Aust. J. Math. Anal. Appl.
Vol. **18** (2021), No. 2, Art. 11, 16 pp.
AJMAA

NUMERICAL SOLUTION OF CERTAIN TYPES OF FREDHOLM-VOLTERRA INTEGRO-FRACTIONAL DIFFERENTIAL EQUATIONS VIA BERNSTEIN POLYNOMIALS

ALIAS B. KHALAF*[,1], AZHAAR H. SALLO[2], AND SHAZAD S. AHMED[3]

Received 26 January, 2021; accepted 6 August, 2021; published 24 September, 2021.

[1]DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, UNIVERSITY OF DUHOK, KURDISTAN REGION, IRAQ.
aliasbkhalaf@uod.ac

[2]DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, UNIVERSITY OF DUHOK, KURDISTAN REGION, IRAQ.
azhaar.sallo@uod.ac

[3]DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, UNIVERSITY OF SULAIMANI, KURDISTAN REGION, IRAQ.
shazad.ahmed@univsul.edu

ABSTRACT. In this article we obtain a numerical solution for a certain fractional order integro-differential equations of Fredholm-Volterra type, where the fractional derivative is defined in Caputo sense. The properties of Bernstein polynomials are applied in order to convert the fractional order integro-differential equations to the solution of algebraic equations. Some numerical examples are investigated to illustrate the method. Moreover, the results obtained by this method are compared with the exact solution and with the results of some existing methods as well.

Key words and phrases: Fractional derivative; Caputo differentiation; Integro- differential equations; Bernstein Polynomials; Fredholm-Volterra types.

2010 *Mathematics Subject Classification.* Primary 34A08. Secondary 33F05.

ISSN (electronic): 1449-5910

*Corresponding author.

1. INTRODUCTION

The fractional calculus takes up a very wide area in applied mathematical fields to study several problems from many fields of sciences such as mathematical physics and biophysics problems [1]. Often the use of the half-order to derivatives and integrals are treated to solve certain electro-chemical problems which is more accurate and useful than the classical approach [16].
Fractional integral and fractional differential equations have been studied in the past few years frequently [4, 5, 6, 8, 12]. Several methods are used to obtain the approximate solutions of such equations. Many types of fractional derivative have been defined and several papers have been devoted recently to study fractional derivatives and their applications in the Caputo sense fractional derivative, see [13, 15]. The study of fractional integro-differential equations of Fredholm-Volterra type is investigated in a very wide range and several mathematicians studied their approximate solutions using several types of methods and polynomials, see [3, 7, 13, 17, 2, 20]. The Bernstein polynomials [10] is one of the methods for finding the approximate solution of fractional equation, see [15, 17, 19].

In this paper, we analyze the numerical solutions of a class of fractional integro-differential equations involving the Caputo fractional derivative of order $n-1 < \alpha \leq n$ and of Fredholm-Volterra type. We begin with the definition and main properties of the Caputo fractional derivative. For more details we refer to [18, 6].

2. PRELEMINARIES

In this section, we recall some necessary definitions and properties of the Caputo fractional derivative. Moreover, some properties of Bernstein polynomials are given.

Definition 2.1. [18] The fractional derivative of the function $y(x)$ of order $\alpha > 0$ in Caputo sense is defined as:

$$ {}_a^c D_x^{\alpha} y(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)}{(x-t)^{\alpha+1+n}} dt & : \quad n-1 < \alpha < n, \quad n \in N, \\ \frac{d^n}{dx^n} y(x) & : \quad \alpha = n \in N. \end{cases} $$

It is clear that the Caputo derivative, ${}_a^c D_x^{\alpha} y(x) = 0$ whenever $y(x)$ is constant.
If $y(x) = (x-a)^j$, then the Caputo derivative of $y(x)$ is given by the following relation see [18]:

$$ {}_a^c D_x^{\alpha} (x-a)^j = \begin{cases} 0 & for \quad j \in N \cup \{0\} \quad and \quad j < \lceil \alpha \rceil, \\ \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} (x-a)^{j-\alpha} & for \quad j \in N \quad and \quad j \geq \lceil \alpha \rceil \\ & or \quad j \notin N \quad and \quad j > \lfloor \alpha \rfloor. \end{cases} $$

We use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α and the floor function $\lfloor \alpha \rfloor$ to denote the largest integer less than or equal to α. it is also known that Caputo fractional differentiation is a linear operation, that is for any two constants a_1, a_2 and any two functions y_1, y_2, we have [18]
${}_a^c D_x^{\alpha}(a_1 y_1 + a_2 y_2) = a_1({}_a^c D_x^{\alpha}(y_1)) + a_2({}_a^c D_x^{\alpha}(y_2))$.

Definition 2.2. [10]
The $n+1$ Bernstein polynomials $B_{i,n}(x)$ of degree n when $x \in [a,b]$ are defined as:

$$ B_{i,n}(x) = \frac{\binom{n}{i}(x-a)^i (b-x)^{n-i}}{(b-a)^n}, \quad i = 0, 1, 2, ..., n. \tag{2.1} $$

As a special case when $[a,b]=[0,1]$, then it is written as: $B_{i,n}(x)=\binom{n}{i}x^i(1-x)^{n-i}, \quad i=0,1,2,...,n.$

From the fact that $(b-x)^{n-i}=[(b-a)-(x-a)]^(n-i)$ and using the binomial expansion, the expression in Equation 2.1 can be transformed to

$$B_{i,n}(x)=\sum_{j=i}^{n}\frac{(-1)^{j-i}}{(b-a)^j}\binom{n}{i}\binom{n-i}{j-i}(x-a)^j \tag{2.2}$$

We can write it in the form

$$B_{i,n}(x)=\sum_{j=i}^{n}\frac{(-1)^{j-i}}{(b-a)^j}\binom{n}{j}\binom{j}{i}(x-a)^j \tag{2.3}$$

Also, we can write

$$B'_{i,n}(x)=\sum_{j=i}^{n}\frac{(-1)^{j-i}}{(b-a)^j}\binom{n}{j}\binom{j}{i}j(x-a)^{j-1} \tag{2.4}$$

and

$$\int_a^b B_{i,n}(x)dx=\frac{b-a}{n+1} \tag{2.5}$$

From Definition 2.1 and Equation 2.3, we obtain the following lemma.

Lemma 2.1. *If $0<\alpha\in\mathbb{R}\setminus\mathbb{N}$, then the α-fractional derivative of the n^{th} degree Bernstein polynomials in the Caputo sense is given by*

$${}_a^cD_x^\alpha B_{i,n}(x)=\sum_{j=i}^{n}\frac{(-1)^{j-i}}{(b-a)^j}\binom{n}{j}\binom{j}{i}{}_a^cD_x^\alpha(x-a)^j \tag{2.6}$$

Since ${}_a^cD_x^\alpha(x-a)^j=0$ for each $j<\alpha$, so we get

$${}_a^cD_x^\alpha B_{i,n}(x)=\sum_{j=\lceil\alpha\rceil}^{n}\frac{(-1)^{j-i}}{(b-a)^j}\binom{n}{j}\binom{j}{i}\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}(x-a)^{j-\alpha} \tag{2.7}$$

Lemma 2.2. *If $B_{i,n}(x)$ is the n^{th} degree Bernstein polynomial on the closed bounded interval $[a,b]$, $a\le b$, then*

$$\int_a^x(t-a)^kB_{i,n}(t)dt=\sum_{j=i}^{n}\frac{(-1)^{j-i}}{(b-a)^j}\binom{n}{j}\binom{j}{i}\frac{(x-a)^{j+k+1}}{j+k+1} \tag{2.8}$$

Proof. From the fact that

$$\int_a^x(t-a)^{k+j}dt=\frac{(x-a)^{k+j+1}}{k+j+1}.$$

The result follows from Equation 2.3. ∎

Lemma 2.3. *If $B_{i,n}(x)$ is the n^{th} degree Bernstein polynomial on the closed bounded interval $[a,b]$, $a\le b$, then*

$$\int_a^x(t)^kB_{i,n}(t)dt=\sum_{r=0}^{k}\binom{k}{r}a^{k-r}\sum_{j=i}^{n}\frac{(-1)^{j-i}}{(b-a)^j}\binom{n}{j}\binom{j}{i}\frac{(x-a)^{j+r+1}}{j+r+1} \tag{2.9}$$

Proof. We can write

$$(t)^k = [(t-a)+a]^k = \sum_{r=0}^{k} \binom{k}{r} a^{k-r}(t-a)^r$$

Applying Lemma 2.2, we get the result. ∎

The following result is a direct consequence of Lemma 2.3.

Lemma 2.4. *If $B_{i,n}(x)$ is the n^{th} degree Bernstein polynomial on the closed bounded interval $[a, b]$, $a \le b$, then*

$$\int_a^b (t)^k B_{i,n}(t)dt = \sum_{r=0}^{k} \binom{k}{r} a^{k-r}(b-a)^{r+1} \sum_{j=i}^{n} \frac{(-1)^{j-i}}{j+r+1} \binom{n}{j}\binom{j}{i} \tag{2.10}$$

Moreover, if $k = 0$, then

$$\int_a^b B_{i,n}(t)dt = (b-a) \sum_{j=i}^{n} \frac{(-1)^{j-i}}{j+1} \binom{n}{j}\binom{j}{i} \tag{2.11}$$

From Equation 2.5 and Equation 2.11, we obtain that

$$\sum_{j=i}^{n} (-1)^{j-i} \binom{n+1}{j+1}\binom{j}{i} = 1$$

From Lemma 2.1 and Lemma 2.3, we have the following result:

Lemma 2.5. *If $B_{i,n}(x)$ is the n^{th} degree Bernstein polynomial on the closed bounded interval $[a, b]$, $a \le b$ and $n-1 \le \alpha \le n$, then*

$$\int_a^x (t-a)^k \, {}_a^cD_t^\alpha B_{i,n}(t)dt = \sum_{j=\lceil\alpha\rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j}\binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} \times \frac{(x-a)^{j+k-\alpha+1}}{j+k-\alpha+1} \tag{2.12}$$

As a special case,

(2.13)

$$\int_a^x t^k \, {}_a^cD_t^\alpha B_{i,n}(t)dt = \sum_{r=0}^{k} \binom{k}{r} a^{k-r} \sum_{j=\lceil\alpha\rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j}\binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} \times \frac{(x-a)^{j+r-\alpha+1}}{j+r-\alpha+1}$$

If $k = 0$, then

$$\int_a^x {}_a^cD_t^\alpha B_{i,n}(t)dt = \sum_{j=\lceil\alpha\rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j}\binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+2-\alpha)} (x-a)^{j-\alpha+1} \tag{2.14}$$

Also, we have

$$\int_a^b {}_a^cD_t^\alpha B_{i,n}(t)dt = \sum_{j=\lceil\alpha\rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^{\alpha-1}} \binom{n}{j}\binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+2-\alpha)} \tag{2.15}$$

Lemma 2.6. *If $m \le n$, then*

(1) *The m^{th}-derivative of $y(x)$ at $x = a$ is*

$$y^{(m)}(a) = \sum_{i=0}^{m} c_i \frac{(-1)^{m-i} m!}{(b-a)^m} \binom{n}{m}\binom{m}{i} \tag{2.16}$$

(2) *The* m^{th}*-derivative of* $y(x)$ *at* $x=b$ *is*

$$y^{(m)}(b)=\sum_{i=n-m}^{n} c_i\frac{(-1)^{2m+i-n}m!}{(b-a)^m}\binom{n}{m}\binom{m}{n-i} \tag{2.17}$$

Proof. From Equation 3.3 and Equation 2.3, we have

$$y^{(m)}(x)=\sum_{i=0}^{n} c_iB_{i,n}^{(m)}(x)=\sum_{i=0}^{n}c_i\sum_{j=i}^{n}\frac{(-1)^{j-i}}{(b-a)^j}\binom{n}{j}\binom{j}{i}m!\binom{j}{m}(x-a)^{j-m}$$

If we calculate the derivative at $x=a$, then all the terms are zero except $j=m$. Hence, we get

$$y^{(m)}(a)=\sum_{i=0}^{m} c_i\frac{(-1)^{m-i}m!}{(b-a)^m}\binom{n}{m}\binom{m}{i} \tag{2.18}$$

To prove (2), From the fact that $x-a=(b-a)-(b-x)$ and from the definition of $B_{i,n}(x)$ the n^{th}-degree polynomial which is defined in the closed interval $[a,b]$ and for each $i=0,1,2,...,n$, we have

$$B_{i,n}(x)=\frac{\binom{n}{i}(x-a)^i(b-x)^{n-i}}{(b-a)^n}=\frac{\binom{n}{i}[(b-a)-(b-x)]^i(b-x)^{n-i}}{(b-a)^n} \tag{2.19}$$

Hence,

$$B_{i,n}(x)=\frac{\binom{n}{i}}{(b-a)^n}\sum_{k=0}^{i}(-1)^k\binom{i}{k}(b-a)^{i-k}(b-x)^{n+k-i} \tag{2.20}$$

Therefore, the m^{th}-derivative of $y(x)$ is given by

$$\begin{aligned} y^{(m)}(x)&=\sum_{i=0}^{n} c_iB_{i,n}^{(m)}(x)\\ &=\sum_{i=0}^{n}c_i\frac{\binom{n}{i}}{(b-a)^n}\sum_{k=0}^{i}(-1)^k\binom{i}{k}(b-a)^{i-k}(-1)^m m!\binom{n+k-i}{m}(b-x)^{n+k-i-m}\end{aligned} \tag{2.21}$$

If we calculate the m^{th}-derivative at $x=b$, then all the terms will be zero except $n+k-i=m$ that is $k=m+i-n$. Hence, after simplifying we get the result. ∎

3. SOLUTION APPROXIMATION

In this section, we are concerned with the numerical solution of the following linear Fredholm-Volterra integro-fractional differential equation:

$$\begin{aligned}\sum_{k=1}^{q_1} g_k(x)\ {}_a^cD_x^{(\frac{\alpha}{k})}y(x)+g_0(x)y(x)&=f(x)+\mu_0\int_a^x K_0(x,t)\ y(t)dt\\ &+\sum_{m=1}^{q_2}\mu_m\int_a^b K_m(x,t)\ {}_a^cD_t^{\frac{\beta}{m}}y(t)dt\end{aligned} \tag{3.1}$$

Where ${}_a^cD_x^\sigma y(x)$, $\sigma=\frac{\alpha}{k}$ and $\frac{\beta}{m}$, indicates the σ^{th} Caputo fractional derivative of $y(x)$, $g_k(x)$, $k=1,2,...q_1$, $f(x)$ and $K_m(x,t)$ for all $m=0,1,...,q_2$ are given continuous functions, x,t are real variables lies in the closed interval $[a,b$. $q_1,q_2\geq 1,\quad p-1<\alpha\leq p,\quad p,q_1,q_2\in\mathbb{N}$,

$\beta \le \alpha$ and $y(x)$ is the unknown function to be determind.
Subject to the conditions:

$$\sum_{j=1}^{p}(a_{ij}y^{(j-1)}(a) + b_{ij}y^{(j-1)}(b)) = d_i, \qquad i = 1, 2, ..., p. \tag{3.2}$$

The solution of Equation 3.1 and 3.2 is the function $y(x) \sum_{i=0}^{\infty} c_i B_{i,n}(x)$ which can be approximated in terms of n^{th}-degree truncated series of Bernstein polynomial

$$y_n(x) = \sum_{i=0}^{n} c_i B_{i,n}(x) \tag{3.3}$$

From the boundary conditions 3.2 and Lemma 2.6, we get m equations and we shall obtain the other $(n-m)$ equations by substituting the approximate solution $y(x) \approxeq y_n(x) = \sum_{i=0}^{n} c_i B_{i,n}(x)$ in Equation 3.1 to get

$$\begin{aligned} \sum_{i=0}^{n} c_i \Big\{ \sum_{k=1}^{q} g_k(x)\ {}_a^cD_x^{(\frac{\alpha}{k})} B_{i,n}(x) + g_0(x)B_{i,n}(x) - \mu_0 \int_a^x K_0(x,t)\ B_{i,n}(t)dt \\ - \sum_{m=1}^{q} \mu_m \int_a^b K_m(x,t)\ {}_a^cD_t^{\frac{\beta}{m}} B_{i,n}(t)dt \Big\} = f(x). \end{aligned} \tag{3.4}$$

By choosing $n-m$ points x_r such that $x_r = a + \frac{(b-a)r}{n}$ for $0 \le r \le n$ and together with the equations obtained from the boundary condition, we shall get an $(n+1) \times (n+1)$ matrix $\mathcal{A}$ such that $\mathcal{A} \times \mathcal{C}^T = \mathcal{B}$ where $\mathcal{C} = [c_0 \quad c_1 \dots c_n]$ and the known matrix $\mathcal{B}^T = [b_0 \quad b_1 \dots b_n]$. Hence $\mathcal{C}^T = \mathcal{A}^{-1} \times \mathcal{B}$. Substituting the $c_i's$ in Equation 3.3 we get the approximate solution of Equation 3.1.

4. ERROR ANALYSIS AND EXAMPLES

In this section, we discuss the error bound to Equation 3.1 and give an approximate solution of some examples for distinct $n-1 < \alpha \le n$ and $\beta \le \alpha$ and compare it with the exact solution of the equation. Let $y(x)$ and $y_n(x) = \sum_{i=0}^{n} B_{i.n}(x)$ be the exact and approximate solutions for Equation 3.1 respectively. Let $\delta_n(x) = y(x) - y_n(x)$ be the error function. Suppose that

$$\begin{aligned} \sum_{k=1}^{q_1} g_k(x)\ {}_a^cD_x^{(\frac{\alpha}{k})} y_n(x) + g_0(x)y_n(x) - \mu_0 \int_a^x K_0(x,t)\ y_n(t)dt \\ - \sum_{m=1}^{q_2} \mu_m \int_a^b K_m(x,t)\ {}_a^cD_t^{\frac{\beta}{m}} y_n(t)dt = f(x) + \zeta(x) \end{aligned}$$

Hence, using Equation 3.1, we get

$$\begin{aligned} \sum_{k=1}^{q_1} g_k(x)\ {}_a^cD_x^{(\frac{\alpha}{k})} \delta_n(x) + g_0(x)\delta_n(x) - \mu_0 \int_a^x K_0(x,t)\ \delta_n(t)dt \\ - \sum_{m=1}^{q_2} \mu_m \int_a^b K_m(x,t)\ {}_a^cD_t^{\frac{\beta}{m}} \delta_n(t)dt = -\zeta(x) \end{aligned}$$

Suppose that $g_k(x)$ and $K_m(x,t)$ are all bounded continuous functions in $[a, b]$. Let

$$G_k = sup_{a \le x \le b}|g_k(x)| \qquad k = 0, 1, 2, ..., q_1$$

and

$$M_m = sup_{a\leq x,t\leq b}|K_m(x,t)| \qquad m = 1,2,...,q_2$$

Then we obtain an error bound

$$|\zeta(x)| \leq \sum_{k=1}^{q_1} G_k \ |{}_a^cD_x^{(\frac{\alpha}{k})}\delta_n(x)| + (G_0 + (b-a)|\mu_0|M_0||\delta_n|| + \sum_{m=1}^{q_2} |\mu_m|(b-a)M_m \ |{}_a^cD_t^{\frac{\beta}{m}}\delta_n(t)|$$

Here $||\delta_n|| = Sup_{x\in[a,b]}|\delta_n(x)|$. On the other hand if $y \in C^p[a,b]$, it is shown that

$$|{}_a^cD_x^{\alpha}\delta_n(x)| \leq \epsilon\frac{b^{p-\alpha}}{\Gamma(p-\alpha+1)}$$

Where ϵ is a small positive number (see Theorem 6, [11]). Hence we obtain that

(4.1)

$$|\zeta(x)| \leq \sum_{k=1}^{q_1} G_k \ \epsilon\frac{b^{p-\frac{\alpha}{k}}}{\Gamma(p-\frac{\alpha}{k}+1)} + (G_0+(b-a)|\mu_0|M_0||\delta_n|| + \sum_{m=1}^{q_2} |\mu_m|(b-a)M_m \ \epsilon\frac{b^{p-\frac{\beta}{m}}}{\Gamma(p-\frac{\beta}{m}+1)}$$

Therefore, Equation 4.1 presents the error bound for the solution of the Equation 3.1.
Now We consider some examples to illustrate their numerical solutions by using the proposed method.

Example 4.1. Consider the Fredholm-Volterra integro-fractional differential equation

$$(4.2) \qquad {}_0^cD_x^{\alpha}y(x) + \frac{x^2e^x}{3}y(x) = f(x) + e^x\int_0^x t \ {}_0^cD_t^{\beta_1}y(t)dt + \int_0^1 x^2 \ {}_0^cD_t^{\beta_2}y(t)dt$$

Where $f(x) = \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{1}{2}x^2$, $0 < \alpha, \beta_1, \beta_2 \leq 1$ and $0 \leq t, x \leq 1$. Subject to the condition

$$2y(0) + 3y(1) = 3$$

By using Bernstein polynomials of degree n, we approximate the solution as follows:

$$(4.3) \qquad y(x) \approxeq y_n(x) = \sum_{i=0}^{n} c_iB_{i,n}(x), \qquad n \in \mathbb{Z}^+$$

From Equation 4.3, and the above condition, we obtain that

$$(4.4) \qquad 2c_0 + 3c_n = 3$$

Substituting for $y(x)$ of Equation 4.3 in Equation 4.2, we get

$$ {}_0^cD_x^{\alpha}\sum_{i=0}^{n} c_iB_{i,n}(x) + \frac{x^2e^x}{3}\sum_{i=0}^{n} c_iB_{i,n}(x) = f(x)$$

$$(4.5) \qquad +e^x\int_0^x t^2 \ {}_0^cD_t^{\beta_1}\sum_{i=0}^{n} c_iB_{i,n}(t)dt + \frac{\pi}{2}\int_0^1 x^2\sqrt{t} \ {}_0^cD_t^{\beta_2}\sum_{i=0}^{n} c_iB_{i,n}(t)dt$$

Hence,

$$\sum_{i=0}^{n} c_i\Big\{{}_0^cD_x^{\alpha}B_{i,n}(x) + \frac{x^2e^x}{3}B_{i,n}(x) - e^x\int_0^x t^2 \ {}_0^cD_t^{\beta_1}B_{i,n}(t)dt$$

$$(4.6) \qquad -\frac{\pi}{2}\int_0^1 x^2\sqrt{t} \ {}_0^cD_t^{\beta_2}c_iB_{i,n}(t)dt\Big\} = f(x)$$

Applying Equation 2.7, we get

$$\sum_{i=0}^{n} c_i \Bigg\{ \sum_{j=\lceil\alpha\rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} (x)^{j-\alpha} + \frac{x^2 e^x}{3} \sum_{j=i}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} (x)^j$$

$$-e^x \int_0^x t^2 \sum_{j=\lceil\beta_1\rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+1-\beta_1)} (t)^{j-\beta_1} dt$$

$$-\frac{\pi}{2} \int_0^1 x^2 \sqrt{t} \sum_{j=\lceil\beta_2\rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+1-\beta_2)} (t)^{j-\beta_2} dt \Bigg\} = f(x) \tag{4.7}$$

Simplifying Equation 4.7, we get

$$\sum_{i=0}^{n} c_i \Bigg\{ \sum_{j=\lceil\alpha\rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} (x)^{j-\alpha} + \frac{x^2 e^x}{3} \sum_{j=i}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} (x)^j$$

$$-e^x \sum_{j=\lceil\beta_1\rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+1-\beta_1)} \times \frac{x^{j+3-\beta_1}}{j+3-\beta_1}$$

$$-\frac{\pi}{2} x^2 \sum_{j=\lceil\beta_2\rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+1-\beta_2)} \times \frac{1}{j+1.5-\beta_2} \Bigg\} = f(x) \tag{4.8}$$

As a particular case, if we take $n=4$, $\alpha=\frac{1}{2}$ and $\beta_1=1$, $\beta_2=0.5$ and select some $0<x_r<1$ for $r=1,2,3,4$, and considering the coefficients of c_0, c_n in Equation 4.4 as one of the rows of the matrix we get a (5×5) matrix $\mathcal{A}$ such that

$$\mathcal{A}\mathcal{C}=\mathcal{B}$$

and hence,

$$\mathcal{C}=\mathcal{A}^{-1}\mathcal{B}$$

where $\mathcal{C}=[c_0 \quad c_1 \quad c_2 \quad c_3 \quad c_4]$ and after solving we get

$$\mathcal{C}=[1.55(10)^{-15} \quad 0.25 \quad 0.5 \quad 0.75 \quad 1].$$

The approximate solution of Equation 4.2 is

$$y(x)\approx 1.55(10)^{-15}(1-x)^4 + x(1-x)^3 + 3x^2(1-x)^2 + 3x^3(1-x) + x^4$$

The following table describes the relation between the exact (y_{Exact}) and approximate (y_{Approx}) solutions for some selected values of x when $n=4$, $\alpha=0.5$, $\beta_1=1$ and $\beta_2=0.5$. Furthermore we compare the absolute error (AE) with the results found in [9] and [14].

Table 4.1: Exact, approximate solution and absolute error when $\alpha=0.5$, $\beta_1=1$ and $\beta_2=0.5$

x	y_{Exact}	y_{Approx}	AE $n=4$	AE in [9] $n=4$	AE in [14] $n=20$
0	0	0	0	8E-11	9.6658E-7
0.2	0.2	0.2	0	7.2352E-11	9.2302E-7
0.4	0.4	0.4	0	4.5952E-11	7.3108E-7
0.6	0.6	0.6	0	1.2128E-11	3.5394E-7
0.8	0.8	0.8	0	7.8080E-12	1.9065E-7
1	1	0.999999999999999	1E-15	1.0E-10	8.8091E-7

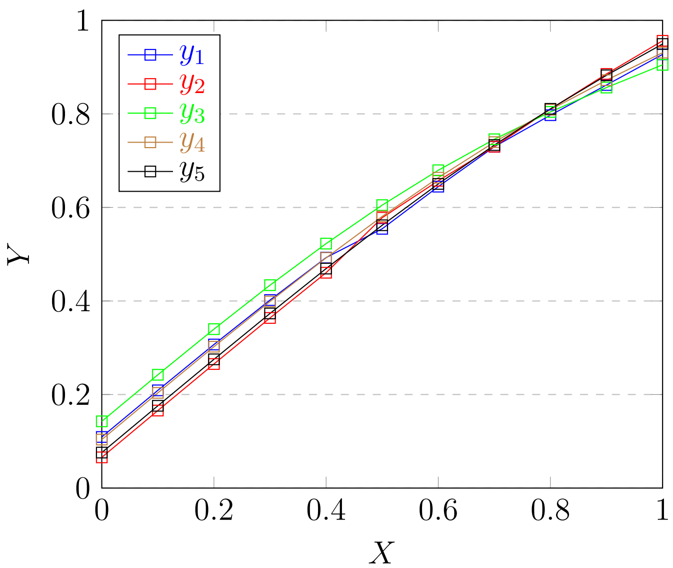
Table 4.2, describes the approximate solution of Equation 4.2 for $n = 5$ and some selected values of α, β_1 and β_2.
y_1, y_2, y_3, y_4 and y_5 represent the approximate solution when $n = 5$, $(\alpha = 0.5, \beta_1 = 0, \beta_2 = 0)$, $(\alpha = 1, \beta_1 = 0, \beta_2 = 0)$, $(\alpha = 0.5, \beta_1 = 0.1, \beta_2 = 0.2)$, $(\alpha = 0.7, \beta_1 = 0.1, \beta_2 = 0.2)$ and $(\alpha = 0.9, \beta_1 = 0.1, \beta_2 = 0.2)$ respectively.

Table 4.2: Approximate solution for $(n = 5)$

x	y_1	y_2	y_3	y_4	y_5
0	0.109268718	0.065627276	0.142674366	0.103663692	0.075878607
0.1	0.208933787	0.165746353	0.242249881	0.203531742	0.175910887
0.2	0.306926515	0.265223878	0.339792232	0.302258349	0.275235533
0.3	0.401989571	0.36369204	0.433805753	0.398890971	0.373314158
0.4	0.492753169	0.460378295	0.522679347	0.49217462	0.469236735
0.5	0.578016758	0.554293587	0.6049312	0.580780949	0.561910913
0.6	0.657030709	0.644420564	0.679453485	0.663537345	0.650251329
0.7	0.729777998	0.729901805	0.745757074	0.739656013	0.733368921
0.8	0.797255897	0.81022804	0.804216246	0.808963067	0.810760244
0.9	0.861757655	0.885426368	0.856313399	0.872127621	0.882496782
1	0.927154188	0.956248482	0.904883756	0.930890872	0.949414262

The following graphs represent the graph of y_i for $i = 1, 2, 3, 4, 5$.

Figure 1: Graphs of approximate solutions for Equation 4.2



Example 4.2. Consider the Fredholm-Volterra integro-fractional differential equation (4.9)

$$ {}_0^cD_x^{\alpha}y(x) + x\ {}_0^cD_x^{\frac{\alpha}{2}}y(x) - 3y(x) = f(x) + 4\int_0^x x\ \ y(t)dt + \int_0^5 t\ {}_0^cD_t^{\beta}y(t)dt - 4\int_0^5 {}_0^cD_t^{\frac{\beta}{2}}y(t)dt $$

Where $f(x) = \frac{(x)^{0.2}}{\Gamma(1.2)}(5x-8)+\frac{(x)^{2.1}}{\Gamma(2.1)}(\frac{20}{7}x-8)-x^5+\frac{7}{3}x^3+16x^2+3-\frac{735}{176}\times\frac{(5)^{2.2}}{\Gamma(1.2)}-\frac{61040}{4557}\times\frac{(5)^{2.1}}{\Gamma(2.1)}$,

$1 < \alpha, \beta \leq 2$ and $0 \leq t, x \leq 5$. Subject to the conditions

$$25y(0) + y(5) + 10y'(0) - y'(5) = -36$$
$$4y(0) + 0.2y(5) + 8y'(0) - 0.2y'(5) = 6.2$$

When $\alpha = \beta = 1.8$, then the exact solution for Equation 4.9 is $y(x) = x^3 - 4x^2 - 1$. The approximate solution of Equation 4.9 is $y(x) \approxeq y_n(x) = \sum_{i=0}^{n} c_i B_{i,n}(x)$. To find the c_i's we substitute and simplify for $y_n(x)$ in Equation 4.9 and get the following equation:

$$\sum_{i=0}^{n} c_i \Bigg\{ \sum_{j=\lceil\alpha\rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^j}\binom{n}{j}\binom{j}{i}\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}(x-a)^{j-\alpha}$$
$$+x\sum_{j=\lceil\frac{\alpha}{2}\rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^j}\binom{n}{j}\binom{j}{i}\frac{\Gamma(j+1)}{\Gamma(j+1-\frac{\alpha}{2})}(x-a)^{j-\frac{\alpha}{2}}$$
$$-3\sum_{j=i}^{n} \frac{(-1)^{j-i}}{(b-a)^j}\binom{n}{j}\binom{j}{i}(x-a)^j - 4x\sum_{j=i}^{n} \frac{(-1)^{j-i}}{(b-a)^j}\binom{n}{j}\binom{j}{i}\frac{(x-a)^{j+1}}{j+1}$$
$$-\sum_{j=\lceil\beta\rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^j}\binom{n}{j}\binom{j}{i}\frac{\Gamma(j+1)}{\Gamma(j+1-\beta)}\times\frac{(b)^{j+2-\beta}}{j+2-\beta}$$
$$+4x\sum_{j=\lceil\frac{\beta}{2}\rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^j}\binom{n}{j}\binom{j}{i}\frac{\Gamma(j+1)}{\Gamma(j+1-\frac{\beta}{2})}\times\frac{(b)^{j+1-\frac{\beta}{2}}}{j+1-\frac{\beta}{2}}\Bigg\} = f(x) \tag{4.10}$$

After solving for the $c_i s$ and obtaining y for $n = 6$ and $\alpha = \beta = 1.8$, we get the following table for the selected values of $x_i = \frac{i}{(b-a)}$ and $0 \leq i \leq 25$.

Table 4.3: Approximate solution for Equation 4.9 for ($n = 6, 8, 10, 12$)

x	y_{exact}	$y_{approx.}$	Abs. error
0	-1	-1	0.0
0.6	-2.224	-2.22399999999999	0.1E-13
1	-4	-3.99999999999999	0.1E-13
1.2	-5.032	-5.03199999999998	0.2E-13
1.4	-6.096	-6.09599999999999	0.1E-13
1.6	-7.144	-7.14399999999999	0.1E-13
2.4	-10.216	-10.216	0.0
3	-10	-10	0.0
3.2	-9.192	-9.19200000000003	0.3E-13
3.4	-7.936	-7.93600000000006	0.6E-13
3.8	-3.888	-3.88800000000016	1.5E-13
4	-1	-1.00000000000023	2.3E-13
4.2	2.528	2.52799999999967	3.4E-13
4.6	11.696	11.6959999999994	5.8E-13
5	24	23.999999999999	0.1E-11

Figures 2, 3 and 4, represent the graphs of the approximate solutions of y_i for $i = 1, 2, 3, 4$ for several values of n, α and β.

Figure 2: Graphs of approximate and exact solutions for Equation 4.9 when $n = 6$

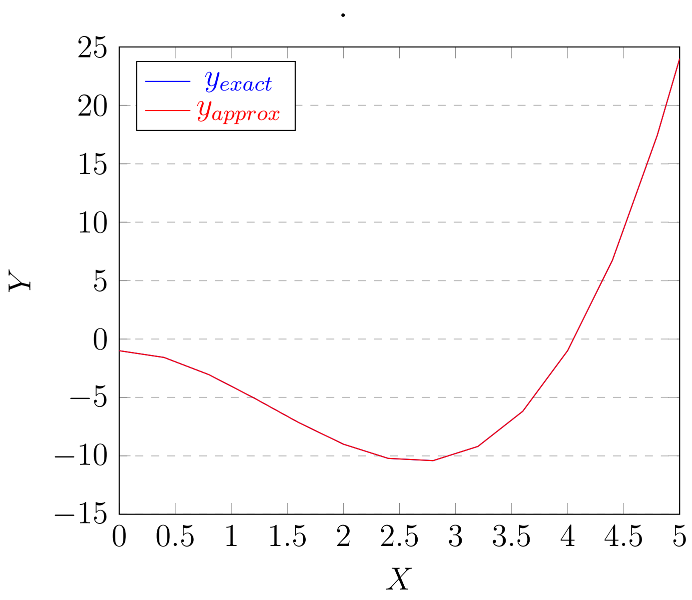


Figure 3: Graphs of approximate solutions for Equation 4.2 ($n = 10$*,* $\alpha = 1.8$*)*

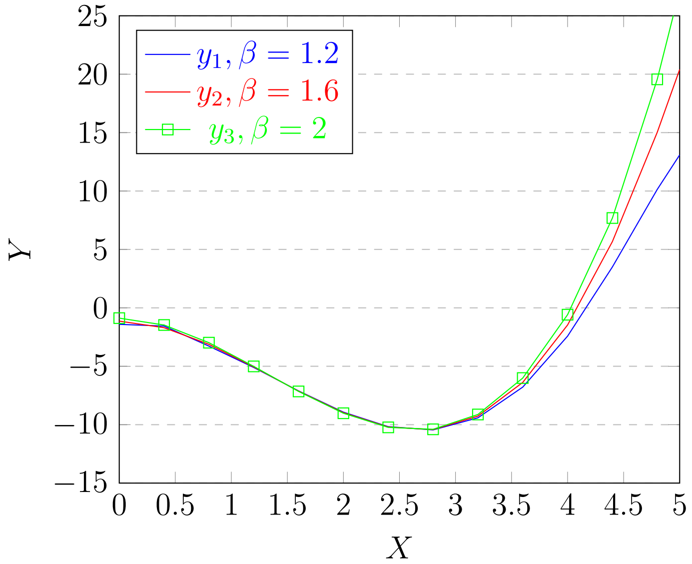
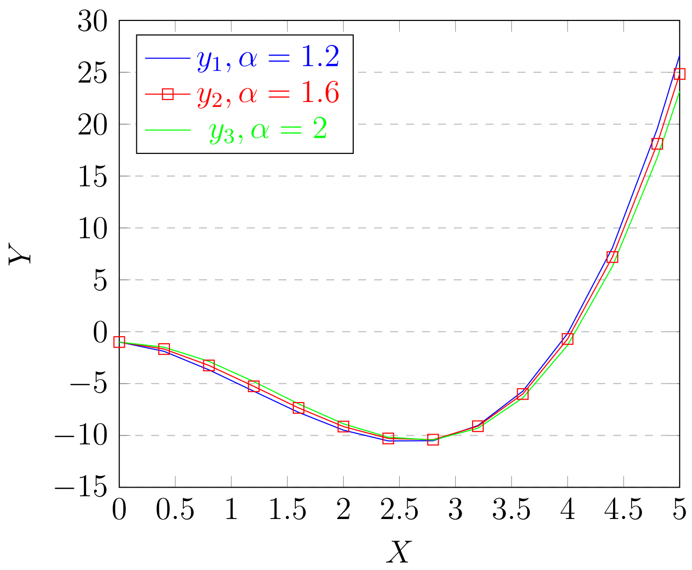


Figure 4: Graphs of approximate solutions for Equation 4.2 ($n = 10$, $\beta = 1.8$)



Example 4.3. Consider the integro differential equation

$$ {}_0^cD_x^\alpha y(x) - \frac{x^2}{99}y(x) = f(x) + \frac{1}{4}\int_0^x (x-t)\ y(t)dt + \frac{1}{7}\int_0^1 xt^2\ {}_0^cD_t^\beta y(t)dt \tag{4.11} $$

Where $f(x) = \frac{\Gamma(4.5)}{\Gamma(2.2)}x^{1.2} - \frac{x}{11}$,

$2 < \alpha \leq 3, 0 < \beta \leq 1$ and $0 \leq t, x \leq 1$. Subject to the conditions

$$ 2y(0) - 3y(1) - y'(0) - 8y'(1) + y''(0) + 4y''(1) = 4 $$
$$ y(0) - \frac{1}{4}y(1) + 2y'(0) + 6y'(1) - y''(0) - y''(1) = 12 $$
$$ 4y(0) + y(1) + 3y'(0) - \frac{1}{2}y'(1) + y''(0) + y''(1) = 8 $$

When $\alpha = 2.3$ and $\beta = 1$, then the exact solution for Equation 4.11 is $y(x) = x^{\frac{7}{2}}$.
The approximate solutions of Equation 4.3 are shown in the table and figures below. From Table 4.6, we compare the absolute errors obtained by generalized hat functions method [2] and CAS wavelets (CASW) for $n = 16$ [20], we can see that the error is smaller whenever x approaches to 1 which is the end point of the interval.
After solving for the c_is and obtaining y for $n = 8, 12, 16$, we get the following table for some selected values x_i.

Table 4.4: Approximate solution for Equation 4.11 for ($n = 8, 10, 12$)

x	$y_1[n=8]$	$y_2[n=12]$	$y_3[n=16]$	Exact
0	-0.027204779	-0.015195009	-0.010202975	0
0.125	-0.010964336	-0.00656962	-0.004372956	0.000690534
0.25	0.007858289	0.006568877	0.006658802	0.0078125
0.375	0.040145738	0.035137011	0.03381492	0.032293078
0.5	0.100145024	0.093386822	0.091349298	0.088388348
0.625	0.204759169	0.198226727	0.196171752	0.193010111
0.75	0.373176938	0.368849727	0.367476493	0.365354467
0.875	0.626643073	0.626486015	0.626494861	0.626654533

The absolute error of the solution of Equation 4.11 is depicted in the following table:

Table 4.5: Absolute error in Equation 4.11 for ($n = 8, 12, 16$)

x	$(n=8)$	$(n=12)$	$(n=16)$
0	0.027204779	0.015195009	0.010202975
0.125	0.01165487	0.007260154	0.00506349
0.25	4.57891E-05	0.001243623	0.001153698
0.375	0.00785266	0.002843933	0.001521842
0.5	0.011756676	0.004998475	0.00296095
0.625	0.011749058	0.005216616	0.003161641
0.75	0.00782247	0.00349526	0.002122026
0.875	1.14603E-05	0.000168518	0.000159672

Figures 5, 6 and 7 represent the graph of y_i for $i = 1, 2, 3, 4$ for some α, β and n.

Figure 5: Graphs of approximate and exact solutions for Equation 4.11 when $n = 8$)

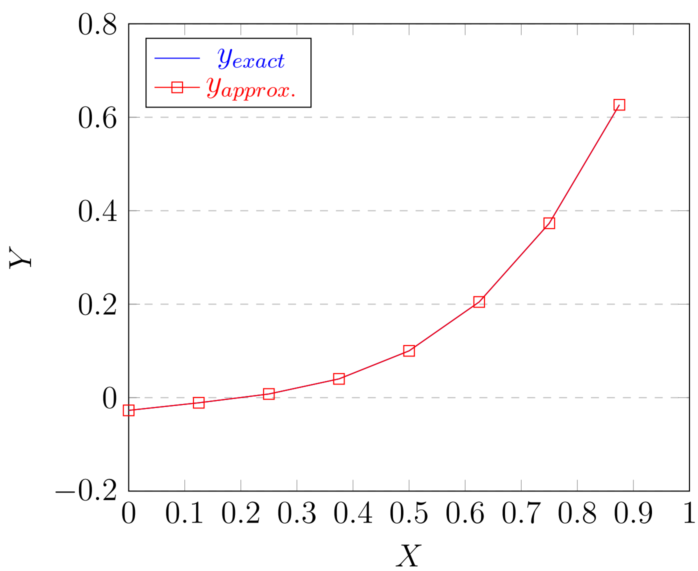


Figure 6: Graphs of approximate solutions for Equation 4.11 ($n = 12$, $\beta = 1$)

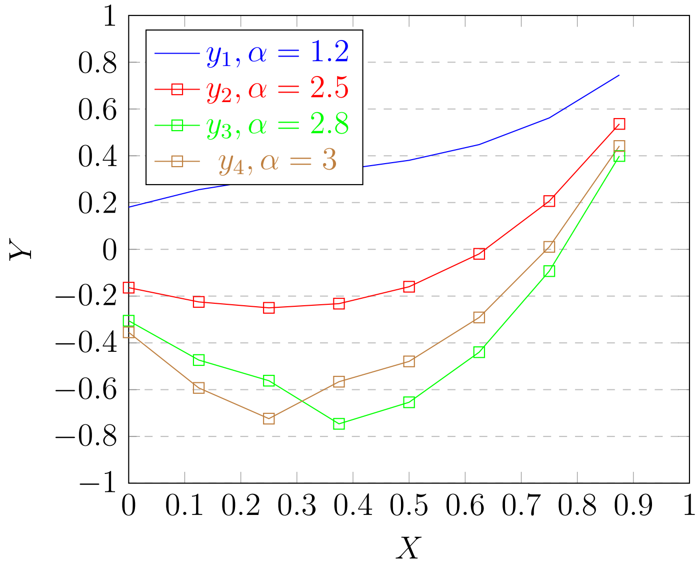
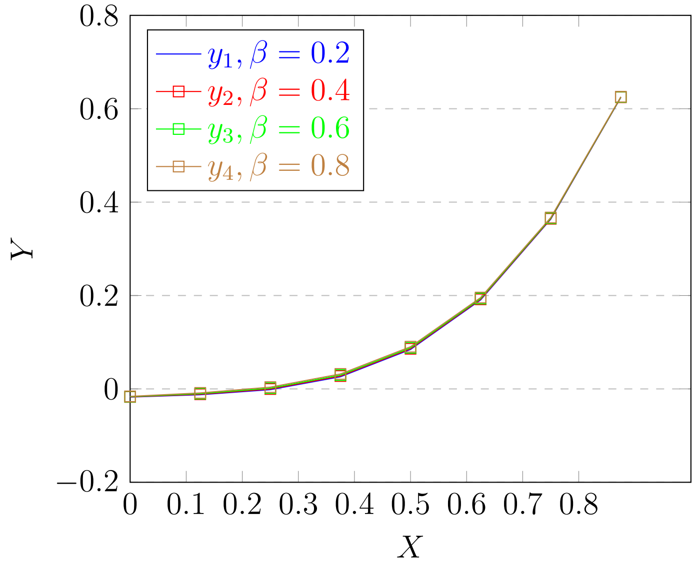


Figure 7: Graphs of approximate solutions for Equation 4.11 ($n = 12$, $\alpha = 2.3$)



Comparing the absolute error with the absolute errors obtained in [20] by CAS wavelet (CASW) method and in [2] obtained by generalized hat function method (GHF).

Table 4.6: The absolute errors of the solution of Equation 4.11 for ($n = 16$)

x	CASW	GHF	Our Method
0	0.0000052528	0	0.010202975
$\frac{1}{8}$	0.00021658	0.000032343	0.00506349
$\frac{2}{8}$	0.00052365	0.000058723	0.001153698
$\frac{3}{8}$	0.00082316	0.000061423	0.001521842
$\frac{4}{8}$	0.0024582	0.00022317	0.00296095
$\frac{5}{8}$	0.0070243	0.00044326	0.003161641
$\frac{6}{8}$	0.044565	0.0064325	0.002122026
$\frac{7}{8}$	0.082364	0.0072324	0.000159672

5. CONCLUSION

In this paper we have considered a numerical tool for solving certain types of Fredholm-Volterra integro-fractional differential equations and we have found an error bound of the solution. Moreover, by using numerical examples, it has been shown that this method is better comparing with the method in [9] and [14]. However, as x tends to b in the interval $[a, b]$ we obtained accurate results comparing with the method in [20] and [2].

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