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ON GENERAL CLASS OF NONLINEAR CONTRACTIVE MAPS AND THEIR PERFORMANCE ESTIMATES

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ABSTRACT. This paper considers two independent general class of nonlinear contractive maps to study the existence properties of nonlinear operators with prior degenerate. The existence properties are proved in the framework of approximate fixed points with the imposition of the general class of contractive conditions in metrical convex spaces without emphasis on completeness or compactness. For computational purposes, the performance estimates and the sensitivity dependence of these conditions are obtained for the Picard operator. Practical examples are also considered to justify the validity of the conditions. The results ensure no term is lost in the operators with prior degenerate and the conditions are strictly larger class when compare with others in the literature.

Key words and phrases: Approximate fixed point; General nonlinear contractive maps; Asymptotic regular; Semi-continuous functions; Prior degenerate; Error estimates.

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1. INTRODUCTION

Several generalizations of Banach's contraction map [2] given by

$$d(Tx,Ty) \leq \alpha d(x,y) \tag{1.1}$$

for $x,y \in X$, X a metric space and $\alpha \in (0,1)$ have been obtained in many ways to study the existence properties of nonlinear operators, for example, see [4, 24]. Many of these generalizations have been unified in the sense that if $\psi : [0,\infty) \to [0,\infty)$ is nondecreasing and upper semi-continuous function from the right, the map T satisfying

$$d(Tx,Ty) \leq \psi(\eta);\ \eta \in [0,\infty) \tag{1.2}$$

is a nonlinear contractive condition. The condition (1.2) is the result of Boyd and Wong [10] if $\psi(\eta) < \eta$ where $\eta = d(x,y)$. In [19], it is proved that if $\psi(\eta) = \alpha(\eta)\eta$, where $\alpha : (0,\infty) \to [0,1)$ is a decreasing function, then the map T satisfying (1.2) also has a unique fixed point. A similar result, where $\alpha(\eta_n) \to 1$ when $\eta_n \to 0$, can be found in [3]. The result of Reich [21] followed from the Banach's contraction map, if $\alpha(\eta) = 1 - \frac{\varphi(\eta)}{\eta}$, where $\varphi : [0,\infty) \to [0,\infty)$ is lower semi-continuous function for which $\phi(0) = 0$, then $\alpha(\eta)$ is increasing in $[0,1)$. Condition (1.2) is also related to the weakly contractive map defined by Alber and Guerre-Delabriere [1] if there is a nondecreasing and lower semi-continuous function φ for which $\psi(\eta) = \eta - \varphi(\eta)$, where $\eta = d(x,y)$. See also [12, 29] for related literature. Going by the work of Rhoades [25], one can easily deduce that the function $\alpha(\eta) = 1 - \frac{\eta-\varphi(\eta)}{\eta}$ with $\varphi(\eta) < \eta$ is similar to the result of Rakotch [19]. It is also evident that if $\psi : [0,\infty)^2 \to [0,\infty)$ is semi-continuous function such that the map T satisfies the inequality

$$d(Tx,Ty) \leq \psi(\eta,\omega), \tag{1.3}$$

where $(\eta,\omega) \in [0,\infty)^2$, then the contractive type conditions of Kannan [18] and Chattajea [11] are embedded in the condition (1.3) if $\psi(\eta,\omega) = a(\eta+\omega)$ for $a \in (0,\frac{1}{2})$. Both inequalities (1.2) and (1.3) are related in such a way that if $\omega = 0$, then $\psi(\eta,0) = \psi(\eta)$ for all $\eta \in [0,\infty)$. Let $\varrho : [0,\infty) \to [0,\infty)$ be a continuous and nondecreasing function with $\varrho(0) = 0$ and $\psi : [0,\infty)^i \to [0,\infty)$, $i = 1(1)5$, be a continuous function for which

$$\varrho\left(d(Tx,Ty)\right) \leq \psi(\tau);\ \tau \in [0,\infty)^i,\ \text{with } \varrho > \psi. \tag{1.4}$$

and a weakly form as:

$$\varrho\left(d(Tx,Ty)\right) \leq \varrho\left(\tau\right) - \varphi(\tau);\ \tau \in [0,\infty)^i. \tag{1.5}$$

or as:

$$\varrho\left(d(Tx,Ty)\right) \leq \alpha\left(\tau\right)\varphi(\tau);\ \tau \in [0,\infty)^i, \alpha \in F. \tag{1.6}$$

A vast amount of literature can be obtained from (1.4)-(1.6); for few recent papers in this regard, see [9, 17, 26, 28]. The efficacy for some of these conditions does not perform effectively as expected from the numerical view. So, this paper aims to introduce two independent nonlinear contractive conditions to study the existence properties as well as the effectiveness of nonlinear operators.

2. PRELIMINARIES

Definition 2.1. [8] A metric space X is said to be metric convex if for each $x,y \in X$ there is a $z \neq x,y$ for which

$$d(x,y) = d(x,z) + d(z,y)$$

Lemma 2.1. [10] *Let X be a convex metric space and $T : X \to X$ be a self map satisfying*

$$d(Tx, Ty) \leq \psi(d(x, y)), \ \ \text{for } x, y \in X.$$

Let $\psi(\eta) = \sup\{d(Tx, Ty) : x, y \in X, \ \eta = d(x, y)\}$. Then,

I. $s \geq 0, \ t \geq 0$ *and* $\eta = s + t < \infty$ *implies* $\psi(\eta) \leq \psi(s) + \psi(t)$*;*
II. $\psi(\eta)$ *is upper semi-continuous from the right of* $[0, \infty)$.

The following definitions generalize some nonlinear contractive conditions in the literature.

Definition 2.2. Let X be a metrical convex space. The map $T : X \longrightarrow X$ is called a general nonlinear contractive map (first kind) if there is $s, t \in [0, \infty)$ such that

$$d(Tx, Ty) \leq \psi(s) + \psi(t), \ \text{for } x, y \in X, \tag{2.1}$$

where $\psi : [0, \infty) \to [0, \infty)$ is upper semi-continuous function.

Further, since ψ is continuous at the origin, for any $\psi_1, \psi_2 \in \Psi$ and $s, t, \eta \in [0, \infty)$ with $\eta = s + t$, there exists $\psi = \psi_1 + \psi_2 \in \Psi$ such that $\psi(\eta) \leq \psi_1(s) + \psi_2(t)$.
In view of this, another contractive condition is defined as follow:

Definition 2.3. Let X be a metrical convex space. The map $T : X \longrightarrow X$ is called a general nonlinear contractive map (second kind) if there is $s, t \in [0, \infty)$ such that

$$d(Tx, Ty) \leq \psi_1(s) + \psi_2(t), \ \text{for } x, y \in X, \tag{2.2}$$

where ψ is upper semi-continuous function.

The goal of Definitions 2.2 and 2.3 is to ensure that no term is lost in the process of approximating nonlinear operators with prior degenerate.

Remark 2.1. Observe that by combining conditions (1.4) and (2.1)(or (2.2)), there results

$$\varrho\left(d(Tx, Ty)\right) \leq \psi(s) + \psi(t); \ s, t \in [0, \infty),$$

or

$$\varrho\left(d(Tx, Ty)\right) \leq \psi_1(s) + \psi_2(t), \ s, t \in [0, \infty),$$

with $\varrho > \psi, \psi_1, \psi_2$, where ϱ is an altering distance. Each of these conditions is more general than the results in [28].

Remark 2.2. If $s = t$ in Definition 2.3, then condition (2.2) is similar to (1.2). If $s \neq t$, then fewer conditions that are facilitated by an operator satisfying Zamfirescu type condition in [5, 22, 23, 27] are embedded in condition (2.2) if $\psi_1(s) = \alpha s$ and $\psi_2(t) = \beta t$ with $\alpha + \beta < 1$. For instance, (i) $\psi_1(d(x, y)) = \alpha d(x, y)$ and $\psi_2(d(y, Tx)) = L d(y, Tx)$ with $\alpha + L < 1$; (ii) $\psi_1(d(x, y)) = \alpha d(x, y)$ and $\psi_2(d(x, Tx)) = 2\alpha d(x, Tx)$ with $\alpha < \frac{1}{3}$; and (iii) $\psi_1(d(x, y)) = \alpha d(x, y)$ and $\psi_2(d(x, Tx)) = \varphi(d(x, Tx))$ with $\alpha < 1$.
(iv) More so, the map T satisfying the Reich operator in [21], for $a, b, c \in \mathbb{R}^+$ with $a+b+c < 1$ may be redefined as

$$d(Tx, Ty) \leq \frac{a + b}{1 - c} d(x, y) + \frac{b + c}{1 - c} d(y, Tx)$$

This inequality is related to the form (2.2) when $\alpha = \dfrac{a + b}{1 - c}$ and $\beta = \dfrac{a + b}{1 - c}$.
(v) The map satisfying the Ciric [14] type conditions could also be embedded in the form (2.2).
(vi) The rational type contraction map defined in [15] is also related to the form (2.2) by letting $\psi_1(d(x, y)) = \alpha d(x, y)$ and $\psi_2(t) = \beta t$, where $t = \dfrac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)}$ with $\alpha + \beta < 1$.

Motivated by the above general nonlinear contractive conditions (2.1) and (2.2), this paper presents a class for each of the general conditions as follows:

Definition 2.4. Let X be a metrical convex space and $T : X \to X$. The map T is called a general (α_i, φ)-weak contractive map, first and second kinds respectively, if it satisfies

$$d(Tx, Ty) \leq \alpha_1(s)\varphi(s) + \alpha_2(t)\varphi(t) \tag{2.3}$$

and

$$d(Tx, Ty) \leq \alpha_1(s)\varphi_1(s) + \alpha_2(t)\varphi_2(t) \tag{2.4}$$

for $x, y \in X$, where φ, φ_1 and φ_2 are lower semi-continuous and nondecreasing functions and $\alpha_1, \alpha_2 \in F = \big\{\alpha_i | \alpha_i : [0, \infty) \to [0, \frac{1}{2}),\ i = 1, 2\big\}$ with the imposition that $\alpha_i(u_n) \to \frac{1}{2}$ implies $u_n \to 0$.

Condition (2.3) is obviously a Geraghty contraction [3] if $t = 0$, $\alpha_1 = \alpha$ and $\varphi(s) = s$.

Definition 2.5. Let X be a metrical convex space and $T : X \to X$. The map T is a general weakly contractive map, first and second kinds, respectively, if it satisfies

$$d(Tx, Ty) \leq s - \varphi(s) + t - \varphi(t) \tag{2.5}$$

$$d(Tx, Ty) \leq s - \varphi_1(s) + t - \varphi_2(t) \tag{2.6}$$

for $x, y \in X$ and $s, t \in [0, \infty)$, where φ, φ_1 and φ_2 are lower semi-continuous and nondecreasing function.

Observe that the condition (2.5) is obtained by taking $\psi(\xi) = \xi - \varphi(\xi)$, for $\xi \in [0, \infty)$.
A similitude of conditions (2.3), (2.4), (2.5) and (2.6) follows from Remark 2.1.
Without preconceiving, it is worthwhile stating that if X is star-shaped, the conditions (2.5) and (2.6) may not be applicable. A counter example is given as follow:

Example 2.1. *Consider the M-shaped* $X = \{(0,0), (1,4), (2,1), (3,4), (4,0)\}$ *furnished with the metric given by*

$$d(x, 0) = \begin{cases} x, & x \neq 0, \\ \\ 0, & x = 0 \end{cases}$$

and $T : X \to X$ *defined by* $T(x_1, x_2) = \begin{cases} (x_1, x_2), & x_1 \leq x_2, \\ (2,\ x_2), & x_1 > x_2 \end{cases}$ *, with* $\varphi(t) = \frac{1}{2}t$.

Condition (2.5) (or (2.6)) is not applicable since for $(1, 2), (3, 2) \in X$, the line segment joining $(1, 2)$ and $(3, 2)$ lies on the region not in X. On the other hand, if we consider the metrical convex hull of X (denoted by coX), then any two points in coX has a line segment contained in coX, and in turn, inequality (2.5) (or (2.6)) is applicable.
The following useful definitions can be found in [5, 23].

Definition 2.6. Let $T : X \to X$, $\varepsilon > 0$, $x_0 \in X$. An element x_0 is called an $\varepsilon-$fixed point of T provided that

$$d(Tx_0, x_0) < \varepsilon$$

The set of all ε-fixed points of $F_\varepsilon(T) = \{x \in X : X \text{ is an } \varepsilon - \text{fixed point of T}\}$. Any map T is said to have an $\varepsilon-$fixed point property if $F_\varepsilon(T) \neq \emptyset$.

Lemma 2.2. *Let (X, d) be a metric space and $T : X \to X$ be a self map such that T is asymptotically regular, that is*

$$d(T^n x, T^{n+1}x) \to 0 \text{ as } n \to \infty,$$

for all $x \in X$. Then T has the ε-fixed point property.

Definition 2.7. Let $S, T : X \to X$ have $\varepsilon-$fixed point property. The map S is called an approximate operator of T if there exists $\rho > 0$ such that

$$d(Sx, Tx) \leq \rho, \; \forall \, x \in X$$

In view of Definition 2.7, if p and q are the $\varepsilon-$fixed points of S and T, respectively, then the following estimate holds

$$d(p, q) \leq \phi(\rho), \; \forall \, p, q \in F_\varepsilon(T).$$

In this case, the diameter of $F_\varepsilon(T)$ is given by

$$\delta(F_\varepsilon(T)) \leq \phi(2\varepsilon),$$

where $\delta(F_\varepsilon(T)) = \sup \{d(p, q) : p, q \in F_\varepsilon(T)\}$.

3. MAIN RESULTS

In this section, some existence properties of nonlinear operator are proved with the imposition of conditions (2.3), (2.5) and (2.6) for few independent inputs and their performance estimates are obtained. The following result is a general nonlinear contractive Geraghty-type.

Theorem 3.1. *Let C be a convex subset of X and $T : C \to C$ is a map satisfying* (2.3) *for which $s = d(x, y)$ and $t = d(x, Tx)$, where $x, y \in C$ and $\alpha_1, \alpha_2 \in F$. Then, T has approximate fixed point. Moreover, if S is the approximate operator of T and $u, v \in F_\varepsilon(T)$ are the approximate fixed points of S, T, respectively. Then, the estimate*

$$\delta(F_\varepsilon(T)) \leq \frac{(\frac{a}{2} + 2)\varepsilon}{1 - \frac{a}{2}}, \;\; \text{for } \varepsilon > 0, \; a \in (0, 1) \tag{3.1}$$

holds.

Proof. Select $x_0 \in X$ and let x_n be a sequence defined by $x_n = T^n x_0$. By the condition of the theorem, there gives

$$\begin{aligned} d(x_n, x_{n+1}) &= d\left(T^n x_0, T^{n+1} x_0\right) \\ &\leq \alpha_1 \left(d(T^{n-1}x_0, T^n x_0)\right) \varphi\left(d\left(T^{n-1}x_0, T^n x_0\right)\right) \\ &\qquad + \alpha_2 \left(d(T^{n-1}x_0, T^n x_0)\right) \varphi\left(d\left(T^{n-1}x_0, T^n x_0\right)\right) \\ &= \left[\alpha_1 \left(d(T^{n-1}x_0, T^n x_0)\right) + \alpha_2 \left(d(T^{n-1}x_0, T^n x_0)\right)\right] \varphi\left(d\left(T^{n-1}x_0, T^n x_0\right)\right) \end{aligned}$$

Since $\alpha_1, \alpha_2 \in F$, this last inequality reduces to

$$d\left(T^n x_0, T^{n+1} x_0\right) \leq \varphi\left(d\left(T^{n-1}x_0, T^n x_0\right)\right)$$

By the property of φ, $\varphi^n \to 0$ as $n \to \infty$. Thus, by Lemma 2.2, we have

$$\lim_{n\to\infty} d\left(T^n x_0, T^{n+1} x_0\right) \to 0$$

Hence, T has approximate fixed point.
More so, let $u, v \in X$ and define $u \approx Su$ and $v \approx Tv$. Let $\rho > 0$ and consider the particular

case $\varphi(\xi) = a\xi$ for $a \in (0,1)$ and $\xi \in [0,\infty)$. By condition of the theorem and Definition 2.6 and 2.7, there results

$$\begin{aligned} d(u,v) &\leq d(Su,Sv) + \rho \\ &\leq \alpha_1(d(u,v))\varphi(d(u,v)) + \alpha_2(d(u,v))\varphi(d(u,Su)) + \rho \\ &= \frac{a}{2}d(u,v) + \frac{a}{2}\varepsilon + \rho \end{aligned}$$

This implies

$$d(u,v) \leq \frac{\frac{a}{2}\varepsilon + \rho}{1 - \frac{a}{2}}$$

By taking $\phi(\rho) = \frac{\frac{a}{2}\varepsilon+\rho}{1-\frac{a}{2}}$, we have

$$\delta(F_\varepsilon(T)) \leq \frac{(\frac{a}{2}+2)\varepsilon}{1-\frac{a}{2}}, \quad \text{for } \varepsilon > 0.$$

As required. ∎

The following result is nonlinear contractive map of rational-type.

Theorem 3.2. *Let C be a convex subset of X and $T : C \to C$ is a map satisfying* (2.5) *for which $s = d(x,y)$ and $t = \dfrac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}$, where $x, y \in C$ and $s,t \in [0,\infty)$. Then, T has approximate fixed point. Moreover, if S is the approximate operator of T and $u,v \in F_\varepsilon(T)$ are the approximate fixed points of S, T, respectively. Then, the estimate*

$$\delta(F_\varepsilon(T)) \leq \frac{2\varepsilon}{a}, \quad \textit{for } \varepsilon > 0, \ a \in (0,1) \tag{3.2}$$

holds.

Proof. Select $x_0 \in X$ and let x_n be a Picard sequence such that $x_n = T^n x_0$. By the condition of the theorem, there gives

$$\begin{aligned} d(x_n, x_{n+1}) &= d\left(T^n x_0, T^{n+1}x_0\right) \\ &\leq d\left(T^{n-1}x_0, T^n x_0\right) - \varphi\left(d\left(T^{n-1}x_0, T^n x_0\right)\right) \\ &\qquad + \frac{d(T^n x_0, T^{n+1}x_0)\left(1 + d(T^{n-1}x_0, T^n x_0)\right)}{1 + d(T^{n-1}x_0, T^n x_0)} \\ &\qquad - \varphi\left(\frac{d(T^n x_0, T^{n+1}x_0)\left(1 + d(T^{n-1}x_0, T^n x_0)\right)}{1 + d(T^{n-1}x_0, T^n x_0)}\right) \\ &= d\left(T^{n-1}x_0, T^n x_0\right) - \varphi\left(d(T^{n-1}x_0, T^n x_0)\right) \\ &\qquad + d(T^n x_0, T^{n+1}x_0) - \varphi\left(d(T^n x_0, T^{n+1}x_0)\right) \end{aligned}$$

Solving further, we obtain

$$\varphi\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) \leq d\left(T^{n-1}x_0, T^n x_0\right) - \varphi\left(d\left(T^{n-1}x_0, T^n x_0\right)\right)$$

Let $l_n = d(T^{n-1}x_0, T^n x_0)$ and consider the antiderivative Γ defined by

$$\Gamma(\zeta) = \int \frac{1}{\varphi(\xi)} d\xi$$

Since $\varphi(l_n) < l_n$, then for $\varphi(l_{n+1}) \geq \varphi(\zeta)$, where $\varphi(l_n) \leq \zeta \leq l_n$, we have

$$\Gamma(l_n) - \Gamma(\varphi(l_n)) = \int_{\varphi(l_n)}^{l_n} \frac{d\xi}{\varphi(\xi)} = \frac{l_n - \varphi(l_n)}{\varphi(\zeta)} \geq \frac{l_n - \varphi(l_n)}{\varphi(l_{n+1})} \geq 1$$

Thus,

$$\Gamma(l_n) - \Gamma(\varphi(l_n)) \geq 1 \;\Rightarrow\; \Gamma(\varphi(l_n)) \leq \Gamma(l_n) - 1 \leq \cdots \leq \Gamma(l_0) - (n+1)$$

This implies

$$\varphi(l_n) \leq \Gamma^{-1}(\Gamma(l_0) - (n+1)) \tag{3.3}$$

Let $\chi(\varsigma) = \Gamma^{-1}(\Gamma(l_0) - (n+1))$. By hypothesis, φ is nondecreasing implies that both Γ and Γ^{-1} are increasing functions, hence, χ is nonincreasing. For fixed $\Gamma(l_0)$, it follows that

$$\lim_{\varsigma\to\infty} \chi(\varsigma) = 0$$

From inequality (3.3), $\varphi(l_n) \leq 0$ and by hypothesis on φ, we have $\varphi(l_n) = 0$.
Therefore,

$$\lim_{n\to\infty} l_{n+1} = \lim_{n\to\infty} d\left(T^n x_0, T^{n+1}x_0\right) \to 0$$

which implies that T has approximate fixed point.
More so, let $u, v \in C$ and define $u \approx Su$ and $v \approx Tv$. Due to the general nature of φ which may be difficult to analyze, we suppose, in particular, that $\varphi(\xi) = a\xi$ for $a \in (0,1)$ and $\xi \in [0,\infty)$. For $\rho > 0$, using condition of the theorem and Definition 2.6 and 2.7, this gives

$$\begin{aligned}
d(u,v) &\leq d(Su,Sv) + \rho \\
&\leq d(u,v) - \varphi(d(u,v)) + \frac{d(v,Sv)(1+d(u,Tu))}{1+d(u,v)} - \varphi\left(\frac{d(v,Sv)(1+d(u,Tu))}{1+d(u,v)}\right) + \rho \\
&\leq (1-a)d(u,v) + (1-a)\frac{\varepsilon(1+\varepsilon)}{1+d(u,v)} + \rho
\end{aligned}$$

This further implies

$$ad^2(u,v) - (\rho - a)d(u,v) - [\varepsilon(1-a)(1+\varepsilon) + \rho] \leq 0 \tag{3.4}$$

Let $e = d(u,v) \geq 0$ and $\rho \leq h = \varepsilon(1-a)(1+\varepsilon) + \rho$ such that $ae^2 - (\rho - a)e - \rho = 0$.
Then,

$$e = \frac{\rho - a \pm \sqrt{(\rho-a)^2 + 4a\rho}}{2a} = \frac{(\rho - a) \pm (\rho + a)}{2a}.$$

Using this in (3.4), we have

$$(ae - \rho)(e+1) \leq 0$$

Since $e + 1 > 0$, then $e - \dfrac{\rho}{a} \leq 0$. Hence,

$$d(u,v) \leq \frac{\rho}{a}, \;\text{ for } \rho > 0,\; a \in (0,1).$$

Letting $\phi(\rho) = \dfrac{\rho}{a}$, by Definition 2.7, we obtain

$$\delta(F_\varepsilon(T)) \leq \frac{2\varepsilon}{a}, \;\text{ for } \varepsilon > 0,\; a \in (0,1).$$

It is not difficult to see that if $\varepsilon \to 0$, there is sufficiently small δ for all $a \in (0,1)$. ∎

Remark 3.1. If $T : C \to C$ satisfies (2.6) for $s = d(x,y)$ and $t = \dfrac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}$, then T has approximate fixed point since both φ_1 and φ_2 belong to the same family.

Theorem 3.3. *Let C be a convex subset of X and $T : C \to C$ is a map satisfying* (2.5) *for which $s = d(x, y)$ and $t = d(x, Tx)$, where $x, y \in X$ and $s, t \in [0, \infty)$. Then, T has approximate fixed point. Furthermore, if S is the approximate operator of T and $u, v \in F_\varepsilon(T)$ are the approximate fixed points of S, T, respectively. Then, the following estimate holds:*

$$\delta(F_\varepsilon(T)) \leq \frac{(3-a)\varepsilon}{a}, \quad \textit{for } \varepsilon > 0. \tag{3.5}$$

Proof. Select $x_0 \in X$ and let x_n be a Picard sequence such that $x_n = T^n x_0$. Then,

$$\begin{aligned} d(x_n, x_{n+1}) &= d\left(T^n x_0, T^{n+1} x_0\right) \\ &\leq d\left(T^{n-1} x_0, T^n x_0\right) - \varphi\left(d\left(T^{n-1} x_0, T^n x_0\right)\right) + d\left(T^{n-1} x_0, T(T^{n-1} x_0)\right) \\ &\quad - \varphi\left(d\left(T^{n-1} x_0, T(T^{n-1} x_0)\right)\right) \\ &= 2d\left(T^{n-1} x_0, T^n x_0\right) - 2\varphi\left(d\left(T^{n-1} x_0, T^n x_0\right)\right) \end{aligned}$$

This further implies that

$$d\left(T^n x_0, T^{n+1} x_0\right) \leq d\left(T^{n-1} x_0, T^n x_0\right) - \varphi^2\left(d\left(T^{n-1} x_0, T^n x_0\right)\right) \leq d\left(T^{n-1} x_0, T^n x_0\right) \tag{3.6}$$

Since $d\left(T^n x_0, T^{n+1} x_0\right) \leq d\left(T^{n-1} x_0, T^n x_0\right)$, then $d\left(T^n x_0, T^{n+1} x_0\right)$ is a non-increasing and non-negative sequence. Let ι be a non-negative real number for which

$$\iota = \lim_{n \to \infty} d\left(T^n x_0, T^{n+1} x_0\right) = \lim_{n \to \infty} d\left(T^{n-1} x_0, T^n x_0\right),$$

By inequality (3.6), we have

$$\iota \leq \iota - \varphi^2(\iota)$$

This implies that $\varphi^2(\iota) \leq 0 \Rightarrow \varphi(\iota) \leq 0$. By the hypothesis on φ, $\varphi(\iota) = 0$, and thus,

$$\iota = \lim_{n \to \infty} d\left(T^n x_0, T^{n+1} x_0\right) \to 0$$

Therefore, T has approximate fixed point.
Furthermore, let $u, v \in X$ and define $u \approx Su$ and $v \approx Tv$. Let $\rho > 0$ and consider the particular case $\varphi(\xi) = a\xi$ for $a \in (0, 1)$ and $\xi \in [0, \infty)$, by condition of the theorem and Definition 2.6 and 2.7, there results

$$\begin{aligned} d(u, v) &\leq d(Su, Sv) + \rho \\ &\leq d(u, v) - \varphi(d(u, v)) + d(u, Su) - \varphi(d(u, Su)) + \rho \\ &= (1-a)\left[d(u, v) + \varepsilon\right] + \rho \end{aligned}$$

This implies

$$d(u, v) \leq \frac{\varepsilon + \rho \sum_{r \geq 0} a^r}{\sum_{r \geq 0} a^r - 1}$$

By taking $\phi(\rho) = \dfrac{\varepsilon + \rho \sum_{r \geq 0} a^r}{\sum_{r \geq 0} a^r - 1}$, we have that

$$\delta(F_\varepsilon(T)) \leq \frac{(3-a)\varepsilon}{a}, \quad \text{for } \varepsilon > 0.$$

It is easily seen that $\delta \to 0$ as $\varepsilon \to 0$. ∎

Theorem 3.4. *Let C be a convex subset of X and $T : C \to C$ is a map satisfying* (2.6) *for which $s = d(x, y)$ and $t = d(x, Tx)$, where $x, y \in X$ and $s, t \in [0, \infty)$. Then, T has approximate* *fixed point. Furthermore, if S is the approximate operator of T and $u, v \in F_\varepsilon(T)$ are the approximate fixed points of S, T, respectively. Then, the following estimate holds:*

$$\delta(F_\varepsilon(T)) \leq \frac{(3-b)\varepsilon}{a}, \quad \textit{for } \varepsilon > 0,\ 0 < a, b < 1. \tag{3.7}$$

Proof. Select $x_0 \in X$ and let x_n be a Picard sequence such that $x_n = T^n x_0$. If $\varphi_1, \varphi_2 \in \Phi$ for which $(\varphi_1 + \varphi_2)t = \varphi(t)$, $t \in [0, \infty)$, then,

$$\begin{aligned} d(x_n, x_{n+1}) &= d\left(T^n x_0, T^{n+1} x_0\right) \\ &\leq d\left(T^{n-1} x_0, T^n x_0\right) - \varphi_1\left(d\left(T^{n-1} x_0, T^n x_0\right)\right) + d\left(T^{n-1} x_0, T(T^{n-1} x_0)\right) \\ &\quad - \varphi_2\left(d\left(T^{n-1} x_0, T(T^{n-1} x_0)\right)\right) \\ &= 2d\left(T^{n-1} x_0, T^n x_0\right) - \varphi\left(d\left(T^{n-1} x_0, T^n x_0\right)\right) \end{aligned}$$

That is,

$$d\left(T^n x_0, T^{n+1} x_0\right) \leq 2d\left(T^{n-1} x_0, T^n x_0\right) - \varphi\left(d\left(T^{n-1} x_0, T^n x_0\right)\right) \tag{3.8}$$

Let $l_{n+1} = d\left(T^n x_0, T^{n+1} x_0\right)$ and $\zeta \in [2l_n, l_{n+1}]$ for $\varphi(l_n) \geq \varphi(\zeta)$, then

$$\Gamma(2l_n) - \Gamma(l_{n+1}) = \int_{l_{n+1}}^{2l_n} \frac{d\zeta}{\varphi(\zeta)} = \frac{2l_n - l_{n+1}}{\varphi(\zeta)} \geq \frac{2l_n - l_{n+1}}{\varphi(l_n)} \geq 1$$

This implies

$$\Gamma(2l_n) - \Gamma(l_{n+1}) \geq 1 \ \Rightarrow\ \Gamma(l_{n+1}) \leq \Gamma(2l_n) - 1 \leq \cdots \leq \Gamma(2l_0) - (n+1)$$

That is,

$$l_n \leq \Gamma^{-1}(\Gamma(2l_0) - n)$$

Let $\chi(\varsigma) = \Gamma^{-1}(\Gamma(2l_0) - n)$. Since φ is nondecreasing, then both Γ and Γ^{-1} are increasing functions imply that χ is nonincreasing. For fixed $\Gamma(2l_0)$, it follows that

$$\lim_{\varsigma \to \infty} \chi(\varsigma) = 0$$

Hence, $l_n \leq 0$ implies that $l_n = 0$.
Therefore,

$$\lim_{n \to \infty} l_{n+1} = \lim_{n \to \infty} d\left(T^n x_0, T^{n+1} x_0\right) \to 0.$$

Furthermore, let $u, v \in X$ and define $u \approx Su$ and $v \approx Tv$. Let $\rho > 0$ and consider the particular case where $\varphi_1(\xi) = a\xi$ and $\varphi_2(\mu) = b\mu$ for $a, b \in (0, 1)$ and $\xi, \mu \in [0, \infty)$.
So,

$$\begin{aligned} d(u, v) &\leq d(Su, Sv) + \rho \\ &\leq d(u, v) - \varphi_1(d(u, v)) + d(u, Su) - \varphi_2(d(u, Su)) + \rho \\ &= (1-a)d(u, v) + (1-b)\varepsilon + \rho \end{aligned}$$

This gives

$$d(u, v) \leq \frac{(1-b)\varepsilon + \rho}{a}$$

By letting $\phi(\rho) = \dfrac{(1-b)\varepsilon + \rho}{a}$, we have that

$$\delta(F_\varepsilon(T)) \leq \frac{(3-b)\varepsilon}{a}, \quad \text{for } \varepsilon > 0,\ a, b \in (0, 1).$$

It is obvious that $\delta \to 0$ as $\varepsilon \to 0$. ∎

Theorem 3.5. *Let C be a convex subset of X and $T : C \to C$ is a map satisfying* (2.5) *for which $s = d(x, y)$ and $t = d(y, Tx)$, where $x, y \in X$ and $s, t \in [0, \infty)$. Then T has approximate fixed point. Furthermore, if S is the approximate operator of T and $u, v \in F_\varepsilon(T)$ are the approximate fixed points of S, T, respectively. Then, the estimate*

$$\delta(F_\varepsilon(T)) \leq \frac{(3-a)\varepsilon}{2a-1}, \ \text{for } \varepsilon > 0, \ \frac{1}{2} < a < 1. \tag{3.9}$$

holds.

Proof. Select $x_0 \in X$ and let x_n be a Picard sequence such that $x_n = T^n x_0$. Then,

$$\begin{aligned} d(x_n, x_{n+1}) &= d\left(T^n x_0, T^{n+1} x_0\right) \\ &\leq d\left(T^{n-1} x_0, T^n x_0\right) - \varphi\left(d\left(T^{n-1} x_0, T^n x_0\right)\right) - d\left(T^n x_0, T(T^{n-1} x_0)\right) \\ &\quad - \varphi\left(d\left(T^n x_0, T(T^{n-1} x_0)\right)\right) \\ &= d\left(T^{n-1} x_0, T^n x_0\right) - \varphi\left(d\left(T^{n-1} x_0, T^n x_0\right)\right) \\ &\leq d\left(T^{n-1} x_0, T^n x_0\right) \end{aligned}$$

This means that $d\left(T^n x_0, T^{n+1} x_0\right)$ is non-increasing and non-negative sequence. Let $l \geq 0$ be a real number such that

$$l = \lim_{n\to\infty} d\left(T^n x_0, T^{n+1} x_0\right) = \lim_{n\to\infty} d\left(T^{n-1} x_0, T^n x_0\right),$$

Then,

$$l \leq l - \varphi(l)$$

This gives $\varphi(l) \leq 0$. By the hypothesis on φ, $\varphi(l) = 0$. This implies that

$$l = \lim_{n\to\infty} d\left(T^n x_0, T^{n+1} x_0\right) \to 0$$

Therefore, T has approximate fixed point.
Furthermore, let $u, v \in X$ and define $u \approx Su$ and $v \approx Tv$. Suppose, in particular, that $\varphi(\xi) = a\xi$ for $a \in (0, 1)$ and $\xi \in [0, \infty)$. For ρ, using the condition of the theorem and Definition 2.6 and 2.7, there results

$$\begin{aligned} d(u, v) &\leq d(Su, Sv) + \rho \\ &\leq d(u, v) - \varphi(d(u, v)) + d(v, Su) - \varphi(d(v, Su)) + \rho \\ &\leq 2\left(d(u, v) - \varphi(d(u, v))\right) + d(u, Su) - \varphi(d(u, Su)) + \rho \\ &= 2\left(d(u, v) - \varphi(d(u, v))\right) + \varepsilon - \varphi(\varepsilon) + \rho \\ &= 2(1-a)d(u, v) + (1-a)\varepsilon + \rho \end{aligned}$$

By resolving, this gives

$$d(u, v) \leq \frac{\varepsilon + \rho \sum_{r\geq 0} a^r}{\sum_{r\geq 0} a^r - 2}.$$

This is only valid for $a \in (\frac{1}{2}, 1)$. Letting $\phi(\rho) = \dfrac{\varepsilon + \rho \sum_{r\geq 0} a^r}{\sum_{r\geq 0} a^r - 2}$, the diameter of the set $F_\varepsilon(T)$ is obtained as:

$$\delta(F_\varepsilon(T)) \leq \frac{(3-a)\varepsilon}{2a-1}, \ \text{ for } \varepsilon > 0 \text{ and } a \in (\frac{1}{2}, 1).$$

∎

Theorem 3.6. *Let C be a convex subset of X and $T : C \to C$ is a map satisfying* (2.6) *for which $s = d(x, y)$ and $t = d(y, Tx)$, where $x, y \in X$ and $s, t \in [0, \infty)$. Then T has approximate fixed point. Furthermore, if S is the approximate operator of T and $u, v \in F_\varepsilon(T)$ are the approximate fixed points of S, T, respectively. Then, the estimate*

$$\delta(F_\varepsilon(T)) \leq \frac{(3-b)\varepsilon}{a+b-1}, \text{ for } \varepsilon > 0, \ \frac{1}{2} < a, b < 1. \tag{3.10}$$

holds.

The proof is similar to the proof of Theorem 3.5.

Remark 3.2. a. Observe that if $a = 1$ in the estimates (3.1), (3.2), (3.5) and (3.9), then the right hand inequalities of the estimates do not exceed 2ε, that is, $\phi(2\varepsilon) \leq 2\varepsilon$. This is not a mere coincident, the number 2ε is the least of the estimates. Similarly for the estimates (3.7) and (3.10) when $a = b = 1$.

b. A similar estimate could be obtained for the functions $\varphi(t) = \dfrac{t}{t+1}$ and $\varphi(t) = c(e^t - 1)$ if $a = 1 - \alpha(t)$, where $\alpha(t) = \dfrac{1}{t+1}$, for $t \geq 0$ and $a = 1 - c$, for $c \in (0, 1)$, respectively.

Remark 3.3. If q is an approximate fixed point of T in Theorem 3.2, 3.3 and 3.4, the error estimate

$$e_n \leq \Gamma^{-1}\left(\Gamma(e_0 - (n-1))\right), \text{ for } n \geq 1.$$

holds, where $\Gamma(t) = \int \frac{dt}{\varphi(t)}$ and Γ^{-1} is its inverse. This is similar to the estimate contained in ([1], Theorem 3.1).

We present the error estimates of Theorem 3.1, 3.5 and 3.6 in the following Theorems:

Theorem 3.7. *Let $T : C \to C$ be a map satisfying* (2.3) *for which $s = d(x, y)$ and $t = d(x, Tx)$ with $F_\varepsilon(T) \neq \emptyset$. Then the Picard iterative process converges to the ε-fixed point q of T with the following estimate:*

$$e_n \leq \frac{\varphi^n(e_0)}{2^n}$$

holds.

Proof. Since T satisfies condition (2.3) for $s = d(x, y)$ and $t = d(x, Tx)$ with $F_\varepsilon(T) \neq \emptyset$. By Theorem 3.1 T has a ε-fixed point q, say. Select $x_0 \in C$ and let x_n be a Picard sequence, for $q \in F_\varepsilon(T)$, we have

$$\begin{aligned} d(x_n, q) &= d(Tx_{n-1}, Tq) \\ &\leq \alpha_1(d(x_{n-1}, q))\varphi(d(x_{n-1}, q)) + \alpha_2(d(x_{n-1}, x_n))\varphi(d(x_{n-1}, x_n)) \end{aligned}$$

Since T is asymptotic regular, $d(x_{n-1}, x_n) \to 0$ implies that $\varphi(d(x_{n-1}, x_n)) \to 0$ as $n \to \infty$. For $\alpha_1 \in F$, this gives

$$d(x_n, q) \leq \frac{1}{2}\varphi(d(x_{n-1}, q))$$

By letting $e_n = d(q, x_n)$, we have

$$e_n \leq \frac{1}{2}\varphi(e_{n-1}) \leq \frac{1}{2^2}\varphi^2(e_{n-2}) \leq \cdots \leq \frac{1}{2^n}\varphi^n(e_0)$$

For obvious reason, e_n is a non-increasing sequence. So, it converges to e, say, and by the hypothesis on φ, $e = 0$. ∎

Remark 3.4. If $t = 0$ in Theorem 3.7 and $\alpha \in F$ in the sense of Geraghty-type condition, then the error estimate $e_n \leq \varphi^n(e_0)$. Clearly,

$$\frac{1}{2^n}\varphi^n(e_0) \leq \varphi^n(e_0).$$

Theorem 3.8. *Let $T : C \to C$ be a map satisfying* (2.5) *for which $s = d(x, y)$ and $t = d(y, Tx)$ with $F_\varepsilon(T) \neq \emptyset$. Then the Picard iterative process converges to the ε-fixed point q of T with the following estimate:*

$$e_n \leq 2\Gamma^{-1}\left[\Gamma(\frac{ne_0}{2}) - (n-1)\right]$$

holds, where $\Gamma(t) = \int \frac{dt}{\varphi(t)}$ and Γ^{-1} is its inverse function.

Proof. Since T satisfies condition (2.5) for $s = d(x, y)$ and $t = d(y, Tx)$ with $F_\varepsilon(T) \neq \emptyset$. By Theorem 3.3 T has a ε-fixed point q, say. Select $x_0 \in C$ and let x_n be a Picard sequence, for $q \in F_\varepsilon(T)$, we have

$$\begin{aligned}
d(x_n, q) &= d(Tx_{n-1}, Tq) \\
&\leq d(x_{n-1}, q) - \varphi(d(x_{n-1}, q)) + d(q, Tx_{n-1}) - \varphi(d(q, Tx_{n-1})) \\
&= d(x_{n-1}, q) - \varphi(d(x_{n-1}, q)) + d(q, x_n) - \varphi(d(q, x_n)) \\
&\leq 2d(x_{n-1}, q) - 2\varphi(d(x_{n-1}, q)) + d(x_{n-1}, x_n) - \varphi(d(x_{n-1}, x_n))
\end{aligned}$$

Since T is asymptotic regular, $d(x_{n-1}, x_n) - \varphi(d(x_{n-1}, x_n)) \to 0$ as $n \to \infty$. Thus,

$$\frac{d(q, x_n)}{2} \leq d(x_{n-1}, q) - \varphi(d(x_{n-1}, q))$$

Let $e_n = d(q, x_n)$ be the estimate at each n-th step so that

$$\varphi(e_{n-1}) \leq e_{n-1} - \frac{1}{2}e_n$$

This implies e_n is a non-increasing sequence. Therefore, it converges to e, say, for which $\varphi(e) \leq 0$. By the hypothesis on φ, $e = 0$.
More so, if there exists $\zeta \in [\frac{e_n}{2}, e_{n-1}]$ such that $\varphi(e_n) \geq \varphi(\zeta)$ for each n, then

$$\Gamma(e_{n-1}) - \Gamma(\frac{e_n}{2}) = \int_{\frac{e_n}{2}}^{e_{n-1}} \frac{dt}{\varphi(t)} = \frac{e_{n-1} - \frac{1}{2}e_n}{\varphi(\zeta)} \geq \frac{e_{n-1} - \frac{1}{2}e_n}{\varphi(e_n)} \geq 1$$

Further, this gives

$$\Gamma(\frac{e_n}{2}) \leq \Gamma(e_{n-1}) - 1 \leq \Gamma(\frac{3e_{n-2}}{2}) - 2 \leq \Gamma(2e_{n-3}) - 3 \leq \cdots \leq \Gamma(\frac{ne_0}{2}) - (n-1).$$

By transitivity, we have the estimate

$$e_n \leq 2\Gamma^{-1}\left[\Gamma(\frac{ne_0}{2}) - (n-1)\right]$$

∎

Theorem 3.9. *Let $T : C \to C$ be a map satisfying* (2.6) *for which $s = d(x, y)$ and $t = d(y, Tx)$ with $F_\varepsilon(T) \neq \emptyset$. Then the Picard iterative process converges to the ε-fixed point q of T with the following estimate:*

$$e_n \leq \Gamma^{-1}\left(\Gamma(2e_0) - (n-1)\right)$$

holds, where $\Gamma(t) = \int \frac{dt}{\varphi(t)}$ and Γ^{-1} is its inverse function.

The proof is immediate and it is left.

Corollary 3.10. *Let C be a convex subset of X and $T : C \to C$ is a map satisfying*

$$d(Tx, Ty) \le \alpha d(x, y) + \beta \frac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)}, \ \ \text{for } x, y \in X \tag{3.11}$$

with $\alpha + \beta < 1$. Then, T has approximate fixed point. Furthermore, if S is the approximate operator T and $p, q \in F_\varepsilon(T)$ are the approximate fixed points of S, T, respectively. Then, the estimate

$$\delta(F_\varepsilon) \le 2\varepsilon(1 - \alpha)^{-1} \text{ for } \varepsilon > 0, \alpha + \beta < 1$$

holds.

Corollary 3.11. *Let $T : C \to C$ be a map satisfying*

$$d(Tx, Ty) \le \alpha d(x, y) + \beta d(x, Tx), \ \ \text{for } x, y \in X \tag{3.12}$$

with $\alpha + \beta < 1$. Then, T has approximate fixed point. Furthermore, if S is the approximate operator T and $p, q \in F_\varepsilon(T)$ are the approximate fixed points of S, T, respectively. Then, the estimate

$$\delta(F_\varepsilon) \le \frac{(\beta + 2)\varepsilon}{1 - \alpha} \text{ for } \varepsilon > 0, \alpha + \beta < 1$$

holds.

Proof. Select $x_0 \in X$ and let x_n be a Picard sequence such that $x_n = T^n x_0$. By letting $s - \varphi_1(s) = (1 - a)s$ with $a = 1 - \alpha$ and $t - \varphi_2(t) = (1 - b)t$ with $b = 1 - \beta$ in Theorem 3.3. The proof follows. ∎

Corollary 3.12. *Let C be a convex subset of X and $T : C \to C$ is a map satisfying*

$$d(Tx, Ty) \le \alpha d(x, y) + \beta d(y, Tx), \ \ \text{for } x, y \in X \tag{3.13}$$

with $\alpha + \beta < 1$. Then, T has approximate fixed point. Furthermore, if S is the approximate operator T and $p, q \in F_\varepsilon(T)$ are the approximate fixed points of S, T, respectively. Then, the estimate

$$\delta(F_\varepsilon) \le \frac{(\beta + 2)\varepsilon}{1 - \alpha - \beta} \text{ for } \varepsilon > 0, \alpha + \beta < 1$$

holds.

The proof is obvious by the application of Theorem 3.5.

Remark 3.5. i. Corollary 3.12 is similar to the result in [23].
ii. The error estimates for Corollary 3.10, 3.11 and 3.12 are, respectively,

$$e_n \le \alpha^n e_o, \ e_n \le \left(\frac{\alpha + \beta}{1 - \beta}\right)^n e_o \text{ and } e_n \le \left(\frac{\alpha}{1 - \beta}\right)^n e_o.$$

Here, the asymptotic error constants are easily accessible unlike estimations of the form $\Gamma^{-1}(\cdot)$ which are inexplicable.

4. EXAMPLES AND PROBLEMS

Example 4.1. *Let $X = [0, 1]$ be endowed with the usual metric $d(x, y) = |x - y|$ and let $T : X \to X$ be defined by the nonlinear operator $Tx = \ln(1 + x)$. Then, T satisfies the hypotheses of Theorems 3.1, 3.3 and 3.4 with fixed point $q = 0$.*

Firstly, observe that the associated function $y(x) = x - Tx$ is degenerate of order one, since $y(0) = 0$, $y'(0) = 0$ and $y''(0) \neq 0$.

Let $x, y \in [0,1]$ with $x > y$ and consider $Tz = \ln(1+z) = \sum_{k\geq 1} (-1)^{k+1}\dfrac{z^k}{k}$ for all $z \in [0,1]$, then

$$Tx - Ty = x - y - \frac{(x^2-y^2)}{2} + \frac{(x^3-y^3)}{3} - \frac{(x^4-y^4)}{4} + \frac{(x^5-y^5)}{5} + \cdots$$

Since $x > y$, both $x^p - y^p \geq (x-y)^p$ and $(x-y)^p \geq (x-y)^{p+1}$ hold for $p > 1$, and imply that

$$\frac{(x-y)^{p+1}}{p+1} - \frac{(x-y)^p}{p} \geq \frac{(x^{p+1}-y^{p+1})}{p+1} - \frac{(x^p-y^p)}{p}.$$

Thus,

(4.1)

$$\begin{aligned}
Tx - Ty &\leq x - y - \frac{(x-y)^2}{2} + \frac{(x-y)^3}{3} - \frac{(x-y)^4}{4} + \frac{(x-y)^5}{5} - \frac{(x-y)^6}{6} + \cdots \\
&= x - y - (x-y)^2 + \frac{(x-y)^2}{2} + \frac{(x-y)^3}{3} + \cdots - \left[\frac{(x-y)^4}{2} + \frac{(x-y)^6}{3} + \cdots\right] \\
&\leq x - y - (x-y)^2 + (1-\lambda(x,y))\left[\frac{(x-y)^2}{2} + \frac{(x-y)^3}{3} + \cdots\right] \\
&\leq x - y - (x-y)^2 + (1-\lambda(x,y))\left(\sum_{k=1}^{\infty}\frac{|x|^k}{k} - x\right),
\end{aligned}$$

for $x > x - y$, where $\lambda(x,y) = \min\{(x-y)^k : k = 1,2,3,\ldots\} < 1$.
Hence, we have

$$\begin{aligned}
d(Tx,Ty) &\leq |x-y| - |x-y|^2 + (1-\lambda)\left|x - \sum_{k=1}^{\infty}\frac{|x|^k}{k}\right| \\
&\equiv s - \varphi_1(s) + t - \varphi_2(t)
\end{aligned}$$

where $\varphi_1(s) \equiv s^2$ and $\varphi_2(t) \equiv \lambda t$, for $s = d(x,y)$ and $t = d(x,Tx)$ (See Theorem 3.4).
This implies that T is a general nonlinear contractive map (2.6) and for any initial seed $x_0 \in [0,1]$, the sequence $x_n = ln(1 + x_{n-1})$ converges to the fixed point $q = 0$.
More so, from inequality (4.1), T is a general nonlinear contractive map (2.3) deduced as:

$$\begin{aligned}
d(Tx,Ty) &\leq \left(\frac{1-|x-y|}{2}\right)|x-y| + \frac{1}{2}(1 - \sum_{k=1}^{\infty}\frac{|x|^k}{k})\left|x - \sum_{k=1}^{\infty}\frac{|x|^k}{k}\right| \\
&\equiv \alpha_1(s)\varphi(s) + \alpha_2(t)\varphi(t)
\end{aligned}$$

where $\alpha_1(s) = \frac{1-s}{2}$, $\alpha_2(t) = \frac{1-t}{2}$, $\varphi(s) \equiv s$, $\varphi(t) \equiv t$, for $s = d(x,y)$ and $t = d(x,Tx)$ (See Theorem 3.1). Since $\alpha_1, \alpha_2 \in F$, it is easily seen that $\alpha_1(s_n), \alpha_2(t_n) \to \frac{1}{2}$ as $s_n, t_n \to 0$.

Remark 4.1. Similarly, Example 4.1 also satisfies the hypothesis of Theorem 3.3 with $\varphi(s) = \dfrac{1}{2}s^2$ and $\varphi(t) = \dfrac{1}{2}t$, for $s = d(x,y)$ and $t = d(x,Tx)$.

Remark 4.2. If $x \leq y$, the results are also valid for the inputs $s = d(x,y)$ and $t = d(y,Ty)$.

In Example 4.1, the error rates for Theorems 3.1, 3.3 and 3.4 are presented in Table 4 with initial seed $x_0 = \frac{1}{4}$. Results in Table 4 show that the sequence x_n in Theorem 3.1 compares favourably to the fixed point $q = 0$ than the Theorems 3.3 and 3.4.

Table 4.1: Comparison of error rates for Example 4.1

n generations	Theorem 3.1 (Also see Theorem 3.7)	Theorem 3.3 (Also see Remark 3.3)	Theorem 3.4 (Also see Remark 3.3)
25	2.6469×10^{-23}	6.2500×10^{-2}	3.5714×10^{-2}
50	7.0065×10^{-46}	3.5088×10^{-2}	1.8868×10^{-2}
100	4.9091×10^{-91}	1.8690×10^{-2}	9.7087×10^{-3}
200	2.4099×10^{-181}	9.6000×10^{-4}	4.9261×10^{-3}

Example 4.2. *Consider the logistic map $Tx = \theta x(1-x)$, where $\theta \in (0,\infty)$ and $x \in \mathbb{R}$ which is often used in the study of chaotic phenomenon, see* [16]. *Here, we are interested in the set of points in the interval $[0,1] \subset \mathbb{R}$. For this reason, the the number θ lies in $(0,4]$. Now, let $T : [0,1] \to [0,1]$ be given by $Tx = 2x(1-x)$ and be furnished with the metric defined by*

$$d(x,y) = \begin{cases} x+y, & x \neq y, \\ 0, & x = y \end{cases}$$

with $\varphi(\eta) = \frac{1}{5}\eta$, $\varphi_1(s) = \frac{1}{3}s$ and $\varphi_2(t) = \frac{1}{2}t$.

Example 4.2 satisfies all the hypotheses of Theorems 3.1, 3.2, 3.3, 3.4 and 3.5. However, Example 4.2 does not satisfy the following weakly contractive maps:

i. $d(Tx,Ty) \leq \eta - \varphi(\eta)$ for $\eta = d(x,y)$;

ii. $d(Tx,Ty) \leq \dfrac{\eta+\omega}{2} - \varphi(\eta,\omega)$ for $\eta = d(x,Tx)$ and $\omega = d(y,Ty)$; and

iii. $d(Tx,Ty) \leq \dfrac{\eta+\omega}{2} - \varphi(\eta,\omega)$ for $\eta = d(x,Ty)$ and $\omega = d(y,Tx)$.

with $\varphi(\eta,\omega) = \frac{1}{5}(\eta+\omega)$. Conditions i. and iii. can be seen in [1, 24] and [12], respectively.

Problem 1. *Are conditions* (2.3) *and* (2.4) *equivalent for $t = d(x,y)$ and $s = d(x,Tx)$?*

Problem 2. *Are conditions* (2.5) *and* (2.6) *equivalent for $t = d(x,y)$ and $s = d(x,Tx)$?*

Problem 3. *Are conditions* (2.5) *and* (2.6) *equivalent for $t = d(x,y)$ and $s = d(y,Tx)$?*

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