$$

where $\hat{v} = D^{l-k}\hat{w} = \frac{1}{B}D^{l-k}w = \frac{1}{B}v$. Hence, using the fact that $B > 0$, one has

$$
\int \varphi d\mu^+_{\{\hat{v},U\}} = -\int_{\{v>0\}} \mathrm{div}(\varphi U)dy = \int \varphi d\mu^+_{\{v,U\}}, \tag{5.109}
$$

Then combining (5.109) and (5.108), and the fact that by Step 8.1, $\hat{w} = w/B$ yields

$$
\int w\phi d\mu^+_{\{v,U\}} = 0. \tag{5.110}
$$

Now using (5.89), one obtains

$$
\int \phi\tilde{w} d\mu^+_{\{\tilde{v},U\}} = \int \phi w d\mu^+_{\{v,U\}} = 0. \tag{5.111}
$$

9. Proceeding as in Step 8 for the measure $\mu^-_{\{\tilde{v},U\}}$ with appropriate adaptations, one obtains

$$\int w\phi d\mu^{-}_{\{v,U\}} = 0. \tag{5.112}$$

Then using Part *(3)* of Theorem 3.3, one obtains

$$\int w\phi d\mu_{\{v,U\}} = 0. \tag{5.113}$$

Now using (5.111), (5.112), and (5.113) and (5.88)-(5.89), one obtains

$$\int \phi\tilde{w}d\mu^{+}_{\{\tilde{v},U\}} = \int \phi w d\mu^{+}_{\{v,U\}} = 0, \int \phi\tilde{w}d\mu^{-}_{\{\tilde{v},U\}} = \int \phi w d\mu^{-}_{\{v,U\}} = 0,$$
$$\int \phi\tilde{w}d\mu_{\{\tilde{v},U\}} = \int \phi w d\mu_{\{v,U\}} = 0.$$

This completes the proof of Theorem 5.19 in the case where $0 \le k < l$ and w such that w does not satisfy: $0 \le w \le 1$.

10. Step 1 yields the proof of Theorem 5.19 when $k = l$. Steps 3-7 and the beginning of Step 8 yield the proof of Theorem 5.19 in the case where $0 \le k < l$ and w satisfies: $0 \le w \le 1$. Steps 8-9 yield the proof of Theorem 5.19 in the case where $0 \le k < l$ and w such that w does not satisfy: $0 \le w \le 1$. Therefore, the proof of Theorem 5.19 is completed. ∎

5.9. **The hypothesis in Theorem 5.19 that spt$_x(\tilde{w}) \setminus$ spt$_x(U)$ is either empty or is a subset of the projection of $\{\tilde{v} = 0\}^o$ into $\mathbb{R}^N$ is necessary.** Here, it will be shown that the conclusion of Theorem 5.19 is not true if spt$_x(\tilde{w}) \setminus$ spt$_x(U)$ is neither empty nor a subset of the projection of $\{\tilde{v} = 0\}^o$ into $\mathbb{R}^N$. This will be shown by showing that the conclusion of Theorem 5.12 does not hold. The setting of Theorem 5.12 will be used.

Take $l = (1, 0, \cdots, 0)$ and $k = 0$. Then $\tilde{v} = \partial_1\tilde{w}$. One builds $(\tilde{w}, \tilde{v}, U)$ so that it satisfies all the assumptions of Theorem 5.12 except that now spt$_x(\tilde{w}) \setminus$ spt$_x(U)$ is neither empty nor a subset of the projection of $\{\tilde{v} = 0\}^o$ into $\mathbb{R}^N$.

1. Let $s' \in (0, 1)$ be such that $s' = q'/2^p$ for some integer $0 < p < n$ and $q' \in I_p$. Let j_1 be an integer such that $0 < j_1 < m+1$. Assume there exists a number $s_n \in \{1, \cdots, r_{j_1}\}$ such that $(q_{j_1} + s_n, i^{s_n}_{j_1}) \in J_{n,j_1}$ and $s' = (q_{j_1} + s_n)/2^n$. Assume that for any $s \in \{0, \cdots, r_{j_1}\}$ such that $(q_{j_1}+s, i^s_{j_1}) \in J_{n,j_1}$, $\Gamma^+_v \cap A_{n,q_{j_1}+s,i^s_{j_1}}$ is a manifold of dimension N. Here, $\Gamma^+_v = \partial\{v > 0\} \setminus \Gamma^s_{v,+}$ with $\Gamma^s_{v,+}$ the singular set corresponding to v introduced in Subsection 3.1. Assume that for any $s \in \{0, \cdots, s_n - 1\}$ such that $(q_{j_1} + s, i^s_{j_1}) \in J_{n,j_1}$, $\{w \le (q_{j_1} + s_n)/2^n\} \cap \overline{A_{n,q_{j_1}+s,i^s_{j_1}}}$ is the union of a subset of $\{v < 0\} \cup \Gamma^+_v$ with nonempty interior and a subset of $\{v > 0\}$ with nonempty interior and their interface, Γ_0, a submanifold of Γ^+_v of dimension N. Let $V(\Gamma_0)$ be an open neighborhood of Γ_0.

By the above, for any $s \in \{0, \cdots, s_n\}$ such that $(q_{j_1} + s, i^s_{j_1}) \in J_{n,j_1}$, one may select $w_{n,q_{j_1}+s,i^s_{j_1}}$ such that the restriction of $v_{n,q_{j_1}+s,i^s_{j_1}}$ to $A^-_{n,q_{j_1}+s,i^s_{j_1}} \cap (\{v > 0\} \setminus V(\Gamma_0))$ is ≥ 0 and $\Gamma^+_{v_{n,q_{j_1}+s,i^s_{j_1}}} \cap A^-_{n,q_{j_1}+s,i^s_{j_1}} \cap (\{v > 0\} \setminus V(\Gamma_0)) = \emptyset$; See §5.1.3 above. Also, by the above, for any $s \in \{0, \cdots, s_n - 2\}$ such that $(q_{j_1} + s, i^s_{j_1}) \in J_{n,j_1}$, one may select $w_{n,q_{j_1}+s,i^s_{j_1}}$ such that the restriction of $v_{n,q_{j_1}+s,i^s_{j_1}}$ to $A^+_{n,q_{j_1}+s,i^s_{j_1}} \cap (\{v > 0\} \setminus V(\Gamma_0))$ is ≥ 0 and $\Gamma^+_{v_{n,q_{j_1}+s,i^s_{j_1}}} \cap A^+_{n,q_{j_1}+s,i^s_{j_1}} \cap (\{v > 0\} \setminus V(\Gamma_0)) = \emptyset$; See §5.1.3 above. Here, $\Gamma^+_{v_{n,q_{j_1}+s,i^s_{j_1}}} = \partial\{v_{n,q_{j_1}+s,i^s_{j_1}} > 0\} \setminus \Gamma^s_{\{v_{n,q_{j_1}+s,i^s_{j_1}},+\}}$, with $\Gamma^s_{\{v_{n,q_{j_1}+s,i^s_{j_1}},+\}}$ the singular set corresponding to $v_{n,q_{j_1}+s,i_{j_1}}$ introduced in Subsection 3.1.

Assume that $\{w = s'\} \cap \overline{A^+_{n,q_{j_1}+s_n-1,i^{s_n}_{j_1}}}$ is a $C^\infty-$hypersurface denoted Γ that is the union of three connected manifolds of dimension N:

$$\Gamma_1 = \{w = s'\} \cap \partial(\{v > 0\} \cap A^+_{n,q_{j_1}+s_n-1,i^{s_n}_{j_1}}) \cap \partial\{v > 0\} \cap \partial A^+_{n,q_{j_1}+s_n-1,i^{s_n}_{j_1}} \cap \partial\{v < 0\},$$
$$\Gamma_2 = \{w = s'\} \cap \{v > 0\} \cap \partial A^+_{n,q_{j_1}+s_n-1,i^{s_n}_{j_1}},$$
$$\Gamma_3 = \{w = s'\} \cap \{v < 0\} \cap \partial A^+_{n,q_{j_1}+s_n-1,i^{s_n}_{j_1}}.$$

Assume that U is such that: (1) for any $s \in \{0, \cdots, s_n - 1\}$ such that $(q_{j_1} + s, i^s_{j_1}) \in J_{n,j_1}$, $\{w \le (q_{j_1} + s_n)/2^n\} \cap \overline{A_{n,q_{j_1}+s,i^s_{j_1}}} \cap (\{v < 0\} \cup (V(\Gamma_0) \cap \{v > 0\}) \cup \Gamma_0) \subset \mathcal{U}^c$ and $\{w \le (q_{j_1} + s_n)/2^n\} \cap \overline{A_{n,q_{j_1}+s,i^s_{j_1}}} \cap (\{v > 0\} \setminus V(\Gamma_0)) \subset \mathcal{U}$; (2) for any $s \in \{s_n, \cdots, r_{j_1}\}$ such that $(q_{j_1} + s, i^s_{j_1}) \in J_{n,j_1}$, $\{w > (q_{j_1} + s_n)/2^n\} \cap A_{n,q_{j_1}+s,i^s_{j_1}} \subset \mathcal{U}$; and (3) for any $j \in \{0, \cdots, m + 1\}$ such that $j \neq j_1$ and for any $s \in \{0, \cdots, r_j\}$ such that $(q_j + s, i^s_j) \in J_{n,j}$, $A_{n,q_j+s,i^s_j} \subset \mathcal{U}$.

2. The following sum will be evaluated

$$\sum_{j=0}^{m+1}\Big(\int \phi w \chi_{A^+_{n,q_j,i^0_j}} d\mu^+_{n,q_j,i^0_j} + \sum_{s=1,(q_j+s,i^s_j)\in J_{n,j}}^{r_j} \int \phi w \chi_{A^-_{n,q_j+s,i^s_j} \cup A^+_{n,q_j+s,i^s_j}} d\mu^+_{n,q_j+s,i^s_j}\Big). \tag{5.114}$$

The proof of Theorem 5.12 will be followed.

2.1 By the construction above, for any $s \in \{0, \cdots, s_n\}$ such that $(q_{j_1} + s, i^s_{j_1}) \in J_{n,j_1}$, $A^-_{n,q_{j_1}+s,i^s_{j_1}} \cap \Gamma^+_v \cap \mathcal{U} = \emptyset$ and $A^-_{n,q_{j_1}+s,i^s_{j_1}} \cap \Gamma^+_{v_{n,q_{j_1}+s,i^s_{j_1}}} \cap \mathcal{U} = \emptyset$. Then by definition of the measure $\mu^+_{n,q_{j_1}+s,i^s_{j_1}}$, one obtains

$$\int \phi w \chi_{A^-_{n,q_{j_1}+s,i^s_{j_1}}} d\mu^+_{n,q_{j_1}+s,i^s_{j_1}} = 0. \tag{5.115}$$

Also by the construction above, for any $s \in \{0, \cdots, s_n - 2\}$ such that $(q_{j_1} + s, i^s_{j_1}) \in J_{n,j_1}$, $A^+_{n,q_{j_1}+s,i^s_{j_1}} \cap \Gamma^+_v \cap \mathcal{U} = \emptyset$ and $A^+_{n,q_{j_1}+s,i^s_{j_1}} \cap \Gamma^+_{v_{n,q_{j_1}+s,i^s_{j_1}}} \cap \mathcal{U} = \emptyset$. Then by definition of the measure $\mu^+_{n,q_{j_1}+s,i^s_{j_1}}$, one obtains

$$\int \phi w \chi_{A^+_{n,q_{j_1}+s,i^s_{j_1}}} d\mu^+_{n,q_{j_1}+s,i^s_{j_1}} = 0. \tag{5.116}$$

By Part *(1)* of Theorem 5.8, one obtains

$$\begin{aligned}
&\int \phi w \chi_{A^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}} d\mu^+_{v_{\{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}\}},U,i^{s_n-1}_{j_1}} = \\
&\int \phi w d[\mu^+_{\{w-(q_{j_1}+s_n)/2^n-(1/2^n)^2,U,\{v>0\}\}} \lfloor \overline{A^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}}] \\
&- \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \operatorname{div}(\phi w U) \chi_{\{v_{\{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}\}}>0\}} \chi_{A^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}} dx d\tau.
\end{aligned} \tag{5.117}$$

By Part *(1)* of Theorem 5.10, one obtains

$$\int \phi w \chi_{A^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}} d\mu^+_{v,U} = -\int \phi w d[\mu^+_{\{w-s',U,\{v>0\}\}} \lfloor \overline{A^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}}]$$
$$+\int \phi w d[\mu^+_{\{w-(q_{j_1}+s_n)/2^n-(1/2^n)^2,U,\{v>0\}\}} \lfloor \overline{A^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}}]$$
$$(5.118) \qquad -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\phi w U)\chi_{\{v>0\}}\chi_{A^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}} dx d\tau.$$

By definition of $\mu^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}$, (5.117), and (5.118), one obtains

$$\int \phi w \chi_{A^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}} d\mu^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}$$
$$= -\int \phi w d[\mu^+_{\{w-s',U,\{v>0\}\}} \lfloor \overline{A^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}}]$$
$$(5.119) \qquad -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v>0\}}-\chi_{\{v_{\{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}\}}>0\}})\chi_{A^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}} dx d\tau.$$

Then by the construction in Step 1 above, (5.115)-(5.119), and the definition of the measure $\mu^+_{\{w-s',U,\{v>0\}\}}$ and its properties, one has

$$\int \phi w \chi_{A^-_{n,q_{j_1}+s_n,i^{s_n}_{j_1}}} d\mu^+_{n,q_{j_1}+s_n,i^{s_n}_{j_1}} + \int \phi w \chi_{A^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}} d\mu^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}$$
$$= -\int \phi w \chi_{\Gamma_2} d\mu^+_{\{w-s',U,\{v>0\}\}}$$
$$(5.120) \qquad -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v>0\}}-\chi_{\{v_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}>0\}})\chi_{A^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}} dx d\tau.$$

2.2 Using the fact that by Step 1 above, U satisfies Points (2) and (3) there, and using (5.115), (5.116), and (5.120) one obtains

$$\sum_{j=0}^{m+1}(\int \phi w \chi_{A^+_{n,q_j,i^0_j}} d\mu^+_{n,q_j,i^0_j} + \sum_{s=1,(q_j+s,i^s_j)\in J_{n,j}}^{r_j} \int \phi w \chi_{A^-_{n,q_j+s,i^s_j}\cup A^+_{n,q_j+s,i^s_j}} d\mu^+_{n,q_j+s,i^s_j})$$
$$= \quad -\int \phi w \chi_{\Gamma_2} d\mu^+_{\{w-s',U,\{v>0\}\}}$$
$$-\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v>0\}}-\chi_{\{v_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}>0\}})\chi_{A^+_{n,q_{j_1}+s_n-1,i^{s_n-1}_{j_1}}} dx d\tau +$$
$$-\sum_{s=s_n,(q_{j_1}+s,i^s_{j_1})\in J_{n,j_1}}^{r_{j_1}} \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v>0\}}-\chi_{\{v_{n,q_{j_1}+s,i^s_{j_1}}>0\}})$$
$$(\chi_{B^-_{n,q_{j_1}+s,i^s_{j_1}}} + \chi_{A^+_{n,q_{j_1}+s,i^s_{j_1}}}) dx d\tau +$$

$$
\begin{aligned}
&- \sum_{j=0,\, j\neq j_1}^{m+1} (\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q_j,i_j^0}>0\}})\chi_{A^+_{n,q_j,i_j^0}} dx d\tau \\
&+ \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \operatorname{div}(\phi w U)(\chi_{\{v>0\}} - \chi_{\{v_{n,q_j+s,i_j^s}>0\}}) \\
&\qquad (\chi_{B^-_{n,q_j+s,i_j^s}} + \chi_{A^+_{n,q_j+s,i_j^s}}) dx d\tau).
\end{aligned} \tag{5.121}
$$

3. Proceeding as in the convergence of Theorem 5.17, as n goes to ∞ one obtains using (5.121),

$$
\begin{aligned}
&\sum_{j=0}^{m+1} (\int \phi w \chi_{A^+_{n,q_j,i_j^0}} d\mu^+_{n,q_j,i_j^0} + \sum_{s=1,(q_j+s,i_j^s)\in J_{n,j}}^{r_j} \int \phi w \chi_{A^-_{n,q_j+s,i_j^s} \cup A^+_{n,q_j+s,i_j^s}} d\mu^+_{n,q_j+s,i_j^s}) \\
&\qquad \to - \int \phi w \chi_{\Gamma_2} d\mu^+_{\{w-s',U,\{v>0\}\}}.
\end{aligned} \tag{5.122}
$$

4. Using the convergence in (5.122) in place of the use of the convergence given by Theorem 5.17 in the proof of Theorem 5.19, one concludes that $\int \phi \tilde{w} d\mu^+_{\{\tilde{v},U\}}$ is not 0. That is, Theorem 5.19 does not hold. This proves that the assumption that $\mathrm{spt}_x(\tilde{w}) \setminus \mathrm{spt}_x(U)$ is either empty or a subset of the projection of $\{\tilde{v} = 0\}^o$ into $\mathbb{R}^N$ is necessary.

6. PROPERTIES OF THE MEASURES OF SECTIONS 3-5 FOR SOME CLASS OF GENERATORS AND A FIRST SET OF APPLICATIONS

The theorem of this section establishes further properties of the measures $\mu^+_{\{v,U\}}$, $\mu^-_{\{v,U\}}$, and $\mu_{\{v,U\}}$ for a special class of generators. Throughout this section $N = 3$. Although in this section $N = 3$, Theorem 6.1 can be obtained with appropriate adaptations for $N = 2$. Let ψ_m, $m \geq 1$, be the sequence of functions defined in $\mathbb{R}^N$ by $\psi_m(x) = \psi(\frac{x}{m})$, where $\psi \in C_c^\infty(\mathbb{R}^N)$ and is such that $\psi(x) = 1$ for $|x| \leq 1$, $\psi(x) = 0$ for $|x| \geq 2$, and $0 \leq \psi(x) \leq 1$ for all $x \in \mathbb{R}^N$.

Theorem 6.1. *Let $N = 3$. Let $0 \leq t_1 < t_2$. Let $u \in C^\infty(\mathbb{R}^N \times (t_1,t_2); \mathbb{R}^N)$ with div $u = 0$. Assume that for each $s \in \mathbb{N}^N$, $D^s u \in C^\infty((t_1,t_2); L^1(\mathbb{R}^N))^N$. Let $\phi \in C_c(t_1,t_2)$.*

(1) Let $i \in \{1,\cdots,N\}$. Let $l, k \in \mathbb{N}^N$ be such that $0 \leq k \leq l$. Then, up to a subsequence,

$$
\begin{aligned}
&\lim_{m\to\infty} \int \phi \psi_m D^k u_i d\mu^+_{\{D^l u_i, D^{l-k} u\}} = 0, \quad \lim_{m\to\infty} \int \phi \psi_m D^k u_i d\mu^-_{\{D^l u_i, D^{l-k} u\}} = 0, \\
&\lim_{m\to\infty} \int \phi \psi_m D^k u_i d\mu_{\{D^l u_i, D^{l-k} u\}} = 0.
\end{aligned}
$$

(2) Let $l \in \mathbb{N}^N$. Then, up to a subsequence,

$$
\begin{aligned}
&\lim_{m\to\infty} \int \phi \psi_m d\mu^+_{\{D^l u_i, D^l(u_i u)\}} = 0, \quad \lim_{m\to\infty} \int \phi \psi_m d\mu^-_{\{D^l u_i, D^l(u_i u)\}} = 0, \\
&\lim_{m\to\infty} \int \phi \psi_m d\mu_{\{D^l u_i, D^l(u_i u)\}} = 0.
\end{aligned}
$$

Proof of Theorem 6.1

If $u \equiv 0$, the result is obvious. Hence, throughout this proof it will be assumed that u is not identically 0. Let $k, l \in \mathbb{N}^N$ with $0 \leq k \leq l$. By Theorem 3.1, the measure $\mu^+_{\{v,U\}}$ resp. $\mu^-_{\{v,U\}}$ and $\mu_{\{v,U\}}$ is concentrated on $\partial\{v > 0\}$ resp. $\partial\{v < 0\}$ and $\partial\{v > 0\} \cup \partial\{v < 0\}$. Hence, for $k = l$, Part *(1)* of the theorem is clear. Therefore, it is assumed that $0 \leq k < l$.

Proof of (1) Let $b_1 < b_2$ be such that spt$(\phi) \subset (b_1, b_2) \subset (t_1, t_2)$. Let $i \in \{1, \cdots, N\}$. For any function w defined in $\mathbb{R}^N \times (t_1, t_2)$, denote by spt$_x w$ the projection of the support of w into $\mathbb{R}^N$.

1. By assumption, for each $s \in \mathbb{N}^N$, $D^s u$ is in $C^\infty((t_1, t_2); L^1(\mathbb{R}^N))^N$. Hence, by Sobolev embeddings, $u \in C^\infty((t_1, t_2); W^{m',q}(\mathbb{R}^N))^N$ for all integers $m' \geq 0$ and all $q \geq 1$. Using the regularity of u and its divergence-free property, one can find a potential vector $\Psi \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$ such that $u =$ curl Ψ, $-\Delta\Psi =$ curl u, and div $\Psi = 0$. Now set $u_p =$ curl $(\psi_p \Psi)$, where ψ_p was introduced at the beginning of this section. By the regularity of u and the properties of ψ_p, $u_p \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$ and for each $s \in \mathbb{N}^N$, $D^s u_p$ is in $C^\infty((t_1, t_2); L^1(\mathbb{R}^N))^N$ for all $p \geq 1$. Moreover, $D^s u_p$ converges in $C^{m''}([a, b]; W^{m',q}(\mathbb{R}^N))^N$ to $D^s u$ when p goes to ∞ for all $s \in \mathbb{N}^N$, all integers $m', \ m'' \geq 0$ and all $q \geq 1$, and all $t_1 < a < b < t_2$. Also, for each integer $p \geq 1$, the projection of the support of u_p into $\mathbb{R}^N$ is compact. Denote this projection by K_p. Let u_{pj}, $j = 1, 2, 3$ denote the components of u_p.

2. Let $p \geq 1$ be fixed. Let m_p be such that for any $m \geq m_p$, $\psi_m \equiv 1$ on K_p. If $k \geq l - k$, then it is clear that spt$_x(D^k u_{pi}) \subset$ spt$_x(D^{l-k} u_p)$. If $k < l - k$, then it is not difficult to show that spt$_x(D^k u_{pi}) \backslash$spt$_x(D^{l-k} u_p)$ is a subset of the projection of $\{D^l u_{pi} = 0\}^o$ into $\mathbb{R}^N$. Then one can apply Theorem 5.19 with $(\tilde{w}, \tilde{v}, U)$ of Theorem 5.19 corresponding to $(D^k u_{pi}, D^l u_{pi}, D^{l-k} u_p)$ of this step, and obtain

$$\int \phi\psi_m D^k u_{pi} d\mu^+_{\{D^l u_{pi}, D^{l-k} u_p\}} = \int \phi D^k u_{pi} d\mu^+_{\{D^l u_{pi}, D^{l-k} u_p\}} = 0.$$

3. Let $m \geq m_p$. Then using Step 2 and Part *(1)* of Theorem 3.3, yields

$$\begin{aligned}
& \int \phi\psi_m D^k u_i d\mu^+_{\{D^l u_i, D^{l-k} u\}} \\
= & \int \phi\psi_m D^k u_i d\mu^+_{\{D^l u_i, D^{l-k} u\}} - \int \phi\psi_m D^k u_{pi} d\mu^+_{\{D^l u_{pi}, D^{l-k} u_p\}} \\
= & -\int_{\{D^l u_i > 0\}} \mathrm{div}(\phi\psi_m D^k u_i D^{l-k} u) dy + \int_{\{D^l u_{pi} > 0\}} \mathrm{div}(\phi\psi_m D^k u_{pi} D^{l-k} u_p) dy \\
= & -\int_{\{D^l u_i > 0\}} \phi\nabla\psi_m D^k u_i \cdot D^{l-k} u dy + \int_{\{D^l u_{pi} > 0\}} \phi\nabla\psi_m D^k u_{pi} \cdot D^{l-k} u_p dy \\
& -\int_{\{D^l u_i > 0\}} \phi\psi_m \nabla D^k u_i \cdot D^{l-k} u dy + \int_{\{D^l u_{pi} > 0\}} \phi\psi_m \nabla D^k u_{pi} \cdot D^{l-k} u_p dy.
\end{aligned} \tag{6.1}$$

Here, a use of the fact that div$(D^{l-k} u_p) = 0$ and div$(D^{l-k} u) = 0$, has been made.

3.1 By assumption, $D^s u \in C^\infty([b_1, b_2]; L^1(\mathbb{R}^N))^N$ for each $s \in \mathbb{N}^N$. Hence, by Sobolev embeddings, one has: $D^{l-k} u \in L^1(b_1, b_2; L^2(\mathbb{R}^N))^N$ and $D^k u_i \in L^\infty(b_1, b_2; W^{1,2}(\mathbb{R}^N))$. Now by Step 1, $D^{l-k} u_p \in L^1(b_1, b_2; L^2(\mathbb{R}^N))^N$. Hence, using Holder inequality, one has: $\phi D^k u_i D^{l-k} u \chi_{\{D^l u_i > 0\}} \in L^1(\mathbb{R}^N \times (b_1, b_2))^N$ and $\phi D^k u_{pi} D^{l-k} u_p \chi_{\{D^l u_{pi} > 0\}} \in L^1(\mathbb{R}^N \times (b_1, b_2))^N$. Then, one obtains

$$
\begin{aligned}
&|\int_{\{D^l u_i>0\}} \phi \nabla \psi_m \cdot D^k u_i D^{l-k} u dy| = \frac{1}{m} |\int_{\{D^l u_i>0\}} \phi \nabla \psi(\frac{x}{m}) \cdot D^k u_i D^{l-k} u dy| \\
\text{(6.2)} \quad \leq \; & \frac{1}{m} \|\nabla \psi\|_{L^\infty(\mathbb{R}^N)} \int_{\{D^l u_i>0\}} |\phi D^k u_i D^{l-k} u| dy \leq \frac{C'}{m},
\end{aligned}
$$

where C' is a positive constant independent of m. Letting m go to ∞ in (6.2), yields

$$
\lim_{m\to\infty} \int_{\{D^l u_i>0\}} \phi \nabla \psi_m \cdot D^k u_i D^{l-k} u dy = 0. \tag{6.3}
$$

Similarly, one obtains

$$
\lim_{m\to\infty} \int_{\{D^l u_{pi}>0\}} \phi \nabla \psi_m \cdot D^k u_{pi} D^{l-k} u_p dy = 0. \tag{6.4}
$$

As m goes to ∞,

$$
\begin{aligned}
&\chi_{\{D^l u_i>0\}} \phi \psi_m \nabla D^k u_i \cdot D^{l-k} u \to \chi_{\{D^l u_i>0\}} \phi \nabla D^k u_i \cdot D^{l-k} u, \\
&\chi_{\{D^l u_{pi}>0\}} \phi \psi_m \nabla D^k u_{pi} \cdot D^{l-k} u_p \to \chi_{\{D^l u_{pi}>0\}} \phi \nabla D^k u_{pi} \cdot D^{l-k} u_p.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&|\chi_{\{D^l u_i>0\}} \phi \psi_m \nabla D^k u_i \cdot D^{l-k} u| \leq |\chi_{\{D^l u_i>0\}} \phi \nabla D^k u_i \cdot D^{l-k} u|, \\
&|\chi_{\{D^l u_{pi}>0\}} \phi \psi_m \nabla D^k u_{pi} \cdot D^{l-k} u_p| \leq |\chi_{\{D^l u_{pi}>0\}} \phi \nabla D^k u_{pi} \cdot D^{l-k} u_p|.
\end{aligned}
$$

By the regularity obtained above, these bounds are in $L^1(\mathbb{R}^N \times (t_1, t_2))$. Then by dominated convergence theorem, up to a subsequence, as m goes to ∞,

$$
\lim_{m\to\infty} \int_{\{D^l u_i>0\}} \phi \psi_m \nabla D^k u_i \cdot D^{l-k} u dy = \int_{\{D^l u_i>0\}} \phi \nabla D^k u_i \cdot D^{l-k} u dy, \tag{6.5}
$$

$$
\lim_{m\to\infty} \int_{\{D^l u_{pi}>0\}} \phi \psi_m \nabla D^k u_{pi} \cdot D^{l-k} u_p dy = \int_{\{D^l u_{pi}>0\}} \phi \nabla D^k u_{pi} \cdot D^{l-k} u_p dy. \tag{6.6}
$$

3.2 One has

$$
\begin{aligned}
& - \int_{\{D^l u_i>0\}} \phi \nabla D^k u_i \cdot D^{l-k} u dy + \int_{\{D^l u_{pi}>0\}} \phi \nabla D^k u_{pi} \cdot D^{l-k} u_p dy \\
= \; & - \int_{\{D^l u_{pi}>0\}} \phi (\nabla D^k u_i \cdot D^{l-k} u - \nabla D^k u_{pi} \cdot D^{l-k} u_p) dy + \\
& \int (\chi_{\{D^l u_{pi}>0\}} - \chi_{\{D^l u_i>0\}}) \phi \nabla D^k u_i \cdot D^{l-k} u dy \\
= \; & - \int \chi_{\{D^l u_{pi}>0\}} \phi (\nabla D^k u_i - \nabla D^k u_{pi}) \cdot D^{l-k} u dy \\
& - \int \chi_{\{D^l u_{pi}>0\}} \phi \nabla D^k u_{pi} \cdot (D^{l-k} u - D^{l-k} u_p) dy +
\end{aligned}
$$

$$\int (\chi_{\{D^l u_{pi}>0\}} - \chi_{\{D^l u_i>0\}})\phi\nabla D^k u_i \cdot D^{l-k} u dy. \tag{6.7}$$

By Holder inequality, one has

$$\begin{aligned}
&|\int \chi_{\{D^l u_i>0\}}\phi(\nabla D^k u_i - \nabla D^k u_{pi}) \cdot D^{l-k} u dy| \\
&\le \|\phi(\nabla D^k u_i - \nabla D^k u_{pi})\|_{L^1(b_1,b_2;W^{1,2}(\mathbb{R}^N))}\|D^{l-k}u\|_{L^\infty(b_1,b_2;W^{1,2}(\mathbb{R}^N))} \qquad (6.8)\\
&|\int \chi_{\{D^l u_i>0\}}\phi\nabla D^k u_{pi} \cdot (D^{l-k} u - D^{l-k} u_p) dy| \\
&\le \|\phi\nabla D^k u_{pi}\|_{L^\infty(b_1,b_2;W^{1,2}(\mathbb{R}^N))}\|D^{l-k}u - D^{l-k}u_p\|_{L^1(b_1,b_2;W^{1,2}(\mathbb{R}^N))} \\
&\le C\|D^{l-k}u - D^{l-k}u_p\|_{L^1(b_1,b_2;W^{1,2}(\mathbb{R}^N))}, \qquad (6.9)
\end{aligned}$$

where C is a positive constant independent of p. Above a use of Step 1 has been made. Using the properties of the approximating sequence u_p obtained in Step 1 above, as p goes to ∞, the left sides of (6.8)- (6.9) go to 0. Hence, the first and second terms in the right side of (6.7) go to 0.

3.3 Set $\Omega_{pi} = \{D^l u_{pi} > 0\}$ and $\Omega_i = \{D^l u_i > 0\}$. Then

$$\begin{aligned}
&\int (\chi_{\{D^l u_{pi}>0\}} - \chi_{\{D^l u_i>0\}})\phi\nabla D^k u_i \cdot D^{l-k} u dy \\
&= \int \chi_{\Omega_{pi}\cap\Omega_i^c}\phi\nabla D^k u_i \cdot D^{l-k} u dy - \int \chi_{\Omega_{pi}^c\cap\Omega_i}\phi\nabla D^k u_i \cdot D^{l-k} u dy. \qquad (6.10)
\end{aligned}$$

Let $y \in \Omega_i^c$. If $D^l u_i(y) < 0$, then by construction of u_p; See Step 1, as p goes to ∞, $\chi_{\{D^l u_{pi}>0\}}(y)$ goes to 0. If $y \in \{D^l u_i = 0\}^o$, then $B(y,\epsilon) \subset \{D^l u_i = 0\}^o$ for some $\epsilon > 0$ sufficiently small. For p large, $\psi_p \equiv 1$ on $B(y,\epsilon)$. Hence, $D^l u_{pi}(y) = D^l u_i(y) = 0$. Therefore, as p goes to ∞, $\chi_{\Omega_{pi}\cap\Omega_i^c}$ converges a.e. to 0. Proceeding similarly, $\chi_{\Omega_i\cap\Omega_{pi}^c}$ converges a.e. to 0. Moreover, by Step 2, $\phi\nabla D^k u_i \cdot D^{l-k} u \in L^1(\mathbb{R}^N \times (b_1, b_2))$. Then by dominated convergence theorem, up to a subsequence, as p goes to ∞, the two integrals in the right side of (6.10) converge to 0. And so, up to a subsequence, as p goes to ∞,

$$\lim_{p\to\infty}\int (\chi_{\{D^l u_{pi}>0\}} - \chi_{\{D^l u_i>0\}})\phi\nabla D^k u_i \cdot D^{l-k} u dy = 0.$$

3.4 Combining the convergence in Steps 3.2-3.3, and using (6.7), one obtains, up to a subsequence, as p goes to ∞,

$$\lim_{p\to\infty}[-\int_{\{D^l u_i>0\}} \phi\nabla D^k u_i \cdot D^{l-k} u dy + \int_{\{D^l u_{pi}>0\}} \phi\nabla D^k u_{pi} \cdot D^{l-k} u_p dy] = 0. \tag{6.11}$$

3.5 Combining the convergence in (6.3)-(6.6) of Step 3.1 and (6.11) in Step 3.4, and using (6.1), one obtains, up to a subsequence, as m goes to ∞ first and then p goes to ∞ second,

$$\lim_{m\to\infty}\int \phi\psi_m D^k u_i d\mu^+_{\{D^l u_i, D^{l-k} u\}} = 0. \tag{6.12}$$

4. Proceeding as in Steps 1-3 above for $\mu^-_{\{D^l u_i, D^{l-k}u\}}$ and $\mu_{\{D^l u_i, D^{l-k}u\}}$, one obtains, up to a subsequence,

$$\lim_{m\to\infty}\int \phi\psi_m D^k u_i d\mu^-_{\{D^l u_i, D^{l-k}u\}} = 0, \tag{6.13}$$

$$\lim_{m\to\infty}\int \phi\psi_m D^k u_i d\mu_{\{D^l u_i, D^{l-k}u\}} = 0. \tag{6.14}$$

5. (6.12)-(6.14) yield Part *(1)* of the theorem.

Proof of (2)

1. Using Leibniz formula yields

$$D^l(u_i u) = \sum_{k\le l} C_{kl} D^k u_i D^{l-k} u, \tag{6.15}$$

where C_{kl} is given by

$$C_{kl} = \frac{l_1!}{k_1!(l_1-k_1)!}\frac{l_2!}{k_2!(l_2-k_2)!}\cdots\frac{l_N!}{k_N!(l_N-k_N)!}. \tag{6.16}$$

Let $0\le k\le l$. Set $v = D^l u_i$. Then using the regularity of u and u_i, and Theorem 3.1 with (v,U) of Theorem 3.1 corresponding to $(D^l u_i, D^{l-k}u)$ of this step, and the fact that $\phi\psi_m D^k u_i$ is Lipschitz continuous function with compact support in $\mathbb{R}^N\times(t_1,t_2)$, one obtains, up to a subsequence, as α goes to 0,

$$\begin{aligned}&\int \phi\psi_m D^k u_i d\mu^+_{\{v, D^{l-k}u\}}\\ &= \lim_{\alpha\to 0}\int \phi\psi_m D^k u_i D^{l-k}u\cdot\nabla v\frac{1}{\alpha}\chi_{\{0<v<\alpha\}}dxd\tau.\end{aligned} \tag{6.17}$$

Using Theorem 3.1 with (v,U) of Theorem 3.1 corresponding to $(D^l u_i, D^l(u_i u))$ of this step, the fact that $\phi\psi_m$ is Lipschitz continuous function with compact support in $\mathbb{R}^N\times(t_1,t_2)$, (6.15), and (6.17), one obtains

$$\begin{aligned}&\int \phi\psi_m d\mu^+_{\{v, D^l(u_i u)\}}\\
&= \lim_{\alpha\to 0}\int \phi\psi_m D^l(u_i u)\cdot\nabla v\frac{1}{\alpha}\chi_{\{0<v<\alpha\}}dxd\tau\\
&= \lim_{\alpha\to 0}\int \sum_{0\le k\le l} C_{kl}\phi\psi_m D^k u_i D^{l-k}u\cdot\nabla v\frac{1}{\alpha}\chi_{\{0<v<\alpha\}}dxd\tau\\
&= \sum_{0\le k\le l} C_{kl}\lim_{\alpha\to 0}\int \phi\psi_m D^k u_i D^{l-k}u\cdot\nabla v\frac{1}{\alpha}\chi_{\{0<v<\alpha\}}dxd\tau\\
&= \sum_{0\le k\le l} C_{kl}\int \phi\psi_m D^k u_i d\mu^+_{\{v, D^{l-k}u\}}.\end{aligned} \tag{6.18}$$

Similarly, one obtains

$$\int \phi\psi_m d\mu^-_{\{v,D^l(u_i u)\}} = \sum_{0\le k\le l} C_{kl} \int \phi\psi_m D^k u_i d\mu^-_{\{v,D^{l-k}u\}}, \tag{6.19}$$

$$\int \phi\psi_m d\mu_{\{v,D^l(u_i u)\}} = \sum_{0\le k\le l} C_{kl} \int \phi\psi_m D^k u_i d\mu_{\{v,D^{l-k}u\}}. \tag{6.20}$$

Then using (6.18)-(6.20) and Part *(1)* of the theorem yields Part *(2)* of the theorem. This completes the proof of Theorem 6.1. ∎

7. PROPERTIES OF THE MEASURES OF SECTIONS 3-5 FOR SOME CLASS OF GENERATORS AND A SECOND SET OF APPLICATIONS

In this section, further properties of the measures $\mu^+_{\{v,U\}}$, $\mu^-_{\{v,U\}}$, and $\mu_{\{v,U\}}$ for special classes of generators are obtained. Throughout this section $N = 3$ and ψ_m, $m \ge 1$, corresponds to the sequence of functions defined in $\mathbb{R}^3$ by $\psi_m(x) = \psi(\frac{x}{m})$, where $\psi \in C^\infty_c(\mathbb{R}^3)$ and is such that $\psi(x) = 1$ for $|x| \le 1$, $\psi(x) = 0$ for $|x| \ge 2$, and $0 \le \psi(x) \le 1$ for all $x \in \mathbb{R}^3$. Also throughout this section, the topological notions of interior of a subset E of $\mathbb{R}^m$, open, compact, etc are with respect to the canonical topology on $\mathbb{R}^m$. Moreover, throughout this section, the notation $y = (x,\tau) = (y_1,\cdots,y_4)$ for the elements of $\mathbb{R}^3 \times (t_1,t_2)$ will also be used. The main results of this section correspond to Theorems 7.1 and 7.2. By an appropriate adaptation of the proofs below, one also obtains Theorems 7.1 and 7.2 for $N = 2$.

Theorem 7.1. *Let $N = 3$. Let $0 \le t_1 < t_2$. Let $u \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R}^N)$ be such that div $u = 0$ and the projection into $\mathbb{R}^N$ of the support of u is compact. Let $\omega =$ curl u. Let $l \in \mathbb{N}^N$. Let $i \in \{1,2,3\}$. Let $\phi \in C_c(t_1,t_2)$. Then*

(1)

$$\int \phi d\mu^+_{\{D^l\omega_i,D^l(\omega_i u)\}} = 0,\ \int \phi d\mu^-_{\{D^l\omega_i,D^l(\omega_i u)\}} = 0,\ \int \phi d\mu_{\{D^l\omega_i,D^l(\omega_i u)\}} = 0.$$

(2)

$$\int \phi d\mu^+_{\{D^l\omega_i,D^l(u_i\omega)\}} = 0,\ \int \phi d\mu^-_{\{D^l\omega_i,D^l(u_i\omega)\}} = 0,\ \int \phi d\mu_{\{D^l\omega_i,D^l(u_i\omega)\}} = 0.$$

Above, $\mu^+_{\{v,U\}}$, $\mu^-_{\{v,U\}}$, and $\mu_{\{v,U\}}$ denote the measures on $\mathbb{R}^N \times (t_1,t_2)$ given by Theorem 3.1 with (v,U) of Theorem 3.1 corresponding to $(D^l\omega_i, D^l(\omega_i u))$ resp. $(D^l\omega_i, D^l(u_i\omega))$ of this theorem.

Proof of Theorem 7.1. If $u \equiv 0$, the result is obvious. Hence, throughout this proof it is assumed that u is not identically 0. Let $l \in \mathbb{N}^N$. Let $\phi \in C_c(t_1,t_2)$. Let K_x denote the projection into $\mathbb{R}^N$ of the support of u. Then the projection into $\mathbb{R}^N$ of the support of ω_i is compact, $i = 1,2,3$. Set $K = K_x \times$ spt(ϕ). By assumption, K is compact.

Proof of Part (1) Proceeding as in the proof of Part *(2)* of Theorem 6.1, one obtains

$$\int \phi\psi_m d\mu^+_{\{D^l\omega_i,D^l(\omega_i u)\}} = \sum_{0\le k\le l} C_{kl} \int \phi\psi_m D^k\omega_i d\mu^+_{\{D^l\omega_i,D^{l-k}u\}}, \tag{7.1}$$

where C_{kl} are given by (6.16). Since K_x is compact, the measures $\phi\mu^+_{\{D^l\omega_i,D^{l-k}u\}}$ are of compact support included in the compact set K; See Theorem 3.1. Taking m so large that $\psi_m \equiv 1$ on K_x, (7.1) yields

$$\int \phi d\mu^+_{\{D^l\omega_i, D^l(\omega_i u)\}} = \sum_{0\leq k\leq l} C_{kl} \int \phi D^k\omega_i d\mu^+_{\{D^l\omega_i, D^{l-k}u\}}. \tag{7.2}$$

Let $k, l \in \mathbb{N}^N$ be such that $0 \leq k \leq l$. By definition of ω, one has $\text{spt}_x(D^k\omega_i) \setminus \text{spt}_x(D^{l-k}u)$ is a subset of $\text{spt}_x(D^k\omega_i)\setminus\text{spt}_x(D^{l-k}\omega)$. Hence, one obtains as in Step 2 of the proof of Theorem 6.1, $\text{spt}_x(D^k\omega_i) \setminus \text{spt}_x(D^{l-k}u)$ is either empty or a subset of $\{D^l\omega_i = 0\}^o$. Therefore, one can apply Theorem 5.19 with $(\tilde{w}, \tilde{v}, U)$ of Theorem 5.19 corresponding to $(D^k\omega_i, D^l\omega_i, D^{l-k}u)$ of this step, and obtain

$$\int \phi D^k\omega_i d\mu^+_{\{D^l\omega_i, D^{l-k}u\}} = 0. \tag{7.3}$$

Then combining (7.1)-(7.3) yields

$$\int \phi d\mu^+_{\{D^l\omega_i, D^l(\omega_i u)\}} = 0,$$

which corresponds to the first statement of Part *(1)* of the theorem. Reasoning as above for $\mu^-_{\{D^l\omega_i, D^l(\omega_i u)\}}$ and $\mu_{\{D^l\omega_i, D^l(\omega_i u)\}}$ yields the second and third statements of Part *(1)* of the theorem.

Proof of Part (2) The case $i = 1$ will be studied. The proofs for $i = 2$ and $i = 3$ can be obtained by an appropriate adaptation of the method introduced here.

By the regularity of u, $\omega_1 \in C^\infty(\mathbb{R}^3 \times (t_1, t_2); \mathbb{R})$. Moreover, since the projection of the support of u into $\mathbb{R}^N$ is compact, the projection of the support of ω_1 into $\mathbb{R}^N$ is compact. Let A be a positive constant such that on K, $\tilde{w} = \omega_1 + A > \alpha > 0$ for some positive α. The method of proof consists of building a vector function $G = (G_1, G_2, G_3)^t \in C^\infty(\mathbb{R}^3 \times (t_1, t_2); \mathbb{R}^3)$ such that

$$\text{div } G = 0, \tag{7.4}$$

$$\int \phi D^l(\tilde{w}G_1) d\mu^+_{\{D^l\omega_1, e_1\}} = 0, \tag{7.5}$$

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\omega_1 u)\}} - \int \phi d\mu^+_{\{D^l\omega_1, D^l(u_1\omega)\}} = \int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}G)\}}. \tag{7.6}$$

Here, e_j, $j = 1, 2, 3$ denote the canonical basis of $\mathbb{R}^3$. Using Theorem 5.19 shows that

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}G)\}} = \int \phi d\mu^+_{\{D^l\omega_1, D^l(\omega_1 G)\}} = 0. \tag{7.7}$$

Here, a use of the fact that $\int \phi d\mu^+_{\{D^l\omega_1, D^l G\}} = 0$; See Step 4 below. Now using Part *(1)* of the theorem, (7.6), and (7.7), one concludes that

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(u_1\omega)\}} = 0, \tag{7.8}$$

which corresponds to the first statement in Part*(2)* of the theorem for the case $i = 1$. Proceeding similarly with appropriate adaptations, one obtains the second and third statements in Part*(2)* of the theorem for the case $i = 1$. As can be seen below, the proof of (7.4)-(7.6) relies on the basic properties of the measures developed in the previous sections and the crucial fact that the first component of the difference of the (convective) vector $\omega_1 u$ and the (stretching) vector $u_1\omega$ is 0.

1. Let $h_2,\ h_3 \in C^\infty(\mathbb{R}^3)$ be such that

$$\partial_2 h_2 + \partial_3 h_3 = 0. \tag{7.9}$$

Set

$$G_2 = \frac{1}{\tilde{w}}(\omega_1 u_2 - u_1\omega_2 + h_2), \quad G_3 = \frac{1}{\tilde{w}}(\omega_1 u_3 - u_1\omega_3 + h_3). \tag{7.10}$$

G_1 will be determined so that (7.4)-(7.6) are satisfied. (7.4) requires that $\partial_1 G_1 = -\partial_2 G_2 - \partial_3 G_3$. Hence G_1 is of the form

$$G_1(x,\tau) = -\int_a^{x_1} (\partial_2 G_2 + \partial_3 G_3)(z, x_2, x_3, \tau)dz + \bar{g}(a, x_2, x_3, \tau), \tag{7.11}$$

where a is a constant and $\bar{g}$ is a function in $C^\infty(\mathbb{R}^3 \times (t_1,t_2); \mathbb{R})$ that are arbitrary. Define g by $g(x,\tau) = \bar{g}(a, x_2, x_3, \tau)$ for every $(x,\tau) \in \mathbb{R}^3 \times (t_1, t_2)$.

Set $g_1 = \int_a^{x_1}(\partial_2 G_2 + \partial_3 G_3)(z, x_2, x_3, \tau)dz$. Let $\mu^+_{\{D^l\omega_1, D^l(\tilde{w}G)\}}$ be the measure associated with the pair $(D^l\omega_1, D^l(\tilde{w}G))$ by Theorem 3.1. By Part *(1)* of Theorem 3.3,

$$\begin{aligned} &\int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}G)\}} = -\int_{\{D^l\omega_1>0\}} \mathrm{div}(\phi D^l(\tilde{w}G))dxd\tau \\ = &-\int_{\{D^l\omega_1>0\}} \phi(\partial_1(D^l(\tilde{w}G_1)) + \partial_2(D^l(\tilde{w}G_2)) + \partial_3(D^l(\tilde{w}G_3)))dxd\tau \\ = &-\int_{\{D^l\omega_1>0\}} \phi(\partial_1(D^l(\tilde{w}(g-g_1))) + \partial_2(D^l(\omega_1 u_2 - u_1\omega_2)) \\ &+\partial_3(D^l(\omega_1 u_3 - u_1\omega_3)))dxd\tau. \end{aligned} \tag{7.12}$$

Here, a use of the fact that $\partial_2 h_2 + \partial_3 h_3 = 0$ has been made. Also, by Part *(1)* of Theorem 3.3,

$$\begin{aligned} &\int \phi d\mu^+_{\{D^l\omega_1, D^l(\omega_1 u)\}} - \int \phi d\mu^+_{\{D^l\omega_1, D^l(u_1\omega)\}} \\ = &-\int_{\{D^l\omega_1>0\}} \mathrm{div}(\phi D^l(\omega_1 u))dxd\tau + \int_{\{D^l\omega_1>0\}} \mathrm{div}(\phi D^l(u_1\omega))dxd\tau \\ = &-\int_{\{D^l\omega_1>0\}} \mathrm{div}(\phi D^l(\omega_1 u - u_1\omega))dxd\tau \\ = &-\int_{\{D^l\omega_1>0\}} \phi(\partial_2(D^l(\omega_1 u_2 - u_1\omega_2)) + \partial_3(D^l(\omega_1 u_3 - u_1\omega_3)))dxd\tau. \end{aligned} \tag{7.13}$$

Hence, using (7.12)-(7.13), it is clear that if one can select g so that

$$\int_{\{D^l\omega_1>0\}} \phi\partial_1(D^l(\tilde{w}(g-g_1)))dxd\tau = 0, \tag{7.14}$$

then the left sides of (7.12)-(7.13) are equal. That is,

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\omega_1 u)\}} - \int \phi d\mu^+_{\{D^l\omega_1, D^l(u_1\omega)\}} = \int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}G)\}}. \tag{7.15}$$

2. Set $v = D^l\omega_1$. Set $\Gamma_v^+ = \partial\{v > 0\} \setminus \Gamma_{v,+}^s$, where $\Gamma_{v,+}^s$ is the singular set corresponding to v introduced in Subsection 3.1. Parts *(2)* and *(4)* of Theorem 3.4 show that the measures $\mu^+_{\{v,D^l(\omega_1 u)\}}$, $\mu^+_{\{v,D^l(u_1\omega)\}}$, and $\mu^+_{\{v,D^l(\tilde{w}G)\}}$ are concentrated on Γ_v^+ and that $\Gamma_v^+ (= \partial\{v > 0\}\setminus\Gamma_{v,+}^s)$ is a $C^\infty-$hypersurface. Hence, without loss of generality, one may assume that $\Gamma_{v,+}^s$ is empty.

Since Γ_v^+ is C^∞-regular, there exist an open neighborhood, $\mathcal{U}$, of Γ_v^+ and a C^∞-function F defined on $\mathcal{U}$ such that for all $y \in \mathcal{U}$, $\nabla_{x,t}F(y) \neq 0$, and

$$\Gamma_v^+ = \{y \in \mathcal{U}|\ F(y) = 0\} \tag{7.16}$$

$$\mathcal{U} \cap \{v > 0\} = \{y \in \mathcal{U}|\ F(y) < 0\}. \tag{7.17}$$

Then the unit exterior normal vector n to Γ_v^+ is given by $n(y) = \frac{\nabla_{x,t}F(y)}{|\nabla_{x,t}F(y)|}$. Here and below, one uses the notation: $\nabla_{x,t}\varphi = (\partial_{x_1}\varphi, \partial_{x_2}\varphi, \partial_{x_3}\varphi, \partial_t\varphi)^t$ for any $\varphi \in C^1(\mathbb{R}^3 \times (t_1, t_2))$. By Part *(1)* of Theorem 3.3 and Part *(1)* of Theorem 3.4,

$$\begin{aligned}\int \phi D^l(\tilde{w}G_1) d\mu^+_{\{v,e_1\}} &= -\int_{\{v>0\}} \phi\partial_1(D^l(\tilde{w}(g - g_1)))dxd\tau \\ &= -\int_{\Gamma_v^+} \phi D^l(\tilde{w}(g - g_1))n_1 d\mathcal{H}^N.\end{aligned} \tag{7.18}$$

Assume that $\partial_1 F \equiv 0$ on Γ_v^+. Then the first component n_1 of the unit exterior normal vector n to $\Gamma_v^+ (= \Gamma_v^+ \setminus \Gamma_{v,+}^s)$ is identically 0 on Γ_v^+. Then one has

$$\int_{\Gamma_v^+} \phi D^l(\tilde{w}(g - g_1))n_1 d\mathcal{H}^N = 0. \tag{7.19}$$

Then using (7.18) and (7.19), the left sides of (7.12)-(7.13) are equal. Hence, (7.6) is satisfied for every G satisfying (7.10) and (7.11) with a and g arbitrary. Then reasoning as in Step 4 below, one obtains

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(u_1\omega)\}} = 0.$$

which corresponds to the first statement of Part *(2)* of the theorem for the case $i = 1$.

Now assume that $\partial_1 F \not\equiv 0$ on Γ_v^+. Set $\tilde{\Gamma}^+ = \Gamma_v^+ \cap \{\partial_1 F \neq 0\}$. Let $y^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \tau^{(0)}) \in \tilde{\Gamma}^+$. Since Γ_v^+ is $C^\infty-$regular, there exist a coordinate system associated with a basis formed of orthogonal vectors with unit length, still denoted (x_1, x_2, x_3, τ), an open neighborhood $\mathcal{U}^{(0)}$ of $y^{(0)}$, an open subset $\mathcal{V}^{(0)}$ of $\mathbb{R}^2 \times (t_1, t_2)$, and a C^∞-regular function f_0 defined on $\mathcal{V}^{(0)}$ such that

$$\begin{aligned}&\mathcal{U}^{(0)} \cap \tilde{\Gamma}^+ = \{(f_0(x_2, x_3, \tau), x_2, x_3, \tau)|\ (x_2, x_3, \tau) \in \mathcal{V}^{(0)}\} \\ &\mathcal{U}^{(0)} \cap \{v > 0\} = \{(x_1, x_2, x_3, \tau) \in \mathcal{U}^{(0)}|\ x_1 < f_0(x_2, x_3, \tau)\}\end{aligned}$$

Using the fact that the projection into $\mathbb{R}^3$ of the support of ω_1 is compact; See the beginning of the proof, one can find $y^{(j)} \in \tilde{\Gamma}^+$, $j = 1, \cdots, p$ such that for $j = 1, \cdots, p$, denoting $\mathcal{U}^{(j)}$, $\mathcal{V}^{(j)}$, and f_j the corresponding open sets and functions satisfying the properties above, one has $\tilde{\Gamma}^+ \subset \cup_{j=1}^p \mathcal{U}^{(j)}$.

Let $\beta_j \in C^\infty(\mathbb{R}^4; \mathbb{R})$, $j = 0, \cdots, p$ be such that

(1) $0 \leq \beta_j \leq 1$;

(2) $\sum_{j=0}^p \beta_j = 1$ on $\mathbb{R}^4$;

(3) spt(β_j) is a compact subset of $\mathcal{U}^{(j)}$, $j = 1, \cdots, p$ and spt$(\beta_0) \subset \mathbb{R}^4 \setminus \tilde{\Gamma}^+$.
Then using the notation $(x_1, x_2, x_3, \tau) = (x_1, x')$,

$$\int_{\Gamma_v^+} \phi D^l(\tilde{w}g) n_1 d\mathcal{H}^N = \sum_{j=1}^{p} \int_{\Gamma_v^+} \beta_j \phi D^l(\tilde{w}g) n_1 d\mathcal{H}^N = \sum_{j=1}^{p} I_j, \tag{7.20}$$

where

$$I_j = \int_{\mathcal{V}^{(j)}} \phi(\tau)(\beta_j D^l(\tilde{w}g) \frac{\partial_1 F}{|\nabla_{x,t} F|})(f_j(x'), x')(1 + |\nabla_{x,t} f_j(x')|^2)^{1/2} dx'. \tag{7.21}$$

Hence

$$\int_{\Gamma_v^+} \phi D^l(\tilde{w}(g - g_1)) n_1 d\mathcal{H}^N = \sum_{j=1}^{p} I_j - \int_{\Gamma_v^+} \phi D^l(\tilde{w} g_1) n_1 d\mathcal{H}^N. \tag{7.22}$$

Denote the projection of a subset E of $\mathbb{R}^N \times (t_1, t_2)$ into (t_1, t_2) by pr$_t(E)$. If $\phi \equiv 0$ on pr$_t(\tilde{\Gamma}^+)$ it is clear that the left side of (7.22) is 0. Hence, it is assumed that $\phi \not\equiv 0$ on pr$_t(\tilde{\Gamma}^+)$. There are five cases: (1) $l = 0$; (2) $\int_{\Gamma_v^+} \phi D^l(\tilde{w}g_1) n_1 d\mathcal{H}^N = 0$; (3) $l \neq 0$, $\int_{\Gamma_v^+} \phi D^l(\tilde{w}g_1) n_1 d\mathcal{H}^N \neq 0$, $\partial_1^{l_1}\omega_1 \not\equiv 0$ on $\tilde{\Gamma}^+$, and $\phi \not\equiv 0$ on $pr_t(\{\partial_1^{l_1}\omega_1 \neq 0\} \cap \tilde{\Gamma}^+)$; (4) $l \neq 0$, $\int_{\Gamma_v^+} \phi D^l(\tilde{w}g_1) n_1 d\mathcal{H}^N \neq 0$, and $\partial_1^{l_1}\omega_1 \equiv 0$ on $\tilde{\Gamma}^+$; and (5) $l \neq 0$, $\int_{\Gamma_v^+} \phi D^l(\tilde{w}g_1) n_1 d\mathcal{H}^N \neq 0$, $\partial_1^{l_1}\omega_1 \not\equiv 0$ on $\tilde{\Gamma}^+$, and $\phi \equiv 0$ on $pr_t(\{\partial_1^{l_1}\omega_1 \neq 0\} \cap \tilde{\Gamma}^+)$.

Cases (1) and (2) will be studied below and in Step 4. Case (3) will be studied in Steps 3 and 4. Case (4) will be studied in Step 5. In Step 6, Case (5) will be studied.

If $l = 0$, then by definition of $\tilde{w}$,

$$\int_{\Gamma_v^+} \phi D^l(\tilde{w}(g - g_1)) n_1 d\mathcal{H}^N = \int_{\Gamma_v^+} \phi(\omega_1 + A)(g - g_1) n_1 d\mathcal{H}^N = A \int_{\Gamma_v^+} \phi(g - g_1) n_1 d\mathcal{H}^N. \tag{7.23}$$

Hence, it is clear that one can build g such that

$$\int_{\Gamma_v^+} \phi g n_1 d\mathcal{H}^N = \int_{\Gamma_v^+} \phi g_1 n_1 d\mathcal{H}^N. \tag{7.24}$$

(7.23)-(7.24) show that the integral in (7.22) is 0. Then reasoning as in Step 4 below, one obtains

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(u_1\omega)\}} = 0.$$

which corresponds to the first statement of Part *(2)* of the theorem for the case $i = 1$.

If $\int_{\Gamma_v^+} \phi D^l(\tilde{w}g_1) n_1 d\mathcal{H}^N = 0$, then one takes $g = 0$. Then

$$\int_{\Gamma_v^+} \phi(g - g_1) n_1 d\mathcal{H}^N = 0. \tag{7.25}$$

Then reasoning as in Step 4 below, one obtains

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(u_1\omega)\}} = 0.$$

which corresponds to the first statement of Part *(2)* of the theorem for the case $i = 1$.

3. In this step Case (3) will be studied. Thus, it is assumed that $l \neq 0$ and $\int_{\Gamma_v^+} \phi D^l(\tilde{w} g_1) n_1 d\mathcal{H}^N \neq 0$. the function g will be build to satisfy

$$\int_{\Gamma_v^+} \phi D^l(\tilde{w}(g - g_1)) n_1 d\mathcal{H}^N = 0 \tag{7.26}$$

in the case where $\partial_1^{l_1}\omega_1 \not\equiv 0$ on $\tilde{\Gamma}^+$ and $\phi \not\equiv 0$ on $\mathrm{pr}_t(\{\partial_1^{l_1}\omega_1 \neq 0\} \cap \tilde{\Gamma}^+)$. In this case, it is clear that $|l| > l_1$. In Step 4, the proof for $i = 1$ of Part *(2)* of the theorem in this case will be concluded.

Recall that $g = \bar{g}(a, x_2, x_3, \tau)$ with a a constant. By Leibniz formula, one has

$$\begin{aligned} D^l(\tilde{w}g) &= \partial_2^{l_2}\partial_3^{l_3}(\partial_1^{l_1}\tilde{w}g) \\ &= C_{0,s}\partial_1^{l_1}\tilde{w}\partial_2^{l_2}\partial_3^{l_3}g + \sum_{0<r<s} C_{r,s}\partial_2^{r_2}\partial_3^{r_3}\partial_1^{l_1}\tilde{w}\partial_2^{l_2-r_2}\partial_3^{l_3-r_3}g + C_{s,s}\partial_2^{l_2}\partial_3^{l_3}\partial_1^{l_1}\tilde{w}g \\ &= \partial_1^{l_1}\tilde{w}\partial_2^{l_2}\partial_3^{l_3}g + \sum_{0<r<s} C_{r,s}\partial_2^{r_2}\partial_3^{r_3}\partial_1^{l_1}\tilde{w}\partial_2^{l_2-r_2}\partial_3^{l_3-r_3}g, \end{aligned} \tag{7.27}$$

on Γ_v^+. Above, $s = (l_2, l_3)$, $r = (r_2, r_3)$, and $C_{r,s}$ are the corresponding coefficients in Leibniz Formula; See (6.16). Also above, a use of the fact that $l \neq 0$ and so on Γ_v^+: $\partial_2^{l_2}\partial_3^{l_3}\partial_1^{l_1}\tilde{w} = D^l\tilde{w} = D^l\omega_1 = 0$, has been made.

Let $h = h(x_2, x_3) \in C^\infty(\mathbb{R}^2; \mathbb{R})$ and $\tilde{g} = \tilde{g}(a, x_2, x_3, \tau) \in C^\infty(\mathbb{R}^2 \times (t_1, t_2); \mathbb{R})$. Then take g to be the function

$$g(a, x_2, x_3, \tau) = h(x_2, x_3)\tilde{g}(a, x_2, x_3, \tau). \tag{7.28}$$

Using Leibniz formula, one has

$$\partial_2^{l_2}\partial_3^{l_3}(h\tilde{g}) = C'_{0,s}\partial_2^{l_2}\partial_3^{l_3}h\tilde{g} + \sum_{0<r\leq s} C'_{r,s}\partial_2^{l_2-r_2}\partial_3^{l_3-r_3}h\partial_2^{r_2}\partial_3^{r_3}\tilde{g} \tag{7.29}$$

$$\begin{aligned} &\partial_2^{l_2-r_2}\partial_3^{l_3-r_3}(h\tilde{g}) \\ &= C'_{0,s_r}\partial_2^{l_2-r_2}\partial_3^{l_3-r_3}h\tilde{g} + \sum_{0<r'\leq s_r} C'_{r',s_r}\partial_2^{l_2-r_2-r'_2}\partial_3^{l_3-r_3-r'_3}h\partial_2^{r'_2}\partial_3^{r'_3}\tilde{g} \end{aligned} \tag{7.30}$$

where $s_r = (l_2 - r_2, l_3 - r_3)$. Using (7.28)-(7.30) in (7.27) yields

$$\begin{aligned} D^l(\tilde{w}g) &= \partial_2^{l_2}\partial_3^{l_3}(\partial_1^{l_1}\tilde{w}g) \\ &= \partial_1^{l_1}\tilde{w}\partial_2^{l_2}\partial_3^{l_3}h\tilde{g} + \partial_1^{l_1}\tilde{w}\sum_{0<r\leq s} C'_{r,s}\partial_2^{l_2-r_2}\partial_3^{l_3-r_3}h\partial_2^{r_2}\partial_3^{r_3}\tilde{g} \\ &\quad + \sum_{0<r<s} C_{r,s}\partial_2^{r_2}\partial_3^{r_3}\partial_1^{l_1}\tilde{w}[\partial_2^{l_2-r_2}\partial_3^{l_3-r_3}h\tilde{g} \\ &\quad + \sum_{0<r'\leq s_r} C'_{r',s_r}\partial_2^{l_2-r_2-r'_2}\partial_3^{l_3-r_3-r'_3}h\partial_2^{r'_2}\partial_3^{r'_3}\tilde{g}] \end{aligned} \tag{7.31}$$

Now let h be such that

$$h(x_2, x_3) = Be^{B_2x_2}e^{B_3x_3}, \tag{7.32}$$

where $B \neq 0$, $B_2 > 0$, and $B_3 > 0$ will be made precise below. Plugging this h in (7.31), one obtains

$$\begin{aligned} & D^l(\tilde{w}g) = \partial_2^{l_2}\partial_3^{l_3}(\partial_1^{l_1}\tilde{w}g) \\ = \; & BB_2^{l_2}B_3^{l_3}e^{B_2x_2}e^{B_3x_3}[\partial_1^{l_1}\tilde{w}\tilde{g} + (\partial_1^{l_1}\tilde{w}\sum_{0<r\leq s} C'_{r,s}B_2^{-r_2}B_3^{-r_3}\partial_2^{r_2}\partial_3^{r_3}\tilde{g} \\ & + \sum_{0<r<s} C_{r,s}B_2^{-r_2}B_3^{-r_3}\partial_2^{r_2}\partial_3^{r_3}\partial_1^{l_1}\tilde{w}(\tilde{g} + \sum_{0<r'\leq s_r} C'_{r',s_r}B_2^{-r'_2}B_3^{-r'_3}\partial_2^{r'_2}\partial_3^{r'_3}\tilde{g}))]. \end{aligned} \tag{7.33}$$

Now set

$$\begin{aligned} \theta = \; & \partial_1^{l_1}\tilde{w}\sum_{0<r\leq s} C'_{r,s}B_2^{-r_2}B_3^{-r_3}\partial_2^{r_2}\partial_3^{r_3}\tilde{g} + \sum_{0<r<s} C_{r,s}B_2^{-r_2}B_3^{-r_3}\partial_2^{r_2}\partial_3^{r_3}\partial_1^{l_1}\tilde{w}(\tilde{g} \\ & + \sum_{0<r'\leq s_r} C'_{r',s_r}B_2^{-r'_2}B_3^{-r'_3}\partial_2^{r'_2}\partial_3^{r'_3}\tilde{g}) \end{aligned} \tag{7.34}$$

Let B_1 be a number to be determined below. Taking

$$B = \frac{B_1}{B_2^{l_2}B_3^{l_3}}, \tag{7.35}$$

in (7.32) and using (7.33) and (7.34), one obtains

$$\begin{aligned} & D^l(\tilde{w}g) = \partial_2^{l_2}\partial_3^{l_3}(\partial_1^{l_1}\tilde{w}g) \\ = \; & BB_2^{l_2}B_3^{l_3}e^{B_2x_2}e^{B_3x_3}(\partial_1^{l_1}\tilde{w}\tilde{g} + \theta) \\ = \; & B_1e^{B_2x_2}e^{B_3x_3}(\partial_1^{l_1}\tilde{w}\tilde{g} + \theta). \end{aligned} \tag{7.36}$$

Using (7.36) and (7.21), one obtains for any $j \in \{1, \cdots, p\}$,

$$\begin{aligned} I_j = \; & \int_{\mathcal{V}(j)} \phi(\tau)(\beta_j D^l(\tilde{w}g)\frac{\partial_1 F}{|\nabla_{x,t}F|})(f_j(x'), x')(1 + |\nabla_{x,t}f_j(x')|^2)^{1/2}dx' \\ = \; & B_1\int_{\mathcal{V}(j)} \phi(\tau)(\beta_j e^{B_2x_2}e^{B_3x_3}(\partial_1^{l_1}\tilde{w}\tilde{g} + \theta)\frac{\partial_1 F}{|\nabla_{x,t}F|})(f_j(x'), x') \\ & \times(1 + |\nabla_{x,t}f_j(x')|^2)^{1/2}dx' \end{aligned} \tag{7.37}$$

By the beginning of this step, one assumed that $\partial_1^{l_1}\omega_1$ is not identically 0 on $\tilde{\Gamma}^+$ and $\phi \not\equiv 0$ on $\text{pr}_t((\{\partial_1^{l_1}\omega_1 \neq 0\} \cap \tilde{\Gamma}^+)$. Using the regularity of ϕ, F, ω_1, and β_j, the construction of the partition of unity β_j and the definition of $\tilde{\Gamma}^+$; See Step 2 above; shows that for some $j_0 \in \{1, \cdots, p\}$, one has either (1) $\partial_1^{l_1}\omega_1 > 0$ on some open set $\mathcal{O} \subset \text{spt}(\beta_{j_0}) \cap \{\partial_1 F > 0\}$ containing an $N-$dimensional submanifold of $\tilde{\Gamma}^+$ and $\phi > 0$ on an open subinterval of $\text{pr}_t(\mathcal{O})$ and $\mathcal{O} \cap \text{spt}(\beta_j) = \emptyset$ for any $j \in \{0, \cdots, p\}$ with $j \neq j_0$; or (2) $\partial_1^{l_1}\omega_1 > 0$ on some open set $\mathcal{O} \subset \text{spt}(\beta_{j_0}) \cap \{\partial_1 F < 0\}$ containing an $N-$dimensional submanifold of $\tilde{\Gamma}^+$ and $\phi > 0$ on an open subinterval of $\text{pr}_t(\mathcal{O})$ and $\mathcal{O} \cap \text{spt}(\beta_j) = \emptyset$ for any $j \in \{0, \cdots, p\}$ with $j \neq j_0$; or (3) $\partial_1^{l_1}\omega_1 < 0$ on some open set $\mathcal{O} \subset \text{spt}(\beta_{j_0}) \cap \{\partial_1 F > 0\}$ containing an $N-$dimensional submanifold of $\tilde{\Gamma}^+$ and $\phi > 0$ on an open subinterval of $\text{pr}_t(\mathcal{O})$ and $\mathcal{O} \cap \text{spt}(\beta_j) = \emptyset$ for any $j \in \{0, \cdots, p\}$ with $j \neq j_0$; or (4) $\partial_1^{l_1}\omega_1 < 0$ on some open set $\mathcal{O} \subset \text{spt}(\beta_{j_0}) \cap \{\partial_1 F < 0\}$

containing an $N-$dimensional submanifold of $\tilde{\Gamma}^+$ and $\phi > 0$ on an open subinterval of $\mathrm{pr}_t(\mathcal{O})$, or any of the above four possibilities with $\phi > 0$ replaced by $\phi < 0$; or none of the above.

If none of the 8 cases invoked above is true, then necessarily $\phi \equiv 0$ on $\mathrm{pr}_t(\{\partial_1^{l_1}\omega_1 \neq 0\} \cap \tilde{\Gamma}^+)$ or $\partial_1^{l_1}\omega_1 \equiv 0$ on $\tilde{\Gamma}^+$. Recall that by Step 2, $\tilde{\Gamma}^+ = \Gamma_v^+ \cap \{\partial_1 F \neq 0\}$. However, this contradicts the assumptions made on ϕ and $\partial_1^{l_1}\omega_1$ above. Hence, this 9th case cannot happen and so the only possibilities are the 8 cases invoked above.

Now assume that Case (1) is true; that is, $\partial_1^{l_1}\omega_1 > 0$ on some open set $\mathcal{O} \subset \mathrm{spt}(\beta_{j_0}) \cap \{\partial_1 F > 0\}$ containing an $N-$dimensional submanifold of $\tilde{\Gamma}^+$ and $\phi > 0$ on an open subinterval of $\mathrm{pr}_t(\mathcal{O})$ and $\mathcal{O} \cap \mathrm{spt}(\beta_j) = \emptyset$ for any $j \in \{0, \cdots, p\}$ with $j \neq j_0$. The other cases can be deduced by appropriately adapting the reasoning developed below.

For $j = 1, \cdots, p$ with $j \neq j_0$ set

$$\alpha_j = |\int_{\mathcal{V}^{(j)}} \phi(\tau)(\beta_j e^{B_2 x_2} e^{B_3 x_3}(\partial_1^{l_1}\tilde{w}\tilde{g} + \theta)\frac{\partial_1 F}{|\nabla_{x,t} F|})(f_j(x'), x') \\ \times (1 + |\nabla f_j(x')|^2)^{1/2} dx'|.$$

Then select $\tilde{g}$ so that $\tilde{g} > c > 0$ on the projection into $\mathbb{R}^2 \times (t_1, t_2)$ of $\mathcal{O}$ with c a constant and for B_2 and B_3 large, one has

$$I'_{j_0} > \sum_{j=1, j\neq j_0}^{p} \alpha_j, \tag{7.38}$$

where

$$I'_j = \int_{\mathcal{V}^{(j)}} \phi(\tau)(\beta_j e^{B_2 x_2} e^{B_3 x_3}(\partial_1^{l_1}\tilde{w}\tilde{g} + \theta)\frac{\partial_1 F}{|\nabla_{x,t} F|})(f_j(x'), x') \\ \times (1 + |\nabla f_j(x')|^2)^{1/2} dx'. \tag{7.39}$$

Above, $\partial_1^{l_1}\tilde{w} = \partial_1^{l_1}\omega_1$ if $l_1 \neq 0$ and $\partial_1^{l_1}\tilde{w} = \omega_1 + A$ if $l_1 = 0$. Let g be given by (7.28) with $\tilde{g}$ the function constructed above with the projection of its support and those of its partial derivatives into $\mathbb{R}^2$ equal $\mathbb{R}^2$ and h given by (7.32), where B is given by (7.35) with B_2 and B_3 large enough to satisfy (7.38) and $B_1 \neq 0$ is determined below. Now using this, (7.37), (7.39), and (7.22) with the above construction, one obtains

$$\int_{\Gamma_v^+} \phi D^l(\tilde{w}(g - g_1)) n_1 d\mathcal{H}^N = B_1 \sum_{j=1}^{p} I'_j - \int_{\Gamma_v^+} \phi D^l(\tilde{w}g_1) n_1 d\mathcal{H}^N. \tag{7.40}$$

Since by (7.38), $\sum_{j=1}^p I'_j \neq 0$, one can take

$$B_1 = \frac{1}{\sum_{j=1}^p I'_j} \int_{\Gamma_v^+} \phi D^l(\tilde{w}g_1) n_1 d\mathcal{H}^N. \tag{7.41}$$

Then using (7.40) yields the first equality in (7.42) below

$$\int_{\Gamma_v^+} \phi D^l(\tilde{w}(g - g_1)) n_1 d\mathcal{H}^N = 0, \quad \int_{\{v>0\}} \phi \partial_1(D^l(\tilde{w}(g - g_1))) dx = 0. \tag{7.42}$$

Using (7.18) and the first equality in (7.42), yields the second equality in (7.42).

4. In this step, the proof of Part *(2)* of the theorem in the following cases will be concluded: (1) $l = 0$; (2) $\int_{\Gamma_v^+} \phi D^l(\tilde{w} g_1) n_1 d\mathcal{H}^N = 0$; and (3) $l \neq 0$, $\int_{\Gamma_v^+} \phi D^l(\tilde{w} g_1) n_1 d\mathcal{H}^N \neq 0$, $\partial_1^{l_1}\omega_1 \not\equiv 0$ on $\tilde{\Gamma}^+$, and $\phi \not\equiv 0$ on $\{\partial_1^{l_1}\omega_1 \neq 0\} \cap \tilde{\Gamma}^+$.

In case (1), using (7.24) and (7.18) of Step 2 yields (7.14) of Step 1. In case (2), using (7.25) and (7.18) of Step 2 yields (7.14) of Step 1. In case (3), (7.42) of Step 3 above yields (7.14) of Step 1. Hence by Step 1, (7.15) is satisfied, and so, one has

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\omega_1 u)\}} - \int \phi d\mu^+_{\{D^l\omega_1, D^l(u_1\omega)\}} = \int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}G)\}}. \tag{7.43}$$

By Part *(1)* of the theorem, one has

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\omega_1 u)\}} = 0. \tag{7.44}$$

Now it will be proved that

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}G)\}} = 0. \tag{7.45}$$

By definition of $\tilde{w}$ and the properties of the above measures, one has

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}G)\}} = \int \phi d\mu^+_{\{D^l\omega_1, D^l(\omega_1 G)\}} + \int \phi d\mu^+_{\{D^l\omega_1, D^l(AG)\}}. \tag{7.46}$$

By the properties of ω_1, its support in x is compact; See the beginning of the proof. Let m be so large that $\psi_m \equiv 1$ on this compact. Then using Part *(1)* of Theorem 3.3, one obtains

$$\begin{aligned} &\int \phi d\mu^+_{\{D^l\omega_1, D^l(AG)\}} = \int \phi\psi_m d\mu^+_{\{D^l\omega_1, D^l(AG)\}} \\ = \; &-\int_{\{D^l\omega_1>0\}} \mathrm{div}(\phi\psi_m D^l(AG))dy = -\int_{\{D^l\omega_1>0\}} A\phi \mathrm{div}(D^l G)dy = 0. \end{aligned} \tag{7.47}$$

Here, a use of the fact that div $G = 0$ has been made.

Now the functions h_2 and h_3 will be selected so that for each $k \in \mathbb{N}^3$ with $k \leq l$, the projection of the support of $D^k G$ into $\mathbb{R}^3$ is $\mathbb{R}^3$. Let $a_1, a_2 > 0$ and $a_3 < 0$ be such that $a_1 + x_1 > 0$, $a_2 + x_2 > 0$, and $a_3 + x_3 < 0$ for all $x \in \mathrm{spt}_x(\omega_1)$. Let α_1 and α_2 be positive numbers. Set

$$h_2 = e^{\alpha_1(a_1+x_1)+\alpha_2(a_2+x_2)-\alpha_2(a_3+x_3)}, \quad h_3 = h_2. \tag{7.48}$$

Then $\partial_2 h_2 + \partial_3 h_3 = \alpha_2 h_2 - \alpha_2 h_3 = 0$. Thus, h_2 and h_3 satisfy (7.9) of Step 1. Let $k \in \mathbb{N}^3$ be such $k \leq l$. For $k = 0$, using the fact that $\mathrm{spt}_x(u)$ is compact and the regularity of u, one obtains for α_1 and α_2 large,

$$\frac{1}{\tilde{w}}((\omega_1 u_2 - u_1\omega_2) + h_2) > 0$$

for all $(x,\tau) \in \mathbb{R}^3 \times (t_1, t_2)$. For $k \neq 0$, one has

$$\begin{aligned} D^k G_2 &= D^k(\frac{1}{\tilde{w}}(\omega_1 u_2 - u_1\omega_2)) + D^k(\frac{h_2}{\tilde{w}}) \\ &= D^k(\frac{1}{\tilde{w}}(\omega_1 u_2 - u_1\omega_2)) + \alpha_1^{k_1}\alpha_2^{k_2}(-\alpha_2)^{k_3}\frac{h_2}{\tilde{w}} + \sum_{0\le r<k} C_{r,k} D^r h_2 D^{k-r}(\frac{1}{\tilde{w}}), \end{aligned} \tag{7.49}$$

where $C_{r,k}$ where introduced in (6.16). For k_3 even, using the fact that $\mathrm{spt}_x(u)$ is compact, and (7.49) one obtains for α_1 and α_2 large, $D^k G_2(x,\tau) > 0$ for all $(x,\tau) \in \mathbb{R}^3 \times (t_1,t_2)$. For k_3 odd, using the fact that $\mathrm{spt}_x(u)$ is compact, and (7.49) one obtains for α_1 and α_2 large, $D^k G_2(x,\tau) < 0$ for all $(x,\tau) \in \mathbb{R}^3 \times (t_1,t_2)$. Therefore, the above shows that for α_1 and α_2 sufficiently large, one has $D^k G_2(x,\tau) \neq 0$ for all $(x,\tau) \in \mathbb{R}^3 \times (t_1,t_2)$ and for all $k \in \mathbb{N}^3$ with $k \le l$. Thus, with the above choice of the parameters, the projection of the support of $D^k G$ into $\mathbb{R}^3$ is $\mathbb{R}^3$ for all $k \in \mathbb{N}^3$ with $k \le l$. This choice of G is the one that will be used throughout the rest of the proof.

Now proceeding as in the proof of Part *(1)* of the theorem with appropriate adaptations with (l,ω_1,u) of Part *(1)* of the theorem replaced by (l,ω_1,G) of this step, one obtains

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\omega_1 G)\}} = 0. \tag{7.50}$$

Then combining (7.46), (7.47), and (7.50), one obtains

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w} G)\}} = 0, \tag{7.51}$$

which corresponds to (7.45). Now combining (7.43), (7.44), and (7.51), one obtains

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(u_1\omega)\}} = 0. \tag{7.52}$$

which corresponds to the first statement in Part *(2)* of the theorem for $i = 1$ in the following cases: (1) $l = 0$; (2) $\int_{\Gamma_v^+} \phi D^l(\tilde{w} g_1) n_1 d\mathcal{H}^N = 0$; and (3) $l \neq 0$, $\int_{\Gamma_v^+} \phi D^l(\tilde{w} g_1) n_1 d\mathcal{H}^N \neq 0$, $\partial_1^{l_1}\omega_1 \not\equiv 0$ on $\tilde{\Gamma}^+$, and $\phi \not\equiv 0$ on $\mathrm{pr}_t(\{\partial_1^{l_1}\omega_1 \neq 0\} \cap \tilde{\Gamma}^+)$. Proceeding as above for the measure $\mu^-_{\{D^l\omega_1, D^l(u_1\omega)\}}$ with appropriate adaptations, one obtains the second statement in Part *(2)* of the theorem for $i = 1$ in Cases (1)-(3) above. Now using Part *(2)* of Theorem 3.1 yields the third statement in Part *(2)* of the theorem for $i = 1$ in Cases (1)-(3) above. In Step 5, the proof of Part *(2)* of the theorem for $i = 1$ in the case (4) will be given.

5. In this step, the proof of Part *(2)* of the theorem for $i = 1$ in the case (4) where $l \neq 0$, $\int_{\Gamma_v^+} \phi D^l(\tilde{w} g_1) n_1 d\mathcal{H}^N \neq 0$, and $\partial_1^{l_1}\omega_1 \equiv 0$ on $\tilde{\Gamma}^+$ will be given. By definition, $\tilde{w} = \omega_1 + A > \alpha > 0$ on K. Hence, l_1 cannot be 0. Since $\partial_1^{l_1}\tilde{w} \equiv 0$ on $\tilde{\Gamma}^+$ and $\tilde{w} = \omega_1 + A > \alpha > 0$, one has necessarily $\partial_1^m \tilde{w} \not\equiv 0$ on $\tilde{\Gamma}^+$ for some nonnegative integer m and $\partial_1^r \tilde{w} \equiv 0$ on $\tilde{\Gamma}^+$ for all $m < r \le l_1$.

5.1 Let G_2 and G_3 be the functions given by (7.10). Let G_1 be the function given by (7.11). Let m be the nonnegative integer introduced above. Let $\tilde{G}_1$ be the function given by

$$\tilde{G}_1(x,\tau) = G_1(x,\tau) + x_1^{l_1-m}\bar{g}_2(b,x_2,x_3,\tau), \tag{7.53}$$

where b is a constant and $\bar{g}_2$ is a function in $C^\infty(\mathbb{R}^3 \times (t_1,t_2);\mathbb{R})$. Define g_2 by $g_2(x,\tau) = \bar{g}_2(b,x_2,x_3,\tau)$ for every $(x,\tau) \in \mathbb{R}^3 \times (t_1,t_2)$. In this step, the function g_2 is built so that

$$\int \phi D^l(\tilde{w}\tilde{G}_1)d\mu^+_{\{D^l\omega_1,e_1\}} = 0. \tag{7.54}$$

Proceeding as in the proof of (7.22) of Step 2, one obtains

$$\int_{\Gamma_v^+} \phi D^l(\tilde{w}(g + x_1^{l_1-m}g_2 - g_1))n_1 d\mathcal{H}^N = \sum_{j=1}^{p} I_j - \int_{\Gamma_v^+} \phi D^l(\tilde{w}g_1)n_1 d\mathcal{H}^N, \tag{7.55}$$

where

$$I_j = \int_{\mathcal{V}(j)} \phi(\tau)(\beta_j D^l(\tilde{w}(g + x_1^{l_1-m}g_2))\frac{\partial_1 F}{|\nabla_{x,t}F|})(f_j(x'), x')(1 + |\nabla_{x,t}f_j(x')|^2)^{1/2}dx'. \tag{7.56}$$

Then using the fact that $\partial_1^m\tilde{w} \not\equiv 0$ on $\tilde{\Gamma}^+$ and $\partial_1^r\tilde{w} \equiv 0$ on $\tilde{\Gamma}^+$ for all $m < r \leq l_1$ and the fact that g and g_2 are independent of x_1, one has on $\tilde{\Gamma}^+$

$$\begin{aligned}
& \partial_1^{l_1}(\tilde{w}(g + x_1^{l_1-m}g_2)) = \partial_1^{l_1}(\tilde{w}g) + \partial_1^{l_1}(\tilde{w}x_1^{l_1-m}g_2) \\
= \;& \partial_1^{l_1}\tilde{w}g + \partial_1^{l_1}(\tilde{w}x_1^{l_1-m})g_2 = \partial_1^{l_1}(\tilde{w}x_1^{l_1-m})g_2 \\
= \;& g_2 \sum_{0\leq s\leq l_1} C_{s,l_1}\partial_1^s\tilde{w}\partial_1^{l_1-s}x_1^{l_1-m} \\
= \;& g_2 \sum_{0\leq s\leq m} C_{s,l_1}\partial_1^s\tilde{w}\partial_1^{l_1-s}x_1^{l_1-m} \\
= \;& g_2 C_{m,l_1}\partial_1^m\tilde{w}(l_1-m)!
\end{aligned} \tag{7.57}$$

Since $\partial_1^m\tilde{w} \not\equiv 0$ on $\tilde{\Gamma}^+$ and $\phi \neq 0$ on $\mathrm{pr}_t(\{\partial_1^m\tilde{w} \neq 0\} \cap \tilde{\Gamma}^+)$, one can proceed as in Step 3 with appropriate adaptations and build g_2 to satisfy

$$\int_{\Gamma_v^+} \phi D^l(\tilde{w}(g + x_1^{l_1-m}g_2 - g_1))n_1 d\mathcal{H}^N = 0. \tag{7.58}$$

By Part *(1)* of Theorem 3.3 and Part *(1)* of Theorem 3.4,

$$\begin{aligned}
& \int \phi D^l(\tilde{w}\tilde{G}_1)d\mu^+_{\{D^l\omega_1,e_1\}} = -\int_{\{v>0\}} \phi\partial_1(D^l(\tilde{w}(g + x_1^{l_1-m}g_2 - g_1)))dxd\tau \\
= \;& -\int_{\Gamma_v^+} \phi D^l(\tilde{w}(g + x_1^{l_1-m}g_2 - g_1))n_1 d\mathcal{H}^N.
\end{aligned} \tag{7.59}$$

Then using (7.59) and (7.58), one obtains (7.54); that is,

$$\int \phi D^l(\tilde{w}\tilde{G}_1)d\mu^+_{\{D^l\omega_1,e_1\}} = 0. \tag{7.60}$$

5.2 Here, one proceeds as in Step 1. Let $\tilde{G}$ denote the vector of components $\tilde{G}_1$, G_2, and G_3. By Part *(1)* of Theorem 3.3,

$$\begin{aligned}
&\int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}\tilde{G})\}} = -\int_{\{D^l\omega_1>0\}} \mathrm{div}(\phi D^l(\tilde{w}\tilde{G}))dxd\tau \\
= &-\int_{\{D^l\omega_1>0\}} \phi(\partial_1(D^l(\tilde{w}\tilde{G}_1)) + \partial_2(D^l(\tilde{w}G_2)) + \partial_3(D^l(\tilde{w}G_3)))dxd\tau \\
= &-\int_{\{D^l\omega_1>0\}} \phi(\partial_1(D^l(\tilde{w}(g + x_1^{l_1-m}g_2 - g_1))) + \partial_2(D^l(\omega_1 u_2 - u_1\omega_2)) \\
&+\partial_3(D^l(\omega_1 u_3 - u_1\omega_3)))dxd\tau. \qquad (7.61)
\end{aligned}$$

Here, a use of the fact that $\partial_2 h_2 + \partial_3 h_3 = 0$ has been made. Also, by Part *(1)* of Theorem 3.3,

$$\begin{aligned}
&\int \phi d\mu^+_{\{D^l\omega_1, D^l(\omega_1 u)\}} - \int \phi d\mu^+_{\{D^l\omega_1, D^l(u_1\omega)\}} \\
= &-\int_{\{D^l\omega_1>0\}} \mathrm{div}(\phi D^l(\omega_1 u))dxd\tau + \int_{\{D^l\omega_1>0\}} \mathrm{div}(\phi D^l(u_1\omega))dxd\tau \\
= &-\int_{\{D^l\omega_1>0\}} \mathrm{div}(\phi D^l(\omega_1 u - u_1\omega))dxd\tau \\
= &-\int_{\{D^l\omega_1>0\}} \phi(\partial_2(D^l(\omega_1 u_2 - u_1\omega_2)) + \partial_3(D^l(\omega_1 u_3 - u_1\omega_3)))dxd\tau. \qquad (7.62)
\end{aligned}$$

Also, by Part *(1)* of Theorem 3.3,

$$\int \phi D^l(\tilde{w}\tilde{G}_1) d\mu^+_{\{D^l\omega_1, e_1\}} = -\int_{\{D^l\omega_1>0\}} \phi\partial_1(D^l(\tilde{w}(g + x_1^{l_1-m}g_2 - g_1)))dxd\tau. \qquad (7.63)$$

Hence, using (7.60)-(7.63), one obtains

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\omega_1 u)\}} - \int \phi d\mu^+_{\{D^l\omega_1, D^l(u_1\omega)\}} = \int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}\tilde{G})\}}. \qquad (7.64)$$

5.3 In this step it will be proved that

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}\tilde{G})\}} = 0. \qquad (7.65)$$

By definition of $\tilde{G}$ and G and the properties of these measures, one has

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}\tilde{G})\}} = \int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}G)\}} + \int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}x_1^{l_1-m}g_2 e_1)\}}. \qquad (7.66)$$

The calculation in (7.57) shows that $\partial_1^{l_1}(\tilde{w}x_1^{l_1-m}g_2) = C_{m,l_1}(l_1-m)!\partial_1^m\tilde{w}g_2$. Hence using Leibniz formula and the properties of the above measures one has,

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}x_1^{l_1-m}g_2 e_1)\}} = C_{m,l_1}(l_1-m)! \sum_{0\le r\le \tilde{l}} C_{r,\tilde{l}} \int \phi D^r\partial_1^m\tilde{w} d\mu^+_{\{D^l\omega_1, D^{\tilde{l}-r}\Psi\}}, \qquad (7.67)$$

where $\Psi = (g_2, 0, 0)^t$, $\tilde{l} = (0, l_2, l_3)$, $C_{r,\tilde{l}}$ are given by (6.16). Since $g_2 \in C^\infty(\mathbb{R}^3 \times (t_1,t_2); \mathbb{R})$ and g_2 is independent of x_1, $\Psi \in C^\infty(\mathbb{R}^3 \times (t_1,t_2); \mathbb{R}^3)$ and div $\Psi = 0$. Moreover, by construction of g_2; See Step 5.1 above, one can select it so that the projection of the support of $D^{l'}\Psi$ into $\mathbb{R}^3$ is equal to $\mathbb{R}^3$ for all $l' = (0, l'_2, l'_3) \in \mathbb{N}^3$.

Let $0 \le r \le \tilde{l}$. Set $s = (m, r_2, r_3)$ then $s \le l$. If $s \ne 0$, then $D^r\partial_1^m\tilde{w} = D^s\tilde{w} = D^s\omega_1$ and $D^{l-s}D^s\tilde{w} = D^l\tilde{w} = D^l\omega_1$. Then one can apply Theorem 5.19 with $(\tilde{w}, \tilde{v}, U)$ of Theorem 5.19 replaced by $(D^s\tilde{w}, D^l\omega_1, D^{\tilde{l}-r}\Psi)$ and then obtains

$$\int \phi D^r\partial_1^m\tilde{w}d\mu^+_{\{D^l\omega_1, D^{\tilde{l}-r}\Psi\}} = 0. \tag{7.68}$$

If $s = 0$, then by definition of $\tilde{w}$,

$$\int \phi D^r\partial_1^m\tilde{w}d\mu^+_{\{D^l\omega_1, D^{\tilde{l}-r}\Psi\}} = \int \phi\omega_1 d\mu^+_{\{D^l\omega_1, D^{\tilde{l}}\Psi\}} + A\int \phi d\mu^+_{\{D^l\omega_1, D^{\tilde{l}}\Psi\}}. \tag{7.69}$$

Using the fact that the projection of the support of ω_1 into $\mathbb{R}^3$ is compact and the fact already established above that the projection of the support of $D^{\tilde{l}}\Psi$ into $\mathbb{R}^3$ is equal to $\mathbb{R}^3$, one can apply Theorem 5.19 with $(\tilde{w}, \tilde{v}, U)$ of Theorem 5.19 replaced by $(\omega_1, D^l\omega_1, D^{\tilde{l}}\Psi)$. One then obtains

$$\int \phi\omega_1 d\mu^+_{\{D^l\omega_1, D^{\tilde{l}}\Psi\}} = 0. \tag{7.70}$$

Now proceeding as in the proof of (7.47) of Step 4, one obtains

$$\int \phi d\mu^+_{\{D^l\omega_1, D^{\tilde{l}}\Psi\}} = 0. \tag{7.71}$$

Using (7.69)-(7.71), one obtains

$$\int \phi D^r\partial_1^m\tilde{w}d\mu^+_{\{D^l\omega_1, D^{\tilde{l}-r}\Psi\}} = 0. \tag{7.72}$$

Then combining (7.67), (7.72), and (7.68) yields

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}x_1^{l_1-m}g_2e_1)\}} = 0. \tag{7.73}$$

Now by (7.51) of Step 4, one obtains

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}G)\}} = 0, \tag{7.74}$$

Combining (7.66), (7.73), and (7.74), one obtains (7.65); that is,

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\tilde{w}\tilde{G})\}} = 0. \tag{7.75}$$

5.4 By Part *(1)* of the theorem, one has

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(\omega_1 u)\}} = 0. \tag{7.76}$$

Combining (7.64), (7.75), and (7.76), one obtains

$$\int \phi d\mu^+_{\{D^l\omega_1, D^l(u_1\omega)\}} = 0. \tag{7.77}$$

which corresponds to the first statement in Part *(2)* of the theorem for $i = 1$ in the case where $l \neq 0$, $\int_{\Gamma_v^+} \phi D^l(\tilde{w} g_1) n_1 d\mathcal{H}^N \neq 0$, and $\partial_1^{l_1}\omega_1 \equiv 0$ on $\tilde{\Gamma}^+$. Proceeding as above for the measure $\mu^-_{\{D^l\omega_1, D^l(u_1\omega)\}}$ with appropriate adaptations, one obtains the second statement in Part *(2)* of the theorem for $i = 1$ in the above case. Now using Part *(2)* of Theorem 3.1 yields the third statement in Part *(2)* of the theorem for $i = 1$ in the above case. This completes the proof of Part *(2)* of the theorem for $i = 1$ in the case where $l \neq 0$, $\int_{\Gamma_v^+} \phi D^l(\tilde{w} g_1) n_1 d\mathcal{H}^N \neq 0$, and $\partial_1^{l_1}\omega_1 \equiv 0$ on $\tilde{\Gamma}^+$.

6. Here, Case (5) will be studied. Case (5) corresponds to: $l \neq 0$, $\int_{\Gamma_v^+} \phi D^l(\tilde{w} g_1) n_1 d\mathcal{H}^N \neq 0$, $\partial_1^{l_1}\omega_1 \not\equiv 0$ on $\tilde{\Gamma}^+$, and $\phi \equiv 0$ on $pr_t(\{\partial_1^{l_1}\omega_1 \neq 0\} \cap \tilde{\Gamma}^+)$. Recall that in Case (5) it is also assumed that $\phi \not\equiv 0$ on $\tilde{\Gamma}^+$; See Step 2 above. Since $\phi \equiv 0$ on $pr_t(\{\partial_1^{l_1}\omega_1 \neq 0\} \cap \tilde{\Gamma}^+)$, one deduces that ϕ is not identically 0 only on $pr_t(\{\partial_1^{l_1}\omega_1 \equiv 0\}$. Then proceeding as in Step 5 above with appropriate adaptations, one obtains the proof for this case.

7. Steps 2-6 yield the proof of Part *(2)* of the theorem for $i = 1$. Proceeding as in Steps 1-6 above for $i = 2$ and $i = 3$ with appropriate adaptations, one obtains the three statements of Part *(2)* of the theorem for $i = 2$ and $i = 3$. This completes the proof of the theorem. ∎

Theorem 7.2. *Let $N = 3$. Let $0 \leq t_1 < t_2$. Let $u \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$ be such that div $u = 0$ and for each $s \in \mathbb{N}^N$, $D^s u \in C^\infty((t_1, t_2); L^1(\mathbb{R}^N))^N$. Let $\omega =$ curl u. Let $l \in \mathbb{N}^N$. Let $i \in \{1, 2, 3\}$. Let $\phi \in C_c(t_1, t_2)$. Then, up to a subsequence,*

(1)

$$\lim_{m\to\infty} \int \phi\psi_m d\mu^+_{\{D^l\omega_i, D^l(\omega_i u)\}} = 0, \quad \lim_{m\to\infty} \int \phi\psi_m d\mu^-_{\{D^l\omega_i, D^l(\omega_i u)\}} = 0,$$
$$\lim_{m\to\infty} \int \phi\psi_m d\mu_{\{D^l\omega_i, D^l(\omega_i u)\}} = 0.$$

(2)

$$\lim_{m\to\infty} \int \phi\psi_m d\mu^+_{\{D^l\omega_i, D^l(u_i\omega)\}} = 0, \quad \lim_{m\to\infty} \int \phi\psi_m d\mu^-_{\{D^l\omega_i, D^l(u_i\omega)\}} = 0,$$
$$\lim_{m\to\infty} \int \phi\psi_m d\mu_{\{D^l\omega_i, D^l(u_i\omega)\}} = 0.$$

Above, $\mu^+_{\{v,U\}}$, $\mu^-_{\{v,U\}}$, and $\mu_{\{v,U\}}$ denote the measures on $\mathbb{R}^N \times (t_1, t_2)$ given by Theorem 3.1 with (v, U) of Theorem 3.1 corresponding to $(D^l\omega_i, D^l(u_i\omega))$ of this theorem.

Proof of Theorem 7.2. Let $l \in \mathbb{N}^N$. Let $i \in \{1, 2, 3\}$. Let $\phi \in C_c(t_1, t_2)$. Let $b_1 < b_2$ be such that spt$(\phi) \subset [b_1, b_2] \subset (t_1, t_2)$.

Proof of Part (1)

1. By assumption, for each $s \in \mathbb{N}^N$, $D^s u \in C^\infty((t_1, t_2); L^1(\mathbb{R}^N))^N$. Hence, by Sobolev embeddings, $u \in C^\infty((t_1, t_2); W^{m',q}(\mathbb{R}^N))^N$ for all integers $m' \geq 0$ and all $q \geq 1$. Using the regularity of u and its divergence-free property, one can find a potential vector $\Psi \in C^\infty(\mathbb{R}^N \times (t_1, t_2); \mathbb{R}^N)$ such that $u =$ curl Ψ, $-\Delta\Psi =$ curl u, and div $\Psi = 0$. Now set

$u_p = \text{curl } (\psi_p \Psi)$, where ψ_p was introduced at the beginning of this section. By the regularity of u and the properties of ψ_p, $u_p \in C^\infty(\mathbb{R}^N \times (t_1,t_2);\mathbb{R}^N)$ and for each $s \in \mathbb{N}^N$, $D^s u_p \in C^\infty((t_1,t_2);L^1(\mathbb{R}^N))^N$ for all $p \geq 1$. Moreover, for each integer $p \geq 1$, the projection of the support of u_p into $\mathbb{R}^N$ is compact. Denote this projection by K_p. Let ω_p denote the curl of u_p. Let u_{pj} resp. ω_{pj}, $j = 1,2,3$ denote the components of u_p resp. ω_p. Then $\omega_p = \psi_p\omega + \nabla\psi_p \times u - \Delta\psi_p\Psi + (\Psi\cdot\nabla)\nabla\psi_p - (\nabla\psi_p\cdot\nabla)\Psi$. Moreover, $D^s\omega_p$ resp. $D^s u_p$ converges in $C^{m''}([a,b];W^{m',q}(\mathbb{R}^N))^N$ to $D^s\omega$ resp. $D^s u$ when p goes to ∞ for all $s \in \mathbb{N}^N$, all integers $m',\ m'' \geq 0$ and all $q \geq 1$, and all $t_1 < a < b < t_2$.

2. Set $v_i = D^l\omega_i$ and $v_{pi} = D^l\omega_{pi}$. Set $\Gamma_i^+ = \partial\{v_i > 0\}\setminus\Gamma^s_{v_i,+}$ and $\Gamma^+_{pi} = \partial\{v_{pi} > 0\}\setminus\Gamma^s_{v_{pi},+}$, where $\Gamma^s_{v_i,+}$ resp. $\Gamma^s_{v_{pi},+}$ is the singular set corresponding to v_i resp. v_{pi} introduced in Subsection 3.1. Let $\mu^+_{\{v_{pi},D^l(\omega_{pi}u_p)\}}$ be the measure corresponding to the pair $(v_{pi}, D^l(\omega_{pi}u_p))$ obtained by applying Theorem 3.1. By Parts *(2)* and *(4)* of Theorem 3.4, the measure $\mu^+_{\{v_i,D^l(\omega_i u)\}}$ is concentrated on the C^∞-hypersurface Γ_i^+ and the measure $\mu^+_{\{v_{pi},D^l(\omega_{pi}u)\}}$ is concentrated on the C^∞-hypersurface Γ^+_{pi}. By Part *(1)* of Theorem 3.3,

$$\int \phi\psi_m d\mu^+_{\{v_i,D^l(\omega_i u)\}} = -\int \phi\chi_{\{v_i>0\}}\text{div}(\psi_m D^l(\omega_i u))dy, \tag{7.78}$$

$$\int \phi\psi_{m'} d\mu^+_{\{v_{pi},D^l(\omega_{pi} u_p)\}} = -\int \phi\chi_{\{v_{pi}>0\}}\text{div}(\psi_{m'} D^l(\omega_{pi} u_p))dy. \tag{7.79}$$

Taking $m' > p$, one may replace $\psi_{m'}$ in (7.79) by the function identically equal to 1. By Part *(1)* of Theorem 7.1 with u of Theorem 7.1 corresponding to u_p of this proof, one obtains for any $m > p$,

$$\int \phi\psi_m d\mu^+_{\{D^l\omega_{pi},D^l(\omega_{pi}u_p)\}} = \int \phi d\mu^+_{\{D^l\omega_{pi},D^l(\omega_{pi}u_p)\}} = 0. \tag{7.80}$$

It will be proved that, up to a subsequence,

$$\lim_{m\to\infty}\int \phi\psi_m d\mu^+_{\{v_i,D^l(\omega_i u)\}} = 0. \tag{7.81}$$

3. Let $m > p$. Using (7.78)-(7.80), one obtains

$$\begin{aligned}
&\int \phi\psi_m d\mu^+_{\{D^l\omega_i,D^l(\omega_i u)\}}\\
=\ &\int \phi\psi_m d\mu^+_{\{D^l\omega_i,D^l(\omega_i u)\}} - \int \phi\psi_m d\mu^+_{\{D^l\omega_{pi},D^l(\omega_{pi}u_p)\}}\\
=\ &-\int \phi\chi_{\{v_i>0\}}\text{div}(\psi_m D^l(\omega_i u))dy + \int \phi\chi_{\{v_{pi}>0\}}\text{div}(\psi_m D^l(\omega_{pi}u_p))dy\\
=\ &-\int \phi\chi_{\{v_i>0\}}\nabla\psi_m\cdot D^l(\omega_i u)dy + \int \phi\chi_{\{v_{pi}>0\}}\nabla\psi_m\cdot D^l(\omega_{pi}u_p)dy\\
&-\int \phi\chi_{\{v_i>0\}}\psi_m\text{div } D^l(\omega_i u)dy + \int \phi\chi_{\{v_{pi}>0\}}\psi_m\text{div } D^l(\omega_{pi}u_p)dy.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int \phi\psi_m d\mu^+_{\{D^l\omega_i,D^l(\omega_i u)\}} = -\frac{1}{m}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \phi\chi_{\{v_i>0\}}\nabla\psi(\frac{x}{m})\cdot D^l(\omega_i u)dxd\tau \\
&+\frac{1}{m}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \phi\chi_{\{v_{pi}>0\}}\nabla\psi(\frac{x}{m})\cdot D^l(\omega_{pi} u_p)dxd\tau \\
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \phi(\chi_{\{v_i>0\}}-\chi_{\{v_{pi}>0\}})\psi_m \text{div } D^l(\omega_i u)dxd\tau \\
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \phi\chi_{\{v_{pi}>0\}}\psi_m \text{div}(D^l(\omega_i u)-D^l(\omega_{pi}u_p))dxd\tau.
\end{aligned} \tag{7.82}$$

4. By assumption, for each $s \in \mathbb{N}^N$, $D^s u \in C^\infty((t_1,t_2);L^1(\mathbb{R}^N))^N$. Hence, by Sobolev embeddings and Leibniz formula: $D^l(\omega_i u) \in L^1(\mathbb{R}^N \times (t_1,t_2))^N$, div$(D^l(\omega_i u)) \in L^1(\mathbb{R}^N \times (t_1,t_2))$, $D^l(\omega_{pi}u_p) \in L^1(\mathbb{R}^N \times (t_1,t_2))^N$, and div$(D^l(\omega_{pi}u_p)) \in L^1(\mathbb{R}^N \times (t_1,t_2))$. Here a use of the regularity of Ψ given by the regularity of u, and the properties of ψ_p has been made.

Now the first and second integral in (7.82) satisfy

$$\begin{aligned}
&\frac{1}{m}|\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \phi\chi_{\{v_i>0\}}\nabla\psi(\frac{x}{m})\cdot D^l(\omega_i u)dxd\tau| \le \frac{1}{m}\|\nabla\psi\|_{L^\infty(\mathbb{R}^N)}\int|\phi D^l(\omega_i u)|dy, \\
&\frac{1}{m}|\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \phi\chi_{\{v_{pi}>0\}}\nabla\psi(\frac{x}{m})\cdot D^l(\omega_{pi} u_p)dxd\tau| \le \frac{1}{m}\|\nabla\psi\|_{L^\infty(\mathbb{R}^N)}\int|\phi D^l(\omega_{pi} u_p)|dy.
\end{aligned}$$

By the L^1 integrability established at the beginning of this step, the right sides of the last two inequalities above converge to 0 as m goes to ∞, and so do the left sides. Therefore, as m goes to ∞, the first and second integral in (7.82) go to 0.

As m goes to ∞,

$$\begin{aligned}
&\phi(\chi_{\{v_i>0\}}-\chi_{\{v_{pi}>0\}})\psi_m \text{div } D^l(\omega_i u) \to \phi(\chi_{\{v_i>0\}}-\chi_{\{v_{pi}>0\}})\text{div } D^l(\omega_i u), \\
&\phi\chi_{\{v_{pi}>0\}}\psi_m \text{div}(D^l(\omega_i u)-D^l(\omega_{pi}u_p)) \to \phi\chi_{\{v_{pi}>0\}}\text{div}(D^l(\omega_i u)-D^l(\omega_{pi}u_p)), \\
&|\phi(\chi_{\{v_i>0\}}-\chi_{\{v_{pi}>0\}})\psi_m \text{div } D^l(\omega_i u)| \le |\phi(\chi_{\{v_i>0\}}-\chi_{\{v_{pi}>0\}})\text{div } D^l(\omega_i u)|, \\
&|\phi\chi_{\{v_{pi}>0\}}\psi_m \text{div}(D^l(\omega_i u)-D^l(\omega_{pi}u_p))| \le |\phi\chi_{\{v_{pi}>0\}}\text{div}(D^l(\omega_i u)-D^l(\omega_{pi}u_p))|.
\end{aligned}$$

By the L^1 integrability established at the beginning of this step, the functions in the right sides of the last two inequalities above are in $L^1(\mathbb{R}^N \times (t_1,t_2))$. Therefore, by convergence dominated theorem, as m goes to ∞, the third resp. fourth integral in (7.82) converges to

$$-\int \phi(\chi_{\{v_i>0\}}-\chi_{\{v_{pi}>0\}})\text{div } D^l(\omega_i u)dy \tag{7.83}$$

$$\text{resp. } -\int \phi\chi_{\{v_{pi}>0\}}\text{div}(D^l(\omega_i u)-D^l(\omega_{pi}u_p))dy. \tag{7.84}$$

5. Now p will be let go to ∞. Set $\Omega_i = \{v_i > 0\}$ and $\Omega_{pi} = \{v_{pi} > 0\}$. Then one writes,

$$\begin{aligned}&\int \phi(\chi_{\{v_i>0\}} - \chi_{\{v_{pi}>0\}})\text{div } D^l(\omega_i u)dy \\ =\ & \int \phi\chi_{\Omega_i\cap\Omega_{pi}^c}\text{div } D^l(\omega_i u)dy - \int \phi\chi_{\Omega_{pi}\cap\Omega_i^c}\text{div } D^l(\omega_i u)dy. \end{aligned} \tag{7.85}$$

Let $y \in \Omega_i^c$. If $D^l\omega_i(y) < 0$, then by construction of ω_p; See Step 1, as p goes to ∞, $\chi_{\{D^l\omega_{pi}>0\}}(y)$ goes to 0. If $y \in \{D^l\omega_i = 0\}^o$, then $B(y,\epsilon) \subset \{D^l\omega_i = 0\}^o$ for some $\epsilon > 0$ sufficiently small. For p large, $\psi_p \equiv 1$ on $B(y,\epsilon)$. Hence, $D^l\omega_{pi}(y) = D^l\omega_i(y) = 0$. Therefore, as p goes to ∞, $\chi_{\Omega_{pi}\cap\Omega_i^c}$ converges a.e. to 0. Proceeding similarly, $\chi_{\Omega_i\cap\Omega_{pi}^c}$ converges a.e. to 0. Moreover, both integrands in (7.85) are bounded by $|\text{div } D^l(\omega_i u)|$. By the L^1 integrability obtained at the beginning of Step 4, this function is in $L^1(\mathbb{R}^N \times (t_1,t_2); \mathbb{R})$. Therefore, by dominated convergence theorem, up to a subsequence as p goes to ∞, both integrals in the right side of (7.85) go to 0.

Using Leibniz formula,

$$\begin{aligned}&\int \phi\chi_{\{v_{pi}>0\}}\text{div}(D^l(\omega_i u) - D^l(\omega_{pi}u_p))dy \\ =\ & \sum_{0\le k\le l} C_{kl}\int \phi\chi_{\{v_{pi}>0\}}\text{div}(D^k\omega_i D^{l-k}u - D^k\omega_{pi}D^{l-k}u_p)dy \\ =\ & \sum_{0\le k\le l} C_{kl}\int \phi\chi_{\{v_{pi}>0\}}(\nabla D^k\omega_i\cdot D^{l-k}u - \nabla D^k\omega_{pi}\cdot D^{l-k}u_p)dy \\ =\ & \sum_{0\le k\le l} C_{kl}\int \phi\chi_{\{v_{pi}>0\}}(\nabla(D^k\omega_i - D^k\omega_{pi})\cdot D^{l-k}u + \\ & \qquad \nabla D^k\omega_{pi}\cdot(D^{l-k}u - D^{l-k}u_p))dy,\end{aligned} \tag{7.86}$$

where C_{kl} are the coefficients given in (6.16). Here, a use of the fact that both u and u_p are divergence-free has been made. The last integral in (7.86) is bounded by

$$\begin{aligned}&\sum_{0\le k\le l} C_{kl}\int |\phi|(|\nabla(D^k\omega_i - D^k\omega_{pi})||D^{l-k}u| + |\nabla D^k\omega_{pi}||D^{l-k}u - D^{l-k}u_p|)dy \\ \le\ & \sum_{0\le k\le l} C_{kl}\|\phi\|_{L^\infty(t_1,t_2)}(\|\nabla(D^k\omega_i - D^k\omega_{pi})\|_{L^1(b_1,b_2;L^q(\mathbb{R}^N))}\|D^{l-k}u\|_{L^\infty(b_1,b_2;L^{q'}(\mathbb{R}^N))} + \\ & \|\nabla D^k\omega_{pi}\|_{L^\infty(b_1,b_2;L^{q'}(\mathbb{R}^N))}\|D^{l-k}u - D^{l-k}u_p\|_{L^1(b_1,b_2;L^q(\mathbb{R}^N))})\end{aligned} \tag{7.87}$$

where $q > 1$ and q' its conjugate; that is, $\frac{1}{q} + \frac{1}{q'} = 1$. By the regularity of u and the convergence obtained at the end of Step 1, $\|\nabla D^k\omega_{pi}\|_{L^\infty(b_1,b_2;L^{q'}(\mathbb{R}^N))}$ is bounded by a constant independent of p, $\|D^{l-k}u\|_{L^1(b_1,b_2;L^{q'}(\mathbb{R}^N))}$ is finite independent of p, and $\|\nabla(D^k\omega_i - D^k\omega_{pi})\|_{L^1(b_1,b_2;L^q(\mathbb{R}^N))}$ and $\|D^{l-k}u - D^{l-k}u_p\|_{L^1(b_1,b_2;L^q(\mathbb{R}^N))}$ converge to 0 as p goes to ∞. Hence, as p goes to ∞, the integral in the right side of (7.87) converges to 0, and so does the integral in the left side. Then the integral in (7.86) converges to 0.

6. Combining the convergence as m goes to ∞ of the third resp. fourth integral in (7.82) to the integral in (7.83) resp. (7.84) obtained in Step 4, and the convergence up to a subsequence as p goes to ∞ of the integrals in (7.85) and (7.86) obtained in Step 5, one concludes that up to a subsequence, as m goes to ∞ first and p goes to ∞ second, the third resp. fourth integral in (7.82) converges to 0. The first half of Step 4 shows that as m goes to ∞, the first and second integrals in (7.82) go to 0. Combining these convergence, one concludes that, up to a subsequence, the integral in (7.82) converges to 0; that is,

$$\lim_{m\to\infty} \int_{\mathbb{R}^N\times(t_1,t_2)} \phi\psi_m d\mu^+_{\{v_i, D^l(\omega_i u)\}} = 0.$$

This completes the proof of the first statement in Part (1) of the theorem. Proceeding as above for the measure $\mu^-_{\{v_i,D^l(u_i\omega)\}}$ with appropriate adaptations, one obtains the second statement in Part (1) of the theorem. Now using Part *(2)* of Theorem 3.1 yields the third statement in Part (1) of the theorem.

Proof of Part (2)

1. Here, the notations in the proof of Part (1) of the theorem will be used. Let $\mu^+_{\{v_{pi},D^l(u_{pi}\omega_p)\}}$ be the measure corresponding to the pair $(v_{pi}, D^l(u_{pi}\omega_p))$ obtained by applying Theorem 3.1. By Parts *(2)* and *(4)* of Theorem 3.4, the measure $\mu^+_{\{v_i,D^l(u_i\omega)\}}$ is concentrated on the C^∞-hypersurface Γ_i^+ and the measure $\mu^+_{\{v_{pi},D^l(u_{pi}\omega_p)\}}$ is concentrated on the C^∞-hypersurface Γ^+_{pi}. By Part *(1)* of Theorem 3.3,

$$\int \phi\psi_m d\mu^+_{\{v_i,D^l(u_i\omega)\}} = -\int \phi\chi_{\{v_i>0\}}\text{div}(\psi_m D^l(u_i\omega))dy, \tag{7.88}$$

$$\int \phi\psi_{m'} d\mu^+_{\{v_{pi},D^l(u_{pi}\omega_p)\}} = -\int \phi\chi_{\{v_{pi}>0\}}\text{div}(\psi_{m'} D^l(u_{pi}\omega_p))dy. \tag{7.89}$$

Taking $m' > p$, one may replace $\psi_{m'}$ in (7.89) by the function identically equal to 1. By Part *(2)* of Theorem 7.1 with u of Theorem 7.1 corresponding to u_p of this proof, one obtains for any $m > p$,

$$\int \phi\psi_m d\mu^+_{\{D^l\omega_{pi},D^l(u_{pi}\omega_p)\}} = \int \phi d\mu^+_{\{D^l\omega_{pi},D^l(u_{pi}\omega_p)\}} = 0. \tag{7.90}$$

It will be proved that, up to a subsequence,

$$\lim_{m\to\infty} \int \phi\psi_m d\mu^+_{\{v_i,D^l(u_i\omega)\}} = 0. \tag{7.91}$$

2. Let $m > p$. Using (7.88)-(7.90), one obtains

$$\begin{aligned}
&\int \phi\psi_m d\mu^+_{\{D^l\omega_i,D^l(u_i\omega)\}} \\
=\ & \int \phi\psi_m d\mu^+_{\{D^l\omega_i,D^l(u_i\omega)\}} - \int \phi\psi_m d\mu^+_{\{D^l\omega_{pi},D^l(u_{pi}\omega_p)\}} \\
=\ & -\int \phi\chi_{\{v_i>0\}}\text{div}(\psi_m D^l(u_i\omega))dy + \int \phi\chi_{\{v_{pi}>0\}}\text{div}(\psi_m D^l(u_{pi}\omega_p))dy \\
=\ & -\int \phi\chi_{\{v_i>0\}}\nabla\psi_m\cdot D^l(u_i\omega)dy + \int \phi\chi_{\{v_{pi}>0\}}\nabla\psi_m\cdot D^l(u_{pi}\omega_p)dy \\
& -\int \phi\chi_{\{v_i>0\}}\psi_m\text{div } D^l(u_i\omega)dy + \int \phi\chi_{\{v_{pi}>0\}}\psi_m\text{div } D^l(u_{pi}\omega_p)dy.
\end{aligned}$$

Hence,

$$\int \phi\psi_m d\mu^+_{\{D^l\omega_i, D^l(u_i\omega)\}} = -\frac{1}{m}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\phi\chi_{\{v_i>0\}}\nabla\psi(\frac{x}{m})\cdot D^l(u_i\omega)dxd\tau$$
$$+\frac{1}{m}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\phi\chi_{\{v_{pi}>0\}}\nabla\psi(\frac{x}{m})\cdot D^l(u_{pi}\omega_p)dxd\tau$$
$$-\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\phi(\chi_{\{v_i>0\}}-\chi_{\{v_{pi}>0\}})\psi_m \text{div } D^l(u_i\omega)dxd\tau$$
$$-\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\phi\chi_{\{v_{pi}>0\}}\psi_m \text{div}(D^l(u_i\omega)-D^l(u_{pi}\omega_p))dxd\tau. \tag{7.92}$$

3. By assumption, for each $s\in\mathbb{N}^N$, $D^s u\in C^\infty((t_1,t_2);L^1(\mathbb{R}^N))^N$. Hence, by Sobolev embeddings and Leibniz formula: $D^l(u_i\omega)\in L^1(\mathbb{R}^N\times(t_1,t_2))^N$, div$(D^l(u_i\omega))\in L^1(\mathbb{R}^N\times(t_1,t_2))$, $D^l(u_{pi}\omega_p)\in L^1(\mathbb{R}^N\times(t_1,t_2))^N$, and div$(D^l(u_{pi}\omega_p))\in L^1(\mathbb{R}^N\times(t_1,t_2))$. Here a use of the regularity of Ψ given by the regularity of u, and the properties of ψ_p has been made.

Now the first and second integral in (7.92) satisfy

$$\frac{1}{m}|\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\phi\chi_{\{v_i>0\}}\nabla\psi(\frac{x}{m})\cdot D^l(u_i\omega)dxd\tau|\le\frac{1}{m}\|\nabla\psi\|_{L^\infty(\mathbb{R}^N)}\int|\phi D^l(u_i\omega)|dy,$$
$$\frac{1}{m}|\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\phi\chi_{\{v_{pi}>0\}}\nabla\psi(\frac{x}{m})\cdot D^l(u_{pi}\omega_p)dxd\tau|\le\frac{1}{m}\|\nabla\psi\|_{L^\infty(\mathbb{R}^N)}\int|\phi D^l(u_{pi}\omega_p)|dy.$$

By the L^1 integrability established at the beginning of this step, the right sides of the last two inequalities above converge to 0 as m goes to ∞, and so do the left sides. Therefore, as m goes to ∞, the first and second integral in (7.92) go to 0.

As m goes to ∞,

$$\phi(\chi_{\{v_i>0\}}-\chi_{\{v_{pi}>0\}})\psi_m\text{div } D^l(u_i\omega)\to\phi(\chi_{\{v_i>0\}}-\chi_{\{v_{pi}>0\}})\text{div } D^l(u_i\omega),$$
$$\phi\chi_{\{v_{pi}>0\}}\psi_m\text{div}(D^l(u_i\omega)-D^l(u_{pi}\omega_p))\to\phi\chi_{\{v_{pi}>0\}}\text{div}(D^l(u_i\omega)-D^l(u_{pi}\omega_p)),$$
$$|\phi(\chi_{\{v_i>0\}}-\chi_{\{v_{pi}>0\}})\psi_m\text{div } D^l(u_i\omega)|\le|\phi(\chi_{\{v_i>0\}}-\chi_{\{v_{pi}>0\}})\text{div } D^l(u_i\omega)|,$$
$$|\phi\chi_{\{v_{pi}>0\}}\psi_m\text{div}(D^l(u_i\omega)-D^l(u_{pi}\omega_p))|\le|\phi\chi_{\{v_{pi}>0\}}\text{div}(D^l(u_i\omega)-D^l(u_{pi}\omega_p))|.$$

By the L^1 integrability established at the beginning of this step, the functions in the right sides of the last two inequalities above are in $L^1(\mathbb{R}^N\times(t_1,t_2))$. Therefore, by convergence dominated theorem, as m goes to ∞, the third resp. fourth integral in (7.92) converges to

$$-\int\phi(\chi_{\{v_i>0\}}-\chi_{\{v_{pi}>0\}})\text{div } D^l(u_i\omega)dy \tag{7.93}$$
$$\text{resp. } -\int\phi\chi_{\{v_{pi}>0\}}\text{div}(D^l(u_i\omega)-D^l(u_{pi}\omega_p))dy. \tag{7.94}$$

4. Now p will be let go to ∞. Set $\Omega_i=\{v_i>0\}$ and $\Omega_{pi}=\{v_{pi}>0\}$. Then one writes,

$$\begin{aligned}&\int \phi(\chi_{\{v_i>0\}} - \chi_{\{v_{pi}>0\}})\text{div } D^l(u_i\omega)dy\\ = \ &\int \phi\chi_{\Omega_i\cap\Omega_{pi}^c}\text{div } D^l(u_i\omega)dy - \int \phi\chi_{\Omega_{pi}\cap\Omega_i^c}\text{div } D^l(u_i\omega)dy.\end{aligned}\tag{7.95}$$

Let $y \in \Omega_i^c$. If $D^l\omega_i(y) < 0$, then by construction of ω_p; See Step 1 of the proof of Part (1), as p goes to ∞, $\chi_{\{D^l\omega_{pi}>0\}}(y)$ goes to 0. If $y \in \{D^l\omega_i = 0\}^o$, then $B(y,\epsilon) \subset \{D^l\omega_i = 0\}^o$ for some $\epsilon > 0$ sufficiently small. For p large, $\psi_p \equiv 1$ on $B(y,\epsilon)$. Hence, $D^l\omega_{pi}(y) = D^l\omega_i(y) = 0$. Therefore, as p goes to ∞, $\chi_{\Omega_{pi}\cap\Omega_i^c}$ converges a.e. to 0. Proceeding similarly, $\chi_{\Omega_i\cap\Omega_{pi}^c}$ converges a.e. to 0. Moreover, both integrands in (7.95) are bounded by $|\text{div } D^l(u_i\omega)|$. By the L^1 integrability obtained at the beginning of Step 3, this function is in $L^1(\mathbb{R}^N \times (t_1,t_2); \mathbb{R})$. Therefore, by dominated convergence theorem, up to a subsequence as p goes to ∞, both integrals in the right side of (7.95) go to 0.

Using Leibniz formula,

$$\begin{aligned}&\int \phi\chi_{\{v_{pi}>0\}}\text{div}(D^l(u_i\omega) - D^l(u_{pi}\omega_p))dy\\ = &\sum_{0\le k\le l} C_{kl}\int \phi\chi_{\{v_{pi}>0\}}\text{div}(D^k u_i D^{l-k}\omega - D^k u_{pi}D^{l-k}\omega_p)dy\\ = &\sum_{0\le k\le l} C_{kl}\int \phi\chi_{\{v_{pi}>0\}}(\nabla D^k u_i\cdot D^{l-k}\omega - \nabla D^k u_{pi}\cdot D^{l-k}\omega_p)dy\\ = &\sum_{0\le k\le l} C_{kl}\int \phi\chi_{\{v_{pi}>0\}}(\nabla(D^k u_i - D^k u_{pi})\cdot D^{l-k}\omega +\\ &\qquad \nabla D^k u_{pi}\cdot(D^{l-k}\omega - D^{l-k}\omega_p))dy,\end{aligned}\tag{7.96}$$

where C_{kl} are the coefficients given in (6.16). Here, a use of the fact that both ω and ω_p are divergence-free has been made. The last integral in (7.96) is bounded by

$$\begin{aligned}&\sum_{0\le k\le l} C_{kl}\int |\phi|(|\nabla(D^k u_i - D^k u_{pi})||D^{l-k}\omega| + |\nabla D^k u_{pi}||D^{l-k}\omega - D^{l-k}\omega_p|)dy\\ \le &\sum_{0\le k\le l} C_{kl}\|\phi\|_{L^\infty(t_1,t_2)}(\|\nabla(D^k u_i - D^k u_{pi})\|_{L^1(b_1,b_2;L^q(\mathbb{R}^N))}\|D^{l-k}\omega\|_{L^\infty(b_1,b_2;L^{q'}(\mathbb{R}^N))} +\\ &\|\nabla D^k u_{pi}\|_{L^\infty(b_1,b_2;L^{q'}(\mathbb{R}^N))}\|D^{l-k}\omega - D^{l-k}\omega_p\|_{L^1(b_1,b_2;L^q(\mathbb{R}^N))})\end{aligned}\tag{7.97}$$

where $q > 1$ and q' its conjugate; that is, $\frac{1}{q} + \frac{1}{q'} = 1$. By the regularity of u and the convergence obtained at the end of Step 1 of the proof of Part (1), $\|\nabla D^k u_{pi}\|_{L^\infty(b_1,b_2;L^{q'}(\mathbb{R}^N))}$ is bounded by a constant independent of p, $\|D^{l-k}\omega\|_{L^1(b_1,b_2;L^{q'}(\mathbb{R}^N))}$ is finite independent of p, and $\|\nabla(D^k u_i - D^k u_{pi})\|_{L^1(b_1,b_2;L^q(\mathbb{R}^N))}$ and $\|D^{l-k}\omega - D^{l-k}\omega_p\|_{L^1(b_1,b_2;L^q(\mathbb{R}^N))}$ converge to 0 as p goes to ∞. Hence, as p goes to ∞, the integral in the right side of (7.97) converges to 0, and so does the integral in the left side. Then the integral in (7.96) converges to 0.

5. Combining the convergence as m goes to ∞ of the third resp. fourth integral in (7.92) to the integral in (7.93) resp. (7.94) obtained in Step 3, and the convergence up to a subsequence as p goes to ∞ of the integrals in (7.95) and (7.96) obtained in Step 4, one concludes that up to a subsequence, as m goes to ∞ first and p goes to ∞ second, the third resp. fourth integral in (7.92) converges to 0. The first half of Step 3 shows that as m goes to ∞, the first and second integrals in (7.92) go to 0. Combining these convergence, one concludes that, up to a subsequence, the integral in (7.92) converges to 0; that is,

$$\lim_{m\to\infty}\int_{\mathbb{R}^N\times(t_1,t_2)}\phi\psi_m d\mu^+_{\{v_i,D^l(u_i\omega)\}}=0. \tag{7.98}$$

This completes the proof of the first statement in Part (2) of the theorem. Proceeding as above for the measure $\mu^-_{\{v_i,D^l(u_i\omega)\}}$ with appropriate adaptations, one obtains the second statement in Part (2) of the theorem. Now using Part *(2)* of Theorem 3.1 yields the third statement in Part (2) of the theorem. The proof of Theorem 7.2 is now completed. ∎

8. APPLICATIONS OF THE MEASURE THEORY OF SECTIONS 3-7 TO THE NAVIER-STOKES EQUATIONS IN SPACE DIMENSION 3

The Navier-Stokes equations correspond to the system

$$\partial_t u+(u\cdot\nabla)u-\nu\Delta u+\nabla p=0 \quad \text{in } \mathbb{R}^3\times(0,T), \tag{8.1}$$

$$\text{div } u=0 \quad \text{in } \mathbb{R}^3\times(0,T) \tag{8.2}$$

complemented with the initial condition

$$u(\cdot,0)=u_0(\cdot) \ \text{ in } \mathbb{R}^3, \tag{8.3}$$

where u_0 is a divergence-free vector field, that is; div $u_0 = 0$ in $\mathbb{R}^3$. Above $u = u(x,t)$ denotes the velocity field at the point $x\in\mathbb{R}^3$ and at time $t\in(0,T)$ with $T>0$ and $p=p(x,t)$ denotes the pressure while ν denotes the kinematic viscosity.

The only global solution known to exist for general initial data is the weak solution of Leray [7, 8]; Consult for instance [9, 10]. The existence and uniqueness of a smooth solution to the Navier-Stokes system on a short time interval is known; Consult for instance [9, 10] where more information about available results and references is given.

In this section, the measure theory of Sections 3-7 to prove the existence, regularity and uniqueness of global solutions of the Navier-Stokes equations in $\mathbb{R}^3$ when the initial velocity $u_0\in W^{q,1}(\mathbb{R}^3)^3$ for all integers $q\geq 0$ and div $u_0=0$ is applied.

In Section 8.1, the vorticity-stream formulation of Navier-Stokes equations and the local existence and regularity results for Navier-Stokes equations (8.1)-(8.3) are given. In Section 8.2, based on the measure theory of [12] and Sections 3-7, measures $\nu^+_{v_i}$, $\nu^-_{v_i}$, and ν_{v_i}, $i=1,2,3$, $v_i=D^l\omega_i$, and l is any $3-$multi-index of nonnegative integers, associated with the vorticity-stream formulation of Navier-Stokes equations are constructed. Here, $\omega=$ curl u. Some basic characterizations of the measures $\nu^+_{v_i}$, $\nu^-_{v_i}$, and ν_{v_i}, $i=1,2,3$ are then obtained. In Section 8.3, some basic properties of the measures associated with the convective terms in the vorticity-stream formulation of the Navier-Stokes equations are obtained. In Section 8.4, some basic properties of the measures associated with the stretching terms in the vorticity-stream formulation of the Navier-Stokes equations are obtained. In Section 8.5, further characterizations of the measures $\nu^+_{v_i}$, $\nu^-_{v_i}$, and ν_{v_i}, $i=1,2,3$ are given. In Section 8.6, based on the above characterizations and properties of these measures, global estimates of solutions of Navier-Stokes equations and their partial derivatives are obtained. In Section 8.7, the proof of the existence, the global regularity, and the uniqueness of global solutions of the Navier-Stokes equations are given. Finally, in Section 8.8, it will be proved that the measure theory introduced and developed in [12] and Sections 3-7 of this paper cannot be applied to obtain corresponding global estimates to those obtained in Section 2 for the full space case, in the periodic case. In other words, the global estimates obtained in Section 2 for the full space case are not true in the periodic case.

8.1. **Vorticity-stream formulation of Navier-Stokes equations.**

8.1.1. *Short time existence theorem and local regularity properties of solutions of Navier-Stokes equations.* Taking the divergence of Navier-Stokes equations (2.1) yields

$$-\Delta p = \text{div } U, \tag{8.4}$$

where U is the vector of components $\text{div}(u_i u)$, $i = 1, 2, 3$. The following short time existence theorem holds.

Theorem 8.1. *Let $u_0 \in W^{q,1}(\mathbb{R}^3)^3$ for all integers $q \geq 0$ with div $u_0 = 0$. Then there exists $T_r > 0$ such that there exists a unique solution u and a unique, up to an additive constant, pressure field p satisfying Navier-Stokes equations (2.1)-(2.3) and the following: for all $T \in (0, T_r)$, $u \in C^\infty([0, T]; W^{q,1}(\mathbb{R}^3))^3$ and $\nabla p \in C^\infty([0, T]; W^{q,1}(\mathbb{R}^3))^3$ for all integers $q \geq 0$. In particular, for all $T \in (0, T_r)$, $u \in C^\infty(\mathbb{R}^3 \times [0, T])^3$ and $p \in C^\infty(\mathbb{R}^3 \times [0, T])$. Moreover, the energy equalities in Parts (2)-(4) of Theorem 2.1 hold for all $s \in [0, T)$, for all $t \in [s, T]$, and for all $T \in (0, T_r)$.*

The proof of this short time existence theorem can be deduced from a combination of the method used to establish the existence of a weak solution [7, 8] together with Sobolev embeddings; Consult for instance [9, 10] for further references and review of Navier-Stokes equations.

Remark 8.2. Let $T \in (0, T_r)$. Theorem 8.1 and Sobolev embeddings show that (u, p) satisfies all of the properties stated in Part *(1)* of Theorem 2.1 on $[0, T]$.

8.1.2. *Vorticity-stream formulation of Navier-Stokes equations.* Let $u_0 \in W^{q,1}(\mathbb{R}^3)^3$ for all integers $q \geq 0$ with div $u_0 = 0$. Assume that $f = 0$. Let $T \in (0, T_r)$. Let $\omega_0 =$ curl u_0. Then using Theorem 8.1, the primitive formulation of Navier-Stokes equations (2.1)-(2.3) is equivalent to the following vorticity-stream formulation

$$\partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u - \nu\Delta\omega = 0 \ \text{ in } \mathbb{R}^3 \times (0, T) \tag{8.5}$$

$$\omega|_{t=0} = \omega_0 \ \text{ in } \mathbb{R}^3, \tag{8.6}$$

with $u =$ curl Ψ and div $\Psi = 0$, with Ψ a potential vector satisfying $-\Delta\Psi = \omega$. Here, $\omega =$ curl u is the vector of $\mathbb{R}^3$ of components $\partial_2 u_3 - \partial_3 u_2$, $\partial_3 u_1 - \partial_1 u_3$, and $\partial_1 u_2 - \partial_2 u_1$.

8.2. **Constructions and characterizations of measures associated with Navier-Stokes equations.**

Theorem 8.3. *($N = 3$) Let u and T_r be the solution and time parameter given by Theorem 8.1. Let $\omega =$ curl u. Let $t_1, t_2 \in [0, T_r)$ with $t_1 < t_2$. Let $l \in \mathbb{N}^3$. Let $i \in \{1, 2, 3\}$. Set $v_i = D^l \omega_i$. Let $\mathcal{T}_i$ be defined by*

$$\mathcal{T}_i(w) = v_i \partial_t w + (D^l(\omega_i u) - D^l(u_i \omega) - \nu \nabla v_i) \cdot \nabla w, \tag{8.7}$$

for any Lipschitz function $w \in C(\mathbb{R}^3 \times (t_1, t_2); \mathbb{R})$. Then
(1) For any real γ and any compact set K of $\mathbb{R}^3 \times (t_1, t_2)$,

$$\int_{\{|v_i-\gamma|<\alpha\}\cap K}|v_i\partial_t v_i|\frac{1}{\alpha}dxd\tau<C,\quad \int_{\{|v_i-\gamma|<\alpha\}\cap K}|D^l(\omega_i u)\cdot\nabla v_i|\frac{1}{\alpha}dxd\tau<C,$$
$$\int_{\{|v_i-\gamma|<\alpha\}\cap K}|D^l(u_i\omega)\cdot\nabla v_i|\frac{1}{\alpha}dxd\tau<C,\quad \int_{\{|v_i-\gamma|<\alpha\}\cap K}|\nabla v_i|^2\frac{1}{\alpha}dxd\tau<C,$$

where C is a positive constant independent of $\alpha\in(0,1)$.
(2) The estimates in (1) yield for any compact set K of $\mathbb{R}^3\times(t_1,t_2)$,

$$\int_{\{|v_i-\gamma|<\alpha\}\cap K}|\mathcal{T}_i(v_i)|\frac{1}{\alpha}dxd\tau<C,\quad \textit{and for } \gamma=0, \int_{\{|v_i|<\alpha\}\cap K}|\mathcal{T}_i(v_i)|\frac{1}{\alpha}dxd\tau<C,$$

where C is a positive constant independent of $\alpha\in(0,1)$.
(3) The estimates in (2) show that, up to a subsequence, as α goes to 0, the following weak convergence in the sense of measures, holds

$$\chi_{\{0<v_i<\alpha\}}\mathcal{T}_i(v_i)\frac{1}{\alpha}\to\nu^+_{v_i},\quad \chi_{\{-\alpha<v_i<0\}}\mathcal{T}_i(v_i)\frac{1}{\alpha}\to\nu^-_{v_i},$$
$$\chi_{\{|v_i|<\alpha\}}\mathcal{T}_i(v_i)\frac{1}{\alpha}\to\nu_{v_i},$$

where $\nu^+_{v_i}$, $\nu^-_{v_i}$, and ν_{v_i} are measures on $\mathbb{R}^3\times(t_1,t_2)$ concentrated respectively on $\partial\{v_i>0\}$, $\partial\{v_i<0\}$, and $\partial\{v_i>0\}\cup\partial\{v_i<0\}$, which are also Radon measures. Moreover,

$$\nu_{v_i}=\nu^+_{v_i}+\nu^-_{v_i}.$$

Proof of Theorem 8.3. The following notations will be used. For a function $w\in C^1(\mathbb{R}^3\times(t_1,t_2);\mathbb{R})$ and for a vector field $V\in C^1(\mathbb{R}^3\times(t_1,t_2);\mathbb{R}^4)$,

$$\nabla_{x,t}w=(\partial_{x_1}w,\partial_{x_2}w,\partial_{x_3}w,\partial_t w)^t,\quad \mathrm{div}_{x,t}V=\partial_{x_1}V_1+\partial_{x_2}V_2+\partial_{x_3}V_3+\partial_t V_4.$$

Let $i\in\{1,2,3\}$. Set $v_i=D^l\omega_i$. Then by Theorem 8.1, $u\in C^\infty(\mathbb{R}^3\times(t_1,t_2);\mathbb{R}^3)$ and so $v_i\in C^\infty(\mathbb{R}^3\times(t_1,t_2);\mathbb{R})$.

Proof of (1)

1. Set $U=(0,0,0,v_i)^t$. Then taking $(v_i,U,\nabla_{x,t})$ in place of (v,U,∇) in Part *(1)* of Theorem 3.1, yields the first estimate in Part*(1)* of Theorem 8.3.

2. Set $U=(D^l(\omega_i u),0)^t$. Then taking $(v_i,U,\nabla_{x,t})$ in place of (v,U,∇) in Part *(1)* of Theorem 3.1, yields the second estimate in Part*(1)* of Theorem 8.3.

3. Set $U=(D^l(u_i\omega),0)^t$. Then taking $(v_i,U,\nabla_{x,t})$ in place of (v,U,∇) in Part *(1)* of Theorem 3.1, yields the third estimate in Part*(1)* of Theorem 8.3.

4. Set $U=(\nabla v_i,0)^t$. Then taking $(v_i,U,\nabla_{x,t})$ in place of (v,U,∇) in Part *(1)* of Theorem 3.1, yields the fourth estimate in Part*(1)* of Theorem 8.3.

5. Steps 1-4 yield the proof of Part *(1)* of Theorem 8.3.

Proof of (2)
Set $U=(D^l(\omega_i u)-D^l(u_i\omega)-\nu\nabla v_i,v_i)^t$. Using the regularity of u one has: $U\in C^\infty(\mathbb{R}^3\times(t_1,t_2);\mathbb{R}^4)$. Then taking $(v_i,U,\nabla_{x,t})$ in place of (v,U,∇) in Part *(1)* of Theorem 3.1, and using

the fact that $\mathcal{T}_i(v_i) = U\cdot\nabla_{x,t}v_i$, yields the first estimate in Part*(2)* of Theorem 8.3. Taking $\gamma = 0$ in the first estimate in Part*(2)* of Theorem 8.3, yields the second estimate in Part*(2)* of Theorem 8.3.

Proof of (3)

The proof of the weak convergence in Part *(3)* is a direct consequence of the fact that for any sequence of functions bounded in L^1_{loc}, one can extract a subsequence converging weakly in the sense of measures to a Radon measure. The weak convergence in Part *(3)* yields $\nu_{v_i} = \nu^+_{v_i} + \nu^-_{v_i}$. This completes the proof of Theorem 8.3. ∎

Theorem 8.4. *($N = 3$) Let u and T_r be the solution and time parameter given by Theorem 8.1. Let $\omega =$ curl u. Let $t_1, t_2 \in [0, T_r)$ with $t_1 < t_2$. Let $l \in \mathbb{N}^3$. Let $i \in \{1, 2, 3\}$. Set $v_i = D^l\omega_i$. Let $\mathcal{T}_i$ be defined by (8.7). Let φ be any Lipschitz function in $C_c(\mathbb{R}^3 \times (t_1, t_2), \mathbb{R})$. Then*

(1)

$$\int \varphi d\nu^+_{v_i} = -\int_{\{v_i>0\}} \mathcal{T}_i(\varphi)dxd\tau$$

(2)

$$\int \varphi d\nu^-_{v_i} = \int_{\{v_i<0\}} \mathcal{T}_i(\varphi)dxd\tau$$

(3)

$$\int \varphi d\nu_{v_i} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \mathcal{T}_i(\varphi)sg(v_i)dxd\tau$$

Proof of Theorem 8.4. The proof of Theorem 3.3 will be followed. Let $G^{(j)}_\alpha$, $j = 1, 2, 3$ denote the sequence of functions introduced in the proof of Theorem 3.3.

Let $i \in \{1, 2, 3\}$. Set $v_i = D^l\omega_i$. Then by Theorem 8.1, $u \in C^\infty(\mathbb{R}^3 \times (t_1, t_2); \mathbb{R}^3)$ and so $v_i \in C^\infty(\mathbb{R}^3 \times (t_1, t_2); \mathbb{R})$. Set $U = (D^l(\omega_i u) - D^l(u_i\omega) - \nu\nabla v_i, v_i)^t$. Then using the regularity of u one has: $U \in C^\infty(\mathbb{R}^3 \times (t_1, t_2); \mathbb{R}^4)$. Moreover,

$$\mathcal{T}_i(\varphi) = U \cdot \nabla_{x,t}\varphi, \tag{8.8}$$

for any Lipschitz function $\varphi \in C_c(\mathbb{R}^3 \times (t_1, t_2), \mathbb{R})$. Using Navier-Stokes equations (8.5) yields

$$\text{div}_{x,t}U = \partial_t D^l\omega_i + \text{div }(D^l(\omega_i u) - D^l(u_i\omega)) - \nu\Delta D^l\omega_i = 0. \tag{8.9}$$

Proof of (1)

Let φ be any Lipschitz function in $C_c(\mathbb{R}^3 \times (t_1, t_2), \mathbb{R})$. Since φ is of compact support and $\varphi U G^{(2)}_\alpha(v_i) \in W^{1,1}(\mathbb{R}^3 \times (t_1, t_2))^4$, one has

$$\begin{aligned} 0 &= \int_{t_1}^{t_2}\int_{\mathbb{R}^3} \text{div}_{x,t}(\varphi U G^{(2)}_\alpha(v_i))dxd\tau \\ &= \int_{t_1}^{t_2}\int_{\mathbb{R}^3} \text{div}_{x,t}(\varphi U)G^{(2)}_\alpha(v_i)dxd\tau + \int_{\{0<v_i<\alpha\}} \frac{U\cdot\nabla_{x,t}v_i}{\alpha}\varphi dxd\tau. \end{aligned}$$

Then

$$\int_{\{0<v_i<\alpha\}} \frac{U\cdot\nabla_{x,t}v_i}{\alpha}\varphi dxd\tau = -\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \text{div}_{x,t}(\varphi U)G_\alpha^{(2)}(v_i)dxd\tau. \tag{8.10}$$

As $\alpha\to 0$, $G_\alpha^{(2)}(v_i)$ converges everywhere to $\chi_{\{v_i>0\}}$. Now $|\text{div}(\varphi U)G_\alpha^{(2)}(v_i)|\le|\text{div}(\varphi U)|$. Using the regularity of U, $\text{div}_{x,t}(\varphi U)$ is in $L^1(\mathbb{R}^3\times(t_1,t_2))$. Therefore using convergence dominated theorem, up to a subsequence, as $\alpha\to 0$

$$\lim_{\alpha\to 0}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \text{div}_{x,t}(\varphi U)G_\alpha^{(2)}(v_i)dxd\tau = \int_{t_1}^{t_2}\int_{\mathbb{R}^3} \text{div}_{x,t}(\varphi U)\chi_{\{v_i>0\}}dxd\tau. \tag{8.11}$$

Hence using (8.8), (8.10)-(8.11), and Part *(3)* of Theorem 8.3 yields, up to a subsequence, as $\alpha\to 0$

$$\int\varphi d\nu_{v_i}^+ = \lim_{\alpha\to 0}\int_{\{0<v_i<\alpha\}} \frac{U\cdot\nabla_{x,t}v_i}{\alpha}\varphi dxd\tau = -\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \text{div}_{x,t}(\varphi U)\chi_{\{v_i>0\}}dxd\tau. \tag{8.12}$$

Now

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \text{div}_{x,t}(\varphi U)\chi_{\{v_i>0\}}dxd\tau &= \int_{t_1}^{t_2}\int_{\mathbb{R}^3} \varphi\text{div}_{x,t}U\chi_{\{v_i>0\}}dxd\tau\\ &+\int_{t_1}^{t_2}\int_{\mathbb{R}^3} U\cdot\nabla_{x,t}\varphi\chi_{\{v_i>0\}}dxd\tau.\end{aligned} \tag{8.13}$$

Then (8.9), and (8.12)-(8.13) yield

$$\int\varphi d\nu_{v_i}^+ = \lim_{\alpha\to 0}\int_{\{0<v_i<\alpha\}} \frac{U\cdot\nabla_{x,t}v_i}{\alpha}\varphi dxd\tau = -\int_{\{v_i>0\}}\mathcal{T}_i(\varphi)dxd\tau,$$

which corresponds to Part *(1)* of Theorem 8.4. Here, the operator $\mathcal{T}_i$ was introduced in (8.7).

Proof of (2)-(3)

Taking $G_\alpha^{(3)}$ resp. $G_\alpha^{(1)}$ in place of $G_\alpha^{(2)}$ and proceeding as in the proof of Part *(1)* yields the proof of Part *(2)* resp. Part *(3)* of the theorem. This completes the proof of Theorem 8.4. ∎

8.3. **Properties of the measures associated with the convective term in Navier-Stokes equations.** Let $\psi\in C_c^\infty(\mathbb{R}^N)$ be such that $\psi(x)=1$ for $|x|\le 1$, $\psi(x)=0$ for $|x|\ge 2$, and $0\le\psi(x)\le 1$ for all $x\in\mathbb{R}^N$. Then define ψ_m, $m\ge 1$, by $\psi_m(x)=\psi(\frac{x}{m})$.

Theorem 8.5. *($N=3$) Let u and T_r be the solution and time parameter given by Theorem 8.1. Let $\omega=$ curl u. Let $t_1,t_2\in[0,T_r)$ with $t_1<t_2$. Let $l\in\mathbb{N}^3$. Let $i\in\{1,2,3\}$. Let $\phi\in C_c(t_1,t_2)$. Then, up to a subsequence, as $m\to\infty$*

$$\begin{aligned}&\lim_{m\to\infty}\int\phi\psi_m d\mu^+_{\{D^l\omega_i,D^l(\omega_i u)\}}=0,\ \lim_{m\to\infty}\int\phi\psi_m d\mu^-_{\{D^l\omega_i,D^l(\omega_i u)\}}=0,\\ &\lim_{m\to\infty}\int\phi\psi_m d\mu_{\{D^l\omega_i,D^l(\omega_i u)\}}=0.\end{aligned}$$

Proof of Theorem 8.5. Let $i\in\{1,2,3\}$. Then using Theorem 8.1, one has: $\omega_i\in C^\infty(\mathbb{R}^3\times[t_1,t_2];\mathbb{R})$, $u\in C^\infty(\mathbb{R}^3\times[t_1,t_2];\mathbb{R}^3)$ with div $u=0$, and for any $s\in\mathbb{N}^3$, $D^s\omega_i\in C^\infty([t_1,t_2];L^1(\mathbb{R}^3))$ and $D^su\in C^\infty([t_1,t_2];L^1(\mathbb{R}^3))^3$. Hence, the assumptions of Part (1) of Theorem 7.2 with (ω_i, u) of Theorem 7.2 corresponding to (ω_i, u) of this proof, are satisfied. Therefore, one can apply Part *(1)* of Theorem 7.2 and obtain the convergence in Theorem 8.5. This completes the proof of Theorem 8.5. ∎

8.4. **Properties of the measures associated with the stretching term in Navier-Stokes equations.**

Theorem 8.6. *Let u and T_r be the solution and time parameter given by Theorem 8.1. Let $\omega =$ curl u. Let $t_1, t_2 \in [0, T_r)$ with $t_1 < t_2$. Let $l \in \mathbb{N}^3$. Let $i \in \{1, 2, 3\}$. Let $\phi \in C_c(t_1, t_2)$. Then, up to a subsequence,*

$$\lim_{m\to\infty} \int \phi\psi_m d\mu^+_{\{D^l\omega_i, D^l(u_i\omega)\}} = 0, \ \lim_{m\to\infty} \int \phi\psi_m d\mu^-_{\{D^l\omega_i, D^l(u_i\omega)\}} = 0,$$
$$\lim_{m\to\infty} \int \phi\psi_m d\mu_{\{D^l\omega_i, D^l(u_i\omega)\}} = 0.$$

Proof of Theorem 8.6. Using Theorem 8.1, one has: $u \in C^\infty(\mathbb{R}^3 \times [t_1, t_2]; \mathbb{R}^3)$ with div $u = 0$, and for any $s \in \mathbb{N}^3$, $D^s u \in C^\infty([t_1, t_2]; L^1(\mathbb{R}^3))^3$. Hence, the assumptions of Part (2) of Theorem 7.2 with u of Theorem 7.2 corresponding to u of this proof, are satisfied. Therefore, one can apply Part (2) of Theorem 7.2 and obtain the convergence in Theorem 8.6. This completes the proof of Theorem 8.6. ∎

8.5. **Characterizations of the measures associated with Navier-Stokes equations.**

Theorem 8.7. *$(N = 3)$ Let u and T_r be the solution and time parameter given by Theorem 8.1. Let $t_1, t_2 \in [0, T_r)$ with $t_1 < t_2$. Let $\omega =$ curl u. Let $l \in \mathbb{N}^3$. Let $i \in \{1, 2, 3\}$ and $v_i = D^l\omega_i$. Then the following holds.*

(1)

$$\begin{aligned}
&\nu^+_{v_i} = \partial_t[v_i\chi_{\{v_i>0\}}] + \textit{div}\ [\chi_{\{v_i>0\}}(D^l(\omega_i u) - D^l(u_i\omega) - \nu\nabla v_i)] \\
&\quad \textit{in } \mathcal{M}(\mathbb{R}^3 \times (t_1, t_2)), \\
&\nu^-_{v_i} = -\partial_t[v_i\chi_{\{v_i<0\}}] - \textit{div}\ [\chi_{\{v_i<0\}}(D^l(\omega_i u) - D^l(u_i\omega) - \nu\nabla v_i)] \\
&\quad \textit{in } \mathcal{M}(\mathbb{R}^3 \times (t_1, t_2)), \\
&\nu_{v_i} = \partial_t|v_i| + \textit{div}\ [\textit{sg}(v_i)(D^l(\omega_i u) - D^l(u_i\omega) - \nu\nabla v_i)] \\
&\quad \textit{in } \mathcal{M}(\mathbb{R}^3 \times (t_1, t_2)).
\end{aligned}$$

(2) For any nonnegative function $\phi \in C^1_c(t_1, t_2)$, the measures $\nu^+_{v_i}$, $\nu^-_{v_i}$, and ν_{v_i}, satisfy, up to a subsequence, as $m \to \infty$,

$$\limsup_{m\to\infty} \int \psi_m\phi d\nu^+_{v_i} \leq 0, \qquad \limsup_{m\to\infty} \int \psi_m\phi d\nu^-_{v_i} \leq 0,$$
$$\limsup_{m\to\infty} \int \psi_m\phi d\nu_{v_i} \leq 0.$$

Proof of Theorem 8.7. Let $i \in \{1, 2, 3\}$. Set $v_i = D^l\omega_i$.

Proof of (1)

1. Part *(1)* of Theorem 8.4, yields for any Lipschitz function $\varphi \in C_c(\mathbb{R}^N \times (t_1, t_2), \mathbb{R})$,

$$\int \varphi d\nu_{v_i}^{+} = -\int_{\{v_i>0\}} \mathcal{T}_i(\varphi) dx d\tau \tag{8.14}$$

In particular, Eq. (8.14) holds for any $\varphi \in \mathcal{D}(\mathbb{R}^3 \times (t_1, t_2))$. Thus, by definition of $\mathcal{T}_i$; See (8.7), one obtains

$$\nu_{v_i}^{+} = \partial_t(v_i \chi_{\{v_i>0\}}) + \text{div}[\chi_{\{v_i>0\}}(D^l(\omega_i u) - D^l(u_i \omega) - \nu \nabla v_i)] \quad \text{in } \mathcal{D}'(\mathbb{R}^3 \times (t_1, t_2)). \tag{8.15}$$

Since $\nu_{v_i}^{+}$ is a measure, Eq. (8.15) holds in $\mathcal{M}(\mathbb{R}^3 \times (t_1, t_2))$. This yields the first equality in Part *(1)*.

2. Part *(2)* of Theorem 8.4, yields for any Lipschitz function $\varphi \in C_c(\mathbb{R}^N \times (t_1, t_2), \mathbb{R})$,

$$\int \varphi d\nu_{v_i}^{-} = \int_{\{v_i<0\}} \mathcal{T}_i(\varphi) dx d\tau. \tag{8.16}$$

In particular, Eq. (8.16) holds for any $\varphi \in \mathcal{D}(\mathbb{R}^3 \times (t_1, t_2))$. Thus, by definition of $\mathcal{T}_i$; See (8.7), one obtains

$$\begin{aligned} \nu_{v_i}^{-} &= -\partial_t(v_i \chi_{\{v_i>0\}}) - \text{div}[\chi_{\{v_i<0\}}(D^l(\omega_i u) - D^l(u_i \omega) - \nu \nabla v_i)] \\ &\quad \text{in } \mathcal{D}'(\mathbb{R}^3 \times (t_1, t_2)). \end{aligned} \tag{8.17}$$

Since $\nu_{v_i}^{-}$ is a measure, Eq. (8.17) holds in $\mathcal{M}(\mathbb{R}^3 \times (t_1, t_2))$. This yields the second equality in Part *(1)*.

3. Part *(3)* of Theorem 8.4, yields for any Lipschitz function $\varphi \in C_c(\mathbb{R}^N \times (t_1, t_2), \mathbb{R})$,

$$\int \varphi d\nu_{v_i} = -\int_{t_1}^{t_2} \int_{\mathbb{R}^3} \mathcal{T}_i(\varphi) \text{sg}(v_i) dx d\tau. \tag{8.18}$$

In particular, Eq. (8.18) holds for any $\varphi \in \mathcal{D}(\mathbb{R}^3 \times (t_1, t_2))$. Thus, by definition of $\mathcal{T}_i$; See (8.7), one obtains

$$\begin{aligned} \nu_{v_i} &= \partial_t(v_i \text{sg}(v_i)) + \text{div}[\text{sg}(v_i)(D^l(\omega_i u) - D^l(u_i \omega) - \nu \nabla v_i)] \\ &\quad \text{in } \mathcal{D}'(\mathbb{R}^3 \times (t_1, t_2)). \end{aligned} \tag{8.19}$$

Since ν_{v_i} is a measure, Eq. (8.19) holds in $\mathcal{M}(\mathbb{R}^3 \times (t_1, t_2))$. This yields the third equality in Part *(1)*, and thus, completes the proof of Part *(1)* of the theorem.

Proof of (2)

a. Proof of the first inequality

1. Let $\phi \in C_c^1(t_1, t_2)$ with $\phi \geq 0$. Using Part *(3)* of Theorem 8.3, one has, up to a subsequence, as $\alpha \to 0$,

$$\int \psi_m \phi d\nu_{v_i}^+ = \lim_{\alpha\to 0}[\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \psi_m \phi \frac{1}{\alpha} v_i \partial_t v_i \chi_{\{0<v_i<\alpha\}} dx d\tau +$$
$$\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \psi_m \phi \frac{1}{\alpha}(D^l(\omega_i u) - D^l(u_i\omega)) \cdot \nabla v_i \chi_{\{0<v_i<\alpha\}} dx d\tau$$
$$-\nu \int_{t_1}^{t_2}\int_{\mathbb{R}^3} \psi_m \phi \frac{1}{\alpha} |\nabla v_i|^2 \chi_{\{0<v_i<\alpha\}} dx d\tau]. \tag{8.20}$$

2. In this step, it will be proved that, up to a subsequence, as $\alpha \to 0$,

$$\lim_{\alpha\to 0}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \psi_m \phi \frac{1}{\alpha} v_i \partial_t v_i \chi_{\{0<v_i<\alpha\}} dx d\tau = 0. \tag{8.21}$$

2.1 On $\{0 < v_i < \alpha\}$, one has: $|\frac{1}{\alpha} v_i| \leq 1$, and so

$$|\psi_m \phi \frac{1}{\alpha} v_i \partial_t v_i \chi_{\{0<v_i<\alpha\}}| \leq \psi_m \phi |\partial_t v_i| \chi_{\{0<v_i<\alpha\}} \leq \psi_m \phi |\partial_t v_i|.$$

By Theorem 8.1, $D^s u \in C^\infty([t_1,t_2]; L^1(\mathbb{R}^3))^3$ for every $s \in \mathbb{N}^3$. In particular, $\psi_m \phi \partial_t v_i \in L^1(\mathbb{R}^3 \times (t_1,t_2))$. This function is obviously independent of α.

2.2 Let $y \in \mathbb{R}^3 \times (t_1,t_2)$. If $v_i(y) \neq 0$, then $|v_i(y)| > \eta > 0$ for some positive number η. Then for all $0 < \alpha < \eta$ one has: $(\psi_m \phi \frac{v_i}{\alpha} \partial_t v_i \chi_{\{0<v_i<\alpha\}})(y) = 0$. If $v_i(y) = 0$, then by definition of the function, one has: $(\psi_m \phi \frac{v_i}{\alpha} \partial_t v_i \chi_{\{0<v_i<\alpha\}})(y) = 0$. Then one concludes that as $\alpha \to 0$, $\psi_m \phi \frac{1}{\alpha} v_i \partial_t v_i \chi_{\{0<v_i<\alpha\}}$ converges to 0, everywhere in $\mathbb{R}^N \times (t_1,t_2)$.

2.3 Combining Steps 2.1 and 2.2 above, one can use dominated convergence theorem to obtain, up to a subsequence, as $\alpha \to 0$, the convergence in (8.21).

3. Using Part *(1)* of Theorem 3.3, yields, up to a subsequence, as $\alpha \to 0$,

$$\lim_{\alpha\to 0}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \psi_m \phi \frac{1}{\alpha} D^l(\omega_i u) \cdot \nabla v_i \chi_{\{0<v_i<\alpha\}} dx d\tau = \int \phi \psi_m d\mu^+_{\{D^l\omega_i, D^l(\omega_i u)\}}, \tag{8.22}$$

and

$$\lim_{\alpha\to 0}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \psi_m \phi \frac{1}{\alpha} D^l(u_i\omega) \cdot \nabla v_i \chi_{\{0<v_i<\alpha\}} dx d\tau = \int \phi \psi_m d\mu^+_{\{D^l\omega_i, D^l(u_i\omega)\}}, \tag{8.23}$$

$$\lim_{\alpha\to 0}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \psi_m \phi \frac{1}{\alpha} |\nabla v_i|^2 \chi_{\{0<v_i<\alpha\}} dx d\tau = \int \phi \psi_m d\mu^+_{\{D^l\omega_i, \nabla D^l\omega_i\}}. \tag{8.24}$$

Then using (8.20), (8.21), and (8.22)-(8.24), one obtains

$$\int \psi_m \phi d\nu_{v_i}^+ = \int \phi \psi_m d\mu^+_{\{D^l\omega_i, D^l(\omega_i u)\}} - \int \phi \psi_m d\mu^+_{\{D^l\omega_i, D^l(u_i\omega)\}}$$
$$- \int \phi \psi_m d\mu^+_{\{D^l\omega_i, \nabla D^l\omega_i\}}.$$

Now using the fact that $\mu^+_{\{D^l\omega_i, \nabla D^l\omega_i\}} \geq 0$, one obtains

$$\int \psi_m \phi d\nu^+_{v_i} \leq \int \phi\psi_m d\mu^+_{\{D^l\omega_i, D^l(\omega_i u)\}} - \int \phi\psi_m d\mu^+_{\{D^l\omega_i, D^l(u_i\omega)\}}. \tag{8.25}$$

4. Using Theorem 8.5, shows that, up to a subsequence, as m goes to ∞,

$$\lim_{m\to\infty} \int \phi\psi_m d\mu^+_{\{D^l\omega_i, D^l(\omega_i u)\}} = 0. \tag{8.26}$$

5. Using Theorem 8.6, shows that, up to a subsequence, as m goes to ∞,

$$\lim_{m\to\infty} \int \phi\psi_m d\mu^+_{\{D^l\omega_i, D^l(u_i\omega)\}} = 0. \tag{8.27}$$

6. Then (8.25)-(8.27) show that, up to a subsequence, as $m \to \infty$,

$$\limsup_{m\to\infty} \int \psi_m \phi d\nu^+_{v_i} \leq 0,$$

for any nonnegative $\phi \in C^1_c(t_1,t_2)$. This completes the proof of the first inequality in Part *(2)* of the theorem.

b. Proof of the second inequality

1. Let $\phi \in C^1_c(t_1,t_2)$ with $\phi \geq 0$. Using Part *(3)* of Theorem 8.3, one has, up to a subsequence, as $\alpha \to 0$,

$$\begin{aligned}\int \psi_m \phi d\nu^-_{v_i} = \lim_{\alpha\to 0}[&\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \psi_m \phi \frac{1}{\alpha} v_i \partial_t v_i \chi_{\{-\alpha<v_i<0\}} dx d\tau + \\ &\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \psi_m \phi \frac{1}{\alpha}(D^l(\omega_i u) - D^l(u_i\omega)) \cdot \nabla v_i \chi_{\{-\alpha<v_i<0\}} dx d\tau \\ &-\nu \int_{t_1}^{t_2}\int_{\mathbb{R}^3} \psi_m \phi \frac{1}{\alpha} |\nabla v_i|^2 \chi_{\{-\alpha<v_i<0\}} dx d\tau].\end{aligned} \tag{8.28}$$

2. Proceeding as in Step 2 of the proof of the first inequality in Part *(2)* of the theorem given above, one obtains up to a subsequence, as $\alpha \to 0$,

$$\lim_{\alpha\to 0}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \psi_m \phi \frac{1}{\alpha} v_i \partial_t v_i \chi_{\{-\alpha<v_i<0\}} dx d\tau = 0. \tag{8.29}$$

3. Using Part *(1)* of Theorem 3.3, yields, up to a subsequence, as $\alpha \to 0$,

$$\lim_{\alpha\to 0}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \psi_m \phi \frac{1}{\alpha} D^l(\omega_i u) \cdot \nabla v_i \chi_{\{-\alpha<v_i<0\}} dx d\tau = \int \phi\psi_m d\mu^-_{\{D^l\omega_i, D^l(\omega_i u)\}}, \tag{8.30}$$

$$\lim_{\alpha\to 0}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \psi_m \phi \frac{1}{\alpha} D^l(u_i\omega) \cdot \nabla v_i \chi_{\{-\alpha<v_i<0\}} dx d\tau = \int \phi\psi_m d\mu^-_{\{D^l\omega_i, D^l(u_i\omega)\}}, \tag{8.31}$$

$$\lim_{\alpha\to 0}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \psi_m \phi \frac{1}{\alpha} |\nabla v_i|^2 \chi_{\{-\alpha<v_i<0\}} dx d\tau = \int \phi\psi_m d\mu^-_{\{D^l\omega_i, \nabla D^l\omega_i\}}. \tag{8.32}$$

Then using (8.28)-(8.32), one obtains

$$\int \psi_m \phi d\nu_{v_i}^- = \int \phi\psi_m d\mu^-_{\{D^l\omega_i, D^l(\omega_i u)\}} - \int \phi\psi_m d\mu^-_{\{D^l\omega_i, D^l(u_i\omega)\}} - \int \phi\psi_m d\mu^-_{\{D^l\omega_i, \nabla D^l\omega_i\}}.$$

Now using the fact that $\mu^-_{\{D^l\omega_i, \nabla D^l\omega_i\}} \geq 0$, one obtains

$$\int \psi_m \phi d\nu_{v_i}^- \leq \int \phi\psi_m d\mu^-_{\{D^l\omega_i, D^l(\omega_i u)\}} - \int \phi\psi_m d\mu^-_{\{D^l\omega_i, D^l(u_i\omega)\}}. \tag{8.33}$$

4. Using Theorem 8.5, shows that, up to a subsequence, as m goes to ∞,

$$\lim_{m\to\infty} \int \phi\psi_m d\mu^-_{\{D^l\omega_i, D^l(\omega_i u)\}} = 0. \tag{8.34}$$

5. Using Theorem 8.6, shows that, up to a subsequence, as m goes to ∞,

$$\lim_{m\to\infty} \int \phi\psi_m d\mu^-_{\{D^l\omega_i, D^l(u_i\omega)\}} = 0. \tag{8.35}$$

6. Then (8.33)-(8.35) show that, up to a subsequence, as $m \to \infty$,

$$\limsup_{m\to\infty} \int \psi_m \phi d\nu_{v_i}^- \leq 0,$$

for any nonnegative $\phi \in C_c^1(t_1, t_2)$. This completes the proof of the second inequality in Part *(2)* of the theorem.

c. Proof of the third inequality

1. Using Part *(3)* of Theorem 8.3 showing that $\nu_{v_i} = \nu_{v_i}^+ + \nu_{v_i}^-$ and the proofs of the first and second inequalities obtained above, one obtains, up to a subsequence, as $m \to \infty$,

$$\limsup_{m\to\infty} \int \psi_m \phi d\mu_{v_i} \leq 0,$$

for any nonnegative $\phi \in C_c^1(t_1, t_2)$. This yields the proof of the third inequality in Part *(2)* of the theorem and completes the proof of Part *(2)* and thus, the proof of Theorem 8.7. ∎

8.6. **Fundamental estimates of solutions of Navier-Stokes equations and their partial derivatives in space dimension 3.**

Theorem 8.8. *$(N = 3)$ Let u and T_r be the solution and time parameter given by Theorem 8.1. Let $t_1, t_2 \in [0, T_r)$ with $t_1 < t_2$. Let $\omega =$ curl u. Let $l \in \mathbb{N}^3$. Let $i \in \{1, 2, 3\}$ and $v_i = D^l\omega_i$. Then for all $0 \leq s < t < T_r$,*

(1)

$$\int_{\{D^l\omega_i(\cdot,t)>0\}} D^l\omega_i(x,t)dx \leq \int_{\{D^l\omega_i(\cdot,s)>0\}} D^l\omega_i(x,s)dx.$$

(2)

$$\int_{\{D^l\omega_i(\cdot,t)<0\}}(-D^l\omega_i)(x,t)dx \le \int_{\{D^l\omega_i(\cdot,s)<0\}}(-D^l\omega_i)(x,s)dx.$$

(3)

$$\int_{\mathbb{R}^3}|D^l\omega_i|(x,t)dx \le \int_{\mathbb{R}^3}|D^l\omega_i|(x,s)dx.$$

Proof of Theorem 8.8. Let $i \in \{1,2,3\}$. Set $v_i = D^l\omega_i$.

Proof of (1)

1. By Theorem 8.1, $D^s u \in C^\infty([t_1,t_2];L^1(\mathbb{R}^3))^3$ for every $s \in \mathbb{N}^3$. In particular, $v_i \in C^1([t_1,t_2];L^1(\mathbb{R}^3))$. Using the regularity of v_i and $\chi_{\{v_i>0\}}$, one obtains

$$\partial_t(v_i\chi_{\{v_i>0\}}) = \partial_t v_i\chi_{\{v_i>0\}} \text{ in } \mathcal{M}(\mathbb{R}^3\times(t_1,t_2)). \tag{8.36}$$

Moreover, by the above regularity, $\partial_t v_i\chi_{\{v_i>0\}} \in L^1(\mathbb{R}^3\times(t_1,t_2))$ and so the equality (8.36) holds a.e. in $\mathbb{R}^3\times(t_1,t_2)$. The above regularity also shows that $\int_{\mathbb{R}^3}|\partial_t v_i\chi_{\{v_i>0\}}|(x,t)dx \in L^1(t_1,t_2)$.

2. Let $\phi \in C^1_c(t_1,t_2)$. Using Step 1 and the definition of ψ_m, $|\phi\psi_m\partial_t(v_i\chi_{\{v_i>0\}})| \le |\phi\partial_t v_i|$ and $\phi\partial_t v_i \in L^1(\mathbb{R}^3\times(t_1,t_2))$. Moreover, as $m\to\infty$, $\phi\psi_m\partial_t(v_i\chi_{\{v_i>0\}})$ converges to $\phi\partial_t(v_i\chi_{\{v_i>0\}})$ a.e. in $\mathbb{R}^3\times(t_1,t_2)$. Therefore, using convergence dominated theorem and Step 1 above, up to a subsequence, as $m\to\infty$,

$$\lim_{m\to\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\phi\psi_m\partial_t(v_i\chi_{\{v_i>0\}})dxdt = \int_{t_1}^{t_2}\phi\int_{\mathbb{R}^3}\partial_t v_i\chi_{\{v_i>0\}}dxdt. \tag{8.37}$$

3. Let $\phi \in C^1_c(t_1,t_2)$. Using the regularity in Step 1, $|\partial_t\phi\psi_m v_i\chi_{\{v_i>0\}}| \le |\partial_t\phi v_i|$ and $\partial_t\phi v_i \in L^1(\mathbb{R}^3\times(t_1,t_2))$. By definition of ψ_m, as $m\to\infty$, $\partial_t\phi\psi_m v_i\chi_{\{v_i>0\}}$ converges to $\partial_t\phi v_i\chi_{\{v_i>0\}}$ a.e. in $\mathbb{R}^3\times(t_1,t_2)$. Therefore, using convergence dominated theorem, up to a subsequence, as $m\to\infty$,

$$\lim_{m\to\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\partial_t\phi\psi_m v_i\chi_{\{v_i>0\}}dxdt = \int_{t_1}^{t_2}\partial_t\phi\int_{\mathbb{R}^3}v_i\chi_{\{v_i>0\}}dxdt. \tag{8.38}$$

4. Since

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\phi\psi_m\partial_t(v_i\chi_{\{v_i>0\}})dxd\tau = -\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\partial_t\phi\psi_m v_i\chi_{\{v_i>0\}}dxd\tau,$$

one obtains using Steps 2 and 3,

$$\int_{t_1}^{t_2}\phi\int_{\mathbb{R}^3}\partial_t v_i\chi_{\{v_i>0\}}dxdt = -\int_{t_1}^{t_2}\partial_t\phi\int_{\mathbb{R}^3}v_i\chi_{\{v_i>0\}}dxdt. \tag{8.39}$$

(8.39) yields

$$\int_{t_1}^{t_2} \phi \int_{\mathbb{R}^3} \partial_t v_i \chi_{\{v_i>0\}} dx dt =< \frac{d}{dt}[\int_{\mathbb{R}^3} v_i \chi_{\{v_i>0\}} dx], \phi >, \tag{8.40}$$

for any $\phi \in C_c^\infty(t_1, t_2)$. Hence, one deduces from (8.40) that

$$\frac{d}{dt}[\int_{\mathbb{R}^3} v_i \chi_{\{v_i>0\}} dx] = \int_{\mathbb{R}^3} \partial_t v_i \chi_{\{v_i>0\}} dx \quad \text{in } \mathcal{D}'(t_1, t_2). \tag{8.41}$$

By Step 1,

$$\int_{\mathbb{R}^3} \partial_t v_i \chi_{\{v_i>0\}} dx \in L^1(t_1, t_2).$$

Hence, (8.41) shows that

$$\frac{d}{dt}[\int_{\mathbb{R}^3} v_i \chi_{\{v_i>0\}} dx] \in L^1(t_1, t_2)$$

and that (8.41) holds a.e. on (t_1, t_2). Then using the fact that by Step 1,

$$\int_{\mathbb{R}^3} v_i \chi_{\{v_i>0\}} dx \in L^1(t_1, t_2),$$

one concludes that

$$\int_{\mathbb{R}^3} v_i \chi_{\{v_i>0\}} dx \in W^{1,1}(t_1, t_2).$$

Therefore, by the properties of the Sobolev space $W^{1,1}(t_1, t_2)$, the function g defined by,

$$g = \int_{\mathbb{R}^3} v_i \chi_{\{v_i>0\}} dx \in C([t_1, t_2]),$$

and is absolutely continuous on $[t_1, t_2]$. Moreover, for all $t_1 \leq s < t \leq t_2$, one has

$$\begin{aligned} g(t) - g(s) &= \int_s^t (\frac{d}{dt}[\int_{\mathbb{R}^3} v_i \chi_{\{v_i>0\}} dx]) d\tau \\ &= \int_s^t [\int_{\mathbb{R}^3} (\partial_t v_i \chi_{\{v_i>0\}})(x, \tau) dx] d\tau. \end{aligned} \tag{8.42}$$

5. By Step 1 above, $\partial_t(v_i \chi_{\{v_i>0\}}) \in L^1(\mathbb{R}^3 \times (t_1, t_2))$. Hence, using Part *(1)* of Theorem 8.7,

$$\text{div}[\chi_{\{v_i>0\}}(D^l(\omega_i u) - D^l(u_i \omega) - \nu \nabla v_i)] \in \mathcal{M}(\mathbb{R}^3 \times (t_1, t_2)).$$

Then using again Part *(1)* of Theorem 8.7, one obtains

$$
\begin{aligned}
&\int \phi\psi_m d\nu_{v_i}^+ \\
= &\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \phi\psi_m \partial_t(v_i\chi_{\{v_i>0\}})dxd\tau \\
&+ < \text{div}[\chi_{\{v_i>0\}}(D^l(\omega_i u) - D^l(u_i\omega) - \nu\nabla v_i)], \phi\psi_m > \\
= &-\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \partial_t\phi\psi_m v_i\chi_{\{v_i>0\}}dxd\tau \\
&-\frac{1}{m}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \phi[\chi_{\{v_i>0\}}(D^l(\omega_i u) - D^l(u_i\omega) - \nu\nabla v_i)]\cdot\nabla\psi(\frac{x}{m})dxd\tau. \quad (8.43)
\end{aligned}
$$

6. Let $\phi \in C_c^1(t_1, t_2)$. By the properties of ψ,

$$
\begin{aligned}
&\frac{1}{m}|\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \phi[\chi_{\{v_i>0\}}(D^l(\omega_i u) - D^l(u_i\omega) - \nu\nabla v_i)]\cdot\nabla\psi(\frac{x}{m})dxd\tau| \\
\leq\ &\frac{C}{m}\int_{t_1}^{t_2}|\phi|\int_{\mathbb{R}^3}|D^l(\omega_i u) - D^l(u_i\omega) - \nu\nabla v_i|dxd\tau \quad (8.44)
\end{aligned}
$$

where C is a positive constant independent of m. By Theorem 8.1, $D^s u$ is in $C^\infty([t_1, t_2]; L^1(\mathbb{R}^3))^3$ for every $s \in \mathbb{N}^3$, and so in particular, $\nabla v_i \in L^1(\mathbb{R}^3 \times (t_1, t_2))^3$. Moreover, using Leibniz formula, Sobolev embeddings, and Holder inequality, one has: $D^l(\omega_i u)$ and $D^l(u_i\omega)$ are in $L^1(\mathbb{R}^3 \times (t_1, t_2))^3$. Therefore, the right side in (8.44) goes to 0 as m goes to ∞.

7. Let $\phi \in C_c^\infty(t_1, t_2)$. Steps 3, 5, and 6 yield, up to a subsequence, as $m \to \infty$,

$$
\begin{aligned}
\lim_{m\to\infty}\int \phi\psi_m d\nu_{v_i}^+ &= -\int_{t_1}^{t_2}\partial_t\phi\int_{\mathbb{R}^3} v_i\chi_{\{v_i>0\}}dxdt \\
&= \int_{t_1}^{t_2}\phi\frac{d}{dt}[\int_{\mathbb{R}^3} v_i\chi_{\{v_i>0\}}(x,t)dx]dt. \quad (8.45)
\end{aligned}
$$

8. On the other hand, Part *(2)* of Theorem 8.7, shows that the left side of (8.45) is non-positive for all nonnegative functions $\phi \in C_c(t_1, t_2)$.

Then one deduces using Schwartz lemma on nonnegative distributions that

$$
\frac{d}{dt}[\int_{\mathbb{R}^3} v_i\chi_{\{v_i>0\}}(x,t)dx] \leq 0 \ \text{ in } \mathcal{M}(t_1, t_2). \quad (8.46)
$$

Now since by Step 4,

$$
\frac{d}{dt}[\int_{\mathbb{R}^3} v_i\chi_{\{v_i>0\}}dx] = \int_{\mathbb{R}^3}\partial_t v_i\chi_{\{v_i>0\}}dx \in L^1(t_1, t_2),
$$

one concludes using (8.46) that

$$
\frac{d}{dt}[\int_{\mathbb{R}^3} v_i\chi_{\{v_i>0\}}(x,t)dx] = \int_{\mathbb{R}^3}\partial_t v_i\chi_{\{v_i>0\}}dx \leq 0 \quad \text{a.e. on } (t_1, t_2). \quad (8.47)
$$

9. Using (8.42) and (8.47), one concludes that for all $t_1 \leq s < t \leq t_2$, one has

$$(8.48)\qquad g(t)-g(s)=\int_s^t[\int_{\mathbb{R}^3}(\partial_t v_i\chi_{\{v_i>0\}})(x,\tau)dx]d\tau\le 0.$$

10. Steps 1-9 hold for any $0\le t_1<t_2<T_r$. Therefore, Step 9 yields Part *(1)* of the theorem.

Proof of (2)

1. Proceeding as in Step 1 of the proof of Part *(1)* of the theorem, one concludes that $\partial_t(v_i\chi_{\{v_i<0\}})=\partial_t v_i\chi_{\{v_i<0\}}\in L^1(\mathbb{R}^3\times(t_1,t_2))$ and $\int_{\mathbb{R}^3}|\partial_t v_i\chi_{\{v_i<0\}}|(x,t)dx\in L^1(t_1,t_2)$.

2. Let $\phi\in C_c^1(t_1,t_2)$. Proceeding as in Step 2 of the proof of Part *(1)* of the theorem, one concludes that, up to a subsequence, as $m\to\infty$,

$$(8.49)\qquad \lim_{m\to\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\phi\psi_m\partial_t(v_i\chi_{\{v_i<0\}})dxdt=\int_{t_1}^{t_2}\phi\int_{\mathbb{R}^3}\partial_t v_i\chi_{\{v_i<0\}}dxdt.$$

3. Let $\phi\in C_c^1(t_1,t_2)$. Proceeding as in Step 3 of the proof of Part *(1)* of the theorem, one concludes that, up to a subsequence, as $m\to\infty$,

$$(8.50)\qquad \lim_{m\to\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\partial_t\phi\psi_m v_i\chi_{\{v_i<0\}}dxdt=\int_{t_1}^{t_2}\partial_t\phi\int_{\mathbb{R}^3}v_i\chi_{\{v_i<0\}}dxdt.$$

4. Proceeding as in Step 4 of the proof of Part *(1)* of the theorem, one concludes that,

$$(8.51)\qquad \frac{d}{dt}[\int_{\mathbb{R}^3}v_i\chi_{\{v_i<0\}}dx]=\int_{\mathbb{R}^3}\partial_t v_i\chi_{\{v_i<0\}}dx\in L^1(t_1,t_2).$$

By Step 1, $\int_{\mathbb{R}^3}v_i\chi_{\{v_i<0\}}dx\in L^1(t_1,t_2)$, and so, $\int_{\mathbb{R}^3}v_i\chi_{\{v_i<0\}}dx\in W^{1,1}(t_1,t_2)$. Then, by the properties of the Sobolev space $W^{1,1}(t_1,t_2)$, the function g defined by,

$$g=\int_{\mathbb{R}^3}(-v_i)\chi_{\{v_i<0\}}dx\in C([t_1,t_2])$$

and is absolutely continuous on $[t_1,t_2]$. Moreover, for all $t_1\le s<t\le t_2$, one has

$$(8.52)\qquad \begin{aligned} g(t)-g(s) &= \int_s^t(\frac{d}{dt}[\int_{\mathbb{R}^3}(-v_i)\chi_{\{v_i<0\}}dx])d\tau \\ &= \int_s^t[\int_{\mathbb{R}^3}(\partial_t(-v_i)\chi_{\{v_i<0\}})(x,\tau)dx]d\tau. \end{aligned}$$

5. By Step 1 above, $\partial_t(v_i\chi_{\{v_i<0\}})\in L^1(\mathbb{R}^3\times(t_1,t_2))$. Hence, using Part *(1)* of Theorem 8.7,

$$\text{div}[\chi_{\{v_i<0\}}(D^l(\omega_i u)-D^l(u_i\omega)-\nu\nabla v_i)]\in\mathcal{M}(\mathbb{R}^3\times(t_1,t_2)).$$

Then using again Part *(1)* of Theorem 8.7, one obtains

$$
\begin{aligned}
&\int \phi\psi_m d\nu^-_{v_i} \\
=\ & -\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \phi\psi_m \partial_t(v_i\chi_{\{v_i<0\}})dxd\tau \\
& - < \mathrm{div}[\chi_{\{v_i<0\}}(D^l(\omega_i u) - D^l(u_i\omega) - \nu\nabla v_i)], \phi\psi_m > \\
=\ & \int_{t_1}^{t_2}\int_{\mathbb{R}^3} \partial_t\phi\psi_m v_i\chi_{\{v_i<0\}}dxd\tau \\
& +\frac{1}{m}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\phi[\chi_{\{v_i<0\}}(D^l(\omega_i u) - D^l(u_i\omega) - \nu\nabla v_i)]\cdot\nabla\psi(\frac{x}{m})dxd\tau. \qquad (8.53)
\end{aligned}
$$

6. Let $\phi \in C_c^1(t_1,t_2)$. Proceeding as in Step 6 of the proof of Part *(1)* of the theorem, one concludes that the second term in the right side of (8.53) goes to 0 as m goes to ∞. Hence, using (8.53) and Step 3, up to a subsequence, as $m \to \infty$,

$$
\lim_{m\to\infty}\int \phi\psi_m d\nu^-_{v_i} = \int_{t_1}^{t_2}\phi\frac{d}{dt}[\int_{\mathbb{R}^3}(-v_i)\chi_{\{v_i<0\}}(x,t)dx]dt. \qquad (8.54)
$$

7. On the other hand, Part *(2)* of Theorem 8.7, shows that the left side of (8.54) is non-positive for all nonnegative functions $\phi \in C_c(t_1,t_2)$. Then proceeding as in Steps 7-8 of the proof of Part *(1)* of the theorem, one obtains using Schwartz lemma on nonnegative distributions that

$$
\frac{d}{dt}[\int_{\mathbb{R}^3}(-v_i)\chi_{\{v_i<0\}}(x,t)dx] \le 0 \ \text{ in } \mathcal{M}(t_1,t_2). \qquad (8.55)
$$

Now since by Step 4,

$$
\frac{d}{dt}[\int_{\mathbb{R}^3}(-v_i)\chi_{\{v_i<0\}}dx] = \int_{\mathbb{R}^3}\partial_t(-v_i)\chi_{\{v_i<0\}}dx \in L^1(t_1,t_2),
$$

one concludes using (8.55) that

$$
\frac{d}{dt}[\int_{\mathbb{R}^3}(-v_i)\chi_{\{v_i<0\}}(x,t)dx] = \int_{\mathbb{R}^3}\partial_t(-v_i)\chi_{\{v_i<0\}}dx \le 0 \quad \text{a.e. on } (t_1,t_2). \qquad (8.56)
$$

8. Using (8.52) and (8.56), one concludes that for all $t_1 \le s < t \le t_2$, one has

$$
g(t) - g(s) = \int_s^t[\int_{\mathbb{R}^3}(\partial_t(-v_i)\chi_{\{v_i<0\}})(x,\tau)dx]d\tau \le 0. \qquad (8.57)
$$

9. Steps 1-8 hold for any $0 \le t_1 < t_2 < T_r$. Therefore, Step 8 yields Part *(2)* of the theorem.

Proof of (3)

Using Parts *(1)* and *(2)* of the theorem yields Part *(3)* of the theorem. The proof of Theorem 8.8 is now completed. ∎

8.7. **Proof of Theorem 2.1.** **1.** Part *(3)* of Theorem 8.8 shows that for $i = 1, 2, 3$ and for all $t \in (0, T_r)$

$$\int_{\mathbb{R}^3} |D^l \omega_i|(x,t)dx \leq \int_{\mathbb{R}^3} |D^l \omega_{0i}|(x)dx, \tag{8.58}$$

for any $l \in \mathbb{N}^3$. Hence, one obtains

$$\|D^l \omega(\cdot,t)\|_{W^{1,1}(\mathbb{R}^3)} \leq \|D^l \omega_0\|_{W^{1,1}(\mathbb{R}^3)}, \tag{8.59}$$

for all $l \in \mathbb{N}^3$. Now by Sobolev embeddings, one obtains

$$\|D^l \omega(\cdot,t)\|_{L^q(\mathbb{R}^3)} \leq C\|D^l \omega(\cdot,t)\|_{W^{1,1}(\mathbb{R}^3)} \leq C\|D^l \omega_0\|_{W^{1,1}(\mathbb{R}^3)}, \tag{8.60}$$

for all $l \in \mathbb{N}^3$, and for all $1 \leq q \leq 3/2$. Here, C is a positive constant independent of t, T_r, ν, and ω_0. Using (8.60) and Calderon-Zygmund inequality, one obtains

$$\|D^l \nabla u(\cdot,t)\|_{L^q(\mathbb{R}^3)} \leq C\|D^l \omega(\cdot,t)\|_{L^q(\mathbb{R}^3)} \leq C\|D^l \omega_0\|_{W^{1,1}(\mathbb{R}^3)}, \tag{8.61}$$

for all $l \in \mathbb{N}^3$, and for all $1 < q \leq 3/2$. Here, C is a positive constant independent of t, T_r, ν, and ω_0.

Using (8.61) and Sobolev embeddings, one obtains

$$\|\nabla u(\cdot,t)\|_{W^{m,q}(\mathbb{R}^3)} < C\|\omega_0\|_{W^{m+1,1}(\mathbb{R}^3)}, \tag{8.62}$$

for all integer m and all $1 < q \leq 3/2$. Here, C is a positive constant independent of t, T_r, ν, and ω_0.

Let m be a nonnegative integer and let $3/2 < q < \infty$. Let $\bar{q} = q - \frac{3}{2}$. Then by using (8.62), Sobolev embeddings, and Holder inequality, one obtains for all $0 < t < T_r$,

$$\begin{aligned} \|\nabla u(\cdot,t)\|_{W^{m,q}(\mathbb{R}^3)} &\leq \|\nabla u(\cdot,t)\|^{\bar{q}/q}_{W^{m,\infty}(\mathbb{R}^3)} \|\nabla u(\cdot,t)\|^{3/(2q)}_{W^{m,3/2}(\mathbb{R}^3)} \\ &\leq C\|\omega_0\|_{W^{m+4,1}(\mathbb{R}^3)} \leq C(m,\omega_0), \end{aligned} \tag{8.63}$$

where $C(m,\omega_0) = C\|\omega_0\|_{W^{m+4,1}(\mathbb{R}^3)}$, and C is a positive constant independent of t, T_r, ν, and ω_0. Then it is clear that $C(m,\omega_0)$ is independent of t and T_r (and ν if ω_0 is independent of ν).

The stream equation $-\Delta\Psi = \omega$ and classical elliptic regularity show that $u = \text{curl } \Psi$ satisfies: $\|u(\cdot,t)\|_{W^{1,q}(\mathbb{R}^3)} \leq C\|\omega(\cdot,t)\|_{L^q(\mathbb{R}^3)}$ for all $1 < q \leq 3/2$. Here, C is a positive constant independent of t, T_r, ν, and ω_0. Then by Sobolev embeddings, $\|u(\cdot,t)\|_{L^p(\mathbb{R}^3)} \leq C\|\omega(\cdot,t)\|_{L^q(\mathbb{R}^3)}$ for all $q \leq p \leq 3q/(3-q)$ for all $1 < q \leq 3/2$.

Let $1 < q < 3$. Then by the above and Sobolev's inequality,

$$\|u(\cdot,t)\|_{L^{q^\star}(\mathbb{R}^3)} \leq C\|\nabla u(\cdot,t)\|_{L^q(\mathbb{R}^3)} \leq C(0,\omega_0), \tag{8.64}$$

where $\frac{1}{q^\star} = \frac{1}{q} - \frac{1}{3}$, $C(0,\omega_0) = C\|\omega_0\|_{W^{4,1}(\mathbb{R}^3)}$, and C is a positive constant independent of t, T_r, ν, and ω_0. Then it is clear that $C(0,\omega_0)$ is independent of t and T_r (and ν if ω_0 is independent of ν).

2. Let $s \in \mathbb{N}^3$ with $|s| \geq 1$ and let $1 \leq q < \infty$. Using Eq. (8.4), (8.62)-(8.64), and classical elliptic theory, one concludes that

$$\|D^s p(\cdot,t)\|_{L^q(\mathbb{R}^3)} < C(k,\omega_0)^2, \tag{8.65}$$

for some integer $k > 0$ depending on $|s|$ and q. Here, $C(k,\omega_0)$ was introduced in Step 1 above and is independent of t, T_r (and ν if ω_0 is independent of ν).

3. Let $l \in \mathbb{N}^3$. Applying the operator D^l to Eq. (2.1) yields

$$\partial_t D^l u - \nu\Delta D^l u = -D^l[(u\cdot\nabla)u] - \nabla D^l p \text{ in } \mathbb{R}^3\times(0,T_r). \tag{8.66}$$

Let $1 \le q < \infty$. Using the regularity of p obtained in Step 2, the regularity of u obtained in Step 1, Leibniz formula and Holder inequality, and classical regularity of the heat equation, one obtains

$$\|D^l u(\cdot,t)\|_{L^q(\mathbb{R}^3)} < C(k_1,\omega_0)^2, \quad \|\partial_t D^l u(\cdot,t)\|_{L^q(\mathbb{R}^3)} < C(k_2,\omega_0)^2, \tag{8.67}$$

for some integers $k_1,\ k_2 > 0$ depending on $|l|$ and q. Here, $C(k_1,\omega_0)$ and $C(k_2,\omega_0)$ were introduced in Step 1 above and are independent of t and T_r (and ν if ω_0 is independent of ν).

4. Let $s \in \mathbb{N}^3$ with $|s| \ge 1$ and let $1 \le q < \infty$. Using Eq. (8.4), the regularity of u obtained in Steps 1 and 3, and classical elliptic theory, one concludes that

$$\|\partial_t D^s p(\cdot,t)\|_{L^q(\mathbb{R}^3)} < C(k,\omega_0)^2, \tag{8.68}$$

for some integer $k > 0$ depending on $|s|$ and q. Here, $C(k,\omega_0)$ was introduced in Step 1 above and is independent of t and T_r (and ν if ω_0 is independent of ν).

5. Let $l \in \mathbb{N}^3$. Applying the operator $\partial_t D^l$ to Eq. (2.1) yields

$$\partial_t(\partial_t D^l u) - \nu\Delta\partial_t D^l u = -\partial_t D^l[(u\cdot\nabla)u] - \nabla\partial_t D^l p \quad \text{in } \mathbb{R}^3\times(0,T_r). \tag{8.69}$$

Let $1 \le q < \infty$. Using the regularity of p obtained in Step 4, the regularity of u obtained in Steps 1 and 3, Leibniz formula and Holder inequality, and classical regularity of the heat equation, one obtains

$$\|\partial_t^2 D^l u(\cdot,t)\|_{L^q(\mathbb{R}^3)} < C(k,\omega_0)^2, \tag{8.70}$$

for some integer $k > 0$ depending on $|l|$ and q. Here, $C(k,\omega_0)$ was introduced in Step 1 above and is independent of t and T_r (and ν if ω_0 is independent of ν).

6. Let $l, s \in \mathbb{N}^3$ with $|s| \ge 1$. Let $1 \le q < \infty$. Repeating the process in Steps 1-5 for $\partial_t^2 D^l u$, $\partial_t^3 D^l u$, $\cdots$, one obtains for any nonnegative integers r_1, r_2

$$\|\partial_t^{r_1} D^l u(\cdot,t)\|_{L^q(\mathbb{R}^3)} < C(k_1,\omega_0)^2, \tag{8.71}$$

$$\|\partial_t^{r_2} D^s p(\cdot,t)\|_{L^q(\mathbb{R}^3)} < C(k_2,\omega_0)^2, \tag{8.72}$$

for some integer $k_1 > 0$ depending on $|l|$, q, and r_1, and $k_2 > 0$ depending on $|s|$, q, and r_2. Above, $C(k_1,\omega_0)$ and $C(k_2,\omega_0)$ were introduced in Step 1 above. By Step 1, both $C(k_1,\omega_0)$ and $C(k_2,\omega_0)$ are independent of t and T_r (and ν if ω_0 is independent of ν).

7. Step 6 and Sobolev embeddings show that for all integers $m \ge 0$ and all $1 \le q < \infty$,

$$u \in C^\infty([0,T_r]; W^{m,q}(\mathbb{R}^3))^3, \quad \text{and} \quad \nabla p \in C^\infty([0,T_r]; W^{m,q}(\mathbb{R}^3))^3.$$

Moreover,

$$u \in C^\infty(\mathbb{R}^3 \times [0,T_r])^3, \text{ and } p \in C^\infty(\mathbb{R}^3 \times [0,T_r]).$$

8. The above regularity shows that, in particular,

$$u(\cdot,T_r) \in W^{m,q}(\mathbb{R}^3)^3$$

for all integers $m \geq 0$ and all $q \geq 1$. Moreover, using Eq. (2.2) and the above regularity, one has

$$\text{div } u(\cdot,T_r) = 0.$$

Therefore, using Theorem 8.1 with the initial condition $u_0(\cdot) = u(\cdot,T_r)$, one obtains a new interval $[T_r,T_r')$ on which the conclusions of Theorem 8.1 hold and consequently Remark 8.2 holds. Then proceeding as in the proofs of Theorems 8.3-8.8, one obtains all of the results of these theorems on the interval $[T_r,T_r')$.

9. Repeating Steps 1-8 with the new interval $[T_r,T_r')$, one obtains yet another new interval $[T_r',T_r'')$ on which all of the results of Theorems 8.1 (and consequently Remark 8.2), Theorems 8.3-8.8 are true. Then continuing this process, one concludes that the results of Theorems 8.1 (and consequently Remark 8.2), Theorems 8.3-8.8 are true on $[0,T]$ for all $T > 0$. Then one obtains Theorem 2.1. The proof of Theorem 2.1 is now completed. ∎

The proof above shows the following.

Theorem 8.9. *Let $\nu > 0$. Let u_0 be an initial velocity field such that div $u_0 = 0$ and $u_0 \in W^{q,1}(\mathbb{R}^3)^3$ for all integers $q \geq 0$. Let (u,p) be the corresponding solution obtained in Theorem 2.1. Let m, r_1, r_2, r_3, r_4 be any nonnegative integers and let $1 \leq q < \infty$. Then*

$$\|u\|_{C^{r_1}([0,\infty);W^{m,q}(\mathbb{R}^3))^3} < C(k_1,\omega_0)^2,$$
$$\|\nabla p\|_{C^{r_2}([0,\infty);W^{m,q}(\mathbb{R}^3))^3} < C(k_2,\omega_0)^2,$$

and

$$\|u\|_{C^{r_3}(\mathbb{R}^3\times[0,\infty))^3} < C(k_3,\omega_0)^2,$$
$$\|\nabla p\|_{C^{r_4}(\mathbb{R}^3\times[0,\infty))^3} < C(k_4,\omega_0)^2,$$

for some integer $k_1 > 0$ depending on m and r_1, $k_2 > 0$ depending on m and r_2, $k_3 > 0$ depending on r_3, and $k_4 > 0$ depending on r_4. For $i = 1,\cdots,4$, $C(k_i,\omega_0) = C\|\omega_0\|_{W^{k_i+4,1}(\mathbb{R}^3)}$ and C is a positive constant independent of t, ν, and ω_0.

8.8. **The measure theory of Sections 3-7 cannot be applied to Navier Stokes equations or Euler equations in the periodic case.** The so-called "2+1/2" dimensional flows consists of solving Euler equations in two dimensions for $u = (u_1,u_2)$; See for instance [9, 10] for references and more on this:

$$\partial_t u_i + \text{div } (u_i u) + \partial_i p = 0 \text{ in } \mathbb{R}^2 \times (0,\infty),\ i = 1,2$$
$$\text{div } u = 0 \text{ in } \mathbb{R}^3 \times (0,\infty)$$

and then solve a transport equation for $w = u_3$

$$\partial_t w + \text{div } (wu) = 0 \text{ in } \mathbb{R}^2 \times (0,\infty),\ \ w(\cdot,0) = w_0 \text{ in } \mathbb{R}^2 \tag{8.73}$$

and w, w_0 are independent of x_3.

Based on a work by DiPerna and Lions; See [9], it was shown that smooth global solution to a particular "2+1/2" dimensional Euler equations flows cannot be estimated in $W^{1,p}$, for $1 < p < \infty$, on any time interval $(0, \eta)$ if the initial data is only assumed to be bounded in $W^{1,p}$. The first two components u_1, u_2 in the example showing this, consist of a stationary and periodic (in space) solution of Euler equations in two dimensions and the third component consists of a periodic solution w of Eq. (8.73) ($u_3 = w$) corresponding to u_1, u_2 and a periodic initial data w_0.

Another example corresponds to the DiPerna-Majda explicit solution
$u(x, y, z, t) = (f(y), 0, g(x - tf(y)))$ to the 3D Euler equations (with zero pressure), where f and g are periodic solutions; for example, $f(x) = g(x) = \sin x$, which gives an example of a smooth periodic solution to Euler equations in which the vorticity increases linearly in time.

It will be shown below that the measure theory introduced and developed in [12] and Sections 3-7 of this paper and then applied to the Navier-Stokes equations in §8 above and [13] and to the Euler equations in [14]-[15] cannot be applied in the periodic case.

Assume that $\mathbb{R}^3$ is replaced by Ω a bounded C^∞ domain. Let u and T_r be the solution and time parameter given by the counterpart of Theorem 8.1 for the bounded case. Let $t_1, t_2 \in [0, T_r)$ with $t_1 < t_2$. $l \in \mathbb{N}^3$. By Sobolev embeddings, $D^l u \in C([t_1, t_2]; W^{1,q}(\Omega))^3$ with $q > 1$.

In order to obtain the counterparts of Theorems 8.5 and 8.6 for the bounded case, one needs to prove the counterparts of Theorems 6.1 and 7.2 for the bounded case. However, the proofs of these theorems require the building of sequences $\tilde{u}_p \in C^\infty(\Omega \times (t_1, t_2); \mathbb{R}^3)$ that are divergence-free approximating sequences of $D^l u$ in $L^1(t_1, t_2; W^{1,q}(\Omega))^3$ such that for each integer $p \geq 1$, the projection of the support of $\tilde{u}_p$ into Ω is compact. Such constructions are not possible unless $D^l u \in C([t_1, t_2]; W_0^{1,q}(\Omega))^3$. However, the condition $D^l u \in C([t_1, t_2]; W_0^{1,q}(\Omega))^3$ is not satisfied in general.

By reasoning as above on a typical cell in the periodic case, one obtains the same conclusions as above for the periodic case. The discussion above also shows that, in particular, the global estimates obtained in Section 8.6 above for the full space case are not true for the periodic case.

REFERENCES

[1] R. ADAMS, *Sobolev Spaces*, Acad. Press, 1975.

[2] J. DIEUDONNÉ, *Treatise on Analysis*, Volumes II and III, Acad. Press 1976 and 1972.

[3] L.C. EVANS and R. GARIEPY, *Measure Theory and Fine Property of Functions*, Studies in Advanced Mathematics, CRC Press 1992.

[4] H. FEDERER, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.

[5] D. GILBARG and N. TRUDINGER, *Elliptic Partial Differential Equations*, Springer-Verlag, Berlin, 1998.

[6] E. GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*, Birkhauser, Boston, 1984.

[7] J. LERAY, Etude de diverses équations intégrales nonlinéaires et de quelques problémes que pose l'hydrodynamique, *J. Math. Pures Appl.*, **12** (1933), pp. 1–82.

[8] J. LERAY, Essai sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Math.*, **(63)** (1934), pp. 193–248.

[9] P.-L. LIONS, *Mathematical Topics in Fluid Mechanics. Volume 1: Incompressible Models*, Oxford Lecture Series in Mathematics and its Applications 3, Clarendon Press, 1996.

[10] A. J. MAJDA and A. L. BERTOZZI, *Vorticity and Incompressible Flow*, Cambridge Texts in Applied Mathematics, Cambridge University Press, 2002.

[11] E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton Math Series, Vol. 30 (1970).

[12] M. TIDRIRI, Construction and study of new measures associated with a given pair of functions, *J. Stat. Phys.*, (2016) 162, pp. 577-602.

[13] M. TIDRIRI, Incompressible mechanics, Part 3: Existence, global regularity, and uniqueness of solutions of Navier-Stokes equations in space dimension 3 when the initial data and the external forces are regular, *ResearchGate*: 09/2017, DOI: 10.13140/RG.2.2.32445.36324.

[14] M. TIDRIRI, Incompressible mechanics, Part 2: Existence, global regularity, and uniqueness of solutions of Euler equations in space dimension 3 when the initial data are regular, *ResearchGate*: 09/2017, DOI: 10.13140/RG.2.2.14409.85609.

[15] M. TIDRIRI, Incompressible mechanics, Part 4: Existence, global regularity, and uniqueness of solutions of Euler equations in space dimension 3 when the initial data and the external forces are regular, *ResearchGate*: 09/2017, DOI: 10.13140/RG.2.2.21540.17286.