Aust. J. Math. Anal. Appl.
Vol. **18** (2021), No. 2, Art. 12, 21 pp.
AJMAA

ALGORITHMS FOR NONLINEAR PROBLEMS INVOLVING STRICTLY PSEUDOCONTRACTIVE MAPPINGS

MATHEW OLAJIIRE AIBINU[1], SURENDRA COLIN THAKUR[2], SIBUSISO MOYO[3]

Received 9 April, 2021; accepted 10 August, 2021; published 11 October, 2021.

[1] INSTITUTE FOR SYSTEMS SCIENCE & KZN E-SKILL COLAB, DURBAN UNIVERSITY OF TECHNOLOGY, DURBAN 4000, SOUTH AFRICA.

[1] DSI-NRF CENTRE OF EXCELLENCE IN MATHEMATICAL AND STATISTICAL SCIENCES (COE-MASS), JOHANNESBURG, SOUTH AFRICA.
moaibinu@yahoo.com, mathewa@dut.ac.za

[2] KZN E-SKILL COLAB, DURBAN UNIVERSITY OF TECHNOLOGY, DURBAN 4000, SOUTH AFRICA.
thakur@dut.ac.za

[3]INSTITUTE FOR SYSTEMS SCIENCE & OFFICE OF THE DVC RESEARCH, INNOVATION & ENGAGEMENT MILENA COURT, DURBAN UNIVERSITY OF TECHNOLOGY,, DURBAN 4000, SOUTH AFRICA.
dvcrie@dut.ac.za

ABSTRACT. The puzzles in approximating a fixed point of nonlinear problems involving the class of strictly pseudocontractive mappings are conquered in this paper through viscosity implicit rules. Using generalized contraction mappings, a new viscosity iterative algorithm which is implicit in nature is proposed and analysed in Banach spaces for the class of strictly pseudocontractive mappings. The computations and analysis which are used in the proposed scheme are easy to follow and this gives rooms for a broad application of the scheme. It is obtained that the proposed iterative algorithm converges strongly to a fixed point of a μ-strictly pseudocontractive mapping which also solves a variational inequality problem. The result is also shown to hold for finite family of strictly pseudocontractive mappings. A numerical example is given to show the skillfulness of the proposed scheme and its implementation.

Key words and phrases: Implicit rule; Strictly pseudocontractive; Generalized contraction.

2010 *Mathematics Subject Classification.* Primary 47H06, 47J05. Secondary 47J25, 47H10, 47H09.

ISSN (electronic): 1449-5910

The first author acknowledges with thanks the postdoctoral fellowship and financial support from the DSI-NRF Center of Excellence in Mathematical and Statistical Sciences (CoE-MaSS). Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS..

1. INTRODUCTION

Let K be a nonempty, closed and convex subset of a real Banach space E. $T : K \to K$ is said to be a μ-strictly pseudocontractive mapping if there exists a fixed constant $\mu \in (0,1)$ such that

$$\langle T(u) - T(v), j(u-v)\rangle \leq \|u-v\|^2 - \mu\|(I-T)u - (I-T)v\|^2, \tag{1.1}$$

for some $j(u-v) \in J(u-v)$ and for every $u, v \in K$, where I is the identity operator (See e.g [1]). Equivalence of (1.1) in a restated form is given by

$$\langle (I-T)(u) - (I-T)(v), j(u-v)\rangle \geq \mu\|(I-T)u - (I-T)v\|^2.$$

A recent research interest to many authors is the viscosity implicit iterative algorithms for finding a common element of the set of fixed points for nonlinear operators and also the set of solutions of variational inequality problems (See [2, 3, 4, 5, 6, 7, 8, 9] and the references therein). Following the ideas of Attouch [10], in 2000, Moudafi [11] introduced the viscosity approximation method for nonexpansive mapping in Hilbert spaces. Refinements in Hilbert spaces and extensions to Banach spaces were obtained by Xu [12]. Recently, Xu et al. [7] introduced the implicit midpoint procedure (1.2).

$$x_{n+1} = \theta_n f(x_n) + (1-\theta_n)T\left(\frac{x_n + x_{n+1}}{2}\right), n \in \mathbb{N}, \tag{1.2}$$

where $\{\theta_n\}_{n=1}^{\infty} \subset (0,1)$, f is a contraction on K and the nonexpansive mapping $T : K \to K$ is also defined on K, which is a nonempty closed convex subset of a real Hilbert space H. It was established that the implicit midpoint sequence (1.2) converges strongly to a fixed point p of a nonexpansive mapping T, which also solves the variational inequality

$$\langle (I-f)p, x-p\rangle \geq 0,\ \forall\, x \in F(T). \tag{1.3}$$

Yao et al. [9] extended the work of Xu et al. [7] and studied the implicit midpoint sequence

$$x_{n+1} = \theta_n^1 f(x_n) + \theta_n^2 x_n + \theta_n^3 T\left(\frac{x_n + x_{n+1}}{2}\right),\ n \in \mathbb{N}, \tag{1.4}$$

where $\left\{\theta_n^1\right\}_{n=1}^{\infty} \subset (0,1)$, $\left\{\theta_n^2\right\}_{n=1}^{\infty} \subset [0,1)$ and $\left\{\theta_n^3\right\}_{n=1}^{\infty} \subset (0,1)$ are real sequences satisfying $\theta_n^1 + \theta_n^2 + \theta_n^3 = 1$ for all $n \in \mathbb{N}$. Under certain conditions on the parameters and denoting the set of fixed points of T by $F(T)$, it was shown that (1.4) converges strongly to $p = P_{F(T)}f(p)$. In other words, the implicit midpoint sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.4) converges in norm to a fixed point p of a nonexpansive mapping T, which is also the unique solution of the variational inequality (1.3). Choosing $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$, Ke and Ma [8] worked further in Hilbert spaces and extended the results of Xu et al. [7] and Yao et al. [9] by proposing the following two viscosity implicit iterative algorithms:

$$x_{n+1} = \theta_n f(x_n) + (1-\theta_n)T\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right),\ n \in \mathbb{N},$$

and

$$x_{n+1} = \theta_n^1 f(x_n) + \theta_n^2 x_n + \theta_n^3 T\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right),\ n \in \mathbb{N}. \tag{1.5}$$

Yan et al. [13] established the main results of Ke and Ma [8] in uniformly smooth Banach spaces. The sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.5) is proved to converge strongly to a fixed point p of a nonexpansive mapping T, which solves the variational inequality

$$\langle (I-f)p, J(x-p)\rangle \geq 0,\ \text{for all } x \in F(T),$$

where J is a normalized duality mapping. Other mappings which are of the same class as nonexpansive mappings but which are more general and with more broad applications are known, e.g asymptotically nonexpansive and pseudocontractive mappings. Some recent studies on the application of implicit procedures for asymptotically nonexpansive mappings include Zhao et al. [14], Xiong and Lan [4], Yan and Cai [6] and Aibinu et al. [15]. Also, there are reports on approximating a fixed point of pseudocontractive mappings by the implicit procedures. Liou [16] used implicit and explicit iterations to compute the fixed points of strictly pseudocontractive mappings in Hilbert spaces. Song and Pei [2] studied semi-implicit midpoint rule in Hilbert spaces. Pertinent studies on implicit iterative algorithms in Banach spaces for pseudocontractive mappings include Argyros et al. [17], Cheng and Su [18] as well as Saluja and Nashine [19].

Motivated by the previous works, the goal of this study is to seek for a way of improving on the existing results in this direction. A new viscosity iterative algorithm which is implicit in nature is proposed and analysed in Banach spaces for the class of strictly pseudocontractive mappings. Precisely, for a nonempty closed convex subset K of a uniformly smooth Banach space E and for real sequences $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$, $\left\{\{\theta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0,1]$ and $\left\{\{\beta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0,1]$ with $\beta_n^1, \beta_n^3 \neq 0$ such that $\displaystyle\sum_{i=1}^{3} \theta_n^i = 1$ and $\displaystyle\sum_{i=1}^{3} \beta_n^i = 1$, a new viscosity iterative algorithm is introduced from an arbitrary $x_1 \in K$ as follows

$$x_{n+1} = \theta_n^1 f(x_n) + \theta_n^2 x_n + \theta_n^3 S_n(\delta_n x_n + (1-\delta_n)x_{n+1}), \tag{1.6}$$

where $S_n x = \beta_n^1 Q(x) + \beta_n^2 x + \beta_n^3 T(x)$, $f : K \to K$ is a generalized contraction, $Q : K \to K$ is a contraction and $T : K \to K$ is a μ-strictly pseudocontractive mapping. The iterative sequence which is given by (1.6) generalizes the existing schemes. The computations and analysis which are used in this proposed scheme are easy to follow and this gives rooms for a broad application of the scheme. The strong convergence of the proposed sequence to a fixed point p of T is obtained and it is shown to be a solution to some variational inequality problems. A numerical example is given to show the skillfulness of the proposed scheme and its implementation.

2. PRELIMINARIES

Let K be a nonempty closed convex subset of a Banach space E and T a self-mapping on K. We shall denote the set of fixed points of T by $F(T) := \{p \in K : Tp = p\}$. Recall that $T : K \to K$ is said to be L-Lipschitzian if for all $x, y \in K$, there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|.$$

If $0 < L < 1$, then T is a contraction and it called nonexpansive mapping if $L = 1$.

Let (X, d) be a metric space and K a subset of X. A mapping $f : K \to K$ is said to be a Meir-Keeler contraction if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for each $x, y \in K$, with $\epsilon \leq d(x,y) < \epsilon + \delta$, we have $d(f(x), f(y)) < \epsilon$. A mapping $f : E \to E$ is called a (ψ, L)*-contraction* if $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an L-function and $d(f(x), f(y)) < \psi(d(x,y))$, for all $x, y \in E,\ x \neq y$.

We have the following interesting results about the Meir-Keeler contraction.

Proposition 2.1. *Let E be a Banach space, K a convex subset of E and $f : K \to K$ a Meir-Keeler contraction. Then $\forall \epsilon > 0$, there exists $c \in (0,1)$ such that*

$$\|f(x) - f(y)\| \leq c\|x - y\|$$

for all $x, y \in K$ with $\|x - y\| \geq \epsilon$ (See [20]).

We shall also need the following Lemmas in the sequel.

Lemma 2.2. *Let K be a nonempty closed and convex subset of a uniformly smooth Banach space E. Let $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$ and $f : K \to K$ be a generalized contraction mapping. Then $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$ for $t \in (0,1)$, converges strongly to $p \in F(T)$, which solves the variational inequality (See* [21]*):*

$$\langle f(p) - p, J(z-p)\rangle \leq 0, \; \forall \, z \in F(T).$$

Lemma 2.3. *Let K be a nonempty closed and convex subset of a uniformly smooth Banach space E. Let $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$ and $f : K \to K$ be a generalized contraction mapping. Assume that $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$ for $t \in (0,1)$, converges strongly to $p \in F(T)$ as $t \to 0$. Suppose that $\{x_n\}$ is a bounded sequence such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ (See* [21]*). Then*

$$\limsup_{n\to\infty} \langle f(p) - p, J(x_n - p)\rangle \leq 0.$$

Lemma 2.4. *Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in a Banach space E and $\{t_n\}$ be a sequence in $[0,1]$ with $0 < \liminf\limits_{n\to\infty} t_n \leq \limsup\limits_{n\to\infty} t_n < 1$. Suppose that $u_{n+1} = (1-t_n)u_n + t_n v_n$ for all $n \geq 0$ and $\limsup\limits_{n\to\infty} (\|u_{n+1} - u_n\| - \|v_{n+1} - v_n\|) \leq 0$. Then $\lim\limits_{n\to\infty} \|u_n - v_n\| = 0$ (See* [22]*).*

Lemma 2.5. *Assume $\{a_n\}$ is a sequence of nonnegative real sequence such that*

$$a_{n+1} = (1-\sigma_n)a_n + \sigma_n \delta_n, \; n > 0,$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence such that

(i) $\sum\limits_{n=1}^{\infty} \sigma_n = \infty$,

(ii) $\limsup\limits_{n\to\infty} \delta_n \leq 0$ *or* $\sum\limits_{n=1}^{\infty} \sigma_n |\delta_n| < \infty$.

Then, $\lim\limits_{n\to\infty} a_n = 0$ (See [23]*).*

Proposition 2.6. *Let K be a nonempty convex subset of a Banach space E, $T : K \to K$ a nonexpansive mapping and $f : K \to K$ a Meir-Keeler contraction. Then Tf and $fT : K \to K$ are Meir-Keeler contractions (See* [24]*).*

In this paper, the generalized contraction mappings will refer to Meir-Keeler or (ψ, L)-contraction contractions. It is assumed from the definition of (ψ, L)-contraction that the L-function is continuous, strictly increasing and $\lim\limits_{t\to\infty} \phi(t) = \infty$, where $\phi(t) = t - \psi(t)$ for all $t \in \mathbb{R}^+$. Whenever there is no confusion, $\phi(t)$ and $\psi(t)$ will be written as $\phi \, t$ and $\psi \, t$, respectively.

3. MAIN RESULTS

The sequence (1.6) is well defined in uniformly smooth Banach spaces. Indeed, let $c_Q \in [0,1]$ be the contractive constant of Q. Firstly, it is shown that for all $y, z \in K$, $\|S_n(y) - S_n(z)\| \leq \|y - z\|$.

$$
\begin{aligned}
& \langle S_n(y) - S_n(z), J(y-z)\rangle \\
= \ & \left\langle \beta_n^1 Q(y) + \beta_n^2 y + \beta_n^3 T(y) - \beta_n^1 Q(z) - \beta_n^2 z - \beta_n^3 T(z), J(y-z)\right\rangle \\
= \ & \left\langle \beta_n^1 (Q(y) - Q(z)) + \beta_n^2 (y-z) + \beta_n^3 (T(y) - T(z)), J(y-z)\right\rangle \\
= \ & \beta_n^1 \langle Q(y) - Q(z),\ J(y-z)\rangle + \beta_n^2 \langle y - z,\ J(y-z)\rangle \\
& + \beta_n^3 \langle T(y) - T(z),\ J(y-z)\rangle \\
\leq \ & \beta_n^1 \langle Q(y) - Q(z),\ J(y-z)\rangle + \beta_n^2 \langle y - z,\ J(y-z)\rangle \\
& + \beta_n^3 \left(\|y-z\|^2 - \mu\|(I-T)y - (I-T)z\|^2\right),
\end{aligned}
$$

by the μ-strictly pseudocontractive mapping property. Therefore,

$$
\begin{aligned}
\|S_n(y) - S_n(z)\|\|y-z\| \ & \leq \ \beta_n^1 \|Q(y) - Q(z)\|\|y-z\| + \beta_n^2 \|y-z\|^2 \\
& \quad + \beta_n^3 \left(\|y-z\|^2 - \mu\|(I-T)y - (I-T)z\|^2\right) \\
& \leq \ \beta_n^1 c_Q \|y-z\|^2 + \beta_n^2 \|y-z\|^2 \\
& \quad + (1 - \beta_n^1 - \beta_n^2)\left(\|y-z\|^2 - \mu\|(I-T)y - (I-T)z\|^2\right) \\
& \leq \ \beta_n^1 \|y-z\|^2 + \beta_n^2 \|y-z\|^2 \text{ (since } c_Q \in [0,1]) \\
& \quad + (1 - \beta_n^1 - \beta_n^2)\left(\|y-z\|^2 - \mu\|(I-T)y - (I-T)z\|^2\right) \\
& = \ \|y-z\|^2 - (1 - \beta_n^1 - \beta_n^2)\mu\|(I-T)y - (I-T)z\|^2 \\
& \leq \ \|y-z\|^2.
\end{aligned}
$$

Thus,

$$
\|S_n(y) - S_n(z)\| \ \leq \ \|y-z\|. \tag{3.1}
$$

Next is to show that for all $v \in K$, the mapping defined by

$$
x \mapsto T_v(x) := \theta_n^1 f(v) + \theta_n^2 v + \theta_n^3 S_n(\delta_n v + (1-\delta_n)x)
$$

for all $x \in K$ is a contraction with a contractive constant $(1-\epsilon) =: \delta \in (0,1)$. Clearly, for all $y, z \in K$,

$$
\begin{aligned}
\|T_v(y) - T_v(z)\| \ & = \ \theta_n^3 \left\|S_n\left(\delta_n v + (1-\delta_n)y\right) - S_n\left(\delta_n v + (1-\delta_n)z\right)\right\| \\
& \leq \ \theta_n^3 \left\|(\delta_n v + (1-\delta_n)y) - (\delta_n v + (1-\delta_n)z)\right\| \\
& \leq \ \theta_n^3 (1-\delta_n)\|y-z\| \\
& \leq \ (1-\delta_n)\|y-z\| \\
& \leq \ (1-\epsilon)\|y-z\| \\
& = \ \delta\|y-z\|.
\end{aligned} \tag{3.2}
$$

Thus, (1.6) is well defined since T_v is a contraction and by Banach contraction principile, T_v a fixed point. Observe that for each $n \in \mathbb{N},\ x \in F(T) \cap F(Q) \Rightarrow x \in F(S_n)$. So, $F(T) \cap F(Q) \subset F(S_n) \neq \emptyset$. Indeed, suppose $x \in F(T) \cap F(Q)$, then

$$
\begin{aligned}
S_n x \ & = \ \beta_n^1 Q(x) + \beta_n^2 x + \beta_n^3 T(x) \\
& = \ \beta_n^1 x + \beta_n^2 x + \beta_n^3 x \\
& = \ (\beta_n^1 + \beta_n^2 + \beta_n^3)x \\
& = \ x.
\end{aligned} \tag{3.3}
$$

Thus, $x \in F(S_n)$.

Next, the proof of the following lemmas which are useful in establishing our main result are given.

Lemma 3.1. *Let E be a uniformly smooth Banach space and K be a nonempty closed convex subset of E. Let $T : K \to K$ be a μ-strictly pseudocontractive mapping and suppose that $f : K \to K$ is a generalized contraction and $Q : K \to K$ is a contraction with $F(T) \cap F(Q) \neq \emptyset$. $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$, $\left\{\{\theta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0,1]$ and $\left\{\{\beta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0,1]$ with $\beta_n^1, \beta_n^3 \neq 0$ are real sequences. For an arbitrary $x_1 \in K$, the iterative sequence which is by (1.6) is bounded.*

Proof. The sequence $\{x_n\}_{n=1}^{\infty}$ is shown to be bounded. Let $z_n := \delta_n x_n + (1-\delta_n)x_{n+1}$ and observe that $p \in F(T) \cap F(Q)$ implies that $p \in F(S_n)$ (See (3.3)). Therefore by (3.1),

$$\|S_n z_n - p\| \leq \|z_n - p\|.$$

Recall that $\phi(t) := t - \psi(t)$ for all $t \in \mathbb{R}^+$. Then,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\theta_n^1 f(x_n) + \theta_n^2 x_n + \theta_n^3 S_n z_n - p\| \\
&= ||\theta_n^1 \left(f(x_n) - f(p)\right) + \theta_n^1 \left(f(p) - p\right) + \theta_n^2 (x_n - p) + \theta_n^3 (S_n z_n - p)|| \\
&\leq \theta_n^1 \|f(x_n) - f(p)\| + \theta_n^1 \|f(p) - p\| + \theta_n^2 \|x_n - p\| + \theta_n^3 \|S_n z_n - p\| \\
&\leq \theta_n^1 \psi \|x_n - p\| + \theta_n^1 \|f(p) - p\| + \theta_n^2 \|x_n - p\| + \theta_n^3 \|z_n - p\| \\
&\leq \theta_n^1 \psi \|x_n - p\| + \theta_n^1 \|f(p) - p\| + \theta_n^2 \|x_n - p\| \\
&\quad + \theta_n^3 \left(\delta_n \|x_n - p\| + (1 - \delta_{n+1}) \|x_{n+1} - p\|\right) \\
&\leq \theta_n^1 \psi \|x_n - p\| + \theta_n^1 \|f(p) - p\| + \theta_n^2 \|x_n - p\| \\
&\quad + \theta_n^3 \delta_n \|x_n - p\| + \theta_n^3 (1 - \delta_{n+1}) \|x_{n+1} - p\|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\left(1 - \theta_n^3(1-\delta_n)\right) \|x_{n+1} - p\| &\leq \left(\theta_n^1 \psi + \theta_n^2 + \theta_n^3 \delta_n\right) \|x_n - p\| + \theta_n^1 \|f(p) - p\| \\
&= \left(\theta_n^1 \psi + (1 - \theta_n^1 - \theta_n^3) + \theta_n^3 \delta_n\right) \|x_n - p\| + \theta_n^1 \|f(p) - p\| \\
&= \left(1 - \theta_n^3(1-\delta_n) - \theta_n^1(1-\psi)\right) \|x_n - p\| + \theta_n^1 \|f(p) - p\| \\
&= \left(1 - \theta_n^3(1-\delta_n) - \theta_n^1 \phi\right) \|x_n - p\| + \theta_n^1 \|f(p) - p\|.
\end{aligned}$$

Observe that $1 - \theta_n^3(1-\delta_n) > 0$ since $\left\{\{\theta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3}$, $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$. Therefore, it leads to

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \frac{1 - \theta_n^3(1-\delta_n) - \theta_n^1 \phi}{1 - \theta_n^k(1-\delta_n)} \|x_n - p\| \\
&\quad + \frac{\theta_n^1}{1 - \theta_n^3(1-\delta_n)} \|f(p) - p\| \qquad (3.4) \\
&= \left(1 - \frac{\theta_n^1 \phi}{1 - \theta_n^3(1-\delta_n)}\right) \|x_n - p\| + \frac{\theta_n^1}{1 - \theta_n^3(1-\delta_n)} \|f(p) - p\| \\
&= \left(1 - \frac{\theta_n^1 \phi}{1 - \theta_n^3(1-\delta_n)}\right) \|x_n - p\| + \frac{\theta_n^1 \phi}{1 - \theta_n^3(1-\delta_n)} . \, \phi^{-1} \|f(p) - p\| \\
&\leq \max\left\{\|x_n - p\|,\ \phi^{-1} \|f(p) - p\|\right\}.
\end{aligned}$$

Thus, by the induction, it is obtained that

$$\|x_{n+1} - p\| \leq \max\left\{\|x_1 - p\|,\ \phi^{-1} \|f(p) - p\|\right\}.$$

This implies that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded and hence $\{S_n(\delta_n x_n + (1-\delta_n)x_{n+1})\}_{n=1}^{\infty}$ and $\{f(x_n)\}_{n=1}^{\infty}$ are also bounded. Certainly, for $p \in F(T) \cap F(Q)$,

$$\begin{aligned} \|S_n(\delta_n x_n + (1-\delta_n)x_{n+1})\| &= \|S_n(\delta_n x_n + (1-\delta_n)x_{n+1}) - p + p\| \\ &\leq \|S_n(\delta_n x_n + (1-\delta_n)x_{n+1}) - S_n p\| + \|p\| \\ &\leq \|\delta_n x_n + (1-\delta_n)x_{n+1} - p\| + \|p\| \\ &\leq \delta_n\|x_n - p\| + (1-\delta_n)\|x_{n+1} - p\| + ||p|| \\ &\leq \max\left\{||x_1 - p||,\ \phi^{-1}||f(p) - p||\right\} + ||p|| \text{ (by induction).} \end{aligned}$$

The boundedness of $\{S_n\}_{n=1}^{\infty}$ implies that Q and T are also bounded since S_n is defined in term of Q and T. Moreover,

$$\begin{aligned} \|f(x_n)\| &= \|f(x_n) - f(p) + f(p)\| \leq \psi\|x_n - p\| + \|f(p)\| \\ &\leq \max\left\{\psi||x_1 - p||,\ \psi\phi^{-1}||f(p) - p||\right\} + ||f(p)|| \text{ (by induction).} \end{aligned}$$

∎

Lemma 3.2. *Let E be a uniformly smooth Banach space and K a nonempty closed convex subset of E. Let $Q : K \to K$ be a contraction, $T : K \to K$ a μ-strictly pseudocontractive mapping and $\{\delta_n\}_{n=1}^{\infty}$ is a real sequences in $(0,1)$. Define $z_n := \delta_n x_n + (1-\delta_n)x_{n+1}$ and let $M_1 = \max\left\{\sup_n \|T(z_n) - z_n\|,\ \sup_n \|Q(z_n) - z_n\|\right\}$. Then*

$$\begin{aligned} \|S_{n+1}z_{n+1} - S_n z_n\| &\leq \delta_n\|x_{n+1} - x_n\| + (1-\delta_{n+1})\|x_{n+2} - x_{n+1}\| \\ &\quad + \left(|\beta_{n+1}^1 - \beta_n^1| + |\beta_{n+1}^3 - \beta_n^3|\right) M_1 \textit{ for all } n \in \mathbb{N}. \end{aligned}$$

Proof. It is known that $\{z_n\}_{n=1}^{\infty}$ is bounded since $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence. Notice that

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|\delta_{n+1}x_{n+1} + (1-\delta_{n+1})x_{n+2} - (\delta_n x_n + (1-\delta_n)x_{n+1})\| \\ &= \|\delta_{n+1}x_{n+1} + (1-\delta_{n+1})x_{n+2} - \delta_n x_n - (1-\delta_n)x_{n+1}\| \\ &= \|(x_{n+2} - x_{n+1}) - \delta_{n+1}(x_{n+2} - x_{n+1}) + \delta_n(x_{n+1} - x_n)\| \\ &= \|\delta_n(x_{n+1} - x_n) + (1-\delta_{n+1})(x_{n+2} - x_{n+1})\| \\ &\leq \delta_n\|x_{n+1} - x_n\| + (1-\delta_{n+1})\|x_{n+2} - x_{n+1}\|. \end{aligned} \tag{3.5}$$

Then,

$$\begin{aligned} \|S_{n+1}z_{n+1} - S_n z_n\| &= \|S_{n+1}z_{n+1} - S_{n+1}z_n + S_{n+1}z_n - S_n z_n\| \\ &\leq \|z_{n+1} - z_n\| + \|\beta_{n+1}^1 Q(z_n) + \beta_{n+1}^2 z_n + \beta_{n+1}^3 T(z_n) \\ &\quad - \beta_n^1 Q(z_n) - \beta_n^2 z_n - \beta_n^3 T(z_n)\| \\ &= \|z_{n+1} - z_n\| + \|\beta_{n+1}^1 Q(z_n) + (1 - \beta_{n+1}^1 - \beta_{n+1}^3)z_n \\ &\quad + \beta_{n+1}^3 T(z_n) - \beta_n^1 Q(z_n) - (1 - \beta_n^1 - \beta_n^3)z_n - \beta_n^3 T(z_n)\| \\ &= \|z_{n+1} - z_n\| + \|\beta_{n+1}^1(Q(z_n) - z_n) + z_n \\ &\quad + \beta_{n+1}^3(T(z_n) - z_n) - \beta_n^1(Q(z_n) - z_n) \\ &\quad - z_n - \beta_n^3(T(z_n) - z_n)\| \\ &= \|z_{n+1} - z_n\| + \|(\beta_{n+1}^1 - \beta_n^1)(Q(z_n) - z_n) \\ &\quad + (\beta_{n+1}^3 - \beta_n^3)(T(z_n) - z_n)\| \\ &\leq \delta_n\|x_{n+1} - x_n\| + (1-\delta_{n+1})\|x_{n+2} - x_{n+1}\| \\ &\quad + \left(|\beta_{n+1}^1 - \beta_n^1| + |\beta_{n+1}^3 - \beta_n^3|\right) M_1. \end{aligned} \tag{3.6}$$

∎

Theorem 3.3. *Let E be a uniformly smooth Banach space and K a nonempty closed convex subset of E. Let T be a μ-strictly pseudocontractive self-mapping defined on K while $f : K \to K$ is a generalized contraction and Q is a contraction defined on K with $F(T) \cap F(Q) \neq \emptyset$. Suppose the real sequences $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$, $\left\{\{\theta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0,1]$ and $\left\{\{\beta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0,1]$ with $\beta_n^1, \beta_n^3 \neq 0$ satisfy the following conditions:*

(i) $\displaystyle\sum_{i=1}^{3} \theta_n^i = 1,\ \sum_{n=1}^{\infty} \theta_n^1 = \infty,$

(ii) $\displaystyle\lim_{n\to\infty} |\theta_{n+1}^2 - \theta_n^2| = 0,\ 0 < \liminf_{n\to\infty} \theta_n^2 \leq \limsup_{n\to\infty} \theta_n^2 < 1,$

(iii) $\displaystyle\sum_{i=1}^{3} \beta_n^i = 1,\ \lim_{n\to\infty} |\beta_{n+1}^1 - \beta_n^1| = 0,\ \lim_{n\to\infty} |\beta_{n+1}^3 - \beta_n^3| = 0,$

(iv) $0 < \epsilon \leq \delta_n \leq \delta_{n+1} < 1$ *for all* $n \in \mathbb{N}$.

Then, for an arbitrary $x_1 \in K$, the iterative sequence $\{x_n\}_{n=1}^{\infty}$ defined by (1.6) converges strongly to a fixed point p of T, which solves the variational inequality

$$\langle (I-f)p, J(x-p)\rangle \geq 0,\ \textit{for all } x \in F(T)\cap F(Q). \tag{3.7}$$

Proof. Observe that one can write the iterative sequence (1.6) as:

$$\begin{aligned} x_{n+1} &= \theta_n^1 f(x_n) + \theta_n^2 x_n + \theta_n^3 S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right) \\ &= \theta_n^2 x_n + (1-\theta_n^2)\frac{\theta_n^1 f(x_n) + \theta_n^3 S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)}{1-\theta_n^2}. \end{aligned}$$

Since $\displaystyle\sum_{i=1}^{3}\theta_n^i = 1$ by condition (i), it could be obtained that

$$\begin{aligned} x_{n+1} &= (1-\theta_n^1-\theta_n^3)x_n + (\theta_n^1+\theta_n^3)\frac{\theta_n^1 f(x_n) + \theta_n^3 S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)}{1-\theta_n^2} \\ &= (1-\theta_n^1-\theta_n^3)x_n + (\theta_n^1+\theta_n^3)w_n, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} w_n &:= \frac{\theta_n^1 f(x_n) + \theta_n^3 S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)}{1-\theta_n^2} \\ &= \frac{\theta_n^1}{1-\theta_n^2} f(x_n) + \frac{\theta_n^3}{1-\theta_n^2} S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right) \\ &= \frac{\theta_n^1}{\theta_n^1+\theta_n^3} f(x_n) + \frac{\theta_n^3}{\theta_n^1+\theta_n^3} S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right),\ n \in \mathbb{N}. \end{aligned} \tag{3.9}$$

Notice that $\{x_n\}_{n=1}^{\infty}$, $\{f(x_n)\}_{n=1}^{\infty}$ and $\{T\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)\}_{n=1}^{\infty}$ are bounded sequences. Furthermore, since the $\limsup_{n\to\infty} \theta_n^2 < 1$ by the condition (iii), there exists $n_0 \in \mathbb{N}$ and $\eta < 1$ such that

$$1-\theta_n^2 > 1-\eta\ \forall\, n \geq n_0. \tag{3.10}$$

The consequence of (3.9) and (3.10) is that $\{w_n\}_{n=1}^{\infty}$ is bounded.

Next is to show that $\lim\limits_{n\to\infty}||w_n - x_n|| = 0$.
The first step is to show that $\limsup\limits_{n\to\infty}(||w_{n+1} - w_n|| - ||x_{n+1} - x_n||) \le 0$. Observe that

$$
\begin{aligned}
w_{n+1} - w_n &= \frac{\theta^1_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}} f(x_{n+1}) + \frac{\theta^3_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}} S_{n+1}\left(\delta_{n+1}x_{n+1} + (1-\delta_{n+1})x_{n+2}\right) \\
&\quad - \left(\frac{\theta^1_n}{\theta^1_n + \theta^3_n} f(x_n) + \frac{\theta^3_n}{\theta^1_n + \theta^3_n} S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)\right) \\
&= \frac{\theta^1_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}} (f(x_{n+1}) - f(x_n)) + \left(\frac{\theta^1_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}} - \frac{\theta^1_n}{\theta^1_n + \theta^3_n}\right) f(x_n) \\
&\quad + \frac{\theta^3_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}} \Bigg(S_{n+1}\left(\delta_{n+1}x_{n+1} + (1-\delta_{n+1})x_{n+2}\right) \\
&\quad - S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)\Bigg) \\
&\quad + \left(\frac{\theta^3_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}} - \frac{\theta^3_n}{\theta^1_n + \theta^3_n}\right) S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right) \\
&= \frac{\theta^1_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}} (f(x_{n+1}) - f(x_n)) + \left(\frac{\theta^1_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}} - \frac{\theta^1_n}{\theta^1_n + \theta^3_n}\right) f(x_n) \\
&\quad + \frac{\theta^3_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}} \Bigg(S_{n+1}\left(\delta_{n+1}x_{n+1} + (1-\delta_{n+1})x_{n+2}\right) \\
&\quad - S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)\Bigg) \\
&\quad + \left(\frac{\theta^1_{n+1} + \theta^3_{n+1} - \theta^1_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}} - \frac{\theta^1_n + \theta^3_n - \theta^1_n}{\theta^1_n + \theta^3_n}\right) S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right) \\
&= \frac{\theta^1_{n+1}}{\theta^1_{n+1} + \theta^k_{n+1}} (f(x_{n+1}) - f(x_n)) + \left(\frac{\theta^1_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}} - \frac{\theta^1_n}{\theta^1_n + \theta^3_n}\right) f(x_n) \\
&\quad + \frac{\theta^3_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}} \Bigg(S_{n+1}\left(\delta_{n+1}x_{n+1} + (1-\delta_{n+1})x_{n+2}\right) \\
&\quad - S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)\Bigg) \\
&\quad + \left(\frac{\theta^1_n}{\theta^1_n + \theta^3_n} - \frac{\theta^1_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}}\right) S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right) \\
&= \frac{\theta^1_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}} (f(x_{n+1}) - f(x_n)) \\
&\quad + \left(\frac{\theta^1_n}{\theta^1_n + \theta^3_n} - \frac{\theta^1_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}}\right) \left(S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right) - f(x_n)\right) \\
&\quad + \frac{\theta^3_{n+1}}{\theta^1_{n+1} + \theta^3_{n+1}} \Bigg(S_{n+1}\left(\delta_{n+1}x_{n+1} + (1-\delta_{n+1})x_{n+2}\right) \\
&\quad - S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)\Bigg).
\end{aligned}
$$

Therefore,

$$\begin{aligned} ||w_{n+1}-w_n|| &\leq \frac{\theta^1_{n+1}\psi}{\theta^1_{n+1}+\theta^3_{n+1}}\|x_{n+1}-x_n\| + \left|\frac{\theta^1_n}{\theta^1_n+\theta^3_n} - \frac{\theta^1_{n+1}}{\theta^1_{n+1}+\theta^3_{n+1}}\right| \\ &\quad \times\|S_n\left(\delta_n x_n+(1-\delta_n)x_{n+1}\right)-f(x_n)\| + \frac{\theta^3_{n+1}}{\theta^1_{n+1}+\theta^3_{n+1}} \\ &\quad \times\|S_n\left(\delta_{n+1}x_{n+1}+(1-\delta_{n+1})x_{n+2}\right)-S_n\left(\delta_n x_n+(1-\delta_n)x_{n+1}\right)\|. \end{aligned}$$

Applying Lemma 3.2 leads to

$$\begin{aligned} ||w_{n+1}-w_n|| &\leq \frac{\theta^1_{n+1}\psi}{\theta^1_{n+1}+\theta^3_{n+1}}\|x_{n+1}-x_n\| \\ &\quad + \left|\frac{\theta^1_n}{\theta^1_n+\theta^3_n} - \frac{\theta^1_{n+1}}{\theta^1_{n+1}+\theta^3_{n+1}}\right| \|S_n\left(\delta_n x_n+(1-\delta_n)x_{n+1}\right)-f(x_n)\| \\ &\quad +\frac{\theta^3_{n+1}}{\theta^1_{n+1}+\theta^3_{n+1}}\left[\,\delta_n\|x_{n+1}-x_n\|+(1-\delta_{n+1})\|x_{n+2}-x_{n+1}\|\right. \\ &\quad \left.+\left(|\beta^1_{n+1}-\beta^1_n|+|\beta^3_{n+1}-\beta^3_n|\right)M_1\,\right] \\ &= \frac{\theta^1_{n+1}\psi+\theta^3_{n+1}\delta_n}{\theta^1_{n+1}+\theta^3_{n+1}}\|x_{n+1}-x_n\| + \frac{\theta^3_{n+1}(1-\delta_{n+1})}{\theta^1_{n+1}+\theta^3_{n+1}}\|x_{n+2}-x_{n+1}\| \\ &\quad + \left|\frac{\theta^1_n}{\theta^1_n+\theta^3_n} - \frac{\theta^1_{n+1}}{\theta^1_{n+1}+\theta^3_{n+1}}\right| \|S_n\left(\delta_n x_n+(1-\delta_n)x_{n+1}\right)-f(x_n)\| \\ &\quad +\frac{\theta^1_{n+1}\psi}{\theta^1_{n+1}+\theta^3_{n+1}}\left(|\beta^1_{n+1}-\beta^1_n|+|\beta^3_{n+1}-\beta^3_n|\right)M_1. \end{aligned} \tag{3.11}$$

To evaluate $||x_{n+2}-x_{n+1}||$, let $M^1 := \sup_n\left\{\|x_n - S_n\left(\delta_n x_n+(1-\delta_n)x_{n+1}\right)\|\right\}$, $M^2 := \sup_n\left\{\|S_n\left(\delta_n x_n+(1-\delta_n)x_{n+1}\right)-f(x_n)\|\right\}$ and $M_2 =: \max\left\{M^1, M^2\right\}$.

$$\begin{aligned} x_{n+2}-x_{n+1} &= \theta^1_{n+1}f(x_{n+1})+\theta^2_{n+1}x_{n+1}+\theta^3_{n+1}S_{n+1}\left(\delta_{n+1}x_{n+1}+(1-\delta_{n+1})x_{n+2}\right) \\ &\quad -\left(\theta^1_n f(x_n)+\theta^2_n x_n+\theta^3_n S_n\left(\delta_n x_n+(1-\delta_n)x_{n+1}\right)\right) \\ &= \theta^1_{n+1}\left(f(x_{n+1})-f(x_n)\right)+(\theta^1_{n+1}-\theta^1_n)f(x_n)+\theta^2_{n+1}(x_{n+1}-x_n) \\ &\quad +(\theta^2_{n+1}-\theta^2_n)x_n+\left(\theta^3_{n+1}-\theta^3_n\right)S_n\left(\delta_n x_n+(1-\delta_n)x_{n+1}\right) \\ &\quad +\theta^3_{n+1}\left(S_{n+1}\left(\delta_{n+1}x_{n+1}+(1-\delta_{n+1})x_{n+2}\right)-S_n\left(\delta_n x_n+(1-\delta_n)x_{n+1}\right)\right) \\ &= \theta^1_{n+1}\left(f(x_{n+1})-f(x_n)\right)+(\theta^1_{n+1}-\theta^1_n)f(x_n)+\theta^2_{n+1}(x_{n+1}-x_n) \\ &\quad +(\theta^2_{n+1}-\theta^2_n)x_n+\left((\theta^1_n-\theta^1_{n+1})-(\theta^2_{n+1}-\theta^2_n)\right)S_n\left(\delta_n x_n+(1-\delta_n)x_{n+1}\right) \\ &\quad +\theta^3_{n+1}\left(S_{n+1}\left(\delta_{n+1}x_{n+1}+(1-\delta_{n+1})x_{n+2}\right)-S_n\left(\delta_n x_n+(1-\delta_n)x_{n+1}\right)\right). \end{aligned}$$

Consequently,

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\leq \left(\theta^1_{n+1}\psi+\theta^2_{n+1}\right)\|x_{n+1}-x_n\| \\
&\quad +|\theta^1_n-\theta^1_{n+1}|\|S_n\left(\delta_n x_n+(1-\delta_n)x_{n+1}\right)-f(x_n)\| \\
&\quad +|\theta^2_{n+1}-\theta^2_n|\|x_n-S_n\left(\delta_n x_n+(1-\delta_n)x_{n+1}\right)\| \\
&\quad +\theta^3_{n+1}\|S_{n+1}\left(\delta_{n+1}x_{n+1}+(1-\delta_{n+1})x_{n+2}\right)-S_n\left(\delta_n x_n+(1-\delta_n)x_{n+1}\right)\| \\
&\leq \left(\theta^1_{n+1}\psi+\theta^2_{n+1}\right)\|x_{n+1}-x_n\|+\left(|\theta^1_n-\theta^1_{n+1}|+|\theta^2_{n+1}-\theta^2_n|\right)M_2 \\
&\quad +\theta^3_{n+1}\left[\delta_n\|x_{n+1}-x_n\|+(1-\delta_{n+1})\|x_{n+2}-x_{n+1}\|\right. \\
&\quad \left.+\left(|\beta^1_{n+1}-\beta^1_n|+|\beta^3_{n+1}-\beta^3_n|\right)M_1\right] \text{ (by Lemma 3.2)} \\
&= \left(\theta^1_{n+1}\psi+\theta^2_{n+1}+\theta^3_{n+1}\delta_n\right)\|x_{n+1}-x_n\| \\
&\quad +\left(|\theta^1_n-\theta^1_{n+1}|+|\theta^2_{n+1}-\theta^2_n|\right)M_2 \\
&\quad +\theta^3_{n+1}\left(|\beta^1_{n+1}-\beta^1_n|+|\beta^3_{n+1}-\beta^3_n|\right)M_1 \\
&\quad +\theta^3_{n+1}(1-\delta_{n+1})\|x_{n+2}-x_{n+1}\|. \qquad (3.12)
\end{aligned}$$

Let

$$\begin{aligned}
B_n &= \frac{1}{1-\theta^3_{n+1}(1-\delta_{n+1})}\left(|\theta^1_n-\theta^1_{n+1}|+|\theta^2_{n+1}-\theta^2_n|\right)M_2 \\
&\quad +\theta^3_{n+1}\left(|\beta^1_{n+1}-\beta^1_n|+|\beta^3_{n+1}-\beta^3_n|\right)M_1,
\end{aligned}$$

since $1-\theta^3_{n+1}(1-\delta_{n+1})>0$, (3.12) gives,

$$\|x_{n+2}-x_{n+1}\| \leq \frac{\theta^1_{n+1}\psi+\theta^2_{n+1}+\theta^3_{n+1}\delta_n}{1-\theta^3_{n+1}(1-\delta_{n+1})}\|x_{n+1}-x_n\|+B_n. \qquad (3.13)$$

Substituting (3.13) into (3.11) gives

$$\begin{aligned}
\|w_{n+1}-w_n\| &\leq \left[\frac{\theta^1_{n+1}\psi+\theta^3_{n+1}\delta_n}{\theta^1_{n+1}+\theta^3_{n+1}}+\frac{\theta^3_{n+1}(1-\delta_{n+1})}{\theta^1_{n+1}+\theta^3_{n+1}}\times\frac{\theta^1_{n+1}\psi+\theta^2_{n+1}+\theta^3_{n+1}\delta_n}{1-\theta^3_{n+1}(1-\delta_{n+1})}\right] \\
&\quad \times\|x_{n+1}-x_n\|+\left|\frac{\theta^1_n}{\theta^1_n+\theta^3_n}-\frac{\theta^1_{n+1}}{\theta^1_{n+1}+\theta^3_{n+1}}\right|M_2+\frac{\theta^3_{n+1}(1-\delta_{n+1})}{\theta^1_{n+1}+\theta^3_{n+1}}B_n \\
&\quad +\frac{\theta^1_{n+1}\psi}{\theta^1_{n+1}+\theta^3_{n+1}}\left(|\beta^1_{n+1}-\beta^1_n|+|\beta^3_{n+1}-\beta^3_n|\right)M_1 \\
&= \frac{\theta^1_{n+1}\psi+\theta^3_{n+1}\delta_n+\theta^3_{n+1}(1-\delta_{n+1})\theta^2_{n+1}}{[\theta^1_{n+1}+\theta^3_{n+1}][1-\theta^3_{n+1}(1-\delta_{n+1})]}\|x_{n+1}-x_n\| \\
&\quad +\left|\frac{\theta^1_n}{\theta^1_n+\theta^3_n}-\frac{\theta^1_{n+1}}{\theta^1_{n+1}+\theta^3_{n+1}}\right|M_2+\frac{\theta^3_{n+1}(1-\delta_{n+1})}{\theta^1_{n+1}+\theta^3_{n+1}}B_n \\
&\quad +\frac{\theta^1_{n+1}\psi}{\theta^1_{n+1}+\theta^3_{n+1}}\left(|\beta^1_{n+1}-\beta^1_n|+|\beta^3_{n+1}-\beta^3_n|\right)M_1
\end{aligned}$$

$$\begin{aligned}
&= \left(1-\frac{\theta^1_{n+1}(1-\psi)+\theta^3_{n+1}(\delta_{n+1}-\delta_n)}{[\theta^1_{n+1}+\theta^3_{n+1}][1-\theta^3_{n+1}(1-\delta_{n+1})]}\right)\|x_{n+1}-x_n\| \\
&\quad +\left|\frac{\theta^1_n}{\theta^1_n+\theta^3_n}-\frac{\theta^1_{n+1}}{\theta^1_{n+1}+\theta^3_{n+1}}\right|M_2+\frac{\theta^3_{n+1}(1-\delta_{n+1})}{\theta^1_{n+1}+\theta^3_{n+1}}B_n \\
&\quad +\frac{\theta^1_{n+1}\psi}{\theta^1_{n+1}+\theta^3_{n+1}}\left(|\beta^1_{n+1}-\beta^1_n|+|\beta^3_{n+1}-\beta^3_n|\right)M_1 \\
&= \left(1-\frac{\theta^1_{n+1}\phi+\theta^3_{n+1}(\delta_{n+1}-\delta_n)}{[\theta^1_{n+1}+\theta^3_{n+1}][1-\theta^3_{n+1}(1-\delta_{n+1})]}\right)\|x_{n+1}-x_n\| \\
&\quad +\left|\frac{\theta^1_n}{\theta^1_n+\theta^3_n}-\frac{\theta^1_{n+1}}{\theta^1_{n+1}+\theta^3_{n+1}}\right|M_2+\frac{\theta^3_{n+1}(1-\delta_{n+1})}{\theta^1_{n+1}+\theta^3_{n+1}}B_n \\
&\quad +\frac{\theta^1_{n+1}\psi}{\theta^1_{n+1}+\theta^3_{n+1}}\left(|\beta^1_{n+1}-\beta^1_n|+|\beta^3_{n+1}-\beta^3_n|\right)M_1 \\
&\leq \left(1-\frac{\theta^1_{n+1}\phi}{\theta^1_{n+1}+\theta^3_{n+1}}\right)\|x_{n+1}-x_n\| \\
&\quad +\left|\frac{\theta^1_n}{\theta^1_n+\theta^3_n}-\frac{\theta^1_{n+1}}{\theta^1_{n+1}+\theta^3_{n+1}}\right|M_2+\frac{\theta^3_{n+1}(1-\delta_{n+1})}{\theta^1_{n+1}+\theta^3_{n+1}}B_n \\
&\quad +\frac{\theta^1_{n+1}\psi}{\theta^1_{n+1}+\theta^3_{n+1}}\left(|\beta^1_{n+1}-\beta^1_n|+|\beta^3_{n+1}-\beta^3_n|\right)M_1,
\end{aligned}$$

since $\theta^1_{n+1}\phi+\theta^3_{n+1}(\delta_{n+1}-\delta_n)>\theta^1_{n+1}\phi$ and $[\theta^1_{n+1}+\theta^3_{n+1}][1-\theta^3_{n+1}(1-\delta_{n+1})]<\theta^1_{n+1}+\theta^3_{n+1}$. It then follows that

$$\begin{aligned}
\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\| &\leq -\frac{\theta^1_{n+1}\phi}{\theta^1_{n+1}+\theta^3_{n+1}}\|x_{n+1}-x_n\| \\
&\quad +\left|\frac{\theta^1_n}{\theta^1_n+\theta^3_n}-\frac{\theta^1_{n+1}}{\theta^1_{n+1}+\theta^3_{n+1}}\right|M_2+\frac{\theta^3_{n+1}(1-\delta_{n+1})}{\theta^1_{n+1}+\theta^3_{n+1}}B_n \\
&\quad +\frac{\theta^1_{n+1}\psi}{\theta^1_{n+1}+\theta^3_{n+1}}\left(|\beta^1_{n+1}-\beta^1_n|+|\beta^3_{n+1}-\beta^3_n|\right)M_1,
\end{aligned}$$

and thus,

$$\limsup_{n\to\infty}\left(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|\right)\leq 0. \tag{3.14}$$

Invoking Lemma 2.4 gives

$$\lim_{n\to\infty}\|w_n-x_n\|=0.$$

Obviously from (3.8), one can obtain that

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|(1-\theta^1_n-\theta^3_n)x_n+(\theta^1_n+\theta^3_n)w_n-x_n\| \\
&\leq (\theta^1_n+\theta^3_n)\|w_n-x_n\|\to 0 \text{ as } n\to\infty.
\end{aligned} \tag{3.15}$$

Next is to show that $\lim_{n\to\infty}\|x_n-S_nx_n\|=0$. From (1.6), one can have that

$$\begin{aligned}
\|x_n - S_nx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S_nx_n\| \\
&\leq \|x_{n+1} - x_n\| + \theta_n^1\|f(x_n) - S_nx_n\| + \theta_n^2\|x_n - S_nx_n\| \\
&\quad +\theta_n^3\|S_n(\delta_nx_n + (1-\delta_n)x_{n+1}) - S_nx_n\| \\
&\leq \|x_{n+1} - x_n\| + \theta_n^1\|f(x_n) - S_nx_n\| + (1-\theta_n^1-\theta_n^3)\|x_n - S_nx_n\| \\
&\quad +\theta_n^3\|\delta_nx_n + (1-\delta_n)x_{n+1} - x_n\| \\
&\leq \|x_{n+1} - x_n\| + \theta_n^1\|f(x_n) - S_nx_n\| + (1-\theta_n^1-\theta_n^3)\|x_n - S_nx_n\| \\
&\quad +\theta_n^3(1-\delta_n)\|x_{n+1} - x_n\|.
\end{aligned}$$

$$\begin{aligned}
(\theta_n^1+\theta_n^3)\|x_n - S_nx_n\| &\leq (1+\theta_n^3(1-\delta_n))\|x_{n+1} - x_n\| + \theta_n^1\|f(x_n) - S_nx_n\| \\
\|x_n - S_nx_n\| &\leq \frac{1+\theta_n^3(1-\delta_n)}{\theta_n^1+\theta_n^3}\|x_{n+1} - x_n\| + \frac{\theta_n^1}{\theta_n^1+\theta_n^3}\|f(x_n) - S_nx_n\| \\
&= \frac{1+\theta_n^3(1-\delta_n)}{1-\theta_n^2}\|x_{n+1} - x_n\| + \frac{\theta_n^1}{1-\theta_n^2}\|f(x_n) - S_nx_n\| \\
&\leq \frac{1+\theta_n^3(1-\delta_n)}{1-\eta}\|x_{n+1} - x_n\| \\
&\quad +\frac{\theta_n^1}{1-\eta}\|f(x_n) - S_nx_n\| \to 0 \text{ as } n\to\infty,
\end{aligned} \tag{3.16}$$

by the condition (ii) and since $1-\eta > 0$ (3.10).

Define a sequence $\{x_t\}$ by $x_t = tf(x_t) + (1-t)S_nx_t$ for $t \in (0,1)$. Lemma 2.2 establishes that $\{x_t\}$ converges strongly to $p \in F(T)\cap F(Q)$, which solves the variational inequality:

$$\langle f(p) - p, J(x-p)\rangle \leq 0,\ \forall\, x \in F(T)\cap F(Q),$$

equivalently,

$$\langle (I-f)p, J(x-p)\rangle \geq 0,\ \forall\, x \in F(T)\cap F(Q).$$

It is claimed that

$$\limsup_{n\to\infty}\langle f(p) - p,\ J(x_{n+1}-p)\rangle \leq 0,$$

where $p \in F(T)\cap F(Q)$ is the unique fixed point of the generalized contraction $P_{F(T)\cap F(Q)}f(p)$ (Proposition 2.6), that is, $p = P_{F(T)\cap F(Q)}f(p)$.

Since $\lim_{n\to\infty}\|x_n - S_nx_n\| = 0$ by (3.16), it follows from Lemma 2.3 that

$$\limsup_{n\to\infty}\langle f(p) - p, J(x_n-p)\rangle \leq 0.$$

Moreover, since the duality map is continuous and $\|x_{n+1} - x_n\| \to 0$ by (3.15), it is obtained that,

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(p) - p,\ J(x_{n+1}-p)\rangle &= \limsup_{n\to\infty}\langle f(p) - p,\ J(x_{n+1}-x_n+x_n-p)\rangle \\
&= \limsup_{n\to\infty}\langle f(p) - p,\ J(x_n-p)\rangle \leq 0.
\end{aligned} \tag{3.17}$$

Finally, it is shown that $x_n \to p \in F(T)$ as $n\to\infty$.
Assume that the sequence $\{x_n\}_{n=1}^{\infty}$ does not converge strongly to $p \in F(T)$. Therefore, there

exists $\epsilon > 0$ and a subsequence $\left\{x_{n_j}\right\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\|x_{n_j} - p\| \geq \epsilon$, for all $j \in \mathbb{N}$. Thus by Proposition 2.1, for this ϵ, there exists $c \in (0,1)$ such that

$$\|f(x_{n_j}) - f(p)\| \leq c\|x_{n_j} - p\|.$$

$$\begin{aligned}
||x_{n_{j+1}} - p||^2 &= \theta^1_{n_j}\left\langle f(x_{n_j}) - f(p), J(x_{n_{j+1}} - p)\right\rangle + \theta^1_{n_j}\left\langle f(p) - p, J(x_{n_{j+1}} - p)\right\rangle \\
&\quad + \theta^2_{n_j}\left\langle x_{n_j} - p, J(x_{n_{j+1}} - p)\right\rangle \\
&\quad + \theta^3_{n_j}\left\langle S_n\left(\delta_{n_j}x_{n_j} + (1-\delta_{n_j})x_{n_{j+1}}\right) - p, J(x_{n_{j+1}} - p)\right\rangle \\
&\leq c\theta^1_{n_j}||x_{n_j} - p||\,||x_{n_j+1} - p|| + \theta^1_{n_j}\left\langle f(p) - p, J(x_{n_{j+1}} - p)\right\rangle \\
&\quad + \theta^2_{n_j}||x_{n_j} - p||\,||x_{n_{j+1}} - p|| \\
&\quad + \left(\theta^3_{n_j}\delta_{n_j}\|x_{n_j} - p\| + \theta^3_{n_j}(1-\delta_{n_j})\|x_{n_{j+1}} - p||\right)||x_{n_{j+1}} - p|| \\
&\leq \left(c\theta^1_n + \theta^2_{n_j}\right)||x_{n_j} - p||\,||x_{n_{j+1}} - p|| + \theta^1_{n_j}\left\langle f(p) - p, J(x_{n_{j+1}} - p)\right\rangle \\
&\quad + \left(\theta^3_{n_j}\delta_{n_j}\|x_{n_j} - p\| + \theta^3_{n_j}(1-\delta_{n_j})\|x_{n_{j+1}} - p||\right)||x_{n_{j+1}} - p|| \\
&\leq \frac{1}{2}\left(c\theta^1_{n_j} + \theta^2_{n_j} + \theta^3_{n_j}\delta_{n_j}\right)(||x_{n_j} - p||^2 + ||x_{n_{j+1}} - p||^2) \\
&\quad + \theta^1_{n_j}\left\langle f(p) - p, J(x_{n_{j+1}} - p)\right\rangle + \theta^3_n(1-\delta_{n_j})\|x_{n_{j+1}} - p||^2
\end{aligned}$$

$$\begin{aligned}
2||x_{n_{j+1}} - p||^2 &\leq \left(1 - \theta^1_{n_j}(1-c) - \theta^3_{n_j}(1-\delta_{n_j})\right)(||x_{n_j} - p||^2 + ||x_{n_{j+1}} - p||^2) \\
&\quad + 2\theta^1_{n_j}\left\langle f(p) - p, J(x_{n_{j+1}} - p)\right\rangle + 2\theta^3_{n_j}(1-\delta_{n_j})\|x_{n_{j+1}} - p||^2 \\
&= \left(1 - \theta^1_{n_j}(1-c) - \theta^3_{n_j}(1-\delta_{n_j})\right)||x_{n_j} - p||^2 \\
&\quad + \left(1 - \theta^1_{n_j}(1-c) + \theta^3_{n_j}(1-\delta_{n_j})\right)||x_{n_{j+1}} - p||^2 \\
&\quad + 2\theta^1_{n_j}\left\langle f(p) - p, J(x_{n_{j+1}} - p)\right\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(1 + \theta^1_{n_j}(1-c) - \theta^3_{n_j}(1-\delta_{n_j})\right)||x_{n_{j+1}} - p||^2 \\
\leq\ &\left(1 - \theta^1_{n_j}(1-c) - \theta^3_{n_j}(1-\delta_{n_j})\right)||x_{n_j} - p||^2 \\
&+ 2\theta^1_{n_j}\left\langle f(p) - p, J(x_{n_{j+1}} - p)\right\rangle,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
||x_{n_{j+1}} - p||^2 &\leq \frac{1 - \theta^1_{n_j}(1-c) - \theta^3_{n_j}(1-\delta_{n_j})}{1 + \theta^1_{n_j}(1-c) - \theta^3_{n_j}(1-\delta_{n_j})}||x_{n_j} - p||^2 \\
&\quad + \frac{2\theta^1_{n_j}}{1 + \theta^1_{n_j}(1-c) - \theta^3_{n_j}(1-\delta_{n_j})}\left\langle f(p) - p, J(x_{n_{j+1}} - p)\right\rangle \\
&= \left(1 - \frac{2\theta^1_{n_j}(1-c)}{1 + \theta^1_{n_j}(1-c) - \theta^3_{n_j}(1-\delta_{n_j})}\right)||x_{n_j} - p||^2 \\
&\quad + \frac{2\theta^1_{n_j}}{1 + \theta^1_{n_j}(1-c) - \theta^3_{n_j}(1-\delta_{n_j})}\left\langle f(p) - p, J(x_{n_{j+1}} - p)\right\rangle. \qquad (3.18)
\end{aligned}$$

By applying Lemma 2.5 to (3.17) and (3.18), one can deduce that $x_{n_j} \to p$ as $j \to \infty$. This is a contradiction. Hence, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in F(T)$. ∎

3.1. **Extension to a finite family of strictly pseudocontractive mappings.** The result of Theorem 3.3 can be extended to a finite family of μ-strictly pseudocontractive mappings by using the lemma given below.

Lemma 3.4. [25] *Let K be a nonempty convex subset of a real smooth Banach space E and let $\lambda_i > 0$ $(i = 1, 2, ..., N)$ such that $\sum_{i=1}^{N} \lambda_i = 1$. Let $\{T_i\}_{i=1}^{N}$ be a finite family of μ_i-strictly pseudocontractive mappings and let $T = \sum_{i=1}^{N} \lambda_i T_i$. Then, we have the following:*

(i) *$T : K \to K$ is μ-strictly pseudocontractive mapping with $\mu = \min\{\mu_i : 1 \leq i \leq N\}$.*
(ii) *If $\cap_{i=1}^{N} F(T_i) \neq \emptyset$ then $F(T) = \cap_{i=1}^{N} F(T_i)$.*

The next following result then comes readily.

Theorem 3.5. *Let E be a uniformly smooth Banach space and K a nonempty closed convex subset of E. Let T_i be a finite family μ_i-strictly pseudocontractive self-mapping defined on K and Q a contraction defined on K with $\cap_{i=1}^{N} F(T_i) \cap F(Q) \neq \emptyset$. Let $f : K \to K$ be a generalized contraction and suppose that the real sequences $\{\delta_n\}_{n=1}^{\infty} \subset (0, 1)$, $\left\{\{\theta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0, 1]$ and $\left\{\{\beta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0, 1]$ with $\beta_n^1, \beta_n^3 \neq 0$ satisfy the following conditions:*

(i) $\sum_{i=1}^{3} \theta_n^i = 1,\ \sum_{n=1}^{\infty} \theta_n^1 = \infty,$
(ii) $\lim_{n\to\infty} |\theta_{n+1}^2 - \theta_n^2| = 0,\ 0 < \liminf_{n\to\infty} \theta_n^2 \leq \limsup_{n\to\infty} \theta_n^2 < 1,$
(iii) $\sum_{i=1}^{3} \beta_n^i = 1,\ \lim_{n\to\infty} |\beta_{n+1}^1 - \beta_n^1| = 0,\ \lim_{n\to\infty} |\beta_{n+1}^3 - \beta_n^3| = 0,$
(iv) $0 < \epsilon \leq \delta_n \leq \delta_{n+1} < 1$ *for all* $n \in \mathbb{N}$.

Then, for an arbitrary $x_1 \in K$, the iterative sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_{n+1} = \theta_n^1 f(x_n) + \theta_n^2 x_n + \theta_n^3 S_n(\delta_n x_n + (1 - \delta_n)x_{n+1}),$$

where $S_n x = \beta_n^1 Q(x) + \beta_n^2 x + \beta_n^3 \sum_{i=1}^{N} \lambda_i T_i(x)$, converges strongly to a fixed point $p \in \cap_{i=1}^{N} F(T_i)$ which solves the variational inequality

Proof. Define $T = \sum_{i=1}^{N} \lambda_i T_i$, it suffices to show that T is a μ-strictly pseudocontractive mapping with $F(T) = \cap_{i=1}^{N} F(T_i)$. Indeed by Lemma 3.4, T is a μ-strictly pseudocontractive mapping with $\mu = \min\{\mu_i : 1 \leq i \leq N\}$. Therefore, the conclusion holds by following the steps of proof for Theorem 3.3. ∎

Remark 3.1. The following results are readily obtained as corollaries of Theorem 3.3.

Corollary 3.6. *Let E be a uniformly smooth Banach space and K a nonempty closed convex subset of E. Let T be a nonexpansive self-mapping defined on K, $f : K \to K$ a generalized contraction and Q is a contraction defined on K with $F(T) \cap F(Q) \neq \emptyset$. Suppose that the* *real sequences* $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$, $\left\{\{\theta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0,1]$ *and* $\left\{\{\beta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0,1]$ *with* $\beta_n^1, \beta_n^3 \neq 0$ *satisfy the following conditions:*

(i) $\sum_{i=1}^{3} \theta_n^i = 1,\ \sum_{n=1}^{\infty} \theta_n^1 = \infty,$
(ii) $\lim_{n\to\infty} |\theta_{n+1}^2 - \theta_n^2| = 0,\ 0 < \liminf_{n\to\infty} \theta_n^2 \leq \limsup_{n\to\infty} \theta_n^2 < 1,$
(iii) $\sum_{i=1}^{3} \beta_n^i = 1,\ \lim_{n\to\infty} |\beta_{n+1}^1 - \beta_n^1| = 0,\ \lim_{n\to\infty} |\beta_{n+1}^3 - \beta_n^3| = 0,$
(iv) $0 < \epsilon \leq \delta_n \leq \delta_{n+1} < 1$ *for all* $n \in \mathbb{N}$.

Then, for an arbitrary $x_1 \in K$, *the iterative sequence* $\{x_n\}_{n=1}^{\infty}$ *defined by (1.6) converges strongly to a fixed point* p *of* T, *which solves the variational inequality (3.7).*

Proof. The class of μ-strictly pseudocontractive mappings is more general than the class of nonexpansive mappings. Hence, the conclusion follows from Theorem 3.3. ∎

Corollary 3.7. *Let* E *be a uniformly smooth Banach space and* K *a nonempty closed convex subset of* E. *Let* T_i *be a* μ*-strictly pseudocontractive self-mapping defined on* K *and* Q *a contraction defined on* K *with* $\cap_{i=1}^{N} F(T_i) \cap F(Q) \neq \emptyset$. *Let* $f : K \to K$ *a generalized contraction and assume that the real sequences he real sequences* $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$, $\left\{\{\theta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0,1]$ *and* $\{\lambda_n\} \subset (0,1)$ *satisfy the following conditions:*

(i) $\sum_{i=1}^{3} \theta_n^i = 1,\ \sum_{n=1}^{\infty} \theta_n^1 = \infty,$
(ii) $\lim_{n\to\infty} |\theta_{n+1}^2 - \theta_n^2| = 0,\ 0 < \liminf_{n\to\infty} \theta_n^2 \leq \limsup_{n\to\infty} \theta_n^2 < 1,$
(iii) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0,$
(iv) $0 < \epsilon \leq \delta_n \leq \delta_{n+1} < 1$ *for all* $n \in \mathbb{N}$.

Then, for an arbitrary $x_1 \in K$, *define the iterative sequence* $\{x_n\}_{n=1}^{\infty}$ *by*

$$x_{n+1} = \theta_n^1 f(x_n) + \theta_n^2 x_n + \theta_n^3 S_n(\delta_n x_n + (1-\delta_n)x_{n+1}), \tag{3.19}$$

where $S_n x = \lambda_n Q(x) + (1-\lambda_n)T(x)$. *Then the iterative sequence* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a fixed point* p *of* T, *which solves the variational inequality (3.7).*

Proof. Take $\beta_n^2 = 0$ in (1.6), then $\lambda_n = \beta_n^1$ and $(1-\lambda_n) = \beta_n^3$. Thus, the desire result follows from Theorem 3.3. ∎

4. NUMERICAL EXAMPLE FOR ILLUSTRATION OF CONVERGENCE ANALYSIS

An example of a μ-strictly pseudocontractive mapping is presented in this section. This is used to illustrate the convergence analysis of the main theorem in this paper.

Example 4.1.

Let E be the real line $\mathbb{R}$ with absolute value norm and define $T : \mathbb{R} \to \mathbb{R}$ by

$$Tx = |x| = \begin{cases} -x, & x \in (-\infty, 0], \\ x, & x \in (0, \infty). \end{cases} \tag{4.1}$$

Clearly, $F(T) = [0, \infty)$. It is imperative to ascertain that T is a μ-strictly pseudocontractive mapping with $\mu \in (0,1)$.

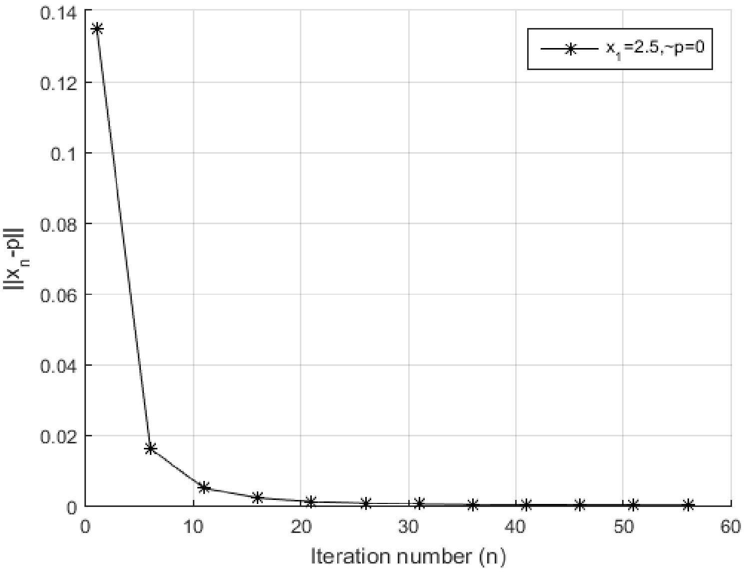

Figure 1: Iteration for $x_1 = 2.5$ and $p = 0$.

Case (i) : Notice that for all $x, y \in (-\infty, 0]$,

$$\begin{aligned} \langle (I-T)(x) - (I-T)(y), x-y \rangle &= 2\langle x-y, x-y \rangle \\ &= 2|x-y|^2 \\ &\geq \mu_1 |(I-T)(x) - (I-T)(y)|^2, \end{aligned}$$

for $\mu_1 \leq \frac{1}{4}$.

Case (ii) : For all $x, y \in (0, \infty)$,

$$\begin{aligned} \langle (I-T)(x) - (I-T)(y), x-y \rangle &= 2\langle 0-0, x-y \rangle \\ &= |0-0|^2 \\ &= \mu_2 |(I-T)(x) - (I-T)(y)|^2, \end{aligned}$$

for $\mu_2 > 0$.

Case (iii) : For all $x \in (-\infty, 0]$ and $y \in (0, \infty)$,

$$\begin{aligned} \langle (I-T)(x) - (I-T)(y), x-y \rangle &= 2\langle x-0, x-y \rangle \\ &= 2|x-0|^2 \\ &\geq \mu_1 |(I-T)(x) - (I-T)(y)|^2, \end{aligned}$$

for $\mu_1 \leq \frac{1}{4}$. Define $\mu := \min\{\mu_1, \mu_2\}$, T is thus a μ-strictly pseudocontractive mapping.

The convergence analysis of Theorem 3.3 will be applied in obtaining a fixed point of the mapping T in Example 4.1. Let $\left\{\theta_n^1\right\}_{n=1}^{\infty} := \left\{\frac{3}{5} - \frac{1}{5n}\right\}_{n=1}^{\infty}$, $\left\{\theta_n^2\right\}_{n=1}^{\infty} := \left\{\frac{1}{5n}\right\}_{n=1}^{\infty}$ and $\left\{\theta_n^3\right\}_{n=1}^{\infty} := \left\{\frac{2}{5}\right\}_{n=1}^{\infty}$. Clearly $\sum\limits_{i=1}^{3} \theta_n^i = 1$ and each sequence satisfy the conditions of Theorem 3.3. Moreover, defined $\left\{\beta_n^1\right\}_{n=1}^{\infty} := \left\{\frac{1}{2} - \frac{1}{2n}\right\}_{n=1}^{\infty}$, $\left\{\beta_n^2\right\}_{n=1}^{\infty} := \left\{\frac{1}{2n}\right\}_{n=1}^{\infty}$, $\left\{\beta_n^3\right\}_{n=1}^{\infty} := \left\{\frac{1}{2}\right\}_{n=1}^{\infty}$ and take $\{\delta_n\}_{n=1}^{\infty} := \left\{\frac{n}{2n+1}\right\}_{n=1}^{\infty}$. The real value functions $f : \mathbb{R} \to \mathbb{R}$ and $Q : \mathbb{R} \to \mathbb{R}$ are respectively defined by $f(x) = \frac{1}{5}x$ and $Q(x) = \frac{1}{4}x$. Figures 1, 2, 3 & 4 display the convergence of the iterative sequences to some given fixed points of T with different starting points.

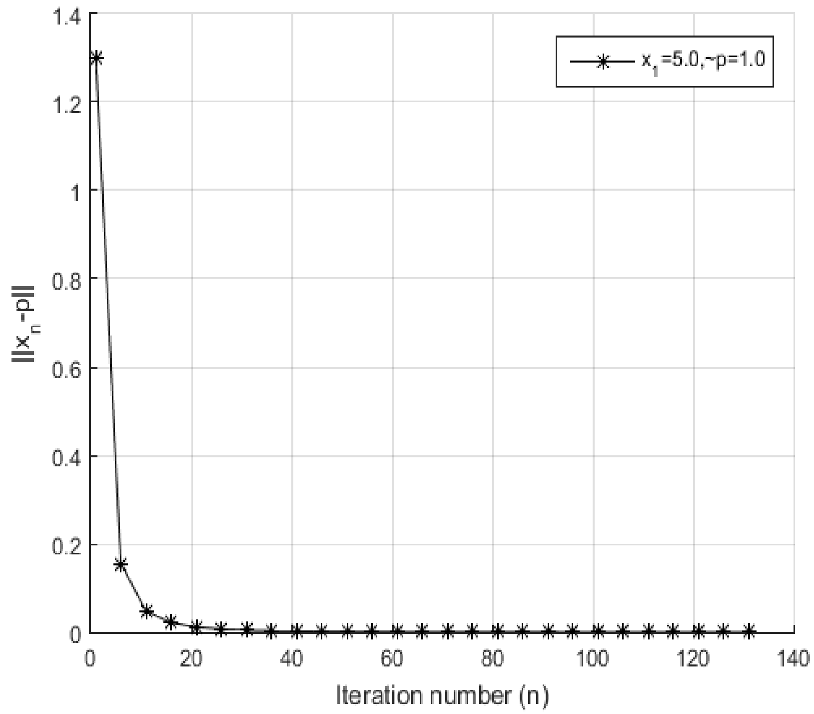

Figure 2: Iteration for $x_1 = 5.0$ *and* $p = 1.0$.

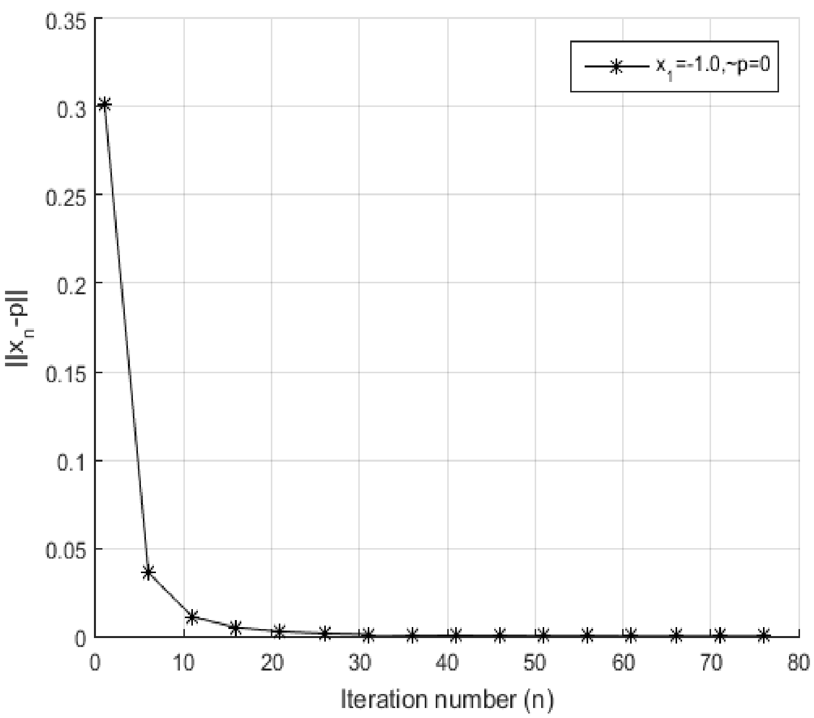

Figure 3: Iteration for $x_1 = -1.0$ *and* $p = 0$.

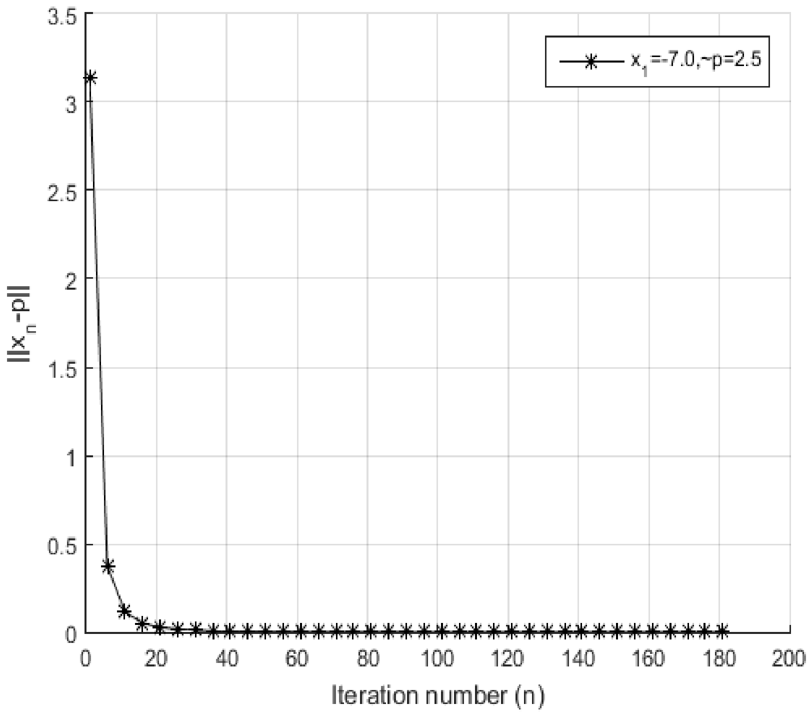

Figure 4: Iteration for $x_1 = -7.0$ and $p = 2.5$.

5. CONCLUSION

This study has contributed immersely to the exploration on how to find a fixed point of nonlinear problems which involve the class of strictly pseudocontractive mappings. The customary riddles in computations and analysis of approximating a fixed of nonlinear problems involving the class of strictly pseudocontractive mappings are elucidated by using the technique which are easy to follow. This gives rooms for a broad application of the scheme which was proposed in this paper. The obtained results was shown to hold for finite family of strictly pseudocontractive and this is an additional prestige to the proposed scheme and techniques which were used for the computations and analysis. The skillfulness of the proposed scheme and its implementation are displayed through a numerical example.

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