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A NOTE ON SCHUR'S LEMMA IN BANACH FUNCTION SPACES

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ABSTRACT. In this small note, in a self contained presentation, we show the validity of Schur's type lemma in the framework of Banach function spaces.

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1. MOTIVATION

Schur's lemma is one of the most basic facts about integral operators, roughly speaking it says that if the integral kernel K satisfy some a priori integral conditions, then the integral operator T_K is a bounded operator on L^p cf. [7, Theo. 1.8] (see also [5, 11]). This result is a very effective test whenever we want to prove the boundedness of integral operators in Lebesgue spaces. Moreover, some sophisticated ideas are based on this result, like the Folland-Stein theorem [8].

The development of this important result started in 1911, when Schur [12] proved a matrix version of the lemma for a positive decreasing kernel in ℓ_2. Afterward, in 1926, Hardy, Littlewood, and Pólya [6] extended this result to L^p for $1 < p < \infty$ and a decreasing kernel. For the case of positive operators, necessary and sufficient conditions for the L^p boundedness were given in 1963 by Aronszajn, Mulla, and Szptycki [1] as well as by Gagliardo [4] in 1965.

On the other hand, Banach function spaces (BFS for short) are a modern tool from analysis which allow to consider concrete problems in a more abstract framework. This theory began in the 1950's with the works of Ellis and Halperin [3], Luxemburg [9], and Zaanen [13]. We can see the theory of BFS as a useful way to characterize and understand spaces of measurable functions in a unified manner. We mention the following important function spaces in harmonic analysis which are BFS: Lebesgue, Lorentz, Orlicz, and Musielak–Orlicz, to name a few. The fundamentals of this class of spaces can be found in [2] (cf. [10] for BFS and variable exponent Lebesgue spaces).

In this note we will show that, under appropriate conditions, we have a Schur's type lemma in the framework of BFS.

2. BANACH FUNCTION SPACES

In this section, besides recalling the notion of BFS, for the convenience of the reader and to make the exposition self-contained, we state and prove some results in the theory of BFS (the standard reference is Bennett and Sharpley [2]).

Let (Ξ, μ) be a measure space, $\mathfrak{M}(\Xi, \mu)$ the space of extended real-valued measurable functions on Ξ and $\mathfrak{M}^+(\Xi, \mu)$ the space of measurable functions on Ξ with range in $[0, \infty]$. We will denote the characteristic function of a measurable set $E \subset \Xi$ by χ_E.

Definition 2.1. A mapping $\rho : \mathfrak{M}^+ \longrightarrow [0, \infty]$ is called a *Banach function norm* if, for all functions f, g, f_n $(n \in \mathbb{N})$ in $\mathfrak{M}^+$, for all constants $a \geq 0$, and for all measurable subsets E of Ξ, the following properties hold:

(A1) $\quad \rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\quad \rho(af) = a\rho(f), \quad \rho(f + g) \leqslant \rho(f) + \rho(g)$,

(A2) $\quad 0 \leqslant g \leqslant f$ a.e. $\Rightarrow \rho(g) \leqslant \rho(f)$ (the lattice property),

(A3) $\quad 0 \leqslant f_n \uparrow f$ a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$ (the Fatou property),

(A4) $\quad \mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty$,

(A5) $\quad \mu(E) < \infty \Rightarrow \displaystyle\int_E f(x)\,\mathrm{d}\mu(x) \leqslant C_E \rho(f)$

with $C_E \in (0, \infty)$ which may depend on E and ρ but is independent of f.

Here, functions differing only on a set of measure zero are identified. The set $\mathsf{X}(\Xi)$ of all functions $f \in \mathfrak{M}$ for which $\rho(|f|) < \infty$ is called a *Banach function space*. For each $f \in \mathsf{X}(\Xi)$, the norm of f is defined by

$$\|f\|_{\mathsf{X}} := \rho(|f|). \tag{2.1}$$

We are going to show that the set $\mathsf{X}(\Xi)$ under the natural linear space operations and under the norm (2.1) becomes a Banach space. By $\mathfrak{M}_0(\Xi,\mu)$ we denote the class of functions in $\mathfrak{M}(\Xi,\mu)$ that are finite μ-a.e.

Theorem 2.1. *Let* X *be a Banach function space generated by a Banach function norm* ρ*. Then* $(\mathsf{X},\|\cdot\|_{\mathsf{X}})$ *is a linear space. Moreover, the inclusions*

$$\mathfrak{S} \subset \mathsf{X} \hookrightarrow \mathfrak{M}_0 \tag{2.2}$$

hold, where $\mathfrak{S}$ *is the set of simple functions on* Ξ*.*

Proof. Since μ is a σ-finite measure, locally integrable functions are μ-a.e. finite. Hence, $\mathsf{X} \subset \mathfrak{M}_0$. Due to the property (A5) from Definition 2.1, we have for $E \subset \Xi$ such that $\mu(E) < \infty$,

$$\int\limits_E f \,\mathrm{d}\mu \leqslant C_E \|f\|_{\mathsf{X}}\,.$$

Which means that all functions in X are locally integrable on Ξ. Thus, since $\mathfrak{M}_0$ is a vector space, and the operations are closed on X in virtue of (A1), we conclude that X is a vector space. It follows immediately from (A1) that X is a normed space. By (A4) $\chi_E \in \mathsf{X}$ for every set E such that $\mu(E) < \infty$. Consequently, by the linearity of X we get $\mathfrak{S} \subset \mathsf{X}$. It remains to show that the embedding $\mathsf{X} \hookrightarrow \mathfrak{M}_0$ is continuous. Assume that a sequence $\{f_n\}_{n=1}^{\infty}$ satisfies $f_n \to f$ in X. Then by (2.1) $\rho(|f_n - f|) \to 0$ as $n \to \infty$. Given $\varepsilon > 0$ and a set $E \subset \Xi$ such that $\mu(E) < \infty$, we get from (A5) that

$$\mu(\{x \in E \,:\, |f_n - f| > \varepsilon\}) \leqslant \frac{1}{\varepsilon} \int\limits_E |f_n - f| \,\mathrm{d}\mu(x) \leqslant \frac{C_E}{\varepsilon} \rho(|f_n - f|) \to 0$$

as $n \to \infty$, since C_E is independent of n. Therefore, $f_n \to f$ in measure on every set of finite measure, in other words $f_n \to f$ in $\mathfrak{M}_0$. ∎

We shall now show that the Fatou lemma, familiar from the theory of Lebesgue integration, holds for every BFS. The key ingredient of the proof is the Fatou property (A3).

Lemma 2.2. (Fatou lemma for BFS) *Let* X *be a Banach function space and assume that* $f_n \in \mathsf{X}$, $n \in \mathbb{N}$ *and* $f_n \to f$ μ*-a.e. for some* $f \in \mathfrak{M}^+$*. Assume further that* $\liminf_{n\to\infty} \|f\|_{\mathsf{X}} < \infty$*. Then* $f \in \mathsf{X}$ *and*

$$\|f\|_{\mathsf{X}} \leqslant \liminf_{n\to\infty} \|f_n\|_{\mathsf{X}}\,.$$

Proof. Denote $g_n(x) = \inf_{m \geqslant n} |f_m(x)|$. Then $0 \leqslant g_n \uparrow |f|$ μ-a.e., whence by (A2) and (A3)

$$\|f\|_{\mathsf{X}} = \lim_{n\to\infty} \|g_n\|_{\mathsf{X}} \leqslant \liminf_{n\to\infty}\inf_{m\geqslant n} \|f_m\|_{\mathsf{X}} = \liminf_{n\to\infty} \|f_n\|_{\mathsf{X}} < \infty.$$

Hence, $f \in \mathsf{X}$ and $\|f\|_{\mathsf{X}} \leqslant \liminf_{n\to\infty} \|f_n\|_{\mathsf{X}}$. ∎

To prove that every BFS is complete we use the notion of Riesz-Fischer property and Fatou's lemma.

Definition 2.2. We say that a normed linear space $(\mathsf{X},\|\cdot\|_{\mathsf{X}})$ has the Riesz-Fischer property if for each sequence $\{u_n\}_{n\in\mathbb{N}}$ such that

$$\sum_{n=1}^{\infty} \|u_n\|_{\mathsf{X}} < \infty, \tag{2.3}$$

there exists an element $u \in \mathsf{X}$ such that $\sum_{n=1}^{\infty} u_n = u$ in X, that is

$$\lim_{n\to\infty} \left\| \sum_{k=1}^{n} u_k - u \right\|_{\mathsf{X}} = 0.$$

Theorem 2.3. *A normed linear space is complete if and only if it has the Riesz-Fischer property.*

Theorem 2.4. *Every Banach function space has the Riesz-Fischer property.*

Proof. Let X be a BFS, let $\{f_n\}_{n\in\mathbb{N}}$ and suppose that

$$\sum_{n=1}^{\infty} \|f_n\|_{\mathsf{X}} < \infty. \tag{2.4}$$

We denote, for every $n \in \mathbb{N}$,

$$g_n = \sum_{k=1}^{n} |f_k| \quad \text{and} \quad g = \sum_{k=1}^{\infty} |f_k|,$$

so that $g_n \uparrow g$. Since

$$\|g_n\|_{\mathsf{X}} \leqslant \sum_{k=1}^{n} \|f_k\|_{\mathsf{X}} \leqslant \sum_{k=1}^{\infty} \|f_k\|_{\mathsf{X}}, \quad n \in \mathbb{N}$$

it follows from (2.4) and Lemma 2.2 that g belongs to X. By the embedding $\mathsf{X} \hookrightarrow \mathfrak{M}_0$, the series $\sum_{n=1}^{\infty} |f_n(x)|$ converges pointwise μ-a.e. and hence so does $\sum_{n=1}^{\infty} f_n(x)$. We set

$$f = \sum_{n=1}^{\infty} f_n \quad \text{and} \quad h_n = \sum_{k=1}^{n} f_k$$

then, $h_n \to f$ μ-a.e. Hence for any $m \in \mathbb{N}$, we have $h_n - h_m \to f - h_m$ μ-a.e. as $n \to \infty$. Furthermore,

$$\liminf_{n\to\infty} \|h_n - h_m\|_{\mathsf{X}} \leqslant \liminf_{n\to\infty} \sum_{k=m+1}^{n} \|f_k\|_{\mathsf{X}} = \sum_{k=m+1}^{\infty} \|f_k\|_{\mathsf{X}} \to 0$$

as $m \to \infty$, since $\sum_{n=1}^{\infty} \|f_n\|_{\mathsf{X}} < \infty$. Thus, by Fatou's Lemma, we get $f - h_m \in \mathsf{X}$, therefore also $f \in \mathsf{X}$, and $\|f - h_m\|_{\mathsf{X}} \to 0$ as $m \to \infty$. This implies that, for every $m \in \mathbb{N}$,

$$\|f\|_{\mathsf{X}} \leqslant \|f - h_m\|_{\mathsf{X}} + \|h_m\|_{\mathsf{X}} \leqslant \|f - h_m\|_{\mathsf{X}} + \sum_{k=1}^{\infty} \|f_k\|_{\mathsf{X}}.$$

By letting $m \to \infty$, we get

$$\|f\|_{\mathsf{X}} \leqslant \sum_{n=1}^{\infty} \|f_n\|_{\mathsf{X}}. \quad \blacksquare$$

If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathfrak{M}^+$ by

$$\rho'(g) := \sup \left\{ \int_{\Xi} f(x) g(x) \, \mathrm{d}\mu(x) \ : \ f \in \mathfrak{M}^+, \ \rho(f) \leqslant 1 \right\}, \quad g \in \mathfrak{M}^+. \tag{2.5}$$

Now we prove that ρ' is itself a Banach function norm.

Theorem 2.5. *Let ρ be a Banach function norm. Then ρ' is a Banach function norm.*

Proof. Suppose $\rho(f) \leqslant 1$. Then (2.2) implies that $f(x) < \infty$ μ-a.e. If moreover $g = 0$ μ-a.e. then

$$\int_{\Xi} fg \,\mathrm{d}\mu = 0,$$

hence, by (2.5), $\rho'(g) = 0$. If $\rho'(g) = 0$, then

$$\int_{\Xi} fg \,\mathrm{d}\mu = 0, \text{ for all } f \in \mathfrak{M}^+ \text{ with } \rho(f) \leq 1.$$

If $E \subset \Xi$ is a μ-measurable set with $0 < \mu(E) < \infty$, then $0 < \rho(\chi_E) < \infty$ by properties (A1) and (A4) of ρ. The function $f = \chi_E/\rho(\chi_E)$ satisfies $\rho(f) = 1$, therefore

$$\big(\rho(\chi_E)\big)^{-1} \int_E g \,\mathrm{d}\mu = \int_{\Xi} fg \,\mathrm{d}\mu = 0.$$

Thus $g = 0$ μ-a.e. on E. Because E was chosen arbitrarily, we get that $g = 0$ μ-a.e. The positive homogeneity and the triangle inequality for ρ' can be easily verified, which shows (A1). The property (A2) follows from the definition of ρ'.

We shall show (A3). Let $\{g_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}^+$ and assume that $0 \leqslant g_n \uparrow g$ μ-a.e. for some $g \in \mathfrak{M}^+$. It is clear that ρ' satisfy (A2). Thus for every $m, n \in \mathbb{N}$, $m \leqslant n$, $\rho'(g_m) \leqslant \rho'(g_n) \leqslant \rho'(g)$. We can, with no loss of generality, assume $\rho'(g_n) < \infty$ for every $n \in \mathbb{N}$. Let ε be any number satisfying $\varepsilon < \rho'(g_n)$. By (2.5), there is a function $f \in \mathfrak{M}^+$ with $\rho(f) \leqslant 1$ such that $\int fg \,\mathrm{d}\mu > \varepsilon$. Now $0 \leqslant fg_n \uparrow fg$ μ-a.e., so the monotone convergence theorem implies $\int fg_n \uparrow \int fg$. Hence, there is $n_0 \in \mathbb{N}$ such that $\int fg_n > \varepsilon$ for all $n \geqslant n_0$. Thus, by (2.5), we obtain

$$\rho'(g_n) > \varepsilon \quad \text{for all } n \geqslant n_0.$$

Consequently, we get $\rho'(g_n) \uparrow \rho'(g)$, thus ρ' enjoys the property (A3).

In order to verify (A4) for ρ' we use (A5) for ρ, and vice versa. Let $E \subset \Xi$ satisfying $\mu(E) < \infty$, then by (A5) for ρ, there is a constant $C_E < \infty$ for which

$$\int_{\Xi} \chi_E f \,\mathrm{d}\mu \leqslant C_E \rho(f) \qquad (f \in \mathfrak{M}^+).$$

Together with (2.5), this gives $\rho'(\chi_E) \leqslant C_E < \infty$, proving (A4) for ρ'.

Finally, let $E \subset \Xi$ be such that $\mu(E) < \infty$. If $\mu(E) = 0$, then $\int_E g \,\mathrm{d}\mu = 0$, hence (A5) holds automatically. When $\mu(E) > 0$, we have by (A4) for ρ that $\rho(\chi_E) < \infty$ and by (A1) for ρ that $\rho(\chi_E) > 0$. We set $C'_E = \rho(\chi_E)$ and $f = \chi_E/\rho(\chi_E)$. Then $\rho(f) = 1$, whence, for any $g \in \mathfrak{M}^+$, we obtain from (2.5)

$$\int_E g \,\mathrm{d}\mu = C'_E \int_{\Xi} fg \,\mathrm{d}\mu \leqslant C'_E \rho'(g).$$

Proving (A5) for ρ'. The proof is complete. ∎

The Banach function space $\mathsf{X}'(\Xi)$ determined by the Banach function norm ρ' is called the *associate space*, also known as the *Köthe dual*, of the space $\mathsf{X}(\Xi)$. The celebrated Lorentz-Luxemburg theorem, cf. [2, Theorem 2.7], states that any Banach function space $\mathsf{X}(\Xi)$ coincides with its second associated space $\mathsf{X}''(\Xi)$, thus we can define the Banach norm of $f \in \mathsf{X}(\Xi)$ as

$$\|f\|_{\mathsf{X}} = \sup_{\|g\|_{\mathsf{X}'}=1} \int_{\Xi} |f(x)g(x)| \,\mathrm{d}\mu(x), \quad g \in \mathsf{X}'(\Xi). \tag{2.6}$$

Lemma 2.6. (Hölder's inequality) *Let* $\mathsf{X}(\Xi)$ *be a Banach function space with associate space* $\mathsf{X}'(\Xi)$*. If* $f \in \mathsf{X}(\Xi)$ *and* $g \in \mathsf{X}'(\Xi)$*, then* fg *is summable and*

$$\int_{\Xi} |f(x)g(x)| \, \mathrm{d}\mu(x) \leqslant \|f\|_{\mathsf{X}} \|g\|_{\mathsf{X}'} . \tag{2.7}$$

Proof. Assume first that $\|f\|_{\mathsf{X}} = 0$, then $f = 0$ μ-a.e. on Ξ. In this case both side of (2.7) are zero. When $\|f\|_{\mathsf{X}} > 0$, then

$$\left\| \frac{f}{\|f\|_{\mathsf{X}}} \right\|_{\mathsf{X}} = 1.$$

Thus, by definition of X', we get, by (2.6), that

$$\int_{\Xi} \left| \left(\frac{f}{\|f\|_{\mathsf{X}}} \right) g \right| \mathrm{d}\mu(x) \leqslant \|g\|_{\mathsf{X}'} .$$

And so,

$$\int_{\Xi} |fg| \, \mathrm{d}\mu(x) \leqslant \|f\|_{\mathsf{X}} \|g\|_{\mathsf{X}'} ,$$

which ends the proof. ∎

3. SCHUR'S TYPE LEMMA

In the sequel, by $A \lesssim B$ we mean that there exists some constant $k > 0$ such that $A \leqslant kB$.

Theorem 3.1. *Let* (Ξ, μ) *be a* σ*-finite measure space and* T_K *be the integral operator with a positive and* $\Xi \times \Xi$ *measurable kernel* K *given by*

$$T_K : \mathsf{X}(\Xi) \ni f \mapsto \int_{\Xi} K(x,y) f(y) \, \mathrm{d}\mu(y) \in \mathsf{Y}(\Xi),$$

where $\mathsf{X}(\Xi)$ *and* $\mathsf{Y}(\Xi)$ *are Banach function spaces. Suppose that there exists a strictly positive function* h *and* $\alpha \in (0,1)$ *such that*

$$\left\| K(x,\cdot)^{\alpha} h(\cdot) \right\|_{\mathsf{X}'} \lesssim h(x) \tag{3.1}$$

and

$$\left\| K(\cdot,y)^{1-\alpha} h(\cdot) \right\|_{\mathsf{Y}} \lesssim h(y). \tag{3.2}$$

Then $T_K : \mathsf{X} \hookrightarrow \mathsf{Y}$.

Proof. Since T is a homogeneous operator, to prove the boundedness of the operator it suffices to show that $\|Tf\|_{\mathsf{Y}} \lesssim 1$ for all $f \in \mathsf{X}$ with $\|f\|_{\mathsf{X}} = 1$. Using the characterization of the Banach function norm via the dual space, the Hölder inequality, Fubini's theorem and conditions (3.1)-(3.2), we have

$$\begin{aligned}
\|Tf\|_{\mathsf{Y}} &= \sup_{\|g\|_{\mathsf{Y}'}=1} \int_{\Xi} |Tf(x)g(x)| \,\mathrm{d}\mu(x) \\
&\leqslant \sup_{\|g\|_{\mathsf{Y}'}=1} \int_{\Xi} |g(x)| \,\mathrm{d}\mu(x) \int_{\Xi} K(x,y)|f(y)| \,\mathrm{d}\mu(y) \\
&\lesssim \sup_{\|g\|_{\mathsf{Y}'}=1} \int_{\Xi} |g(x)| \big\|K(x,\cdot)^{\alpha} h(\cdot)\big\|_{\mathsf{X}'} \big\|K(x,\cdot)^{1-\alpha} h(\cdot)^{-1} |f(\cdot)|\big\|_{\mathsf{X}} \,\mathrm{d}\mu(x) \\
&\lesssim \sup_{\|g\|_{\mathsf{Y}'}=1} \int_{\Xi} |g(x)| h(x) \,\mathrm{d}\mu(x) \sup_{\|\varphi\|_{\mathsf{X}'}=1} \int_{\Xi} K(x,y)^{1-\alpha} h(y)^{-1} |f(y)\varphi(y)| \,\mathrm{d}\mu(y) \\
&= \sup_{\|g\|_{\mathsf{Y}'}=1} \sup_{\|\varphi\|_{\mathsf{X}'}=1} \int_{\Xi} h(y)^{-1} |f(y)\varphi(y)| \,\mathrm{d}\mu(y) \int_{\Xi} K(x,y)^{1-\alpha} h(x) |g(x)| \,\mathrm{d}\mu(x) \\
&\lesssim \sup_{\|g\|_{\mathsf{Y}'}=1} \sup_{\|\varphi\|_{\mathsf{X}'}=1} \int_{\Xi} h(y)^{-1} |f(y)\varphi(y)| \big\|K(\cdot,y)^{1-\alpha} h(\cdot)\big\|_{\mathsf{Y}} \|g\|_{\mathsf{Y}'} \,\mathrm{d}\mu(y) \\
&\lesssim \sup_{\|g\|_{\mathsf{Y}'}=1} \sup_{\|\varphi\|_{\mathsf{X}'}=1} \|g\|_{\mathsf{Y}'} \int_{\Xi} |f(y)\varphi(y)| \,\mathrm{d}\mu(y) \\
&\lesssim \sup_{\|g\|_{\mathsf{Y}'}=1} \sup_{\|\varphi\|_{\mathsf{X}'}=1} \|g\|_{\mathsf{Y}'} \|\varphi\|_{\mathsf{X}'} \|f\|_{\mathsf{X}} \\
&\lesssim 1,
\end{aligned}$$

which ends the proof. ∎

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