Aust. J. Math. Anal. Appl.
Vol. **18** (2021), No. 2, Art. 13, 9 pp.
AJMAA

MULTISTAGE ANALYTICAL APPROXIMATE SOLUTION OF QUASI-LINEAR DIFFERENTIAL- ALGEBRAIC SYSTEM OF INDEX TWO

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Received 3 February, 2021; accepted 3 June, 2021; published 11 October, 2021.

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ABSTRACT. In this paper, a new Multistage Transform Method (MSDTM) has been proposed by utilizing a well-known transformation technique, the Differential Transform Method (DTM), to solve Differential Algebraic Equations (DAEs) with index 2. The advantage of the proposed scheme is that it does not require an index reduction and extends the convergence domain of the solution. Some examples for various types of problems are carried out to show the ability of MSDTM in solving DAEs. The results obtained are in good agreement with the existing literature which demonstrates the effectiveness and efficiency of the proposed method.

Key words and phrases: Differential Algebraic Equations; Differential Transform Method; Multistage Differential Transform Method.

2010 *Mathematics Subject Classification.* Primary 34A09,40A05. Secondary 34A38, 34A25.

ISSN (electronic): 1449-5910

The authors would like to thank the School of Mathematical Sciences for providing computing facilities.

1. INTRODUCTION

There are many physical problems which are naturally described by a system of DAEs, and these problems have a wide range of applications in various branches of science and engineering, such as mechanical systems, chemical processes, optimal control, electric circuit design and dynamical systems [1],[2],[3]. Chen [3] has used the DAEs to handle adjective dispersive transport problems. Several numerical schemes were highlighted in solving DAEs [4][5]. In this work, our focus is on DAEs with index-2. Numerical solutions for DAEs were first introduced by [6]. DAEs are very important in real applications, as they appear in many aspects of life and physical phenomena. For example, it plays a major role in mathematical modeling, such as electrical circuits, electrical networks, robotic systems, multi-body systems, and aerospace engineering etc.[7],[8],[9]. In general, DAEs are a combination of differential equations and algebraic equations. To sort this kind of problems, we have an important quantity, namely Differential Index. It is defined as the number of times needed to differentiate the DAEs in order to get a system of ODEs. In higher index equations, the index is greater than one, and in our case, the index is two. This type of equation is not easy to solve by numerical methods or by analytical methods, since many of the solutions previously relied on index reduction. However, the disadvantage in applying this method is due to an index reduction, as the solution may be non-physical. Recently, researchers [2],[5],[10], have focused on solving index-2 DAEs. These problems sometimes do not have exact solutions and are difficult to solve by numerical methods [11],[12]. Therefore, approximate analytical methods will be used. In this article, an Euler Lagrange equation will be used, and it is given by the following form :

$$u'(x) = f(x) - G^T\lambda;\ g(x) = 0, \tag{1.1}$$

where $f : R^n \times R^n \to R^n$ and $g(x) : R^n \to R^n$; where $G = g'(x)$. The Jacobian of g(x) and G^T are nonsingular matrices, and $g'(x)$ is a full row rank matrix. An important subclass of System (1.1) consists of DAEs arising from the simulation of constrained Mechanical Multibody Systems. The mechanical system is of great priority in application, especially in the fields of factories and robotic systems [3],[5]. The difficulty in solving this type of problem has motivated our current work in this article by developing a scheme to facilitate computation and helps in filling the mentioned gap.

Definition 1.1 (Differential index): The index along the solution path is defined as the minimum number of differentiation of the System (1.1) that is required to reduce the system to a set of ODEs.

2. DIFFERENTIAL TRANSFORM METHOD

The DTM was first introduced by Zhou [9], who solved linear and non-linear electrical circuit problems using DTM[13]. He developed this method for partial differential equations and [14] applied it for a system of DAEs. A review of the DTM is given below.

Definition 2.1 If a function $u(t)$ is analytical with respect to t in the domain of interest, then

$$U(k) = \frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0}, \tag{2.1}$$

is the transformed function equation of $u(t)$ [15].

Definition 2.2[15] The differential inverse transforms of the set $\{U\left(k\right)\frac{n}{k=0}\}$ is defined by

$$u(t)=\sum_{k=0}^{\infty}U(k)(t-t_0)^k. \tag{2.2}$$

Substituting Eq.(1.1) into Eq.(2.1), we deduce that

$$u(t)=\sum_{k=0}^{\infty}\frac{(t-t_0)^k}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0}. \tag{2.3}$$

From Eq. (2.3), it is easy to see that the concept of DTM is obtained from the expansion of the energy chain. To illustrate the proposed application of DTM to solve systems of ordinary differential equations, consider the nonlinear system of DAEs [15]

$$\frac{du(t)}{dt}=\ f\ \left(u(t),t\right);\ t\geq t_0, \tag{2.4}$$

where $f\left(u\left(t\right),t\right)$ is a nonlinear smooth function system supplied with initial conditions:

$$u\left(t_0\right)=u_0\left(t\right). \tag{2.5}$$

The DTM establishes that the solution of Eq. (2.4) which can be written as:

$$u\left(t\right)=\sum_{k=0}^{\infty}U\left(k\right)\left(t-t_0\right)^k, \tag{2.6}$$

where $U(0),U(1),U(2),\ldots$ are unknowns to be determined by the DTM. Applying DTM to the initial conditions in Eq.(2.5) and System (2.4) respectively, we obtain the transformed initial conditions given by

$$U\left(t_0\right)=u_0\left(t\right), \tag{2.7}$$

and the recursion system becomes

$$\left(1+k\right)U\left(k+1\right)=F\left(U\left(0\right),\ldots,U\left(k\right),k\right),\ k=0,1,2,\ldots, \tag{2.8}$$

where $F\left(U\left(0\right),\ldots,U\left(k\right),k\right)$ is the differential of $f\left(u\left(t\right),t\right)$.

By utilizing Eq.(2.7) and Eq.(2.8), the unknowns $U(k);\ k=0,1,2,\ldots,$ can be determined. Then, the differential inverse transformation of the set of values $\{U(k)\}_{k=0}^{m}$ gives the approximate solution

$$u\left(t\right)=\sum_{k=0}^{m}U\left(k\right)\left(t-t_0\right)^k, \tag{2.9}$$

where m is the approximation order of the solution (2.5). The exact solution of problem (2.4)-(2.5) is then given by $u\left(t\right)=\sum\limits_{k=0}^{\infty}U\left(k\right)t^k$.

3. FORMULATION OF MSDTM

The basic definitions and theories of MsDTM are given by [16],[17]. Due to the fact that the DTM failed to provide convergent approximate analytical solutions over large time intervals, the MsDTM has been introduced to enhance the convergence over the interest interval. Generally, MsDTM requires small time steps, but are able to provide solutions with high accuracy especially for initial value problems [18]. In addition, since it is based on the DTM, it does not depend on a perturbation parameter, trial function, or Lagrangian multiplier as required by other analytical methods. In the next section, the MsDTM is then proposed to solve a system of DAEs of index 2.

4. ADOMIAN POLYNOMIALS AND DTM

In this section, Adomian Polynomials (APs) and DTM will be reviewed in terms of their relationship properties. In general, a nonlinear term $N(u)$ in a differential equation can be decomposed in terms of APs [15],[12].

Theorem 1. Let f be an analytical function with $f\left(u\right) = \sum_{0}^{\infty} f_k\left(U_0,\ \ldots,\ U_k\right)$, where f_k are the AP and $u = \sum_{0}^{\infty} U_k$. If $U_k = u_k\left(x\right)\ t^k$, then $f\left(u\right) = \sum_{0}^{\infty} f_k\left(U_0,\ \ldots,\ U_k\right)$.

The proof of Theorem 1 is given in [19]. The following definitions and theorems are new, and their proofs follow directly from Theorem 1 and the properties of power series.

Definition 4.1 Let $f \in R^n$ be an analytical function with$f\left(u\right) = \left(f_1\left(u\right),\ \ldots,\ f_n\left(u\right)\right)\ ^T$ and let $f_{t,k}$ be the $k-$th AP corresponding to the component $f_i\left(u\right)$. Then $f_k\ = \left(f_{1,\ k}\ ,\ \ldots, f_{n,\ k}\right)$ is called the vector of the kth AP corresponding to $f(u)$.

Definition 4.2 Let $A \in R^{nxn}$ be analytical with $A\ \ (u) = (A_{ij}\ \ (u))$ and let $A_{ij,k}$ be the kth AP corresponding the entry $A_{ij}\left(u\right)$. Then

$$A_k = \left(A_{ij,1}, ..., A_{ij,k}\right)^T$$

corresponding to $A(u)$.

Theorem 2. Let $f(u), g(u), h\left(u\right) \in R^n$ and $A\left(u\right), H\left(u\right) \in R^{nxn}$ be analytical, then the vectors and matrices of the kth AP corresponding to $f\left(u\right),\ g\left(u\right),\ h\left(u\right),\ A\left(u\right)$ and $H\ \ (u)$ satisfy:

(1) $h\left(u\right) = f\ \ (u) + \alpha g\left(u\right)$ then $h_k = f_k + \alpha g_k;\ \alpha$ is scalar,

(2) $h\left(u\right) = f\ \left(u\right)^T,$ then $h_k = f_{k.}^T,$

(3) $h\left(u\right) = A\ \ (u)\, f\left(u\right)$ then $h_k = \sum_{t=0}^{k} A_{k-i}\ f_{i.}$

(4) $h\left(u\right) = A(u)^T$ then $H_k = A_{k.}^T$

(5) $h\left(u\right) = A^{-1}\left(u\right) f\left(u\right)$ then $h_k A_0^{-1}(f_k - \sum_{t=0}^{k-1} A_{k-i-1}\ f_{i.})$

5. QUASILINEAR DAE OF INDEX-2

The DAE of index-2 is given by

$$u'\left(x\right) = f\left(x\right) - G^T,\ 0 = g\left(x\right), \tag{5.1}$$

where $f:\ R^n \times R^n \to R^n,\ g: R^n \to R^n$ and $G = g'(u\,(x))$. The Jacobian of g and GG^T is a nonsingular matrix and full row rank.

By using the results of Theorem 2 and applying the DTM, the following series are obtained

(5.2) $$\sum\nolimits_{k=1}^{\infty} k_{n_k} t^{k^{-1}} = \sum\nolimits_{k=0}^{\infty} (G_{K_{-L}} \lambda_L);\ \sum\nolimits_{k=0}^{\infty} g_k(x) t^k = 0.$$

Equating like terms in Eq. (5.1), the algebraic recursive system is then created, given by

(5.3) $$\begin{aligned} k u_k\,(x) &= f_{k-1}\ - \sum\nolimits_{i=0}^{k-1} G^T_{K_{-1-l}} \lambda_l, \\ G_0 u_k\,(x) &= s_k, \\ s_k &= -g_k\,(x) + G_0 u_k(x). \end{aligned}$$

Multiplying Eq. (5.2) by G_0, we obtain

(5.4) $$\begin{aligned} k s_k &= G_0 f_{k-1}\ -\ G_0 \sum_{i=0}^{k-1} G^T_{K_{-1-l}} l, \\ K1 &= \left(G_0 G_0^T\right)^{-1} \left(G_0 f_{k1} - k_{sk} - G_0 \sum\nolimits_{l=0}^{k-2} G_{k-ll}\right). \end{aligned}$$

Lastly, from Eq. (5.2), $u_{k(x)}$ for $k \geq 1$, can be determined as

(5.5) $$u_k\,(x) = \frac{1}{k} f_{k-1} - \frac{1}{k} \sum\nolimits_{i=0}^{k-1} G^T_{K_{-1-l}} l.$$

Finally,

(5.6) $$u(x) = \sum\nolimits_{k=0}^{\infty} u_k t^k;\ \lambda(t) = \sum\nolimits_{k=0}^{\infty} \lambda_k t^k,$$

where k is the order of approximation of $u(x)$.

6. APPLICATION OF MSDTM

In this section, the MsDTM is applied for solving a DAE problem to illustrate the advantages and the accuracy of the MsDTM. The following example will be solved by MsDTM to illustrate the solution procedure of the proposed method in solving a quasi-linear DAEs of index-2;

(6.1) $$\begin{aligned} u'\,(t) &= (-u_2\,(t) - 2u_1\,(t)\,u_1\,(t) - 2u_2\,(t)) + 2\begin{pmatrix} u_1\,(t) \\ u_2\,(t) \end{pmatrix} \lambda, \\ g\,(u\,(t)) &= u_1^2\,(t) + u_2^2\,(t) - 1, \end{aligned}$$

and its Jacobian $G\,(u\,(t)) = (2u_1\,(t)\,, 2u_2\,(t))$ is full row rank $r = 1$. For consistent initial conditions, $u(0) = (0, 1)^T$. The exact solution for this example is $u\,(t) = (sin\ t, cos\ t)^T, u\,(t) =\ 1.$

Figure 1, Figure 2 and Figure 3 show the approximation solution for components $u_1\,(t)\,,\ u_2\,(t)$ and λ. Figure 4, Figure 5 and Figure 6 display the errors of $u_1\,(t)\,, u_2\,(t)$ and λ.

Figures 1-6 illustrate the approximate solution of MsDTM which show that it is close to the exact solution of a DAE for $u_1(t)$ and $u_2(t)$. The above results demonstrate the MSDTM's efficiency which involves only a few terms to obtain the approximate solutions. The results illustrate the accuracy of the MsDTM by applying it to solve the nonlinear DAEs.

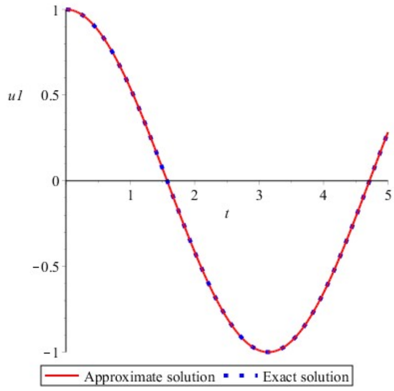

Figure 1: Approximate solution of component $u_1(t)$ with the exact solution

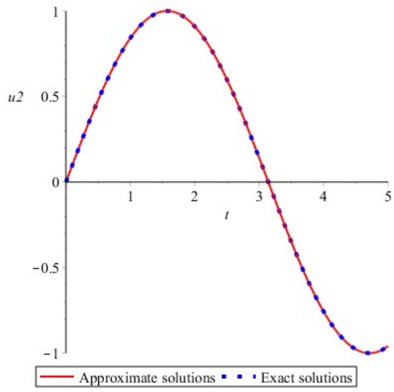

Figure 2: Approximate solution of component $u_2(t)$ with the exact solution

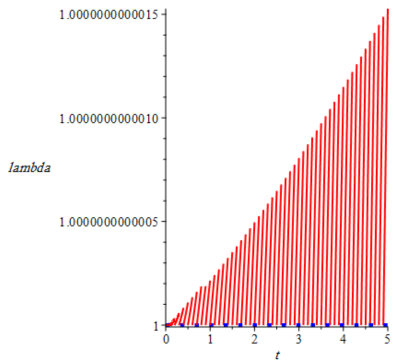

Figure 3: Approximate solution of λ with the exact solution

7. CONCLUSIONS

In this work, the MsDTM has been successfully applied to find an approximate solution for a quasi-linear DAE system of index 2. The results have demonstrated that the proposed method has efficiently solved DAEs of index 2. It is observed that MsDTM is an effective tool for solving DAEs, where the solution obtained is in good agreement with the exact solution. In

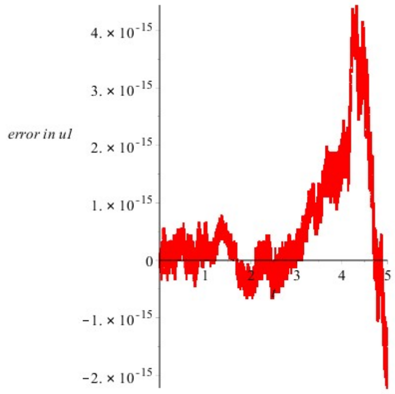

Figure 4: Error of component $u_1(t)$ using MsDTM approximation

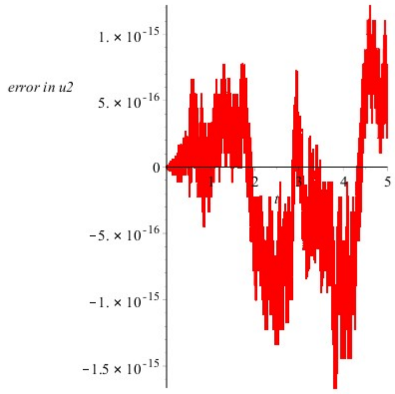

Figure 5: Error of component $u_2(t)$ using MsDTM approximation

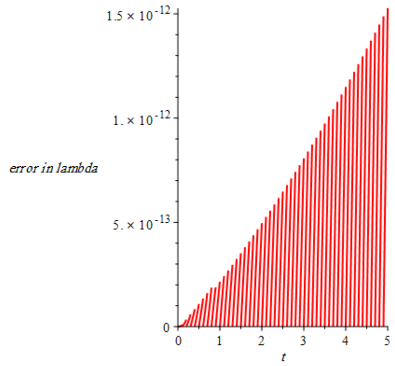

Figure 6: Error of λ using MsDTM approximation

many cases, the series solutions obtained using MsDTM can be written in an exact closed form. In addition, the proposed method reduces the computational difficulties of analytical methods or traditional methods and only involve simple calculations.

REFERENCES

[1] H. M. M. JARADAT, Numerical solution of linear differential-algebraic equations using Chebyshev polynomials, *International Mathematical Forum*, **39**, (2008), pp. 1933–1943.

[2] P. KUNKEL and V. MEHRMANN, *Differential-Algebraic Equations: Analysis and Numerical Solution*, *European Mathematical Society*, European Mathematical Society Publishing House, Switzerland, 2006.

[3] S. V. GAIDOMAK, On the numerical solution of a quasilinear algebraic-differential system, *Differential Equations*, **45**, (2009), pp. 249–256.

[4] R. MARZ and C. TISCHENDORF, Solving more general index-2 differential algebraic equations, *Computers & Mathematics with Applications*, **28**, (1994), pp. 77–105.

[5]] V. BRASEY and E. HAIRER, Half-Explicit Runge-Kutta methods for Differential-Algebraic Systems of Index 2, *SIAM Journal on Numerical Analysis*, **30**, (1993), pp. 538–552.

[6] C. W. GEAR, The Simultaneous Numerical Solution of Differential- Algebraic Equations, *IEEE Trans. Circuit Theory*, **18**, (1971), pp. 89–95.

[7] B. BENHAMMOUDA, H. VAZQUEZ-LEAL, L. HERNANDEZ-MARTINEZ, Modified Differential Transform Method for Solving the Model of Pollution for a System of Lakes, *Discrete Dynam. Nat. Soc.*, **2014**, (2014), 645726, doi:10.1155/2014/645726.

[8] Z. M. ODIBAT, C. BERTELLE, M. A. AZIZ- ALAOUI, G. H. E. DUCHAMP, A Multi-Step Differential Transform Method and Application to Non-chaotic or Chaotic Systems, *Comput. Math. Appl.*, **59**(2010), pp. 1462–1472.

[9] J. K. ZHOU, *Differential Transformation and Its Applications for Electrical Circuits*, Huazhong University Press, Wuhan, 1986.

[10] B. S. ATTILI, Numerical Treatment of Differential-Algebraic Equations with Index 2 by Reducing the Index, *AIP Conference Proceedings*, **936**, (2007), https://doi.org/10.1063/1.2790226.

[11] B. ATTILI, I. TABASH, M.DHAHERI, S. SALHI and S. HASSAN, The Differential Transform Method for the Numerical Treatment of Differential-Algebraic Equations with Index 2, *Journal of Dynamical Systems and Geometric Theories*, **7**, (2009), pp. 147–159, `https://doi.org/10.1080/1726037X.2009.10698570`.

[12] M. M. HOSSEINI, Adomian decomposition method for solution of differential-algebraic equations, *Journal of Computational & Applied Mathematics*, **197**, (2006), pp. 495–501.

[13] S. MOMANI and S. E. VEDAT, Solutions of non-linear oscillators by the modified differential transform method, *Computers & Mathematics with Applications*, **55**, (2008), pp. 833–842, `https://doi.org/10.1016/j.camwa.2007.05.009`.

[14] N. BILDIK and A. KONURALP, Two-dimensional differential transform method, Adomian's decomposition method, and variational iteration method for partial differential equations, *International Journal of Computer Mathematics*, **83**,(2006), pp. 973–987, `https://doi.org/10.1080/00207160601173407`.

[15] A. M. WAZWAZ, A new algorithm for calculating adomian polynomials for nonlinear operators, *Applied Mathematics and Computation*, **111**, (2000), pp. 33–51.

[16] A. GOKDOGAN, M. MERDAN and A. YILDIRIM, A multistage differential transformation method for approximate solution of Hantavirus infection model, *Communications in Nonlinear Science and Numerical Simulation*, **17**, (2012), pp. 1–8.

[17] E. R. EL-ZAHAR, Applications of Adaptive Multi Step Differential Transform Method to Singular Perturbation Problems Arising in Science and Engineering, *Applied Mathematics & Information*

Sciences, **9**, (2015), pp. 223–232.

[18] S. S. MOTSA, P. G. DIAMINI and M. KHUMALO, A new multistage spectral relaxation method for solving chaotic initial value systems, *Nonlinear Dynamics*, **72**, (2013), pp.265–283, DOI: 10.1007/s11071-012-0712-8.

[19] G. ADOMIAN and R. RACH, Transformation of series, *Appl. Math. Lett.*, **4**, (1991), pp. 69-7l, `https://doi.org/10.1016/0893-9659(91)90058-4`.