Aust. J. Math. Anal. Appl.
Vol. **18** (2021), No. 2, Art. 17, 21 pp.
AJMAA

A REVIEW ON MINIMALLY SUPPORTED FREQUENCY WAVELETS

K. PALLAVI[1], M. C. LINEESH[1], A. NOUFAL[2]

Received 12 June, 2021; accepted 26 October, 2021; published 12 November, 2021.

[1]DEPARTMENT OF MATHEMATICS, NATIONAL INSTITUTE OF TECHNOLOGY CALICUT, KERALA 673601, INDIA.
pavikrishnakumar@gmail.com
lineesh@nitc.ac.in

[2]DEPARTMENT OF MATHEMATICS, COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY, KERALA 682022, INDIA.
noufal@cusat.ac.in

ABSTRACT. This paper provides a review on Minimally Supported Frequency (MSF) wavelets that includes the construction and characterization of MSF wavelets. The characterization of MSF wavelets induced from an MRA is discussed and the nature of the low-pass filter associated with it is explained. The concept of wavelet set and dimension function is introduced to study this class of wavelets. Along with MSF wavelets, s-elementary wavelets and unimodular wavelets are also considered due to the similarity in definitions. Examples and illustrations are provided for more clarity.

Key words and phrases: MSF wavelets; S-elementary wavelets; Unimodular wavelets; Wavelet sets; Dimension function.

2010 *Mathematics Subject Classification.* MSC 46-02, MSC 42C40.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Joseph Fourier's remarkable works on heat conduction problems fascinated the scientific community in many different ways since the eighteenth century. His idea was used by mathematicians and physicists in many different areas for developing new theories and technologies. During the detailed studies, people noticed some practical difficulties such as the inability of local analysis of signals in signal processing and related areas. As a solution to this issue of Fourier analysis, wavelet analysis has emerged since the nineteenth century. The vast applications of wavelets in fields such as signal processing, image compression, medicine and similar fields have been an inspiration behind the growing interest in this subject. The basic idea in wavelet theory is that using the dilates and integer translates of a single function in $L^2(\mathbb{R})$, we construct a basis for $L^2(\mathbb{R})$. Wavelets first appeared in the thesis by *Alfred Haar* in 1909. In 1985, it was *Meyer* who constructed a wavelet that is continuously differentiable but lacks the property of having compact support. Many advancements were made in this field during this period, one major achievement was in 1988 by *Meyer* and *Mallat* with the formulation of the mathematical aspects of multiresolution analysis. These major mathematical foundations coined by these researchers was followed by a remarkable work of *Daubechies* in 1988 with the construction of a set of orthonormal wavelet basis functions. Details of these discussions about wavelets and the formations of wavelet bases could be seen in [19, 8, 30, 18].

Classification of wavelets has always been of great interest, and the importance of each wavelet depends on the purpose we are dealing with. It is the reason for the classification of wavelets based on the support of their Fourier transform and that leads to Minimally Supported Frequency (MSF) wavelets or unimodular wavelets or s-elementary wavelets. MSF wavelets and unimodular wavelets are defined in the same manner. The difference in their names is due to the different authors who studied the subject independently in almost the same period. Although s-elementary wavelets seem to have a similar definition, the difference is in the way they are defined. s-elementary wavelets are defined using wavelet sets whereas MSF and unimodular wavelets use the support of their Fourier transform. In both cases, the modulus of the Fourier transform of these wavelets is the characteristic function of a measurable set.

Five sections of this paper study the subject in some detail, major results are examined and included without proof but, a brief outline is mentioned for clarity wherever it is necessary. The introductory part and the few results and definitions that are used in this paper constitutes the first two sections. An introduction to MSF, s-elementary and unimodular wavelets along with the construction and characterization of MSF wavelets is covered in the third section. The fourth section is devoted to MRA MSF wavelets. MSF wavelets induced by an MRA and the nature of the corresponding scaling function associated with the MRA is the highlight of the fourth section. The concept of Dimension function which plays a major role in the study of MSF wavelets is introduced and the low-pass filter associated with MRA MSF wavelets is also discussed in this section along with a narration about the topological property of the path connectedness of s-elementary wavelets. Some examples are studied for a better understanding of the subject and is included in the fifth section.

2. BASIC CONCEPTS IN WAVELETS

It is a common practice among mathematicians to use an orthonormal basis for the Hilbert space to decompose any complicated function in terms of simple functions, which are known and easy to handle. Also, we need to overcome the drawbacks like the inability to analyse the function locally and uncertainty in handling the position and frequency simultaneously of the Fourier transform. In the same manner, we need to reconstruct the original function from the coefficients. The process of decomposing the function is termed as **analysis** and the reconstruction process is called **synthesis**. Here we provide a few basic definitions and results which are used in this paper. We use the letter, $\mathcal{M}$ to denote the Lebesgue measure from this point onwards.

Definition 2.1. [18] A function $\psi \in L^2(\mathbb{R})$ is an **orthonormal wavelet** if $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, where $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$. Here $2^{j/2}$ is introduced so that norm is unaffected by the dilation and translation i.e, $\| \psi_{j,k} \|_2 = \| \psi \|_2$.

2.1. **MRA wavelets.** MRA acts as one of the main tools in constructing wavelets.

Definition 2.2. [18] A **multiresolution analysis (MRA)** consists of a sequence of closed subspaces $\{V_j\},\ j \in \mathbb{Z}$, of $L^2(\mathbb{R})$ satisfying

(1) $V_j \subset V_{j+1}\, for\ all\ j \in \mathbb{Z}$

(2) $f(\cdot) \in V_j\ if\ and\ only\ if\ f(2\cdot) \in V_{j+1} for\ all\ j \in \mathbb{Z}$

(3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$

(4) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$

(5) function $\phi \in V_0$ exists, such that $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

The function ϕ is called a **scaling function** of the given MRA. Once an MRA is obtained, corresponding wavelets can be constructed as follows.

To construct an orthonormal wavelet from an MRA, let W_j be the orthogonal complement of V_j in V_{j+1}. Since $\frac{1}{2}\phi(\frac{\cdot}{2}) \in V_{-1} \subset V_0$, expressing it in terms of the basis functions we have

$$\hat{\phi}(2\zeta) = \hat{\phi}(\zeta)m_0(\zeta) \tag{2.1}$$

where $m_0(\zeta)$ is a 2π-periodic function in $L^2([-\pi,\pi))$ satisfying the condition

$$|m_0(\zeta)|^2 + |m_0(\zeta+\pi)|^2 = 1, \quad a.e.\ \zeta \in \mathbb{R} \tag{2.2}$$

and is called the low-pass filter associated with the scaling function ϕ. From this low-pass filter, we can construct an orthonormal wavelet ψ for $L^2(\mathbb{R})$ if and only if

$$\hat{\psi}(2\zeta) = e^{\iota\zeta}\overline{m_0(\zeta+\pi)}\hat{\phi}(\zeta), \ \ a.e.\ \zeta \in\ \mathbb{R}. \tag{2.3}$$

Thus, $\{V_j : j \in \mathbb{Z}\}$ will generate an MRA if there exists a function $\phi \in L^2(\mathbb{R})$ such that the system $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0. The wavelet ψ constructed in this manner is known as **MRA wavelet**. From equations (2.1), (2.2) and (2.3) we have

$$|\hat{\phi}(2\zeta)|^2 + |\hat{\psi}(2\zeta)|^2 = |\hat{\phi}(\zeta)|^2, \ \ a.e.\ \zeta \in\ \mathbb{R}. \tag{2.4}$$

On iterating equation (2.4), we obtain the relation

$$|\hat{\phi}(\zeta)|^2 = |\hat{\phi}(2^N\zeta)|^2 + \sum_{j=1}^{N} |\hat{\psi}(2^j\zeta)|^2, \ \ for\ every\ N \geq 1. \tag{2.5}$$

Since $|\hat{\phi}(\zeta)| \geq 1$ and $\left\{\sum_{j=1}^{N} |\hat{\psi}(2^j\zeta)|^2 :\ N = 2, 3, \cdots\right\}$ is an increasing sequence of real numbers bounded by 1, $\lim\limits_{N\to\infty} \sum\limits_{j=1}^{N} |\hat{\psi}(2^j\zeta)|^2$ exists which in turn implies the existence of the limit

$\lim_{N\to\infty} |\hat{\phi}(2^N\zeta)|^2$. By a change of variable, $\lim_{N\to\infty}\int_{\mathbb{R}} |\hat{\phi}(2^N\zeta)|^2 d\zeta = 0$ and Fatou's Lemma gives $\lim_{N\to\infty} |\hat{\phi}(2^N\zeta)|^2 = 0$. By equation (2.4), we have $|\hat{\phi}|$ related with the wavelet ψ as follows:

$$|\hat{\phi}(\zeta)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\zeta)|^2 \;\; for\ a.e.\ \zeta \in \mathbb{R}. \tag{2.6}$$

Theorem 2.1. [18] *A function $\phi \in L^2(\mathbb{R})$ is a scaling function for an MRA if and only if there exists a 2π-periodic function m_0 such that*

$$\hat{\phi}(2\zeta) = m_0(\zeta)\hat{\phi}(\zeta) \;\; for\ a.e. \zeta \in \mathbb{R} \tag{2.7}$$

and

$$\sum_{k\in\mathbb{Z}} |\hat{\phi}(\zeta + 2k\pi)|^2 = 1 \;\; for\ a.e. \zeta \in [-\pi, \pi) \tag{2.8}$$

$$\lim_{j\to\infty} |\hat{\phi}(2^{-j}\zeta)| = 1 \;\; for\ a.e. \zeta \in \mathbb{R}. \tag{2.9}$$

Although MRA wavelets are those classes of wavelets that are discussed all over, there exist wavelets that are not induced from an MRA which are reported in the literature. Journe wavelets and Lemarie wavelets are a few among them.

Theorem 2.2 (Characterization of orthonormal system)**.** [18] *If $\psi \in L^2(\mathbb{R})$, then $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system if and only if $\sum_{k\in\mathbb{Z}} |\hat{\psi}(\zeta + 2k\pi)|^2 = 1$ for $a.e.\ \zeta \in \mathbb{R}$.*

3. MSF WAVELETS FROM MEYER'S EQUATIONS

To study MSF wavelets it would be appropriate to begin with Meyer's equations which provide equivalent conditions for an orthonormal wavelet. These equations first appeared in [21] without proof and details of the proof published later in [15]. It makes use of the basic orthonormality properties, convergence of Cesaro sums, Lebesgue differentiation Theorem and Poisson summation formula.

Theorem 3.1 (Meyer's equations)**.** [15] *$\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet for $L^2(\mathbb{R})$ if and only if ψ satisfies*

a) $\sum_{k\in\mathbb{Z}} |\hat{\psi}(\zeta + 2k\pi)|^2 = 1\, a.e.\ \zeta \in \mathbb{R}$

b) $\sum_{k\in\mathbb{Z}} \hat{\psi}(\zeta + 2k\pi)\hat{\psi}^*(2^j(\zeta + 2k\pi)) = 0$ *for* $j \geq 1\ a.e.\ \zeta \in \mathbb{R}$

c) $\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^{-j}\zeta)|^2 = 1\, a.e.\ \zeta \in \mathbb{R}$

d) $\sum_{l\geq 0} \hat{\psi}(2^l(\zeta + 2m\pi))\hat{\psi}^*(2^l\zeta) = 0$ *for* $m \in 2\mathbb{Z} + 1\ a.e.\ \zeta \in \mathbb{R}$

where $\hat{\psi}^*$ is the complex conjugate of $\hat{\psi}$.

Meyer's equations help in determining whether a function in $L^2(\mathbb{R})$ is an orthonormal wavelet or not. But the problem faced here is that we need to verify all four equations which are infinite sums. Thus there arises the problem of convergence of the series. This problem can be settled if we could get a "nice" form for $\hat{\psi}$ such that the equations are converted to finite ones. If $|\hat{\psi}|$ is the characteristic function of some measurable subset of $\mathbb{R}$, then this problem is reduced.

From the characterization of an orthonormal system given by Theorem (2.2), we observe that if ψ is an orthonormal wavelet, then clearly $|\hat{\psi}(\zeta)| \leq 1$ $a.e.$ $\zeta \in \mathbb{R}$. By the orthonormality of ψ, we have $\| \psi \|_2 = 1$ and by Plancheral's Theorem $\| \hat{\psi} \|_2^2 = 2\pi$. Thus $|\hat{\psi}(\zeta)| \leq 1$ almost everywhere on $\mathbb{R}$ together with $\| \hat{\psi} \|_2^2 = 2\pi$ leads to the fact that $\mathcal{M}(Supp\,\hat{\psi}) \geq 2\pi$. So it is natural to consider the case when $\mathcal{M}(Supp\,\hat{\psi}) = 2\pi$. By method of contradiction, one can show that on $Supp\,\hat{\psi}$, $|\hat{\psi}(\zeta)| = 1$. For this, we assume that $0 < |\hat{\psi}(\zeta)| < 1$ on a set E of positive measure. Then,

$$2\pi = \mathcal{M}(Supp\,\hat{\psi}) = \int_{Supp\,\hat{\psi}} 1 \; d\zeta \; > \; \int_{\mathbb{R}} |\hat{\psi}(\zeta)|^2 d\zeta \; = \| \hat{\psi} \|_2^2 = \; 2\pi \tag{3.1}$$

which is a contradiction. For the converse, we assume that $|\hat{\psi}(\zeta)| = 1$ on $Supp\,\hat{\psi}$ which in turn is equivalent to $|\hat{\psi}| = \chi_E$ where $E = Supp\,\hat{\psi}$. The orthonormality of $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ thus gives

$$\mathcal{M}(E) = \int_E d\zeta = \int_{\mathbb{R}} |\hat{\psi}(\zeta)|^2 d\zeta = \| \hat{\psi} \|_2^2 = 2\pi \| \psi \|_2^2 = \; 2\pi \tag{3.2}$$

Thus we have for $\psi \in L^2(\mathbb{R})$ which is an orthonormal wavelet, $\mathcal{M}(Supp\,\hat{\psi})$ is atleast 2π. Equality holds here if and only if $|\hat{\psi}| = \chi_E$ for some measurable set $E \subset \mathbb{R}$ with $\mathcal{M}(E) = 2\pi$. In such cases, the wavelet ψ is called a **Minimally Supported Frequency wavelet (MSF wavelet)** and the set E is called a **wavelet set**. From this, the name 'minimally supported frequency wavelets' is clear. $\mathcal{M}(Supp\,\hat{\psi}) = 2\pi$ is the minimal condition that can be attained. MSF wavelets was introduced in [13] and as 'unimodular wavelets' in [15]. Unimodular wavelets are defined using the functions $\psi \in L^2(\mathbb{R})$ such that $|\hat{\psi}(\zeta)| = 1$ on $Supp\,\hat{\psi}$ termed as unimodular functions in [15].

When ψ is an MSF wavelet, the Meyer's equations in Theorem (3.1) eventually gets reduced to just two equations. By Meyer's equations one side proof is trivial. Hence it is enough to show that if both conditions *(a)* and *(c)* of Theorem (3.1) are satisfied together with the assumption that for $\psi \in L^2(\mathbb{R})$ such that $|\hat{\psi}| = \chi_K$ for some measurable set $K \subset \mathbb{R}$, then ψ is an MSF wavelet. It is shown in [15] that for any unimodular function ψ in $L^2(\mathbb{R})$, the equation *(a)* implies *(d)* and *(c)* implies *(b)* which in turn proves ψ to be an orthonormal wavelet by Theorem (3.1). If for a fixed ζ, such that $2^j\zeta \in K$, then $|\hat{\psi}(2^j\zeta)| = 1$. By *(a)* part of Theorem (3.1), $\hat{\psi}(2^j\zeta + 2k\pi) = 0$ for nonzero k which in turn implies *(d)*. In a similar way, if for a fixed ζ for which $\zeta + 2k\pi \in Supp\,\hat{\psi}$, $|\hat{\psi}(\zeta + 2k\pi)| = 1$. This together with the assumption of *(c)* would give us for nonzero j, $\hat{\psi}(2^j(\zeta + 2k\pi)) = 0$ and which in turn makes each term in *(b)* to be zero. Thus we have the following characterization of unimodular wavelets.

Theorem 3.2. [15] *A unimodular function ψ in $L^2(\mathbb{R})$ is a wavelet if and only if*

a) $\sum_{k \in \mathbb{Z}} |\hat{\psi}(\zeta + 2k\pi)|^2 = 1$ $a.e.$ $\zeta \in \mathbb{R}$

b) $\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^{-j}\zeta)|^2 = 1$ $a.e.$ $\zeta \in \mathbb{R}$.

A set-theoretic equivalence of this Theorem which is very much helpful to check whether a function is an MSF wavelet or not is given here. This Theorem is also helpful in the construction of MSF wavelets.

Theorem 3.3. [13] *Let E be a measurable set in $\mathbb{R}$ and let $\psi \in L^2(\mathbb{R})$ be a function such that $|\hat{\psi}| = \chi_E$. Then ψ is an orthonormal wavelet if and only if*

a) $\{E + 2k\pi : k \in \mathbb{Z}\}$ *is a partition of* $\mathbb{R}$ $a.e.$ *and*
b) $\{2^j E : j \in \mathbb{Z}\}$ *is a partition of* $\mathbb{R}$ $a.e.$

s-elementary wavelets were defined in terms of a set and are found in chapter 4 of [10]. A **wavelet set** E is a measurable subset of $\mathbb{R}$, if the characteristic function of E is the Fourier transform of a wavelet and are called **s-elementary wavelets**, where the 's' is used to denote **set**. But whether this class of wavelets and MSF wavelets are the same is not yet known. The properties and construction of MSF wavelets or s-elementary wavelets are studied mainly in terms of wavelet sets. Hence any study that deals with these wavelets would be incomplete without the mention of wavelet sets. An introduction to wavelet sets could be seen along with MSF wavelets or s-elementary wavelets in [10, 13, 16, 17].

The importance of the characterization of wavelet sets is found in [10]. For this, two notions such as translation congruent modulo 2π and dilation congruent modulo 2 are defined.

Definition 3.1. [10] Measurable sets A and B are translation congruent modulo 2π if there exits a measurable bijection $\tau : A \longrightarrow B$ such that $\tau(x) - x = 2m\pi$ for $m \in \mathbb{Z}$ and for each $x \in A$. Equivalently, there exists a measurable partition $\{A_n : n \in \mathbb{Z}\}$ of A such that the 2π integer translates of this collection forms a partition of B.

Definition 3.2. [10] Measurable sets A and B are dilation congruent modulo 2 if there exists a measurable bijection $\delta : A \longrightarrow B$ such that for each $s \in A$ there exists an integer n such that $\delta(s) = 2^n s$. Equivalently, there exists a measurable partition $\{A_n : n \in \mathbb{Z}\}$ of A such that $\{2^n A_n\}$ is a measurable partition of B.

Using these two concepts, one of the most important characterizations of wavelet set given in [10] is as follows.

Theorem 3.4. [10] *A measurable set $E \subset \mathbb{R}$ is a wavelet set if and only if E is both translation congruent to $[0, 2\pi)$ modulo 2π and dilation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi)$ modulo 2.*

A study on the wavelet set of a unimodular wavelet is done in [15]. The $Supp\, \hat{\psi}$ denoted by E is partitioned into two sets namely E^+ and E^-, where $E^+ = E \cap (0, \infty)$ and $E^- = E \cap (-\infty, 0)$. Here also characterization of unimodular wavelets is done using the concepts of translations and dilations which are almost the same as defined in definitions (3.1), (3.2). One difference is that these definitions which are made here are more general whereas in [15] the sets are restricted. In [15], for any real number a, the function τ_a is defined by $\tau_a : \mathbb{R} \longrightarrow [a, a + 2\pi)$ as $\tau_a(x) = x + 2k\pi$ where for each $x \in \mathbb{R}$, there exists a unique integer $k(x)$ such that $a \leq x + 2k(x)\pi < a + 2\pi$ and for $a > 0$, define $\delta_a : (0, \infty) \longrightarrow [a, 2a)$ as $\delta_a(x) = 2^{j(x)}x$, where for each $x \in \mathbb{R}$, there exists a unique integer $j(x)$ such that $a \leq 2^{j(x)}x < 2a$. For $a < 0$, define $\delta_a(x) = -\delta_{-a}(-x)$ for $x < 0$. Using Meyer's equations it is proved that a function $\psi \in L^2(\mathbb{R})$ is a unimodular wavelet if and only if for some $a \in \mathbb{R}$, τ_a is one to one on E except on a set of measure zero and $\mathcal{M}([a, a + 2\pi) - \tau_a E) = 0$ and for some $a > 0$ and $b < 0$, δ_a, δ_b are one to one on E^+ and E^- except on a set of measure zero respectively. Here on closely observing this with Theorem (3.4), we can see that both characterizations are the same. This can be obtained if we consider the translation τ_0 on E and the dilations δ_π on E^+ and $\delta_{-\pi}$ on E^-, which are the same as mentioned in Theorem (3.4).

This characterization of unimodular wavelets whose wavelet set E are a finite union of intervals is given in [15]. This is done using the fact that in this case $\mathcal{M}([a, a+2\pi) - \tau_a E) = 0$ if and only if $\tau_a E = [a, a + 2\pi)$ and τ_a is one to one on a set of measure zero which in turn happens if and only if τ_a is one to one on E except at finitely many points. In such case, $\tau_a E = [a, a+2\pi)$ and τ_a is one to one on E. Similar results are obtained for δ_a.

Three cases are being discussed in [15] depending on the structure of both E^+ and E^-. The first case deals with both E^+ and E^- to be intervals, using these results it is established that E is precisely of the form

$$E = [2a - 4\pi, a - 2\pi] \bigcup [a, 2a]$$

for some $0 < a < 2\pi$. Second case is when E^+ consists of disjoint intervals and $E^- = -E^+$. Here we discuss the case of symmetric wavelet set, then E^+ is of the form

$$E^+ = \left[\frac{2^j\pi}{2^{j+1}-1}, \pi\right] \bigcup \left[2^j\pi, 2^j\pi + \frac{2^j\pi}{2^{j+1}-1}\right]$$

where $j \in \mathbb{Z}^+$. This in turn says that E is the union of four intervals. Third case is when E^+ is the disjoint union of two intervals and E^- is an interval. Thus, the precise form of E^- and E^+ are given by

$$E^- = \left[-2\left(1 - \frac{2p+1}{2^{j+1}-1}\right)\pi, -\left(1 - \frac{2p+1}{2^{j+1}-1}\right)\pi\right]$$

and

$$E^+ = \left[\frac{2(p+1)\pi}{2^{j+1}-1}, \frac{2(2p+1)\pi}{2^{j+1}-1}\right] \bigcup \left[\frac{2^{j+1}(2p+1)\pi}{2^{j+1}-1}, \frac{2^{j+2}(p+1)\pi}{2^{j+1}-1}\right]$$

for $j \geq 2$ and $1 \leq p \leq 2^j - 2$.

Once we have the characterization of MSF wavelets, we will turn in to the construction of these wavelets. One such important result in this construction can be found in [16].

Theorem 3.5. [13] *For $\psi \in L^2(\mathbb{R})$ with $|\hat{\psi}| = \chi_E$, ψ is a wavelet if and only if there exists a partition $\{I_l\}_{l\in\mathbb{Z}}$ of $I = [-2\pi, -\pi) \bigcup [\pi, 2\pi)$, a partition $\{E_l\}_{l\in\mathbb{Z}}$ of E and two sequences $\{j_l\}_{l\in\mathbb{Z}}$, $\{k_l\}_{l\in\mathbb{Z}} \subset \mathbb{Z}$ such that*

a) $E_l = 2^{j_l} I_l,\ l \in \mathbb{Z}$
b) $\{E_l + 2k_l\pi\}_{l\in\mathbb{Z}}$ *forms a partition of* I.

Here we note that I is the support of the Fourier transform of the Shannon wavelet and using Theorem (3.5) we obtain MSF wavelets from Shannon wavelet. The proof of this is obtained by using the basic ideas of wavelets but is a bit constructive in nature. If we assume (a) and (b) of this Theorem to hold, then using $|\hat{\psi}| = \chi_E$ and our assumptions, together with Theorem (3.2) leads ψ to be a wavelet. For the converse, we assume ψ to be a wavelet, then by Theorem (3.3), we have $\{E+2k\pi\}_{k\in\mathbb{Z}}$ and $\{2^j E\}_{j\in\mathbb{Z}}$ to be partitions of $\mathbb{R}$. Let $E_n = E \cap (2^n I)$ and $J_n = 2^{-n}E_n$ for $n \in \mathbb{Z}$. Since $\{2^n I\}_{n\in\mathbb{Z}}$ and $\{2^{-n}E\}_{n\in\mathbb{Z}}$ are partitions of $\mathbb{R}$, $\{E_n\}_{n\in\mathbb{Z}}$ is a partition of E and $\{J_n\}_{n\in\mathbb{Z}}$ forms a partition of I. For each integer n, define $E_{n,j} = E_n + (I + 2j\pi)$ and $I_{n,j} = J_n + (2^{-n}(I + 2j\pi))$ for $j \in \mathbb{Z}$. This $\{E_{n,j}\}_{n,j\in\mathbb{Z}}$ forms a partition of I, by taking $(n, j) = l$ and $j = -k_l$ completes the proof.

An example that illustrates this construction has been given here.

Example 3.1. [13] *Let $I = [-2\pi, -\pi) \bigcup [\pi, 2\pi)$. We choose a partition for I and let it be $\{I_l\}$ defined by $I_1 = [-2\pi, -\frac{4}{3}\pi)$, $I_2 = [-\frac{4}{3}\pi, -\pi)$, $I_3 = [\pi, \frac{4}{3}\pi)$, $I_4 = [\frac{4}{3}\pi, 2\pi)$, $I_l = \emptyset$. Also, we choose $\{j_l\}$ to be the sequence defined by $j_1 = -1$, $j_2 = 1$, $j_3 = 1$, and $j_4 = -1$ and let $E_l = \{2^{j_l} I_l\}$. We have $E_1 = 2^{j_1} I_1 = [-\pi, -\frac{2}{3}\pi)$, $E_2 = [-\frac{8}{3}\pi, -2\pi)$, $E_3 = [2\pi, \frac{8}{3}\pi)$, and $E_4 = [\frac{2}{3}\pi, \pi)$. Let $\{k_l\}$ be the sequence defined such that $\{2^{j_l} I_l + 2k_l\pi\}_{l\in\mathbb{Z}}$ is a partition of I. Thus if we choose some ζ in $E_1 + 2k_1\pi$, then this ζ has to belong to $[-2\pi, -\pi) \bigcup [\pi, 2\pi)$. If $\zeta \in [-2\pi, -\pi)$ then we have*

$$-2\pi \leq -\pi + 2k_1\pi, \qquad \frac{-2}{3}\pi + 2k_1\pi \leq -\pi.$$

Since k_1 must be an integer satisfying these equations, we get no such k_1 exists. Thus we have if $\zeta \in E_1 + 2k_1\pi$ then for $\zeta \in [\pi, 2\pi)$, we get two similar equations which gives us $k_1 = 1$. In a similar manner we get $k_2 = 2,\ k_3 = -2$ and $k_4 = -1$. But by Theorem (3.5)

$$E = \bigcup_{l\in\mathbb{Z}} 2^{j_l} I_l = [-\frac{8}{3}\pi, -2\pi) \bigcup [-\pi, -\frac{2}{3}\pi) \bigcup [\frac{2}{3}\pi, \pi) \bigcup [2\pi, \frac{8}{3}\pi)$$

give rise to an MSF wavelet. Here I is the support of the Shannon wavelet and E is that of the Lemarie-Meyer wavelet. Thus we have constructed Lemarie-Meyer wavelet from Shannon wavelet.

Characterization of all those orthonormal wavelets whose Fourier transform has support contained in $S_\alpha = [\frac{-8}{3}\alpha, 4\pi - \frac{4}{3}\alpha],\ 0 < \alpha < 2\pi$ is obtained in [16]. Also, the support need not be symmetric about the origin. For $\alpha = \pi$, we have the Lemarie-Meyer wavelet. The main Theorem which deals with the equivalence of orthonormal wavelets proved in [16] is as follows.

Theorem 3.6. [16] *For $\psi \in L^2(\mathbb{R})$, has support contained in $S_\alpha = [\frac{-8\alpha}{3}, 4\pi - \frac{4\alpha}{3}],\ 0 < \alpha \leq \pi$ and $b = |\hat{\psi}|$. Then ψ is an orthonormal wavelet if and only if*

a) $b^2(\zeta) + b^2(\frac{\zeta}{2}) = 1\ a.e.\ \zeta \in [4\pi - \frac{8\alpha}{3}, 4\pi - \frac{4\alpha}{3}]$

b) $b(\zeta) = 1\ a.e.\ \zeta \in [2\pi - \frac{2\alpha}{3}, 4\pi - \frac{8\alpha}{3}]$

c) $b^2(\zeta) + b^2(\zeta + 2\pi) = 1\ a.e.\ \zeta \in [-\frac{4\alpha}{3}, -\frac{2\alpha}{3}]$

d) $b(\zeta) = b(\frac{\zeta}{2} + 2\pi)\ a.e.\ \zeta \in [-\frac{8\alpha}{3}, -\frac{4\alpha}{3}]$

e) $\hat{\psi}(\zeta) = e^{\iota p(\zeta)} b(\zeta)$ *with* $p(\zeta)$ *satisfying* $p(\zeta) + p(2(\zeta - 2\pi)) - p(\zeta - 2\pi) = (2n(\zeta) + 1)\pi\ a.e.\ \zeta \in D_\alpha \cap (Supp\ b) \cap (\frac{1}{2} Supp\ b)$, *where* $D_\alpha = [2\pi - \frac{4\alpha}{3}, 2\pi - \frac{2\alpha}{3}]$ *and* $n(\zeta)$ *is an integer-valued measurable function.*

f) $b(\zeta) = 0\ a.e.\ \zeta \in [\frac{-2\alpha}{3}, 2\pi - \frac{4\alpha}{3}] = H_\alpha$

This remarkable result simply means that if $b = |\hat{\psi}|$ is any measurable function whose support is contained in S_α, then all such wavelets ψ have a Fourier transform that can be expressed on the rest of S_α in terms of b, proof of this depends mainly on Theorem (3.1) and is divided into three sections. At first, it is obtained that for such a ψ whose Fourier transform has support contained in $S_\alpha,\ 0 < \alpha \leq \pi$, this support is precisely contained in $\overline{S_\alpha - H_\alpha}$. The concept of dimension function (refer definition (4.2)) is used for this. To prove this part, J_α is taken to be the union of D_α and H_α and is assumed to be any interval of length 2π. Then using the definition of dimension function, $\int_{J_\alpha} D_\psi(\zeta) d\zeta = 2\pi$. By making use of these definitions of $H_\alpha,\ S_\alpha,\ D_\alpha$, we get

$$D_\psi(\zeta) = 1 - \sum_{j=-\infty}^{0} |\hat{\psi}(2^j\zeta)|^2\ \ a.e.\ \zeta \in H_\alpha \tag{3.3}$$

and

$$D_\psi(\zeta) = 1 - |\hat{\psi}(2(\zeta - \pi)|^2\ \ a.e.\ \zeta \in D_\alpha. \tag{3.4}$$

These two equations gives $D_\psi(\zeta) \leq 1$ and clearly $D_\psi(\zeta) \geq 0$. Since $\int_{J_\alpha} D_\psi(\zeta) d\zeta = 2\pi$, we have $D_\psi(\zeta) = 1$ on J_α almost everywhere. Hence using equations (3.3) and (3.4) on

H_α, $\sum_{j=-\infty}^{0} |\hat{\psi}(2^j\zeta)|^2 = 0$ almost everywhere and $|\hat{\psi}\left(2(\zeta-\pi)\right)| = 0$ almost everywhere on D_α which in turn gives us $\hat{\psi}(\zeta) = 0$ on H_α almost everywhere. Then using the Meyer's equations the remaining part of the Theorem is proved by considering different intervals and eliminating those terms that does not lie in this support. The last part of the proof is to prove the converse part. So if we are assuming all these six conditions, then to show that the given wavelet is an MSF wavelet we use Theorem (3.2). This also makes use of the support of $\hat{\psi}$ and each infinite sums gets reduced to finite ones. Examples are provided to illustrate this.

From Theorem (3.6), we get two results given by equations (3.5) and (3.6). From condition *(a)*, $b^2(\zeta)+b^2(\frac{\zeta}{2}) = 1$ $a.e.$ $\zeta \in [4\pi-\frac{8\alpha}{3}, 4\pi-\frac{4\alpha}{3}]$. Then $\frac{\zeta}{2} \in D_\alpha$ $a.e.$ which in turn is equivalent to $b^2(2\zeta)+b^2(\zeta) = 1$ $a.e.$ $\zeta \in D_\alpha$. Also, from condition *(c)*, for $\zeta - 2\pi \in [-\frac{4\alpha}{3}, -\frac{2\alpha}{3}]$, we get $b^2(\zeta-2\pi)+b^2(\zeta) = 1$ $a.e.$ $\zeta \in D_\alpha$. Thus we get

$$b(2\zeta) = \sqrt{1-b^2(\zeta)} = b(\zeta-2\pi) \quad a.e.\ \zeta \in D_\alpha \tag{3.5}$$

Similarly using condition *(d)*, $b(\zeta) = b(\frac{1}{2}\zeta+2\pi)$ $a.e.$ $\zeta \in [-\frac{8\alpha}{3}, -\frac{4\alpha}{3}]$ which in turn gives $(\frac{1}{2}\zeta+2\pi) \in D_\alpha$ $a.e.$ Thus we get,

$$b(2(\zeta-2\pi)) = b(\zeta)\ a.e.\ \zeta \in D_\alpha \tag{3.6}$$

Theorem (3.6) is very helpful in the construction of orthonormal wavelets as shown in the following example. To explain this, we construct Shannon wavelet using Theorem (3.6). We take $\alpha = \pi$ and $b(\zeta) = \chi_{[\pi,\frac{4}{3}\pi]}$ on $[\frac{2}{3}\pi, \frac{4}{3}\pi]$. Now from condition *(a)*, we get

$$b(\zeta) = \begin{cases} 1 & \text{if} \quad \zeta \in [\frac{4}{3}\pi, 2\pi) \\ 0 & \text{if} \quad \zeta \in [2\pi, \frac{8}{3}\pi] \end{cases} \tag{3.7}$$

and thus we could extend $b(\zeta)$ to $[\frac{4}{3}\pi, \frac{8}{3}\pi]$. Similarly, using condition *(c)* we have

$$b(\zeta) = \begin{cases} 1 & \text{if} \quad \zeta \in [\frac{-4}{3}\pi, -\pi) \\ 0 & \text{if} \quad \zeta \in [-\pi, \frac{-2}{3}\pi] \end{cases} \tag{3.8}$$

on $[\frac{-4}{3}\pi, \frac{-2}{3}\pi]$. Using condition *(d)*, we could extend this to the interval $[\frac{-8}{3}\pi, \frac{-4}{3}\pi]$ as

$$b(\zeta) = \begin{cases} 0 & \text{if} \quad \zeta \in [\frac{-8}{3}\pi, -2\pi) \\ 1 & \text{if} \quad \zeta \in [-2\pi, \frac{-4}{3}\pi]. \end{cases} \tag{3.9}$$

Thus we have extended $b(\zeta)$ to $S_\pi = [\frac{-8\pi}{3}, \frac{8\pi}{3}]$. Summarize the above as follows:

$$b(\zeta) = \begin{cases} 0 & \text{if} \quad \zeta \in [\frac{-8}{3}\pi, -2\pi) \cup [-\pi, \frac{-2}{3}\pi] \cup [\frac{2}{3}\pi, \pi) \cup [2\pi, \frac{8}{3}\pi] \\ 1 & \text{if} \quad \zeta \in [-2\pi, \frac{-4}{3}\pi] \cup [\frac{-4}{3}\pi, -\pi) \cup [\pi, \frac{4}{3}\pi] \cup [\frac{4}{3}\pi, 2\pi). \end{cases} \tag{3.10}$$

This is same as $\chi_{[-2\pi,-\pi)\cup[\pi,2\pi)}$ on $\mathbb{R}$. Also, we have $[\frac{2}{3}\pi, \frac{4}{3}\pi] \cap Supp\,(b) \cap \frac{1}{2}\,Supp(b) = \{\pi\}$ and hence by condition *(e)*, we choose $\alpha(\zeta) = \frac{\zeta}{2}$. This choice is made in such a way that $\alpha(\zeta)$ is a measurable function satisfying the equation given in condition *(e)* of Theorem (3.6). Thus by Theorem (3.6), $\hat{\psi}(\zeta) = e^{\iota\alpha(\zeta)}b(\zeta)$ is an orthonormal wavelet, $\psi(x) = -2\frac{\sin(2\pi x)+\cos(\pi x)}{\pi(2x+1)}$ which is the Shannon wavelet as in Figure 1.

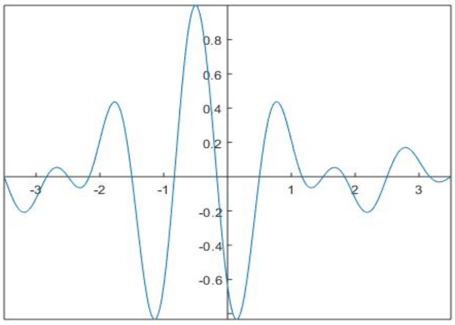


Figure 1: Shannon wavelet

4. MRA MSF WAVELETS

If a scaling function ϕ exists with $\hat{\psi}(\zeta) = e^{\iota\frac{\zeta}{2}}\overline{m_0(\frac{\zeta}{2}+\pi)}\hat{\phi}(\frac{\zeta}{2})$, then ψ is an **MRA wavelet**. Characterization of unimodular wavelets that are associated with MRA is discussed in [20]. Since these wavelets arise from an MRA, a scaling function ϕ exists and is of the form $\hat{\phi} = \chi_M$ where M is a finite disjoint union of closed intervals. The equivalent conditions for such a ϕ to be a scaling function of an MRA are that $\{M + 2k\pi\}$ forms a partition of $\mathbb{R}$, the dilate $\frac{M}{2}$ of M is contained in M and M contains a neighbourhood of zero. The wavelet ψ that is generated by such a scaling function ϕ will be of the form $|\hat{\psi}(\zeta)| = \chi_E$ where $E = 2M \setminus M$. This characterization is true only when M is a finite disjoint union of closed intervals. When M is an infinite union of intervals, this result is not valid which is shown by taking

$$M = \bigcup_{k+1}^{\infty}\left[\frac{-1}{2^k}\left(2-\frac{1}{2^k}\right)\pi, \frac{-\pi}{2^k}\right] \cup \left[0, \frac{5}{4}\pi\right] \cup \bigcup_{k=1}^{\infty}\left[\left(2-\frac{1}{2^k}\right)\pi, \left(2-\frac{1}{2^{k+1}}(2-\frac{1}{2^{k+1}})\right)\pi\right].$$

This choice of M violates the condition that M contains a neighbourhood of zero. Second statement of this characterization can be easily proved by combining the fact that an MRA wavelet ψ is related to ϕ using the relation

$$\hat{\psi}(\zeta) = e^{i\frac{\zeta}{2}}\overline{m_0\left(\frac{\zeta}{2}+\pi\right)}\hat{\phi}\left(\frac{\zeta}{2}\right)$$

and we have the following equality

$$|m_0(\zeta)|^2 + |m_0(\zeta+\pi)|^2 = 1.$$

Since $M \subset 2M$, we have $|\hat{\psi}(\zeta)| = \chi_{2M\setminus M}$.

A natural question that raises in our mind is whether it is possible to have the set M as disjoint union of n closed intervals for any natural number n. A negative answer to this is given in [20] by proving that for $n = 2$ such a set is impossible. The case when M is a single interval is considered first. Since it must contain a neighbourhood of the origin and is of measure 2π, we obtain that M is precisely of the form $[-a, 2\pi - a]$ for $0 < a < 2\pi$. The wavelet ψ obtained from this scaling function ϕ will be $\hat{\psi}(\zeta) = \theta(\zeta)\chi_E(\zeta)$ where $E = [-2a, -a] \cup [2\pi - a, 4\pi - 2a]$ and $|\theta(\zeta)| = 1$. This is same as the general form of those unimodular wavelets associated with an MRA whose Fourier transform has a support consisting of two intervals given in [15]. To check whether M could be the disjoint union of two intervals, take $M = [-a, b] \cup [c, d]$ where $0 < b < c < d$. Applying the condition $\frac{1}{2}M \subset M$, we get $d \leq 2b$. Using the fact that $\{M + 2k\pi : k \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$ and simple computations gives the inequality $-a + 2(k+1)\pi = d \leq 2b = 2c - 4k\pi$ which in turn gives $-a + 6k\pi + \pi \leq 2c < 2d = -2a + 4k\pi + 4\pi$ which in turn produces the inequality $2k\pi - 2\pi < -a < 0$ which is not possible for a positive integer k. Thus M cannot be the disjoint union of two closed intervals. In a similar manner, consider M as the union of three intervals and then M is of the form $M = [-a, -b] \cup [-c, 2\pi - a] \cup [2\pi - b, 2\pi - c]$ where $0 < c < b < a \leq 2c < 4\pi$. This makes E as the union of six intervals. Also, when $M = [-a, b] \cup [c, d] \cup [e, f]$, taking $c - b \geq 2\pi$ leads to a contradiction and thus E is the union of three to five intervals as well, the details could be seen in [20].

Study of MSF wavelets associated with an MRA were done by many authors. A detailed characterization of unimodular wavelets can be found in section four of [20]. It is proved that for any MRA unimodular wavelet ψ with scaling function ϕ is also unimodular and $|\hat{\phi}(\zeta)| = \chi_M$ where $M = \bigcup\limits_{j<0} 2^j E$ is a disjoint union. To prove this, we make use of the equations (2.1), (2.2) and (2.3). Thus we obtain $|\hat{\psi}(\zeta)|^2 = |\hat{\phi}(\frac{\zeta}{2})|^2 - |\hat{\phi}(\zeta)|^2$. Rest of the proof can be completed easily by using the facts $|\hat{\phi}(\zeta)|^2 = \sum\limits_{j<0} \chi_{2^j E}(\zeta)$ and $\sum\limits_{j\in\mathbb{Z}} \chi_{2^j E}(\zeta) = 1$ which we can obtain from the definition of ϕ.

Use the set theoretic equivalence of Meyer's equations for unimodular wavelets. An equivalent condition for ψ such that $|\hat{\psi}| = \chi_E$ to be an MRA unimodular wavelet is given as $\bigcup_{l\in\mathbb{Z}} \bigcup_{j<0} (2^j E + 2l\pi) = \mathbb{R}$. The properties that must be satisfied by the support of the Fourier transform of the scaling function and the result which implies this scaling function to be unimodular is used to prove this.

An important characterization of MRA MSF wavelet was made in [13]. If $\psi \in L^2(\mathbb{R})$ is an MSF wavelet, then $|\hat{\psi}| = \chi_E$ where $\mathcal{M}(E) = 2\pi$. Define $E^s = \bigcup_{j=1}^{\infty} (2^{-j}E)$, by Theorem (3.3) the sets in this union are all mutually disjoint almost everywhere and also note that

$$\mathcal{M}(E^s) = 2\pi. \tag{4.1}$$

Now if we consider ψ to be an MRA wavelet, then we have the relation

$$|\hat{\phi}(\zeta)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\zeta)|^2 \ a.e.\ \zeta \in \mathbb{R} \tag{4.2}$$

which implies

$$|\hat{\phi}| = \chi_{E^s}. \tag{4.3}$$

By the orthonormality of $\left\{\phi_{0,k} : k \in \mathbb{Z}\right\}$, we get

$$\sum_{k\in\mathbb{Z}} |\hat{\phi}(\zeta + 2k\pi)|^2 = 1 \ a.e.\ \zeta \in [-\pi, \pi). \tag{4.4}$$

Thus from (4.1),(4.3) and (4.4), we get

$$\mathcal{M}\left(E^s \cap (E^s + 2k\pi)\right) = 2\pi\delta_{k,0} \quad k \in \mathbb{Z}. \tag{4.5}$$

In fact, this condition is proved to be a sufficient condition for an MSF wavelet to be associated with an MRA. Quadrature mirror filter (QMF) method explained in [12] is used to achieve this. We assume $\hat{\psi}(\zeta) = |\hat{\psi}(\zeta)|e^{\iota\alpha(\zeta)}$ where α being a real valued measurable function on E along with the criteria (4.5). Also, using equation (4.3), we define $\hat{\phi}(\zeta) = |\hat{\phi}(\zeta)|e^{\iota\beta(\zeta)} = e^{\iota\beta(\zeta)}\chi_{E^s}$. Since the sets in that union are all disjoint for almost every $\zeta \in E^s$, by the definition of E^s and by Theorem (3.3), it is possible to find a unique $l \geq 1$ such that $\zeta \in 2^{-l}E$ and thereby we define $\beta(\zeta) = 2^{1-l}\alpha(2^l\zeta)$. Thus by equation (4.4), we can see that the integer translates of ϕ forms an orthonormal system. We define the low-pass filter m_0 on E^s as given in equation (4.6) and extend it to a 2π-periodic function on $\mathbb{R}$ using equation (4.5) which is our assumption. Here m_1 is also defined and extended in a similar manner. Thus we have m_0 and m_1 as

$$m_0(\zeta) = \begin{cases} 0 & \text{if} \quad \zeta \in \frac{1}{2}E \subset E^s \\ e^{\iota 2^{1-j}\alpha(2^j\zeta)} & \text{if} \quad \zeta \in 2^{-j}E \quad \text{for unique} \quad j \geq 2 \end{cases} \tag{4.6}$$

$$m_1(\zeta) = \begin{cases} 1 & \text{if} \quad \zeta \in \frac{1}{2}E \subset E^s \\ 0 & \text{if} \quad \zeta \in 2^{-1}E^s \end{cases} \tag{4.7}$$

and since the supports of both m_0 and m_1 are disjoint,

$$m_0(\zeta)\overline{m_1(\zeta)} + m_0(\zeta+\pi)\overline{m_1(\zeta+\pi)} = 0 \ a.e. \ \zeta \in [-\pi, \pi). \tag{4.8}$$

Taking into consideration the two cases $\zeta \in \frac{1}{2}E$ and $\zeta \notin \frac{1}{2}E$, it can be easily shown that

$$\hat{\psi}(2\zeta) = m_1(\zeta)\hat{\phi}(\zeta) \ a.e. \ \zeta \in \mathbb{R}. \tag{4.9}$$

The equation (2.1) being equivalent to equation (4.10) given by,

$$\hat{\phi}(\zeta) = \prod_{j=1}^{\infty} m_0(2^{-j}\zeta) \ a.e. \ \zeta \in \mathbb{R} \tag{4.10}$$

the later is proved using some simple calculations. By the orthonormality of $\left\{\phi_{0,k} : k \in \mathbb{Z}\right\}$ given by equation (4.4) and by equation (2.1) we have

$$|m_0(\zeta)|^2 + |m_0(\zeta+\pi)|^2 = 1 \ a.e. \ \zeta \in [-\pi, \pi). \tag{4.11}$$

Proceeding in a similar manner, the orthonormality of $\left\{\psi_{0,k} : k \in \mathbb{Z}\right\}$ and by equation (4.9) we obtain

$$|m_1(\zeta)|^2 + |m_1(\zeta+\pi)|^2 = 1 \ a.e. \ \zeta \in [-\pi, \pi) \tag{4.12}$$

which in turn gives $m_0, m_1 \in L^\infty([-\pi, \pi))$. In fact, from equation (4.11) we get $m_1(\zeta) = \lambda(\zeta)\overline{m_0(\zeta+\pi)}$ where $\lambda(\zeta)$ is 2π-periodic and $\lambda(\zeta) + \lambda(\zeta+\pi) = 0 \ a.e. \ \zeta \in [-\pi, \pi)$. Thus there exists a 2π-periodic function ν such that $\lambda(\zeta) = e^{\iota\zeta}\nu(2\zeta), \ \zeta \in [-\pi, \pi)$ and $m_1(\zeta) = e^{\iota\zeta}\nu(2\zeta)\overline{m_0(\zeta+\pi)}$ where equation (4.12) implies $|\nu(\zeta)|^2 = 1 \ a.e. \ \zeta \in [-\pi, \pi)$. Thus we have shown that (m_0, m_1) is a Quadrature Mirror Filter (QMF) pair and the last part of the proof is devoted in showing that ϕ is a scaling function for $V_0 = \text{span}\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ which is done by proving $\overline{V_0} = V_0$. This makes use of equations (4.9) and (4.4) and orthonormality.

One useful characterization of MRA MSF wavelets that is given in [13] is the following.

Theorem 4.1. [13] *For an MSF wavelet ψ with $|\hat{\psi}| = \chi_E$, the following conditions are equivalent:*

a) *ψ is associated with an MRA.*

b) $\mathcal{M}\left(E^s \cap (E^s + 2k\pi)\right) = 2\pi\delta_{k0} \quad k \in \mathbb{Z}$ *where* $E^s = \bigcup_{j=1}^{\infty}(2^{-j}E)$.

c) $\{E^s + 2k\pi : k \in \mathbb{Z}\}$ *forms a partition of* $\mathbb{R}$.

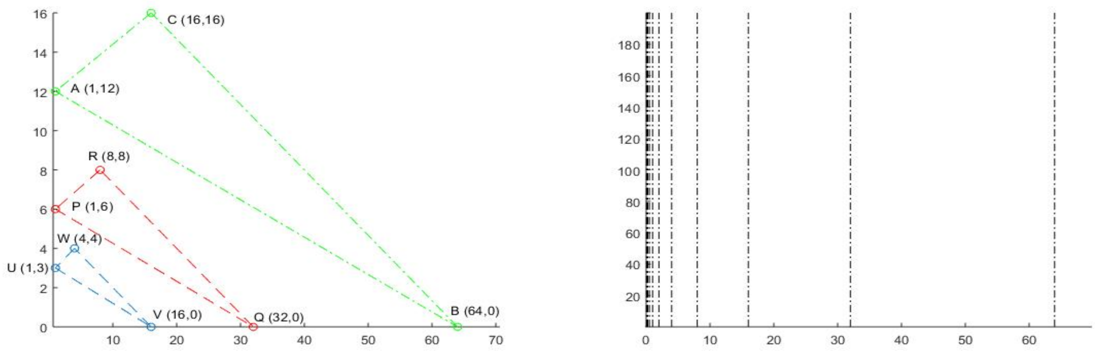
d) $\mathcal{M}\left((E + 4k\pi) \cap (2^{-j}E)\right) = 0$ *for every* $k \in \mathbb{Z}, \quad j \geq 1$.

Outline of the proof of the equivalence of the first two statements has already been mentioned. The equivalence of second and third is trivial and it is enough to show that (*b*) and (*d*) are equivalent. Using the fact that $\{2^j E\}_{j\in\mathbb{Z}}$ forms a partition of $\mathbb{R}$, both the L.H.S of (*b*) and (*d*) are zero and hence equal. The converse part is proved using the fact that $\{E + 2k\pi\}_{k\in\mathbb{Z}}$ forms a partition of $\mathbb{R}$. Using the statement (*d*), it is shown in examples that Lemarie and Journe wavelets are non-MRA wavelets. Also, using Theorem (4.1), Shannon wavelets are shown to be associated with an MRA.

Results proved in [1] could be seen as an extension of the results found in [15], here the symmetric class of MSF wavelets are considered. Because of the symmetric nature, it is enough to consider for E^+. The case where E^+ is the disjoint union of two intervals and the case of just one interval could be seen in [15, 20], a general method is put forward when E^+ is a finite disjoint union of intervals of more than two intervals. If E^+ is just one interval, then the only symmetric wavelet set is the Shannon wavelet set. The idea of MSF polygon is introduced to give a geometric insight to construction and then relates this with wavelet sets.

Definition 4.1. [1] Consider the first quadrant of the Cartesian plane, let D be the set of points P such that for some $m \in \mathbb{N} \cup \{0\}$ and $\lambda \in \mathbb{Z}$, $P \equiv P[\lambda, m] = (2^{-\lambda}, 2^{-\lambda} m)$. Let $\mathbb{P} = (P_1, P_2, \cdots, P_n)$ be an ordered sequence of points in D where $P_j = P[\lambda_j, m_j]$ for $j = 1, 2, \cdots, n$. For $j = 1, 2, \cdots, n-1$, define a_j as the negative of the slope of the straight line through P_j and P_{j+1} giving $a_j = -\frac{m_j 2^{-\lambda_j} - m_{j+1} 2^{-\lambda_{j+1}}}{2^{-\lambda_j} - 2^{-\lambda_{j+1}}}$. Thus the polygon $\mathbb{P}$ formed is said to be an **MSF polygon** if $\lambda_1 = 0$, $4m_1 = 2^{-\lambda_n}(2m_n + 1)$ and $0 < a_0 < a_1 < \cdots < a_n = \frac{1}{2}$.

On observing the definition of MSF polygons, we see that the vertices lie only on the portion of lines $\left\{(2^{-\lambda_j}, y) : \lambda_j \in \mathbb{Z},\, y \in \mathbb{R}\right\}$ shown in Figure (2b) in the first quadrant. We note that the slopes of the straight lines joining any P_j and P_{j+1} are decreasing and lies in $[-\frac{1}{2}, 0]$. We have plotted a few MSF polygons for the case $n = 3$ in Figure (2a). In this Figure, $\triangle ABC$ refers to the case where we choose $m_3 = 1$, $\lambda_1 = 0$, $\lambda_2 = -6$, $m_2 = 0$ and $\lambda_3 = -4$, thus obtaining $m_1 = 12$. For the case $n = 3$ in Figure (2a), we choose $a_3 = \frac{1}{2}$ for each triangle to be an MSF polygon. For $\triangle PQR$, $\lambda_2 = -5$ and $\lambda_3 = -3$ thereby obtaining $m_1 = 6$ and the values of λ_1, m_2, m_3 and a_3 are same as that of $\triangle ABC$. Also, for $\triangle UVW$ the only change made was that we considered $\lambda_3 = -2$ and $\lambda_2 = -4$ and thereby obtaining $m_1 = 3$.



(a) (b)

Figure 2: (a) MSF polygon for $n = 3$ (b) Lines $\left\{(2^{-\lambda_j}, y) : \lambda_j \in \mathbb{Z},\, y \in \mathbb{R}\right\}$

It is shown in [1] that if $\mathbb{P}$ is an MSF polygon and I_j is defined as $I_j = [a_{j-1}, a_j] + m_j$ for $j = 1, 2, \cdots, n$, then $E = E^+ \cup E^-$ where $E^+ = I_1 \cup I_2 \cup \cdots \cup I_n$, is a symmetric wavelet set of $L^2(\mathbb{R})$ and thus E is a disjoint union of $2n$ intervals. Also, distinct MSF polygons give rise to distinct wavelet sets. In the case of $\triangle UVW$, $a_1 = \frac{1}{5}$, $a_2 = \frac{1}{3}$ and thereby obtaining $E^+ = [3, \frac{16}{5}] \cup [\frac{1}{5}, \frac{1}{3}] \cup [\frac{4}{3}, \frac{3}{2}]$. In the third section, corresponding to each n, each n-tuple $\epsilon = (\epsilon_1, \cdots, \epsilon_n) \in \{-1, 1\}^n$ and for each permutation $\tau \in S_{n-1}$ on $1, 2, \cdots, n-1$, there is a family $\mathbb{M}(n, \epsilon, \tau)$ of symmetric wavelet sets of $L^2(\mathbb{R})$ associated with it and a necessary and sufficient condition for a symmetric set E which is a finite union of intervals to be wavelet set in $L^2(\mathbb{R})$ is also provided. An example of a wavelet set which is symmetric and have zero as an accumulation point is included in [1]. Also, it is proved that there exists uncountably many symmetric wavelet sets in $L^2(\mathbb{R})$.

We have four interval symmetric wavelet sets in $\mathbb{R}$ of the form $E_r = E_r^+ \bigcup E_r^-$ where $E_r^+ = \left[\frac{2^r}{2^{r+1}-1}\pi, \pi\right) \bigcup \left[2^r\pi, \frac{2^{2r+1}}{2^{r+1}-1}\pi\right)$ and since being symmetric gives $E_r^- = -E_r^+$, $r \in \mathbb{N}$. For $r = 2$, we get the Journe wavelet set. Also, the generalized Journe wavelet set is given by

$$J_{\beta_r} = [-2^r\pi - \frac{2^r}{2^{r+1}-1}\pi, -2^r\pi + 2^r\beta_r) \bigcup [-\pi + \beta_r, -\frac{2^r}{2^{r+1}-1}\pi) \bigcup [\frac{2^r}{2^{r+1}-1}\pi, \pi + \beta_r) \bigcup [2^r\pi + 2^r\beta_r, 2^r\pi + \frac{2^r}{2^{r+1}-1}\pi) \tag{4.13}$$

where $r \in \mathbb{N}$ and $\beta_r \in (\frac{-\pi}{2^{r+1}-1}, \frac{\pi}{2^{r+1}-1})$. Using the characterization of wavelet set, we define three maps that are measurable bijections, one of this is translation congruent modulo 2π on J_{β_r} to $[-\pi + \beta_r, \pi + \beta_r)$, a dilation on $J_{\beta_r}^+$ to $[\frac{2^r}{2^{r+1}-1}\pi, \frac{2^{r+1}}{2^{r+1}-1}\pi)$ and another dilation on $J_{\beta_r}^-$ to $[-\frac{2^{r+1}}{2^{r+1}-1}\pi, -\frac{2^r}{2^{r+1}-1}\pi)$. This is an equivalent characterization of wavelet set and hence J_{β_r} is shown to be a wavelet set. It is shown that J_{β_1} is an MRA wavelet set and for $r \neq 1$, J_{β_r} is a non-MRA wavelet set, and this result is proved using the concept of dimension function in [25].

Definition 4.2. [6] The **dimension function** of an orthonormal wavelet $\psi \in L^2(\mathbb{R})$ is defined as

$$D_\psi(\zeta) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\zeta + k))|^2$$

Dimension function was introduced by *Guido Weiss*. It is also referred to as multiplicity function in literature, using this notion of dimension function, characterization of wavelets was easily made possible. This definition does not always hold for non-integer dilations in one dimension, but there are two cases where it is valid. The first one having a rational dilation factor and the other being MSF wavelet detail can be seen in [7]. A characterization of MRA wavelet given in [6] has proved that for a function ψ to be an MRA wavelet, it is necessary and sufficient that $D_\psi(\zeta) = 1$ for $a.e.\ \zeta \in \mathbb{R}$. Using this concept of dimension function, it is shown that in [16] any orthonormal wavelet whose Fourier transform has support contained in S_α is an MRA wavelet. Here we show that $D_\psi(\zeta) = 1$ almost everywhere. Since $D_\psi(\zeta)$ is 2π-periodic, restriction is made to $D_\alpha \cup H_\alpha$. Using equation (3.4) and *(f)* of Theorem (3.6), it is shown that $D_\psi(\zeta) = 1$ on D_α almost everywhere. Then using equation (3.3) and H_α, it is possible to show $D_\psi(\zeta) = 1$ on H_α almost everywhere.

As the Fourier transform of these wavelets is characteristic functions that are discontinuous, they lack good regularity properties which makes them less applicable. But these classes of wavelets can be used to derive certain other wavelets with good regularity properties. This could be seen in [17] where we obtain smoother low-pass filters for an MRA by mollifying the low-pass filters of MSF wavelets. This in turn leads to an approximation of MSF wavelets by smoother wavelets of any degree of smoothness. Cohen's characterization in [9] deals with those C^∞ filters which are associated with MRA's. But for an MSF wavelet, filters $m_0 \notin C^r$ and as a result, this characterization cannot be directly used for them. Cohen's characterization is the following.

Theorem 4.2. [18] *Let m_0 be a 2π-periodic function in $C^{r+1}(\mathbb{R})$ for $r = 0, 1, \cdots, \infty$ which satisfies $m_0(0) = 1$. Then m_0 is a low-pass filter for a wavelet if and only if m_0 satisfies $|m_0(\zeta)|^2 + |m_0(\zeta+\pi)|^2 = 1$ $a.e.$ $\zeta \in \mathbb{R}$ and there exists a set $E \subset \mathbb{R}$ such that E is congruent to $[-\pi, \pi]$ modulo 2π.*

Characterization of the low-pass filter of a wavelet is given in [17], this together with Theorem (4.3) leads to results concerning MSF wavelet.

Theorem 4.3. [17] *Suppose that ψ is an MSF MRA wavelet for which $|\hat{\phi}|$ is continuous at zero and $|\hat{\phi}(0)| = 1$. Then the low-pass filter m_0 associated with this wavelet must be of the form $|m_0| = \chi_E$, where $E \subset \mathbb{R}$ is a measurable set such that $E = E + 2\pi$ and $\chi_E(\zeta) + \chi_E(\zeta+\pi) = 1$ $a.e.$ $\zeta \in \mathbb{R}$. Also the set $S = \bigcap_{j=1}^{\infty} 2^j E$ is the support of $\hat{\phi}$, $|\hat{\phi}| = \chi_S$, $\mathcal{M}(S) = 2\pi$, S contains 0 in its interior and satisfies $\sum_{l \in \mathbb{Z}} \chi_S(\zeta + 2l\pi) = 1$ $a.e.$ $\zeta \in \mathbb{R}$ and $|m_0(2^{-j}\zeta)| = 1$ $a.e.$ $\zeta \in S$ and all $j \geq 1$.*

This result can be established easily using the basic facts. The equations (2.1) together with (2.6) implies that $|m_0| = \chi_E$ for some measurable set $E \subset \mathbb{R}$. As m_0 is 2π-periodic, we get $\chi_E(\zeta) + \chi_E(\zeta + \pi) = 1$ $a.e.$ $\zeta \in \mathbb{R}$. Since $\hat{\phi}(\zeta) = \prod_{j=1}^{\infty} m_0(2^{-j}\zeta)$, we obtain $|\hat{\phi}| = \chi_S$ and $|m_0(2^{-j}\zeta)| = 1$ $a.e.$ $\zeta \in S$. Applying Plancherel's Theorem, $\mathcal{M}(S) = 2\pi$ and the orthonormality of $\{\phi(x-k) : k \in \mathbb{Z}\}$ leads to $\sum_{l \in \mathbb{Z}} \chi_S(\zeta + 2l\pi) = 1$ $a.e.$ $\zeta \in \mathbb{R}$. Finally we need only to show that 0 lies in the interior of S. This can be proved by the method of contradiction using the continuity of $|\hat{\phi}|$. The importance of Theorem (4.3) is that it leads to the characterization of the low-pass filter for an MSF wavelet.

Theorem 4.4. [17] *Let m_0 be a 2π-periodic measurable function defined on $\mathbb{R}$ such that m_0 is continuous at 0 and $|m_0(0)| = 1$. Then m_0 is a low-pass filter for an MSF wavelet if and only if $|m_0| = \chi_E$, $E \subset \mathbb{R}$ is a measurable set such that*

$$\chi_E(\zeta) + \chi_E(\zeta + \pi) = 1 \ \ a.e. \ \zeta \in \mathbb{R} \tag{4.14}$$

$$m_0\left(\bigcap_{j=1}^{\infty} 2^j E\right) = 2\pi. \tag{4.15}$$

By Theorem (4.3), one side can be easily obtained. If we assume $|m_0| = \chi_E$ with equations (4.14) and (4.15) we need to show that m_0 is a low-pass filter. For this, we define $\hat{\phi}(\zeta) = \prod_{j=1}^{\infty} m_0(2^{-j}\zeta)$. Then we get $|\hat{\phi}| = \chi_S$ where $S = \bigcap_{j=1}^{\infty} 2^j E$. The orthonormality of $\{\phi(x-k) : k \in \mathbb{Z}\}$ is proved since the sets $S + 2k\pi$ for $k \in \mathbb{Z}$ are disjoint. By the continuity assumption on m_0, we have the continuity of $|\hat{\phi}|$ and $|\hat{\phi}(0)| = 1$. From the definition of $\hat{\phi}$, we obtain $\hat{\phi}(2\zeta) = m_0(\zeta)\hat{\phi}(\zeta)$. Thus by Theorem (2.1), ϕ is a scaling function associated with an MRA and hence m_0 is a low-pass filter for a wavelet.

A procedure to obtain new low-pass filter for an MRA with a smooth scaling function from a given low-pass filter is seen in [17]. Even though the construction is shown for bandlimited MSF wavelets, it is applicable in general to MSF which are not bandlimited also. If ψ is the bandlimited MRA MSF wavelet with low-pass filter $m_0(\zeta) = \sum_{k\in\mathbb{Z}} \chi_F(\zeta + 2k\pi)$, where $F = \bigcup_{l=1}^{n} I_l$ is a finite disjoint union of intervals contained in $(-\pi, \pi)$ and the scaling function ϕ is chosen such that $\hat{\phi}(\zeta) = \prod_{j=1}^{\infty} m_0(2^{-j}\zeta)$. A function m_ϵ is constructed with any desired degree of smoothness. For any positive real number ϵ, $s_\epsilon \in C^r$ $r = 1, 2, \cdots, \infty$ be defined on $\mathbb{R}$ such that $s_\epsilon(x) = 0$ for all $x < -\epsilon$ and for every real number x, $s_\epsilon^2(x) + c_\epsilon^2(x) = 1$, where $c_\epsilon(x) \equiv s_\epsilon(-x)$. Choose $I_l = (a_l, b_l)$ $l = 1, 2 \cdots, n$ and ϵ is so small such that

$$-\pi < a_1 - \epsilon < \cdots < a_1 + \epsilon < b_l - \epsilon < b_l + \epsilon < a_{l+1} - \epsilon < \cdots < b_n + \epsilon < \pi.$$

Assumption is made so as 0 is in some interval (a_{l_0}, b_{l_0}) and $\epsilon \leq \min\{-a_{l_0}, b_{l_0}\}$. Using these s_ϵ and c_ϵ, a 2π-periodic function m_ϵ is defined as

$$m_\epsilon(\zeta) = \sum_{k\in\mathbb{Z}} \left\{ \sum_{l=1}^{n} s_\epsilon(\zeta - a_l + 2k\pi) c_\epsilon(\zeta - b_\epsilon + 2k\pi) \right\}.$$

Then $m_\epsilon(0) = 1$ and $|m_\epsilon(\zeta)|^2 + |m_\epsilon(\zeta+\pi)|^2 = 1$ $a.e.$ $\zeta \in \mathbb{R}$. Choose $E = Supp\,\hat{\phi}$ and ψ being bandlimited gives $\sum_{l\in\mathbb{Z}} \chi_E(\zeta + 2l\pi) = 1$ for all $\zeta \in \mathbb{R}$ and $m_0(2^{-j}\zeta) \neq 0$ for all $\zeta \in E$ and all $j \in \mathbb{N}$. If $m_\epsilon \in C^{r+1}$, a scaling function is obtained from m_ϵ as $\hat{\phi}_\epsilon(\zeta) = \prod_{j=1}^{\infty} m_\epsilon(2^{-j}\zeta)$ and a wavelet ψ_ϵ is also obtained, both belongs to C^r. By showing that $\hat{\phi}_\epsilon$ tends to $\hat{\phi}$ in the L^2-norm and $\hat{\psi}_\epsilon$ tends to $\hat{\psi}$ in the L^2-norm as ϵ goes to zero, we get m_ϵ tends to m_0 in the L^2-norm.

Also, it is shown in [16] that non MRA wavelet cannot be approximated by MRA wavelets in the L^2-norm. Hence if ψ is a wavelet which is not an MRA wavelet then it is isolated from the MRA wavelets in the L^2-norm. This in turn leads to the fact that the collection of all MRA wavelets as a subset of the set of all wavelets is open in the L^2-norm. It is already established that if $\{\psi^n : n = 1, 2, \cdots\}$ is a sequence of MRA wavelets converging in the L^2-norm to a wavelet ψ, then ψ must be an MRA wavelet. The proof of this result is done by showing that the dimension function $D_\psi(\zeta) = 1$ almost everywhere. By using the property of dimension function, for any interval J of length 2π, we have $\int_J D_\psi(\zeta)\ d\zeta = 2\pi$, it is obtained that $\int_0^{2\pi} D_\psi(\zeta)\ d\zeta = 2\pi\|\psi\|_2^2$. Using this, we have

$$\int_0^{2\pi} D_{\psi^n-\psi}(\zeta) d\zeta = 2\pi\|\psi^n - \psi\|_2^2.$$

This when followed by Fatou's Lemma and the inequality

$$D_\psi(\zeta) \leq D_{\psi^n-\psi}(\zeta) + D_{\psi^n}(\zeta) + 2\sqrt{D_{\psi^n-\psi}(\zeta) D_{\psi^n}(\zeta)} \quad a.e \quad \zeta \in \mathbb{R},$$

together with simple calculations leads to the result. This means that the collection of all MRA wavelets is closed as a subset of the set of all wavelets. Thus, it can be concluded that the set of all MRA wavelets as a subset of the set of all wavelets in $L^2(\mathbb{R})$ is both closed and open set and hence is not connected.

Another interesting fact mentioned in [16] is that if we consider S as the set of all wavelets as a subset of the unit sphere in $L^2(\mathbb{R})$, then S is not closed in $L^2(\mathbb{R})$. For this, we consider the wavelet ψ_α with

$$\hat{\psi}_\alpha = \chi_{[-2\alpha,-\alpha]\cup[2\pi-\alpha,4\pi-2\alpha]}, \quad 0 < \alpha < 2\pi.$$

Then as α tends to zero, $\hat{\psi}_\alpha$ goes to $\hat{\psi}_H$ where $\hat{\psi}_H = \chi_{[2\pi,4\pi]}$. But by Theorem (3.3), ψ_H is not a wavelet.

The connectedness of different class of wavelet sets has always been of interest. The connectedness of the collection of s-elementary wavelets has been dealt in the paper [27]. The main result that we have, regarding the connectivity of this class of wavelets is the following.

Theorem 4.5. [27] *The s-elementary wavelets form a path connected subset of* $L^2(\mathbb{R})$.

Here onwards, λ denotes Lebesgue measure on $\mathbb{R}$ and μ denote the measure defined by $\mu(A) = \int \chi_A(x)\frac{d\lambda}{|x|}$. Also, let $M(A)$ be the collection of all measurable subsets B of A such that $\lambda(B) < \infty$ and $\mu(B) < \infty$. To establish the path connectivity, first it is shown that the collection of wavelet sets is path connected in the symmetric difference metric. This metric is defined as $d_\lambda(A, B) = \lambda(A\Delta B)$, for measurable sets A and B of $\mathbb{R}$. For this, the first step is to construct a path of subsets of wavelet sets. Then for each set in the path, we find a wavelet superset which depends continuously on the subset. Define R_t to be a subset of W such that R_t is translation 2π congruent to $[-\pi-t, -\pi)\bigcup[\pi, \pi+t)$ where $t \in [0, \pi]$ and P_t is another subset of W which is 2-dilation congruent to $[-2\pi, -2\pi-t)\bigcup[2\pi-t, 2\pi)$ and $Q_t = [-2\pi+t, -\pi-t)\bigcup[\pi+t, 2\pi-t)$. Then the path of these sets is defined by

$$S_t = \begin{cases} [(Q_t \bigcup R_t) \setminus (\tau(R_t) \bigcap Q_t)] \setminus P_t & if \quad 0 \quad \le t \ \le \pi/2 \\ R_t \setminus P_t & if \quad \pi/2 \ \le t \ \le \pi. \end{cases}$$

The continuity of dilations, translations and set operations concerning this metric justifies the continuity of this path. Then a recursive construction of sets M_i of continuous functions of t such that $S_t \bigcup(\bigcup_{i=0}^{\infty} M_i) = W_t$ is a wavelet set for each t is made. This construction is even shown to be continuous. When the symmetric difference metric is restricted to s-elementary wavelets, it is equivalent to the $L^2(\mathbb{R})$ metric and thus s-elementary wavelets form a path connected subset of $L^2(\mathbb{R})$.

Combining the result of *Auscher* in [2], with the one regarding the convergence of MRA wavelets in [16], we obtain that any orthonormal non-MRA wavelet cannot be approximated by those in Schwartz class. In particular, the Journe wavelet cannot be approximated by those in the Schwartz class. The result from [2] is as follows.

Theorem 4.6. [2] *Suppose* ψ *is an orthonormal wavelet that satisfies* $|\hat{\psi}|$ *is continuous on* $\mathbb{R}$ *and* $|\hat{\psi}(\zeta)| = O\left((1 + |\zeta|)^{-\alpha-1/2}\right)$ *at infinity and for some* $\alpha > 0$. *Then* ψ *is an MRA wavelet.*

Thus we obtain the fact that any sequence of wavelets from the Schwartz class will be MRA wavelets and any sequence of MRA wavelets if it converges to a wavelet, then the limit is also an MRA wavelet.

5. EXAMPLES OF MSF AND NON MSF WAVELETS

There are a variety of examples for MSF wavelets. Some of the frequently mentioned ones are listed below. Here most of them are defined in terms of their Fourier transform. So those wavelets whose modulus of the Fourier transform is a characteristic function of a measurable set are easily seen to be MSF wavelets.

1) Shannon wavelet ψ_s such that $|\hat{\psi}_s| = \chi_I$ with $I = [-2\pi, -\pi) \cup [\pi, 2\pi)$. Using the equivalent conditions for an MRA MSF wavelet, we can easily show that Shannon wavelet is one such wavelet. For this, we denote $I^s = \bigcup_{j=1}^{\infty} 2^{-j} I$. Hence
$$I^s = \left\{[-\pi, \frac{-\pi}{2}) \cup [\frac{\pi}{2}, \pi)\right\} \bigcup \left\{[\frac{-\pi}{2}, \frac{-\pi}{4}) \cup [\frac{\pi}{4}, \frac{\pi}{2})\right\} \bigcup \cdots$$
Thus $I^s = [-\pi, \pi)$ which is of length 2π. So $I^s + 2k\pi = [-\pi + 2k\pi, \pi + 2k\pi)$ and we get $\mathcal{M}(I^s \bigcap I^s + 2k\pi) = 2\pi\delta_{k,0}$. Thus Shannon wavelet is an MRA MSF wavelet by Theorem (4.1).
2) Lemarie wavelet given by $|\hat{\psi}_L| = \chi_L$ where $L = [\frac{-8\pi}{7}, \frac{-4\pi}{7}) \bigcup [\frac{4\pi}{7}, \frac{6\pi}{7}) \bigcup [\frac{24\pi}{7}, \frac{32\pi}{7})$. This can be shown to be a non-MRA wavelet. For this, consider $(L - 4\pi) \bigcap (\frac{1}{2}L)$. We get
$$L - 4\pi = [\frac{-36\pi}{7}, \frac{-32\pi}{7}) \bigcup [\frac{-24\pi}{7}, \frac{-22\pi}{7}) \bigcup [\frac{-4\pi}{7}, \frac{4\pi}{7})$$
and
$$\frac{1}{2}L = [\frac{-4\pi}{7}, \frac{-2\pi}{7}) \bigcup [\frac{2\pi}{7}, \frac{3\pi}{7}) \bigcup [\frac{12\pi}{7}, \frac{16\pi}{7}).$$
Then $[\frac{2\pi}{7}, \frac{3\pi}{7}) \in (L - 4\pi) \bigcap (\frac{1}{2}L)$ and hence its measure is nonzero. This implies that ψ_L is a non-MRA MSF wavelet by Theorem (4.1).
3) Journe wavelet given by ψ_E is a non MRA MSF wavelet. Here $|\hat{\psi}_E| = \chi_E$ where
$$E = [\frac{-32\pi}{7}, -4\pi] \bigcup [-\pi, -\frac{4\pi}{7}] \bigcup [\frac{4\pi}{7}, \pi] \bigcup [4\pi, \frac{32\pi}{7}].$$
We have
$$E - 4\pi = [\frac{-60\pi}{7}, -8\pi] \bigcup [-5\pi, -\frac{32\pi}{7}] \bigcup [\frac{-24\pi}{7}, -3\pi] \bigcup [0, \frac{4\pi}{7}]$$
and
$$\frac{E}{2} = [\frac{-16\pi}{7}, -2\pi] \bigcup [\frac{-\pi}{2}, \frac{-2\pi}{7}] \bigcup [\frac{2\pi}{7}, \frac{\pi}{2}] \bigcup [2\pi, \frac{16\pi}{7}]$$
Thus, $(E - 4\pi) \bigcap \frac{E}{2} = [\frac{2\pi}{7}, \frac{\pi}{2}]$ whose measure is nonzero and hence a non-MRA wavelet.
4) Theorem (3.3) can be used to show that the function ψ_H given by $\hat{\psi}_H = \chi_{[2\pi,4\pi]}$ is not a wavelet as this would violate the condition b of the Theorem. The dyadic dilations of $[2\pi, 4\pi]$ form a partition of $(0, \infty)$ only.
5) Using Theorem (3.5), a positive answer is provided to the question of whether there exists an MSF wavelet which is not bandlimited. A construction of such a wavelet is made possible in [13]. For this, a symmetric set $E = E^+ \bigcup E^-$ where $E^- = -E^+$ is considered. We decompose $I^+ = [\pi, 2\pi)$ to $[\pi, x_0) \bigcup [x_0, 2\pi)$. Let $\{L_j\}$ and $\{R_j\}$ be partitions of $[\pi, x_0)$ and $[x_0, 2\pi)$ respectively. Thus $I^+ = \bigcup_{j=1}^{\infty} (L_j \bigcup R_j)$. We try to obtain an unbounded set E^+. For this, we dilate each L_j by 2^j for some $l_j \geq 1$ with $\lim_{j\to\infty} l_j = \infty$ and dilate each R_j by 2^{-1}. Hence $E^+ = (\bigcup_{j=1}^{\infty} 2^{l_j} L_j) \bigcup (2^{-1} \bigcup_{j=1}^{\infty} R_j)$. So it is enough to find conditions on $\{l_j : j \geq 1\}$ which makes $E^+ = [-2\pi, -\pi)(mod\ 2\pi)$. This is done in such a way that the Lebesgue measure is preserved at each stage. Hence we get for each $j \geq 1$,

$$2^{l_j}\,\mathcal{M}\,(L_j) + 2^{-1}\,\mathcal{M}(R_j) = \mathcal{M}\left(L_j \bigcup R_j\right) = \mathcal{M}(L_j) + \mathcal{M}(R_j). \tag{5.1}$$

Let $\mathcal{M}(L_j) = p_j\,\mathcal{M}(L_j \bigcup R_j)$ and $\mathcal{M}(R_j) = q_j\,\mathcal{M}(L_j \bigcup R_j)$, hence

$$p_j + q_j = 1$$
$$2^{l_j} p_j + 2^{-1} q_j = 1.$$

Thus equation (5.1) is equivalent to above set of equations. This can be solved by converting it into matrix form. Thus on solving, we get $p_j = \frac{1}{2^{l_j+1}-1}$ and $q_j = \frac{2^{l_j+1}-2}{2^{l_j+1}-1}$. Since $\mathcal{M}\left(I^+\right) = \pi$, we have $\sum \mathcal{M}(L_j \bigcup R_j) = \pi$. Since $\displaystyle\sum_{j=1}^{\infty} 2^{-j}\pi = \pi$, we can choose $\mathcal{M}\left(L_j \bigcup R_j\right) = 2^{-j}\pi$. Then we get $\mathcal{M}\left(L_j\right) = \alpha_j$ and $\mathcal{M}\left(R_j\right) = \beta_j$ in terms of l_j. Choose $l_1 \geq 2$ and obtain $l_{j+1} = 2l_j + 1$. This is obtained by defining $L_1 = [\pi, \pi + \alpha_1)$ and $R_1 = [2\pi - \beta_1, 2\pi)$. Also, we define

$$L_j = [\pi + \sum_{n=1}^{j-1} \alpha_n, \pi + \sum_{n=1}^{j} \alpha_n)$$

and

$$R_j = [2\pi - \sum_{n=1}^{j} \beta_n, 2\pi - \sum_{n=1}^{j-1} \beta_n).$$

Since $\bigcup R_j = (x_0, 2\pi]$, we get $x_0 = 2\pi - \displaystyle\sum_{j=1}^{\infty} \beta_j = \pi + \sum_{j=1}^{\infty} \alpha_j$. Now we express $2^{l_j} L_j$ in terms of l_j and simple computation gives the desired result. Thus as j tends to ∞, l_j goes to infinity and hence we get unbounded E^+.

Similar studies have been carried out in higher-dimensional cases as well. Corresponding to the concept of wavelet set in $L^2(\mathbb{R})$, we have the concept of $A-wavelet$ set in $\mathbb{R}^n$, details could be seen in [11]. A method to produce all wavelet sets in $\mathbb{R}^n$ for an expansive matrix with integral entries is provided in [3] and the construction of compact wavelet sets in $\mathbb{R}^n$ could be seen in [14]. MSF multiwavelet of order L in [6] could be considered as an extension of the concept of MSF wavelet in a one-dimensional setting. Analogous version of Meyer's equations, equivalent conditions for an orthonormal wavelet in $L^2(\mathbb{R}^n)$ could be seen in [11]. The definitions (3.1), (3.2) and the set-theoretic equivalent conditions of a wavelet set as in Theorem (3.4) is extended to n-dimensional in [26], this is used in the construction of an unbounded wavelet set in higher dimensions, but the construction in single dimension is appeared in [13]. Classification of orthonormal wavelets whose Fourier transform has support contained in S_α is seen in Theorem (3.6) could not be generalized to the higher dimensional case is shown in [26]. Interconnections of s-elementary wavelets and MSF wavelets have been studied in higher dimensions is found in [27], the major result found in this work is that the collection of all n-dimensional wavelet set is path connected in the symmetric difference metric. Path connectedness of the set of $A-$wavelet collection of sets for an arbitrary expansive matrix A in $\mathbb{R}^d$ is dealt in [23]. Also, in [22], the space of MSF wavelets associated with an MRA and the set of its associated scaling functions is shown to be path-connected and in [4], construction of scaling sets along with MSF orthonormal wavelets in higher dimension could be seen.

Although MSF wavelets or s-elementary wavelets lack good regularity properties, many of the MSF wavelets serve as good counter examples. From MSF wavelets many more interesting non-MSF wavelets can also be constructed which could be seen in [28, 29, 24]. Although it has been proved that for any expansive matrix $A \in \mathbb{GL}_n(\mathbb{R})$ and for any lattice $\Gamma \subset \mathbb{Z}^n$, there exists a wavelet set (A, Γ), the characterization of this pair (A, Γ) in general remains as an open question mentioned in [5]. For the case $n = 2$, such a characterization is made possible. The study regarding the topological properties other than the connectivity of this class of wavelets provides scope for future research. Also, the characterization of MSF wavelets based on the spectral properties of the dilation matrices in higher dimensions can be developed further.

REFERENCES

[1] N. ARCOZZI, B. BEHERA and S. MADAN, Large classes of minimally supported frequency wavelets of $\mathbb{L}^2(\mathbb{R})$ and $\mathbb{H}^2(\mathbb{R})$, *The Journal of Geometric Analysis*, **13**(4) (2003), pp. 557–579, Springer.

[2] P. AUSCHER, Solution of two problems on wavelets, *The Journal of Geometric Analysis*, **5**(2) (1995), pp. 181–236.

[3] L. W. BAGGETT , H. A. MEDINA and K. D. MERRILL, Generalized multi-resolution analyses and a construction procedure for all wavelet sets in $\mathbb{R}^n$, *Journal of Fourier Analysis and Applications*, **5**(6) (1999), pp. 563–573, Springer .

[4] D. BAKIĆ and E. N. WILSON, Scaling sets and orthonormal wavelets with dilations induced by expanding matrices, *Glasnik matematički*, **46**(1) (2011), pp. 189–213, Hrvatsko matematičko društvo i PMF-Matematički odjel, Sveučilišta u Zagrebu.

[5] M. BOWNIK and Z. RZESZOTNIK, Open problems in wavelet theory, *Operator Theory, Operator Algebras and their interactions with Geometry and Topology*, (2020), pp. 77–100, Springer.

[6] M. BOWNIK and Z. RZESZOTNIK and D. SPEEGLE, A characterization of dimension functions of wavelets, *Applied and Computational Harmonic Analysis*, **10**(1) (2001), pp. 71–92, Elsevier.

[7] M. BOWNIK and D. SPEEGLE, The wavelet dimension function for real dilations and dilations admitting non-MSF wavelets, *Approximation theory, X (St. Louis, MO, 2001)*, (2002), pp. 63–85.

[8] C. K. CHUI, *An introduction to wavelets*, Elsevier, (2016).

[9] A. COHEN, Ondelettes, analyses multirésolutions et filtres miroirs en quadrature, *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, **7**(5) (1990), pp. 439–459, Elsevier.

[10] X. DAI and D. R. LARSON, Wandering vectors for unitary systems and orthogonal wavelets, *American Mathematical Soc.*, **640** (1998).

[11] X. DAI, D. R. LARSON and D.M. SPEEGLE, Wavelet sets in $\mathbb{R}^n$, *Journal of Fourier Analysis and Applications*, **3**(4) (1997), pp. 451–456.

[12] I. DAUBECHIES, *Ten Lectures on Wavelets*, (1992), SIAM.

[13] X. FANG and X. WANG, Construction of minimally-supported-frequencies wavelets, *J. Fourier Anal. Appl*, **2** (1966), pp. 315–327.

[14] J. P. GABARDO and X. YU, Construction of wavelet sets with certain self-similarity properties, *The Journal of Geometric Analysis*, **14**(4) (2004), pp. 629–651, Springer .

[15] Y. H. HA, H. KANG , J. LEE and J. K. SEO, Unimodular wavelets for $\mathbb{L}^2$ and the Hardy space $\mathbb{H}^2$, *The Michigan Mathematical Journal*, **41**(2) (1994), pp. 345–361.

[16] E. HERNÁNDEZ and X. WANG and G. WEISS, Smoothing minimally supported frequency (MSF) wavelets: Part I, *J. Fourier Anal. & Appl.*, (1997), Citeseer.

[17] E. HERNÁNDEZ and X. WANG and G. WEISS, Smoothing minimally supported frequency wavelets: Part II, *Journal of Fourier Analysis and Applications*, **3**(1) (1997), pp. 23–41.

[18] E. HERNÁNDEZ and G. WEISS, *A first course on wavelets*, CRC press, (1996).

[19] G. KAISER, *A friendly guide to wavelets*, Springer Science & Business Media., (2010).

[20] H. O. KIM and J. H. PARK, Unimodular wavelets and scaling functions, *Comm. Korean Math. Soc.*, **13**(2) (1998), pp. 289–305.

[21] LEMARIÉ-RIEUSSET and PIERRE GILLES, Analyses multi-résolutions non orthogonales, commutation entre projecteurs et derivation et ondelettes vecteurs à divergence nuIIe, *Revista Matemática Iberoamericana*, **8**(2) (1992), pp. 221–237.

[22] M. LEON, Path Connectedness of Minimally Supported Frequency MRA for Expansive Matrices, *Preprint.*, (1999), Citeseer.

[23] M. T. LEON, Path Connectedness of Orthonormal Basis of Minimally Supported Frequency Wavelets, (2000), Citeseer .

[24] N. K. SHUKLA, Non-MSF A-Wavelets from A-Wavelet sets, *International Journal of Wavelets, Multiresolution and Information Processing*, **11**(01) (2013), pp. 1350002, World Scientific.

[25] N. K. SHUKLA and G. C. S. YADAV, Constructing non-MSF wavelets from generalized Journ'e wavelet sets, *Analysis and Applications*, **9**(2) (2011), pp.225–233, World Scientific .

[26] P. M. SOARDI and D. WEILAND, Single wavelets in n-dimensions, *Journal of Fourier Analysis and Applications*, **4**(3) (1998), pp. 299–315, Springer .

[27] D. SPEEGLE, The s-elementary wavelets are path-connected, *Proceedings of the American Mathematical Society*, **127**(1) (1999), pp. 223–233.

[28] A. VYAS, Construction of non-MSF non-MRA wavelets for $\mathbb{L}^2(\mathbb{R})$ and $\mathbb{H}^2(\mathbb{R})$ from MSF Wavelets, *Bulletin of the Polish Academy of Sciences. Mathematics*, **57**(1) (2009), pp. 33–40.

[29] A. VYAS and R. DUBEY, Non-MSF wavelets from six interval MSF wavelets, *International Journal of Wavelets, Multiresolution and Information Processing*, **9** (2011), pp. 375–385, World Scientific.

[30] P. WOJTASZCZYK, *A Mathematical Introduction to Wavelets*, Cambridge University Press, **37** (1997).