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LOCALLY BICOMPLEX CONVEX MODULE AND THEIR APPLICATIONS

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ABSTRACT. Let X be a locally $\mathbb{BC}$ convex module and $L(X)$ be the family of all continuous bicomplex linear operators on X. In this paper, we study some concepts of $\mathbb{D}$-valued seminorms on locally $\mathbb{BC}$ convex module. Further, we study the bicomplex version of C_o and $(C_o, 1)$ semigroup. The work of this paper is inspired by the work in [2] and [6].

Key words and phrases: Hyperbolic modules; Locally $\mathbb{BC}$-convex modules; C_o and $(C_o, 1)$ semigroup.

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1. INTRODUCTION

Bicomplex numbers have been studied for quite a long time and lot of work has been done in this area. The work on bicomplex numbers probably begin with the work of Italian school of Segre, Spampinato and Scorza Dragoni. At present bicomplex analysis is an active area of research and many paper are being published in this direction. In [15], G. B. Price present the most comprehensive and detail on bicomplex analysis. The Recent book [1] and [13] give the most systematic developments of bicomplex analysis and bicomplex functional analysis.
Topological vector spaces are one of the basic structures investigated in functional analysis. The bicomplex version of topological vector spaces was introduced in [10]. For the study of topological vector spaces we refer the reader to [3], [5], [14], [16] and [17] and references therein provide more information on these applications.
Strongly continuous one parameter semigroups of operators (C_0 Semigroup) have been discussed by many authors under varying restrictions. In 1957, the edition of semigroups and Functional Analysis, by E. Hille and R. S. Phillips, the theory of one-parameter semigroups on Banach space attained its first peak. For the detailed study of the theory of semigroups one can refer to book [5], [8],[6].
Some results on $(C_0, 1)$ and C_0 semigroup were proposed in paper [2] for locally convex space. Now, we summarize some basic properties of Bicomplex numbers. For the details about bicomplex numbers and hyperbolic numbers we can refer to [1],[7], [13],[15],[18],[19].
The set of bicomplex numbers is denoted by $\mathbb{BC}$ and is defined as the commutative ring whose elements are of the form $Z = z_1 + jz_2$, where $z_1 = x_1 + iy_1 \in \mathbb{C}(i)$ and $z_2 = x_2 + iy_2 \in \mathbb{C}(i)$ are complex numbers with imaginary units i and j respectively. Note that $i^2 = j^2 = -1$.
The set $\mathbb{D}$ of hyperbolic numbers is defined as

$$\mathbb{D} = \{\alpha = \alpha_1 + k\alpha_2 : \alpha_1, \alpha_2 \in \mathbb{R} \; with \; k \notin \mathbb{R}\},$$

where k is hyperbolic unit such that $k^2 = 1$.
Since bicomplex numbers are defined as the pair of two complex numbers connected through another imaginary unit, there are several notions of conjugations. Let $Z = z_1 + jz_2 \in \mathbb{BC}$. Then the following three conjugations can be defined in $\mathbb{BC}$:
(i) $\overline{Z} = \overline{z_1} + j\overline{z_2}$, (ii) $Z^{\dagger} = z_1 - jz_2$, (iii) $Z^* = \overline{z_1} - j\overline{z_2}$, where $\overline{z_1}, \overline{z_2}$ denote the usual complex conjugates to z_1, z_2 in $\mathbb{C}(i)$. For bicomplex numbers we have three possible moduli which are defined as follows: (i) $\left|\overline{Z}\right|_j^2 = Z.\overline{Z}$, (ii) $|Z|_i^2 = Z.Z^{\dagger}$, (iii) $|Z|_k^2 = Z.Z^*$.

The hyperbolic numbers e and $e^{\dagger}$ are defined as

$$e = \frac{1+k}{2} \quad \text{and} \quad e^{\dagger} = \frac{1-k}{2}\,.$$

Here, e and $e^{\dagger}$ form a pair of idempotents such that their product is zero and sum is equal to 1. Thus these are the zero divisors and we denote the set of zero divisors of $\mathbb{BC}$ by $\mathbb{NC}$ i.e.,

$$\mathbb{NC} = \left\{Z \mid Z \neq 0,\ {z_1}^2 + {z_2}^2 = 0\right\}.$$

Any bicomplex number $Z = z_1 + jz_2$ can be uniquely written as

$$Z = \beta_1 e + \beta_2 e^{\dagger}, \tag{1.1}$$

where $\beta_1 = z_1 - iz_2$ and $\beta_2 = z_1 + iz_2 \in \mathbb{C}(i)$. Formulae (1.1) is called the idempotent decomposition representation of a bicomplex number Z.
A hyperbolic number $\alpha = \gamma_1 + k\gamma_2$ can be written as

$$\alpha = \alpha_1 e + \alpha_2 e^{\dagger},$$

where $\alpha_1 = \gamma_1 + \gamma_2$, $\alpha_2 = \gamma_1 - \gamma_2$ are real numbers, we say that α is positive if $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$. Thus, the set of positive hyperbolic numbers $\mathbb{D}_+$ is given by

$$\mathbb{D}_+ = \{\alpha = \alpha_1 e + \alpha_2 e^\dagger : \alpha_1 \geq 0, \alpha_2 \geq 0\}.$$

For $P, Q \in \mathbb{D}$, (set of hyperbolic numbers) we define a relation $\leq'$ on $\mathbb{D}$ by $P \leq' Q \iff Q - P \in \mathbb{D}_+$. This relation is reflexive, anti-symmetric as well as transitive and hence defines a partial order on $\mathbb{D}$, (cf. [1]).
A $\mathbb{BC}$-module (or $\mathbb{D}$-module) $\mathbf{B}$ can be written as

$$\mathbf{B} = e\mathbf{B}_1 + e^\dagger \mathbf{B}_2 \ , \tag{1.2}$$

where $\mathbf{B}_1 = e\mathbf{B}$ and $\mathbf{B}_2 = e^\dagger \mathbf{B}$ are $\mathbb{C}(i)$-vector (or $\mathbb{R}$-vector) spaces. The bicomplex modules were introduced in [11], [21]. In this paper, we extend the results of paper [2].

2. $\mathbb{D}$- VALUED SEMINORM ON LOCALLY $\mathbb{BC}$ CONVEX MODULE

In this section, we study some properties of topological vector spaces with $\mathbb{BC}$ scalars. For details on topological vector spaces, we refer to [3], [9], [16].

Definition 2.1. [11] Let X be $\mathbb{BC}$-module and τ be a Hausdorff topology on X such that the operations
(i) $+ : X \times X \to X$ and
(ii) $. : \mathbb{BC} \times X \to X$
are continuous. Then the pair (X, τ) is called a topological $\mathbb{BC}$ module.

Remark 2.1. Let (X, τ) be a topological $\mathbb{BC}$-module. Write

$$X = X_1 e + X_2 e^\dagger$$

where $X_1 = eX$ and $X_2 = e^\dagger X$ are $\mathbb{C}(i)$-vector spaces. Then $\tau_1 = \{e_l G : G \in \tau\}$ is a Hausdorff on X_l for $l = 1, 2$.

Example 2.1. *Every $\mathbb{BC}$-module with $\mathbb{D}$-valued norm (or real-valued norm) is a topological $\mathbb{BC}$-module.*

Let X be a $\mathbb{BC}$-module. Then $X = X_1 e + X_2 e^\dagger$ and $p : X \to \mathbb{D}$ be a $\mathbb{D}$-valued seminorm on X.

$$\begin{aligned} p(x) &= (p_1 e + p_2 e^\dagger)(x_1 e + x_2 e^\dagger) \\ &= p_1(xe)e + p_2(xe^\dagger)e^\dagger, \end{aligned}$$

where $p_1, p_2 : X \to \mathbb{R}$ are real seminorms on X_1 and X_2 respectively.

Definition 2.2. [11] A topological $\mathbb{BC}$-module (X, τ) is a locally bicomplex convex (or $\mathbb{BC}$-convex) module if it has a neighbourhood base at 0 of $\mathbb{BC}$-convex sets.

Definition 2.3. A family of $\mathbb{D}$-valued seminorm $\mathbb{P}$ on locally $\mathbb{BC}$ convex module X is said to be saturated if $\max_{1 \leq \alpha \leq n} p_\alpha \in \mathbb{P}$ where $p_\alpha \in \mathbb{P}$ $(1 \leq \alpha \leq n)$.

Let X be a locally $\mathbb{BC}$ convex module and $\mathbb{P} = \{p_\alpha : \alpha \in I\}$ be a saturated family of continuous $\mathbb{D}$-valued seminorms on X. We set

$$\overline{\mathcal{B}}_{p_\alpha}(0, \epsilon) = \{x \in X : p_\alpha(x) \leq' \epsilon\}, \ \epsilon >' 0, \alpha \in I \ and \, p_\alpha \in \mathbb{P}.$$

Then, $\mathbb{U}_{p_\alpha} = \{\overline{\mathcal{B}}_{p_\alpha}(0, \epsilon) : \alpha \in I, \epsilon >' 0\}$ forms the neighbourhood base at origin for the topology of X.
Let $\overline{\mathcal{B}}_{p_\alpha}(0) = \{x \in X : p_\alpha(x) \leq' 1\}$ and $\mathcal{B}_{p_\alpha}(0) = \{x \in X : p_\alpha(x) <' 1\}$. Then $\overline{\mathcal{B}}_{p_\alpha}(0)$ and $\mathcal{B}_{p_\alpha}(0)$ are $\mathbb{BC}$-convex, $\mathbb{BC}$-balanced and $\mathbb{BC}$-absorbing set[11].

Theorem 2.1. *Suppose X and Y are locally $\mathbb{BC}$ convex-module, whose topologies defined by two families of $\mathbb{D}$-valued seminorm: $\mathcal{P}_1$ on X and $\mathcal{P}_2$ on Y. For a $\mathbb{BC}$-linear operator $T : X \to Y$, the following are equivalent:*

(i) *T is continuous.*
(ii) *T is continuous at 0.*
(iii) *For every $q \in \mathcal{P}_2,\ \exists\ p_1, p_2, ...p_n \in \mathcal{P}_1$ such that $sup\{q(Tx) : x \in \mathcal{B}_{p_1}(0) \cap, ... \cap \mathcal{B}_{p_n}(0)\} <' \infty$.*
(iv) *For every $q \in \mathcal{P}_2,\ \exists\, p_1, p_2, ...p_n \in \mathcal{P}_1$ and $t_1, t_2, ...t_n \geq' 0$ such that $q(Tx) \leq' t_1 p_1(x) + t_2 p_2(x) + ... + t_n p_n(x), \forall x \in X$.*

Proof. **(i)**$\Leftrightarrow$ **(ii)** trivial.
(ii)$\Leftrightarrow$ **(iii)** Assume that T is continuous at 0 and let q is a $\mathbb{D}$-valued seminorm. Since q is continuous, the set $\mathcal{B}_q(0) = \{y \in Y : q(y) <' 1\}$ is a (open) neighbourhood at 0 in Y. Since T is continuous at 0, the preimage $\mathcal{N} = T^{-1}\mathcal{B}_q(0)$, which is given by $\mathcal{N} = \{x \in X : q(Tx) <' 1\}$ is a neighbourhood of 0 in X, then there exist $p_1, ...p_n \in \mathcal{P}_1$ and $\epsilon_1, \epsilon_2, ...\epsilon_n >' 0$ such that $\mathcal{B}_{p_1}(0, \epsilon_1) \cap \mathcal{B}_{p_2}(0, \epsilon_2) \cap ... \cap \mathcal{B}_{p_n}(0, \epsilon_n) \subset \mathcal{N}$, where $\mathcal{B}_p(0, \epsilon) = \{x \in X : p(x) <' \epsilon\}$.
Let $\epsilon = min\{\epsilon_1, ...\epsilon_n\}$. Suppose $x \in \mathcal{B}_{p_1}(0) \cap ... \cap \mathcal{B}_{p_n}(0)$.
Then $\epsilon x \in \mathcal{B}_{p_1}(0, \epsilon) \cap ... \cap \mathcal{B}_{p_n}(0, \epsilon) \subset \mathcal{N} \Rightarrow \epsilon x \in \mathcal{N}$, for $x \in \mathcal{B}_p(0)$. So, we get $q(T(\epsilon x)) <' 1$ $\Rightarrow Sup\{q(Tx) : x \in \mathcal{B}(p)\} \leq \epsilon^{-1} < \infty$.
i.e, $q(T) = Sup\{q(Tx) : x \in \mathcal{B}(p)\} < \infty$.

(iii)$\Rightarrow$ **(iv)** Assume condition (iii)
i.e, $q(T) = sup\{q(Tx) : x \in \mathcal{B}_p(0)\} < \infty$.
let $t = sup\{q(Tx) : x \in \mathcal{B}_{p_1}(0) \cap ... \cap \mathcal{B}_{p_n}(0)\}$. To prove (iv), we show that $q(Tx) \leq' tp_1(x) + ... + tp_2(x)$.
Take $x \in X$, and then for every $\epsilon >' 0$, the vector $x_\epsilon = \frac{x}{p_1(x)+...+p_n(x)+\epsilon}$ satisfies
$p(x_\epsilon) = \frac{p(x)}{p_1(x)+...+p_n(x)+\epsilon} <' 1$.
So, $x_\epsilon \in \mathcal{B}_{p_1}(0) \cap .... \cap \mathcal{B}_{p_n}(0)$, by condition (ii), it follows that $q(Tx_\epsilon) \leq' sup\ q(Tx) = t$. Then, we get

$$\begin{aligned} q(Tx) &= q(Tx_\epsilon)[p_1(x) + ... + p_n(x) + \epsilon] \\ &\leq' t[p_1(x) + ... + p_n(x) + \epsilon]. \end{aligned}$$

Therefore, the inequality $q(Tx) \leq' tq(x)$ holds, for $\epsilon > 0$,

(iv)$\Rightarrow$**(i)** Assume condition (iv) and let us prove that T is continuous. Take some net (x_l) in X which converges to some $x \in X$, we have to show that $(Tx_l) \to Tx$, which means $q(Tx_l - Tx) \to 0, \forall q \in \mathcal{P}_2$. We have

$$q(Tx) \leq' t_1 p_1(x) + ... + t_n p_n(x). \tag{2.1}$$

By linearity, we get $Tx_l - Tx = T(x_l - x)$,
so, eqn (2.1) gives

$$0 \leq q(Tx_l - Tx) \leq' t_1 p_1(x_l - x) + ...t_n p_n(x_l - x) \tag{2.2}$$

since $x_l \to x$, we also know that $p(x_l - x) \to 0, \forall p \in \mathcal{P}_1$ So, from (2.2), we get $q(Tx_l - Tx) \to 0$. ∎

Definition 2.4. Let X be a locally $\mathbb{BC}$ convex module. Then we say that a $\mathbb{BC}$ linear operator $T : X \to X$ is bounded with respect to $\mathbb{P}$, if for any $p_\alpha \in \mathbb{P}$ there exists λ (depend on α,T) with

$\lambda >' 0$ such that

$$p_\alpha(Tx) \leq' \lambda p_\alpha(x),\ \forall x \in X\ and\ \alpha \in I. \tag{2.3}$$

We know that $\overline{\mathcal{B}}_{p_\alpha}(0) = \{x \in X : p_\alpha(x) \leq' 1\}$. Then (2.3) means $T\overline{\mathcal{B}}_{p_\alpha}(0) \subset \lambda\overline{\mathcal{B}}_{p_\alpha}(0),\ \forall\ \alpha \in I$. $L_I(X)$ denote the set which consists of all bounded operators with respect to $\mathbb{P}$.

Since each $\overline{\mathcal{B}}_{p_\alpha}(0)$ is $\mathbb{BC}$-balanced, so if for any $\lambda \in \mathbb{BC}$ with $|\lambda|_k \leq' 1$, then $\lambda\overline{\mathcal{B}}_{p_\alpha}(0) \subset \overline{\mathcal{B}}_{p_\alpha}(0)$. It follows that $T\overline{\mathcal{B}}_{p_\alpha}(0) \subset \lambda\overline{\mathcal{B}}_{p_\alpha}(0) \subset \overline{\mathcal{B}}_{p_\alpha}(0)$. Thus, $T\overline{\mathcal{B}}_{p_\alpha}(0) \subset \overline{\mathcal{B}}_{p_\alpha}(0)$ and

$$\begin{aligned} p_\alpha(Tx) &\leq' |\lambda|_k\, p_\alpha(x) \\ &\leq' p_\alpha(x),\ for\ |\lambda|_k \leq' 1 \\ &\leq' 1 \\ p_\alpha(Tx) &\leq' 1. \end{aligned}$$

Remark 2.2. We can say that the family of $\mathbb{BC}$-linear Operator $T : X \to X$ such that there exist a hyperbolic number λ (depending on α and T) with $T\overline{\mathcal{B}}_{p_\alpha}(0) \subset \lambda\overline{\mathcal{B}}_{p_\alpha}(0),\ \forall \alpha \in I$ is identical with $L_I(X)$.

Furthermore, with addition defined pointwise and multipication by composition, $L_I(X)$ becomes a $\mathbb{BC}$-algebra. For each $p_\alpha \in \{p_\alpha : \alpha \in I\}$, the mapping $p_\alpha : L_I(X) \to \mathbb{D}$ defined as

$$\begin{aligned} p_\alpha(T) &= inf_{\mathbb{D}}\{\lambda : p_\alpha(Tx) \leq' \lambda p_\alpha(x), \forall x \in X\} \\ &= sup_{x \in X}\{p_\alpha(Tx) : p_\alpha(x) \leq' 1\} \end{aligned}$$

is also a $\mathbb{D}$ valued seminorm.

$$So,\ p_\alpha(T) = inf_{\mathbb{D}}\{\lambda : p_\alpha(Tx) \leq' \lambda p_\alpha(x)\}.$$

Setting $T = eT_1 + e^\dagger T_2$ and $\lambda = e\lambda_1 + e^\dagger\lambda_2$ with $\lambda_1, \lambda_2 \in \mathbb{R}^+$.

$$\begin{aligned} p_\alpha(T) &= p_\alpha(eT_1 + e^\dagger T_2) \\ &= inf_{\mathbb{D}}\{\lambda : p_\alpha(T) \leq' \lambda p_\alpha(x)\} \\ &= inf_{\mathbb{D}}\{e\lambda_1 + e^\dagger\lambda_2\} \end{aligned}$$

such that

$$\begin{aligned} p_\alpha(eT_1x_1 + e^\dagger T_2x_2) &\leq' (e\lambda_1 + e^\dagger\lambda_2)(ep_{\alpha,1}(x_1) + e^\dagger p_{\alpha,1}(x_2)) \\ \Rightarrow P_\alpha(T) &= e.inf\lambda_1 + e^\dagger.inf\lambda_2 \\ such\ that\ p_{\alpha,1}(T_1x_1) \leq \lambda_1 p_{\alpha,1}(x_1) \quad and &\quad p_{\alpha,2}(T_2x_2) \leq \lambda_2 p_{\alpha,2}(x_2). \end{aligned}$$

Thus, the hyperbolic semi-norm of T can also be defined as
$p_\alpha(T) = e.p_{\alpha,1}(T_1) + e^\dagger p_{\alpha,2}(T_2)$ where $p_{\alpha,l}(T_l) = inf\{\lambda_l : p_{\alpha,l}(Tx) \leq \lambda_l p_{\alpha,l}(x)\}$, $l = 1, 2$.

Remark 2.3. Let X be a locally $\mathbb{BC}$ convex module. A bicomplex linear Operators T and S are belongs to $L_I(X)$. Then there exist $\mathbb{C}(i)$ linear (or $\mathbb{R}$-linear) Operator T_l and S_l on $L_I(X_l), l = 1, 2$ such that $T = T_1e + T_2e^\dagger$ and $S = S_1e + S_2e^\dagger, TS = T_1S_1e + T_2S_2e^\dagger$ and

$p_\alpha = p_{\alpha,1}e + p_{\alpha,2}e^\dagger$.

$$\begin{aligned}
p_\alpha(TS) &= p_{\alpha,1}(T_1S_1e)e + p_{\alpha,2}(T_2S_2e^\dagger)e^\dagger \\
&\leq' p_{\alpha,1}(T_1e)P_{\alpha,1}(S_1e)e + p_{\alpha,2}(T_2e^\dagger)p_{\alpha,2}(S_2e^\dagger)e^\dagger \\
&= [(p_{\alpha,1}e + p_{\alpha,2}e^\dagger)(T_1e + T_2e^\dagger)][(p_{\alpha,1}e + p_{\alpha,2}e^\dagger)(S_1e + S_2e^\dagger)] \\
&= p_\alpha(T)p_\alpha(S) \\
p_\alpha(Tx) &= p_{\alpha,1}(T_1x_1e)e + p_{\alpha,2}(T_2x_2e^\dagger)e^\dagger \\
&\leq' p_{\alpha,1}(T_1e)p_{\alpha,1}(x_1e)e + p_{\alpha,2}(T_2e^\dagger)p_{\alpha,2}(x_2e^\dagger)e^\dagger \\
&= [(p_{\alpha,1}e + p_{\alpha,2}e^\dagger)(T_1e + T_2e^\dagger)][(p_{\alpha,1}e + p_{\alpha,2}e^\dagger)(x_1e + x_2e^\dagger)] \\
&= p_\alpha(T)p_\alpha(x)
\end{aligned}$$

The family $\{p_\alpha : \alpha \in I\}$ of $\mathbb{D}$-valued seminorms on $L_I(X)$ defined the topology on $L_I(X)$. Under this topology, $L_I(X)$ becomes Hausdorff locally multiplicatively $\mathbb{BC}$-convex topological algebra. Then $L_I(X)$ is complete if locally $\mathbb{BC}$ convex module X is complete iff X_1 and X_2 are complete locally convex spaces.

Definition 2.5. A $\mathbb{BC}$-algebra $L_I(X)$ is said to be locally multiplicatively $\mathbb{BC}$-convex if it has a neighbourhood base $\mathbb{U}_{p_\alpha}$ at 0 such that each $\overline{\mathcal{B}}_{p_\alpha}(0) \in \mathbb{U}_{p_\alpha}$ is $\mathbb{BC}$-convex and $\mathbb{BC}$-balanced (i.e, $\lambda\overline{\mathcal{B}}_{p_\alpha}(0) \subset \overline{\mathcal{B}}_{p_\alpha}(0)$ for $|\lambda|_k \leq' 1$) and satisfies $\overline{\mathcal{B}}^2_{p_\alpha}(0) \subset \overline{\mathcal{B}}_{p_\alpha}(0)$.

Theorem 2.2. *Let X be a $\mathbb{BC}$-module and p_α be a continuous $\mathbb{D}$ valued seminorm on X. Then $N_\alpha = p_\alpha^{-1}(0) = \{x \in X : P_\alpha(x) = 0\}$ is a submodule of X.*

Proof. If $x, y \in X$ and $a, b \in \mathbb{BC}$. Then

$$\begin{aligned}
0 &\leq p_\alpha(ax + by) \\
&\leq |a|_k\, p_\alpha(x) + |b|_k\, p_\alpha(y) \\
&= 0.
\end{aligned}$$

$$\Rightarrow ax + by \in N_\alpha.$$

Hence, $N_\alpha = p_\alpha^{-1}(0)$ is a submodule of X. ∎

Definition 2.6. [12] Let X be a $\mathbb{BC}$-module and $N_\alpha = p_\alpha^{-1}(0)$ be a submodule of X. We write $X = X_1e + X_2e^\dagger$, where X_1 and X_2 are complex linear spaces and $N_\alpha = N_{\alpha,1}e + N_{\alpha,2}e^\dagger$. Then $N_{\alpha,1}$ and $N_{\alpha,2}$ are complex linear subspace of X_1 and X_2 respectively, so that $\frac{X_l}{N_{\alpha,l}}$, $l = 1, 2$ are quotient space over the Complex field.
Consider the set $\frac{X}{N_\alpha} = \{x + N_\alpha : x \in X\}$, where $x_\alpha = x + N_\alpha$ is the coset of x in $\frac{X}{N_\alpha}$.
let $x, y \in X, a \in \mathbb{BC}$.

(i) $(N_\alpha + x) + (N_\alpha + y) = (N_\alpha + x + y)$.
(ii) $a(N_\alpha + x) = N_\alpha + ax$.

With the operations defined above $\frac{X}{N_\alpha}$ form a module over $\mathbb{BC}$ and is called $\mathbb{BC}$ quotient module.

Further, $N_\alpha + x = (N_{\alpha,1} + x)e + (N_{\alpha,2} + x)e^\dagger$ for any $x \in X$, so one can write that

$$\begin{aligned}
\frac{X}{N_\alpha} &= \frac{X_1e + X_2e^\dagger}{N_{\alpha,1}e + N_{\alpha,2}e^\dagger} \\
&= \frac{X_1}{N_{\alpha,1}}e + \frac{X_2}{N_{\alpha,2}e^\dagger}.
\end{aligned}$$

Let X be a locally $\mathbb{BC}$ convex module and p_α is a $\mathbb{D}$-valued seminorm on X. If N_α is the kernel of the $\mathbb{D}$-valued seminorm, then $\frac{X}{N_\alpha} = X_\alpha$ is $\mathbb{BC}$-normed linear module and the $\mathbb{D}$-valued norm on $\frac{X}{N_\alpha}$ is defined as

$$\|x_\alpha\|_{\mathbb{D}} = \|x + N_\alpha\|_{\mathbb{D}} = p_\alpha(x) \; for\; each\; x \in X,$$

where x_α is the coset of x in the $\mathbb{BC}$-quotient module $\frac{X}{N_\alpha}$.

Remark 2.4.

$$\begin{aligned} \|x_\alpha\|_{\mathbb{D}} &= \|x + N_\alpha\|_{\mathbb{D}} \\ &= \|x + N_{\alpha,1}\|_1 e + \|x + N_{\alpha,2}\|_2 e^\dagger \\ &= p_{\alpha,1}(x)e + p_{\alpha,2}(x)e^\dagger \\ &= p_\alpha(x). \end{aligned}$$

Thus, $\|.\|_{\mathbb{D}}$ is a $\mathbb{D}$-valued norm defined on $\mathbb{BC}$-quotient module $\frac{X}{N_\alpha}$.

Theorem 2.3. *$\|.\|_{\mathbb{D}}$ is a $\mathbb{D}$-valued norm defined on $\mathbb{BC}$-quotient module $\frac{X}{N_\alpha} \Leftrightarrow \|.\|_1$ and $\|.\|_2$ are real-valued norm on the quotient spaces $\frac{X_1}{N_{\alpha,1}}$ and $\frac{X_2}{N_{\alpha,2}}$.*

Proof. We have $x = x_1 e + x_2 e^\dagger$ and $N_\alpha = N_{\alpha,1}e + N_{\alpha,2}e^\dagger$.
(i)

$$\begin{aligned} \|x + N_\alpha\|_{\mathbb{D}} &= 0 \Rightarrow x + N_\alpha = 0 \\ \Leftrightarrow \|x_1 + N_{\alpha,1}\|_1 e + \|x_2 + N_{\alpha,2}\|_2 e^\dagger &= 0 \Rightarrow (x_1 + N_{\alpha,1})e + (x_2 + N_{\alpha,2})e^\dagger = 0 \\ \Leftrightarrow \|x_1 + N_{\alpha,1}\|_1 = 0 \Rightarrow x_1 + N_{\alpha,1} = 0 \quad and &\quad \|x_2 + N_{\alpha,2}\|_2 = 0 \Rightarrow (x_2 + N_{\alpha,2}) = 0. \end{aligned}$$

(ii)

$$\begin{aligned} \|a x_\alpha\|_{\mathbb{D}} &= |a|_k \|x_\alpha\|_{\mathbb{D}} \\ \|a(x + N_\alpha)\|_{\mathbb{D}} &= |a|_k \|x + N_\alpha\|_{\mathbb{D}} \\ \Leftrightarrow \|a_1(x_1 + N_{\alpha,1})\|_1 e + \|a_2(x_2 + N_{\alpha,2})\|_2 e^\dagger &= |a_1| \|x_1 + N_{\alpha,1}\|_1 e + |a_2| \|x_2 + N_{\alpha,2}\|_2 e^\dagger \\ \Rightarrow \|a_1(x_1 + N_{\alpha,1})\|_1 = |a_1| \|x_1 + N_{\alpha,1}\|_1 \quad and &\quad \|a_2(x_2 + N_{\alpha,2})\|_2 = |a_2| \|x_2 + N_{\alpha,2}\|_2 \end{aligned}$$

(iii)

$$\begin{aligned} &\|(x + N_\alpha) + (y + N_\alpha)\|_{\mathbb{D}} \le \|x + N_\alpha\|_{\mathbb{D}} + \|y + N_\alpha\|_{\mathbb{D}} \\ &\Leftrightarrow \|(x_1 + N_{\alpha,1}) + (y_1 + N_{\alpha,1})\|_1 e + \|(x_2 + N_{\alpha,2}) + (y_2 + N_{\alpha,2})\|_2 e^\dagger \\ &\le \|x_1 + N_{\alpha,1}\|_1 e + \|y_1 + N_{\alpha,1}\|_1 e + \|x_2 + N_{\alpha,2}\|_2 e^\dagger + \|y_2 + N_{\alpha,2}\|_2 e^\dagger \\ &\Leftrightarrow \|(x_1 + N_{\alpha,1}) + (y_1 + N_{\alpha,1})\|_1 \le \|x_1 + N_{\alpha,1}\|_1 + \|y_1 + N_{\alpha,1}\|_1 \; and \\ &\|(x_2 + N_{\alpha,2}) + (y_2 + N_{\alpha,2})\|_2 e^\dagger \le \|x_2 + N_{\alpha,2}\|_2 + \|y_2 + N_{\alpha,2}\| \end{aligned}$$

∎

The set $L(X_\alpha)$ denote the set of all $\mathbb{BC}$-linear Operator i.e., $L(X_\alpha) = \{T/T : X_\alpha \to X_\alpha\}$. Let X_α is a $\mathbb{BC}$-normed module. We can write $X_\alpha = X_{\alpha,1}e + X_{\alpha,2}e^\dagger$, where $X_{\alpha,1} = eX_\alpha$ and $X_{\alpha,2} = e^\dagger X_\alpha$ are normed linear space.

Definition 2.7. Let X_α be a $\mathbb{BC}$-module with $\mathbb{D}$-valued norm. Let $T_\alpha : X_\alpha \to X_\alpha$ be a map such that

$$T_\alpha(a x_\alpha + b y_\alpha) = aT_\alpha(x_\alpha) + bT_\alpha(y_\alpha), \; \forall x_\alpha, y_\alpha \in X_\alpha, \; \forall a, b \in \mathbb{BC}.$$

Then we say that T_α is a $\mathbb{BC}$-linear Operator on X_α.
The idempotent decomposition of the operator is given as (see[4])

$$T_\alpha = T_{\alpha,1}e + T_{\alpha,2}e^\dagger,$$

where $T_{\alpha,1} : eX_\alpha \to eX_\alpha$ and $T_{\alpha,2} : e^\dagger X_\alpha \to e^\dagger X_\alpha$ are the linear operators.

Let $x, y \in X$ and $a, b \in \mathbb{BC}$. Then π_α is a mapping from locally $\mathbb{BC}$ convex module X onto $\frac{X}{N_\alpha} = X_\alpha$ as

$$\pi_\alpha(x) = x_\alpha = x + N_\alpha.$$

$$\begin{aligned}
\pi_\alpha(x+y) &= (x+y)_\alpha \\
&= x + y + N_\alpha \\
&= (x_1 + y_1 + N_{\alpha,1})e + (x_2 + y_2 + N_{\alpha,2})e^\dagger \\
&= (x_1 + N_{\alpha,1})e + (x_2 + N_{\alpha,2})e^\dagger + (y_1 + N_{\alpha,1})e + (y_2 + N_{\alpha,2})e^\dagger \\
&= (x + N_\alpha) + (y + N_\alpha) = \pi_\alpha(x) + \pi_\alpha(y) \\
\pi_\alpha(ax) &= (ax)_\alpha \\
&= ax + N_\alpha \\
&= (a_1e + a_2e^\dagger) + \big((x_1e + x_2e^\dagger) + (N_{\alpha,1}e + N_{\alpha,2}e^\dagger)\big) \\
&= a(x + N_\alpha) = ax_\alpha = a\pi_\alpha(x)
\end{aligned}$$

Thus, π_α is $\mathbb{BC}$ homomorphism on $\frac{X}{N_\alpha}$.

Remark 2.5. For any X, $\pi_\alpha(x) = x_\alpha = x + N_\alpha$.

$$\begin{aligned}
\Rightarrow x + N_\alpha &= (x_1 + N_{\alpha,1})e + (x_2 + N_{\alpha,2})e^\dagger \\
&= \pi_{\alpha,1}(x)e + \pi_{\alpha,2}(x)e^\dagger.
\end{aligned}$$

So, we can conclude that

$$\pi_\alpha(x) = \pi_{\alpha,1}(x)e + \pi_{\alpha,2}(x)e^\dagger.$$

where $\pi_{\alpha,l}$ are natural Homomorphism of X_l onto $X_{\alpha,l}$ respt., $l = 1, 2$.

Definition 2.8. Let X_α be the $\mathbb{BC}$-normed module. Define a mapping $\pi_\alpha : X \to X_\alpha$, for each α as $\pi_\alpha(x) = x_\alpha$ for each $x \in X$. Clearly, π_α is a continuous and is called a $\mathbb{BC}$ natural homomorphism of X onto X_α.

If $T \in L_I(X)$ implies $T(p_\alpha^{-1}(0)) \subset p_\alpha^{-1}(0)$.

Remark 2.6. Let X_α be a $\mathbb{BC}$ normed module and $T_\alpha = T_{\alpha,1}e + T_{\alpha,2}e^\dagger$ be the operator on X_α defined by $T_\alpha x_\alpha = (Tx)_\alpha$, $x_\alpha \in X_\alpha$.
Now

$$\begin{aligned}
T_\alpha x_\alpha &= (Tx)_\alpha, \\
\Rightarrow T_{\alpha,1}x_{\alpha,1}e + T_{\alpha,2}x_{\alpha,2}e^\dagger &= \big(T_1x_1e + T_2x_2e^\dagger\big)_\alpha \\
\Rightarrow T_{\alpha,1}x_{\alpha,1}e + T_{\alpha,2}x_{\alpha,2}e^\dagger &= (T_1x_1)_\alpha e + (T_2x_2)_\alpha e^\dagger \\
\Rightarrow T_{\alpha,1}x_{\alpha,1} = (T_1x_1)_\alpha \quad &and \quad T_{\alpha,2}x_{\alpha,2} = (T_2x_2)_\alpha
\end{aligned}$$

where $T_{\alpha,l}$ and T_l are the operator on $X_{\alpha,l}$ and X_l respt., $l = 1, 2$.

Proposition 2.4. *let $T \in L_I(X)$, where X is a locally $\mathbb{BC}$-convex module. Then the operator $T_\alpha : X_\alpha \to X_\alpha$, $\alpha \in I$ defined by $T_\alpha x_\alpha = (Tx)_\alpha$, $x_\alpha \in X_\alpha$ is in $L(X_\alpha)$.*

Proof. Let X_α be the $\mathbb{BC}$ normed module and $x_\alpha \in X_\alpha$

$$\begin{aligned} x_\alpha &= y_\alpha \\ x + N_\alpha &= y + N_\alpha \Rightarrow x - y \in N_\alpha \\ \Leftrightarrow x_1 + N_{\alpha,1} = y_1 + N_{\alpha,1} \quad &and \quad x_2 + N_{\alpha,2} = y_2 + N_{\alpha,2} \\ \Rightarrow x_1 - y_1 \in N_{\alpha,1} \quad &and \quad x_2 - y_2 \in N_{\alpha,2}. \end{aligned}$$

Then,

$$\begin{aligned} T_\alpha(x-y)_\alpha &= T_{\alpha,1}(x_1 - y_1)_\alpha e + T_{\alpha,2}(x_2 - y_2)_\alpha e^\dagger \\ &= \left(T_1(x_1 - y_1)e + T_2(x_2 - y_2)e^\dagger\right)_\alpha \\ &= (T(x-y))_\alpha \\ &= 0 \end{aligned}$$

Also,

$$\begin{aligned} T_\alpha x_\alpha - T_\alpha y_\alpha &= (T_{\alpha,1}x_{\alpha,1} - T_{\alpha,1}y_{\alpha,1})\, e + (T_{\alpha,2}x_{\alpha,2} - T_{\alpha,2}y_{\alpha,2})\, e^\dagger \\ &= (T_1x_1 - T_1y_1)_\alpha\, e + (T_2x_2 - T_2y_2)_\alpha\, e^\dagger \\ &= (T_1(x_1 - y_1))_\alpha\, e + (T_2(x_2 - y_2))_\alpha\, e^\dagger \\ &= T_\alpha(x-y)_\alpha = 0. \\ \Rightarrow T_\alpha x_\alpha &= T_\alpha y_\alpha \end{aligned}$$

Thus, T is well defined.
Now, given that $T = T_1e + T_2e^\dagger \in L_I(X)$, so

$$\begin{aligned} p_\alpha(Tx) &= p_{\alpha,1}(T_1x_1)e + p_{\alpha,2}(T_2x_2)e^\dagger \\ &\leq' p_{\alpha,1}(T_1)p_{\alpha,1}(x_1)e + p_{\alpha,2}(T_2)p_{\alpha,2}(x_2)e^\dagger \\ &= p_\alpha(T)p_\alpha(x). \end{aligned}$$

$$\begin{aligned} \|T_\alpha x_\alpha\|_{\mathbb{D}} &= \|(Tx)_\alpha\|_{\mathbb{D}} \\ &= p_\alpha(Tx) \\ &\leq' p_\alpha(T)p_\alpha(x) \\ &= p_\alpha(T)\,\|x_\alpha\|\,,\ \forall x_\alpha \in X_\alpha. \end{aligned}$$

Therefore, T_α is bounded on X_α.
Next, we will check the linearity of T_α. Given that $T \in L_I(X)$ and $a, b \in \mathbb{D}$.

$$\begin{aligned} T_\alpha\,(ax_\alpha + by_\alpha) &= (T(ax+by))_\alpha \\ &= (aT(x) + bT(y))_\alpha \\ &= aT_\alpha x_\alpha + bT_\alpha y_\alpha. \end{aligned}$$

∎

Definition 2.9. Let X_α be a $\mathbb{BC}$-normed module and we defined a $\mathbb{BC}$ linear operator $T_\alpha : X_\alpha \to X_\alpha$ by

$$T_\alpha x_\alpha = (Tx)_\alpha$$

and

$$\|T_\alpha x_\alpha\| = \|(Tx)_\alpha\| = p_\alpha(Tx) \leq' p_\alpha(T)p_\alpha(x) \leq' p_\alpha(x) = \|x_\alpha\|$$

$\Rightarrow T_\alpha$ is bounded.

Remark 2.7. Thus T_α is bounded $\mathbb{BC}$ linear operator.
Further, we can write

$$T_\alpha = T_{\alpha,1}e + T_{\alpha,2}e^\dagger$$

where $T_{\alpha,l}$ are bounded linear operator on $X_{\alpha,l}$, $l = 1, 2$.

Definition 2.10. Let $\overline{X}_\alpha$ be the completion of $\mathbb{BC}$ normed module X_α such that $\overline{X}_\alpha$ form a $\mathbb{BC}$ Banach module.
Then $\overline{T}_\alpha : \overline{X}_\alpha \to \overline{X}_\alpha$ is a bounded $\mathbb{BC}$ linear operator.

For $T \in L_I(X)$, $\overline{T}_\alpha$ is the extended form of T_α such that

$$\left\|\overline{T}_\alpha\right\|_{\mathbb{D}} = \|T_\alpha\|_{\mathbb{D}} = sup\{\|T_\alpha x_\alpha\|_{\mathbb{D}} : \|x_\alpha\|_{\mathbb{D}} \leq' 1\} = p_\alpha(T)$$

The operator norm on T_α is

$$\|T_\alpha\|_{\mathbb{D}} = sup\{\|T_\alpha x_\alpha\|_{\mathbb{D}} : \|x_\alpha\|_{\mathbb{D}} \leq' 1\}$$

Note that this norm is a hyperbolic norm on T_α.
Hence, we can write

$$\|T_\alpha\|_{\mathbb{D}} = \|T_{\alpha,1}\|_1 e + \|T_{\alpha,2}\|_2 e^\dagger$$

where $\|.\|_1$ and $\|.\|_2$ define the usual norm on $T_{\alpha,1}$ and $T_{\alpha,2}$ respt.

Definition 2.11. (Directed Set) Let I be a partially ordered set with the order relation $\geq'$, then I is called as a directed set if for any two elements $a, b \in I$, $\exists$ some $c \in I$ such that $c \geq' a$ and $c \geq' b$.

Let $(I, \leq')$ be a directed set. Then for $\beta \geq \alpha$, $\mathbb{BC}$ operator $\pi_{\alpha\beta} : X_\beta \to X_\alpha$ defined by $\pi_{\alpha\beta}(x_\beta) = x_\alpha$ is a continuous $\mathbb{BC}$ normed module X_β onto X_α. This $\mathbb{BC}$ operator can be extended to a continuous $\mathbb{BC}$ Homomorphism $\pi_{\alpha\beta}$ from the completion $\overline{X}_\beta$ into $\overline{X}_\alpha$.
A projective system of $\mathbb{BC}$ Banach module is a pair $(\overline{X}_\alpha, \pi_{\alpha\beta})$ subject to the following properties:
(i) $(I, \leq')$ be a directed set.
(ii) $(\overline{X}_\alpha)_{\alpha\in I}$ is a family of $\mathbb{BC}$ Banach module.
(iii) $\{\pi_{\alpha\beta} : \pi_{\alpha\beta} : \overline{X}_\beta \to \overline{X}_\alpha, \alpha, \beta \in I, \alpha \leq' \beta\}$ is a famly of continuous $\mathbb{BC}$ Homomorphism such that $\pi_{\alpha\alpha}$ is the identity operator on $\overline{X}_\alpha$ $\forall \alpha \in I$.
(iv) $\pi_{\alpha\gamma} = \pi_{\alpha\beta} o \pi_{\beta\gamma}$ $\forall \alpha, \beta, \gamma \in I$ such that $\alpha \leq' \beta \leq' \gamma$. and its projective limit is denoted by X i.e,

$$X = \lim_{\leftarrow} \overline{X}_\alpha$$

where X is a complete locally bicomplex convex module.
The projective limit of the projective system is defined to be the submodule of the cartesian product $\prod \overline{X}_\alpha$ consisiting of elements which satisfy $\pi_{\alpha\beta}(\overline{x}_\beta) = (\overline{x}_\alpha)$ for $\beta > \alpha$.

Remark 2.8. The operator $\pi_{\alpha\beta}$ can be written as
$\pi_{\alpha\beta} = \pi_{\alpha\beta,1}e + \pi_{\alpha\beta,2}e^\dagger$, where

$$\pi_{\alpha\beta,l} : \pi_{\beta,l} \to \pi_{\alpha,l}, l = 1, 2$$

is an operator.

Let us denote $\overline{X}_\alpha$ with Z_α.

Definition 2.12. : Let $T_\alpha : D(T_\alpha) \subset Z_\alpha \to Z_\alpha$ be a $\mathbb{BC}$-linear operator from $D(T_\alpha) \subset Z_\alpha$ into Z_α. Then $\{T_\alpha : \alpha \in I\}$ is called (saturated) projective family of operators

$$\Leftrightarrow \quad T_\alpha(\pi_{\alpha\beta}x_\beta) = \pi_{\alpha\beta}(T_\beta x_\beta)$$

$$\Leftrightarrow \quad T_{\alpha,1}(\pi_{\alpha\beta}x_\beta) = \pi_{\alpha\beta}(T_{\beta,1}x_\beta)\ and\ T_{\alpha,2}(\pi_{\alpha\beta}x_\beta) = \pi_{\alpha\beta}(T_{\beta,2}x_\beta), for\ x_\beta \in D(T_\beta)\ and\ \beta \geq \alpha.$$

where $T_{\alpha,1}, T_{\alpha,2}$ are linear operator on $Z_{\alpha,1}, Z_{\alpha,2}$ respt. and $T_{\beta,1}, T_{\beta,2}$ are linear operator on $Z_{\beta,1}, Z_{\beta,2}$respt.

Definition 2.13. : A $\mathbb{BC}$ linear operator T on the projective limit $D(T)$ of $(D(T_\alpha) : \alpha \in I)$ can be define by $\pi_\alpha(Tx) = T_\alpha(\pi_\alpha x)$ for $x \in D(T)$ and $\alpha \in I$ and the operator T is called the projective limit of the family of operator $\{T_\alpha : \alpha \in I\}$.
If $T_\alpha \in L(X_\alpha)$ for each α, then $T \in L_I(X)$. Moreover, the family $\{\overline{T_\alpha} : \alpha \in I\}$ associated $T \in L(X_\alpha)$ above is projective and its limit is T.

Remark 2.9.

$$\begin{aligned} \pi_\alpha(Tx) &= T_\alpha(\pi_\alpha x) \\ \Leftrightarrow \pi_{\alpha,1}(T_1x_1) = T_{\alpha,1}(\pi_{\alpha,1}x_1) \quad &and \quad \pi_{\alpha,2}(T_2x_2) = T_{\alpha,2}(\pi_{\alpha,2}x_2) \end{aligned}$$

3. SOME BASIC PROPERTIES OF $(C_0, 1)$ SEMIGROUP

The result in this section are generalization of results of [2].

Definition 3.1. Let X be a locallly $\mathbb{BC}$ convex module and a family $\{T(t), t \in \mathbb{D}^+\}$ of bounded $\mathbb{BC}$ linear operator in X is called a C_0-semigroup if
(i) $T(t+s)x = T(t)(T(s)x)\ \forall\ t, s \in \mathbb{D}^+$ and $x \in X$
(ii) $T(0)x = x\ \forall\ x \in X$.
(iii) $T(t)x \to x$ as $t \to 0\ \forall\ x \in X$
A C_0 semigroup $t \longmapsto T(t)$ is said to be a $(C_0, 1)$ semigroup if $T(t) \in L_I(X), \forall\ t \geq' 0$ and, for each α and $\delta >' 0$, there exist a positive hyperbolic number $\lambda = \lambda(\alpha, T(t) : t \in [0, \delta]_{\mathbb{D}})$ such that

$$T(t)\overline{\mathcal{B}}_{p_\alpha}(0) \subset \lambda\overline{\mathcal{B}}_{p_\alpha}(0)$$

or equivalently $p_\alpha(Tx) \leq' \lambda p_\alpha(x)\ \forall\ 0 \leq' t \leq' \delta$, where $\overline{\mathcal{B}}_{p_\alpha}(0) = \{x \in X : p_\alpha(x) \leq' 1\}$. It is also called a $L_I(X)$-operator semigroup of class $(C_0, 1)$

Theorem 3.1. *: If $\{T(t), t \in \mathbb{D}^+\}$ is a $(C_0, 1)$-semi-group in X. Then the family $\{\overline{T}_\alpha(t), t \in \mathbb{D}^+\}$ is a C_0 semi-group in the $\mathbb{BC}$-Banach module $\overline{X}_\alpha$, for each α.*

Proof. : Here, $\{T(t), t \in \mathbb{D}^+\}$ is a $(C_0, 1)$-semi-group in X.
Let $\overline{T}_{\alpha,1}(t_1) = e.\overline{T}_\alpha(t)$ and $\overline{T}_{\alpha,2}(t_2) = e^\dagger.\overline{T}_\alpha(t)$. Then using [2, thm 2.3, P-168], we see that for each α, $\overline{T}_{\alpha,1}(t_1)$ and $\overline{T}_{\alpha,2}(t_2)$ are C_0 semi-group in the Banach space $\overline{X}_{\alpha,1}$ and $\overline{X}_{\alpha,2}$ respt. Thus,

$$\overline{T}_\alpha(t) = e\overline{T}_\alpha(t) + e^\dagger\overline{T}_\alpha(t)$$

is C_0 semi-group in $\mathbb{BC}$-Banach module $\overline{X}_\alpha$. ∎

Let us denote $\overline{X}_\alpha$ with Z_α and X with Z. Now, Let Z_α be a $\mathbb{BC}$-Banach module and Z be complete locally $\mathbb{BC}$ convex module. A family $\{T_\alpha(t) \in L(Z_\alpha) : \alpha \in I, t \geq' 0\}$ is called a projective family of C_0 semigroups on $\mathbb{BC}$-Banach module iff
(i) for each $t \geq' 0, \{T_\alpha(t) : \alpha \in I\}$ is a projective family.
(ii) for each α, $\{T_\alpha(t) : t \geq' 0\}$ is a C_0 semi-group on the $\mathbb{BC}$-Banach module Z_α.
The limit of such a family is denoted by $\{T(t) : t \geq' 0\}$.

Theorem 3.2. *let $\Gamma = \{T_\alpha(t) : \alpha \in I, t \geq' 0\}$ be a projective family of C_0-semigroup on $\mathbb{BC}$-Banach module Z_α. Then the following statement are equivalent:*
(i) $\{T(t) : t \geq' 0\}$ is a $(C_0, 1)$-semigroup in Z $\Leftrightarrow$ $\{T_1(t) : t \geq 0\}$ and $\{T_2(t) : t \geq 0\}$ are $(C_0, 1)$-semigroup on Z_1 and Z_2 respt.
(ii) $\{T(t) : t \geq' 0\}$ be the limit of $\Gamma \Leftrightarrow \{T_1(t) : t \geq 0\}$ and $\{T_2(t) : t \geq 0\}$ are the limit of Γ_1 and Γ_2 respt,
where $\Gamma_1 = \{T_{\alpha,1}(t) : \alpha \in I, t \geq 0\}$ and $\Gamma_2 = \{T_{\alpha,2}(t) : \alpha \in I, t \geq 0\}$.

Proof. : By using definition 2.7, a $\mathbb{BC}$ linear operator $T(t)$ can be written as $T(t) = T_1(t_1)e + T_2(t_2)e^\dagger$, where $T_1(t_1)$ and $T_2(t_2)$ are linear operator on Z_1 and Z_2 respt.
Let $x \in Z$ with $x = x_1e + x_2e^\dagger$ where $x_1 \in Z_1$, $x_2 \in Z_2$ and $t, s \geq' 0$ with $t = t_1e + t_2e^\dagger$ and $s = s_1e + s_2e^\dagger$ where $t_1, t_2, s_1, s_2 \geq 0$.
Then (i)

$$\begin{aligned} T(t+s)x &= T(t)(T(s)x),\ \forall x \in Z \\ \Leftrightarrow T_1(t+s)xe + T_2(t+s)xe^\dagger &= T_1(t)(T_1(s)x)e + T_2(t)(T_2(s)x)e^\dagger \\ \Leftrightarrow T_1(t+s)x = T_1(t)(T_1(s)x) \quad &and \quad T_2(t+s)x = T_2(t)(T_2(s)x). \end{aligned}$$

(ii)

$$\begin{aligned} T(0)x &= x \\ \Leftrightarrow T_1(0)xe + T_2(0)xe^\dagger &= xe + xe^\dagger \\ \Leftrightarrow T_1(0)x = x \quad &and \quad T_2(0)x = x\ \forall x_1 \in Z_1\ and\ x_2 \in Z_2. \end{aligned}$$

(iii)

$$\begin{aligned} lim_{t\to 0^+} T(t)x &= x,\ t \notin \mathbb{NC} \cup \{0\} \\ \Leftrightarrow lim_{t_1\to 0} T_1(t)xe + lim_{t_2\to 0} T_2(t)xe^\dagger &= xe + xe^\dagger \\ \Leftrightarrow lim_{t_1\to 0} T_1(t)x = x \quad &and \quad lim_{t_2\to 0} T_2(t)x = x. \end{aligned}$$

(iv) there exist a +ve hyperbolic number $\sigma_\alpha = \sigma_{\alpha,1} + \sigma_{\alpha,2}$ where $\sigma_{\alpha,l}$, $l = 1, 2$ is a +ve number such that

$$p_\alpha(T(t)x) \leq' e^{\sigma_\alpha t} p_\alpha(x).$$

$$\Leftrightarrow p_{\alpha,1}(T_1(t)x) \leq e^{\sigma_{\alpha,1}t} p_{\alpha,1}(x)\ and\ p_{\alpha,2}(T_2(t)x) \leq e^{\sigma_{\alpha,2}t} p_{\alpha,1}(x)$$

Therefore, $T_1(t) : t \geq 0$ are $(C_0, 1)$ semigroup on Z_l, $l = 1, 2$ respt.
Thus, $\{T(t) : t \geq' 0\}$ is a $(C_0, 1)$-semigroup in Z $\Leftrightarrow$ $\{T_1(t) : t \geq 0\}$ and $\{T_2(t) : t \geq 0\}$ are $(C_0, 1)$-semigroup on Z_1 and Z_2 respt.

(2) Suppose that $\{T(t) : t \geq' 0\}$ be the limit of $\Gamma \Rightarrow \pi_\alpha(T(t)x) = T_\alpha(\pi_\alpha x)$.

$$\begin{aligned} \pi_\alpha(T(t)x) &= T_\alpha(\pi_\alpha x) \\ \Rightarrow \pi_{\alpha,1}(T_1(t)x)e + \pi_{\alpha,2}(T_2(t)x)e^\dagger &= T_{\alpha,1}(\pi_{\alpha,1}x)e + T_{\alpha,2}(\pi_{\alpha,2}x)e^\dagger \\ \Rightarrow \pi_{\alpha,1}(T_1(t)x) = T_{\alpha,1}(\pi_{\alpha,1}x) \quad &and \quad \pi_{\alpha,2}(T_2(t)x) = T_{\alpha,2}(\pi_{\alpha,2}x) \end{aligned}$$

$\Rightarrow \{T_1(t) : t \geq 0\}$ and $\{T_2(t) : t \geq 0\}$ are the limit of Γ_1 and Γ_2 respt.
Conversely,suppose that $\{T_1(t) : t \geq 0\}$ and $\{T_2(t) : t \geq 0\}$ are the limit of Γ_1 and Γ_2 respt.
$\Rightarrow \pi_{\alpha,1}(T_1(t)x) = T_{\alpha,1}(\pi_{\alpha,1}x)$ and $\pi_{\alpha,2}(T_2(t)x) = T_{\alpha,2}(\pi_{\alpha,2}x)$
Let $x = x_1e + x_2e^\dagger \in Z$ and $\pi_\alpha = \pi_{\alpha,1}e + \pi_{\alpha,2}e^\dagger$ be the natural $\mathbb{BC}$ homomorphism of Z onto

Z_α, where $\pi_{\alpha,l}$ is the natural homomorphism of Z_l onto $Z_{\alpha,l}$, $l = 1, 2$.
Then

$$\begin{aligned}\pi_\alpha(T(t)x) &= \pi_{\alpha,1}(T_1(t)x)e + \pi_{\alpha,2}(T_2(t)x)e^\dagger \\ &= T_{\alpha,1}(\pi_{\alpha,1}x)e + T_{\alpha,2}(\pi_{\alpha,2}x)e^\dagger \\ &= T_\alpha(\pi_\alpha x).\end{aligned}$$

Thus, $\{T(t) : t \geq' 0\}$ is the limit of Γ. ∎

Definition 3.2. Let X be a locally $\mathbb{BC}$ convex module and the $\mathbb{BC}$ linear operator $A : D(A) \subset X \to X$ is called the infinitesimal generator of the semigroup $\{T(t) : t \geq' 0\}$ if it satisfies

$$Ax = lim_{h\to 0^+}\frac{T(h)x - x}{h}, \ \forall x \in X, \ h \notin \mathbb{NC} \cup \{0\}.$$

Remark 3.1. The $\mathbb{BC}$ linear operator A can be written as

$$A = A_1e + A_2e^\dagger.$$

$$\begin{aligned}Ax &= lim_{h\to 0^+}\frac{T(h)x - x}{h} \\ &= lim_{h\to 0^+}\frac{(T_1(h_1)x_1 - x_1)e + (T_2(h_2)x_2 - x_2)e^\dagger}{h_1e + h_2e^\dagger} \\ &= lim_{h_1\to 0^+}\frac{(T_1(h_1)x_1 - x_1)e}{h_1e + h_2e^\dagger} + lim_{h_2\to 0^+}\frac{(T_2(h_2)x_2 - x_2)e^\dagger}{h_1e + h_2e^\dagger} \\ &= lim_{h_1\to 0^+}\frac{(T_1(h_1)x_1 - x_1)e}{h_1} + lim_{h_2\to 0^+}\frac{(T_2(h_2)x_2 - x_2)e^\dagger}{h_2} \\ Ax &= A_1x_1e + A_2x_2e^\dagger.\end{aligned}$$

where A_l are the infinitesimal generator of the semigroup $\{T_l(t_l) : t_l \geq 0\}$, $l = 1, 2$ in complex version.

Proof of the following theorem is in similar lines as in [2, theorem 1], so we omit the proof.

Theorem 3.3. *Let X be a complete locally $\mathbb{BC}$ convex module. Then $(C_0, 1)$ semi-groups on X is in 1-1 correspondence with the projective families of C_0 semigroups on $\mathbb{BC}$ Banach module X_α. Further, if A is the infinitesimal generator of a $(C_0, 1)$ semigroup and $\{A_\alpha\}$ is the family of generators associated with the corresponding C_0 semigroups then $\{A_\alpha\}$ is a projective family and its limit is A.*

Definition 3.3. Let a family $\Gamma = \{T(t) : t \geq' 0\}$ be a C_0 semi-groups in locally $\mathbb{BC}$-convex module X. Then Γ is a $(C_0, 1)$ semigroup $\Leftrightarrow$ there exist sets of hyperbolic numbers $\{M_\alpha : M_\alpha = M_{\alpha,1}e + M_{\alpha,2}e^\dagger, \alpha \in I\}$ and $\{\sigma_\alpha : \sigma_{\alpha,1}e + \sigma_{\alpha,2}e^\dagger, \alpha \in I\}$ such that for $\alpha \in I$

$$p_\alpha(T(t)x) \leq M_\alpha e^{\sigma_\alpha t}p_\alpha(x), \ \forall t \in \mathbb{D}^+, \ x \in X,$$

where $\{M_{\alpha,l} : \alpha \in I\}$ and $\{\sigma_{\alpha,l} : \alpha \in I\}$, $l = 1, 2$ are the sets of real numbers.

Definition 3.1 is equivalent to the following definition.

Definition 3.4. $\{T(t) : t \geq' 0\}$ be a family of continuous $\mathbb{BC}$ linear operators on X. Then $\{T(t) : t \geq' 0\}$ is a $(C_0, 1)$ semigroup $\Leftrightarrow$ if it satisfies the following condition:
(i) $\{T(t) : t \geq' 0\}$ is a C_0 semigroup in X;
(ii) for each continuous $\mathbb{D}$ valued seminorm p on X there exist a positive hyperbolic number σ_p and a continuous $\mathbb{D}$ valued seminorm q on X such that $p(T(t)x) \leq' e^{\sigma_p t}q(x)$ for all $t \geq' 0$ and $x \in X$.

We can write above inequality as

$$\begin{aligned} p(T(t)x) \quad &\leq' \quad e^{\sigma_p t} q(x) \\ \Leftrightarrow p_1(T_1(t_1)x_1) \leq e^{\sigma_{p_1} t_1} q_1(x_1) \quad &and \quad p_2(T_2(t_2)x_2) \leq e^{\sigma_{p_2} t_2} q_2(x_2). \end{aligned}$$

4. CONCLUSION

Using the idempotent representation of bicomplex numbers, most of the results on C_o semigroups of linear operators with complex scalars can be extended to C_o semigroups of linear operators with bicomplex scalars and can be an interesting area of research.

REFERENCES

[1] D. ALPAY, M. E. LUNA-ELIZARRAR$\acute{A}S$, M. SHAPIRO, D. C. STRUPPA, *Basics of Functional Analysis with Bicomplex Scalars and Bicomplex Schur Analysis*, Springer Briefs in Mathematics, 2014.

[2] V. A. BABALOLA, Semigroup of operators on locally convex spaces, *American Mathematical Society*, (1974), No.199, pp. 163-179.

[3] N. BOURBAKI, *Topological Vector Spaces*, Springer, Berlin, 1987.

[4] F. COLOMBO, I. SABADINI and D. C. STRUPPA, Bicomplex holomorphic functional calculus, *Math. Nachr.*, **287**(2013), No.13, pp. 1093-1105.

[5] N. DUNFORD and J. T. SCHWARTZ, *Linear Operators: Part I*, Wiley, New York, London, 1958.

[6] K. J. ENGEL and R. NAGEL, *One Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Maths, Springer-Verlag, New York, 2000.

[7] R. GERVAIS LAVOIE, L. MARCHILDON and D. ROCHAN, Finite-dimensional bicomplex Hilbert spaces, *Adv. Appl. Clifford Algebr.*, **21**(2011), No. 3, pp. 561-581.

[8] E. HILLE and R. S. PHILLIPS, *Functional Analysis and Semigroups*, American Mathematical Society, 1996.

[9] G. KOTHE, *Topological Vector Spaces-I*, Springer-Verlag, New York,1969.

[10] R. KUMAR, R. KUMAR and D. ROCHAN, The fundamental theorems in the framework of bicomplex topological modules, (2011), arXiv:1109.3424v.1.

[11] R. KUMAR and H. SAINI, Topological bicomplex modules, *Adv. Appl. Clifford Algebr*, **26**(2016), No.4, pp. 1249 - 1270.

[12] R. KUMAR and K. SINGH, Bicomplex linear operators on bicomplex Hilbert spaces and Littlewood's subordination theorem, *Adv. Appl. Clifford Algebr.*, **25**(2015), No.3, 591x96610.

[13] M. E. LUNA-ELIZARRAR$\acute{A}$S, M. SHAPIRO, D. C. STRUPPA and A-VAJIAC, *Bicomplex Holomorphic Functions*, Frontiers in Mathematics. Birkhauser, Basel(2015).

[14] L. NARICI, E. BECKENSTEIN, *Topological Vector Spaces*, Marcel Dekker, New York, 1985.

[15] G. B. PRICE, *An Introduction to Multicomplex Spaces and Functions*, Marcel Dekker, New York, 1991.

[16] H. H. SCHAEFER, Topological Vector Spaces, Springer, Berlin, 1971.

[17] K. YOSHIDA, *Functional Analysis*, Springer-Verlag, Berlin, 1968.

[18] D. ROCHAN and M. SHAPIRO, On algebraic properties of bicomplex and hyperbolic numbers, *Anal. univ. Oradea, Fasc. Math.* **11**(2004), pp. 71-110.

[19] M. E. LUNA-ELIZARRARÁS, M. SHAPIRO, D. C. STRUPPA and A. VAJIAC, Bicomplex numbers and their elementary functions, *Cubo*., **14** (2012), No. 2, pp.61-80.

[20] C. YU-HSIEN and H. CHENG HONG, Some application of $(C_0, 1)$-semigroup, *Journal of Applied Mathematics*, 2012.

[21] D. ROCHAN and S. TREMBLAY, Bicomplex quantum mechanics II: The Hilbert space, *Advances in Applied Clifford Algebras*, **16** (2006), No. 2, pp. 135-157.