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SOME PROPERTIES ON A CLASS OF p-VALENT FUNCTIONS INVOLVING GENERALIZED DIFFERENTIAL OPERATOR

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ABSTRACT. This paper aiming to introduce a new differential operator $T^{m}_{w,p,\alpha,\beta,\lambda,\delta}$ in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. We then, introduce a new subclass of analytic function $G^{w}_{m}(\mu, \lambda, \alpha, \beta, \delta, b, p)$. Moreover, we discuss coefficient estimates, growth and distortion theorems, and inclusion properties for the functions belonging to the class $G^{w}_{m}(\mu, \lambda, \alpha, \beta, \delta, b, p)$.

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1. INTRODUCTION

Let $A(p)$ denote the class of function f of the form:

$$f(z) = (z-w)^p + \sum_{n=p+1}^{\infty} a_n (z-w)^n, \tag{1.1}$$

which are analytic and normalized in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For a function f in $A(p)$, we defined the following differential operator

$$T^0_{w,p} f(z) = f(z), \tag{1.2}$$

$$T^1_{w,p,\alpha,\beta,\lambda,\delta} f(z) = (1 - p(p-1)\delta - p\beta(\lambda-\alpha)) f(z) + \beta(\lambda-\alpha)(z-w) f'(z) + \delta(z-w)^2 f''(z), \tag{1.3}$$

and for $n = 1, 2, 3, ...,$

$$\begin{aligned} T^2_{w,p,\alpha,\beta,\lambda,\delta} f(z) &= (1 - p(p-1)\delta - p\beta(\lambda-\alpha))(T^1_p f(z)) \\ &+ \beta(\lambda-\alpha)(z-w)(T^1_p f(z))' + \delta(z-w)^2 (T^1_p f(z))''. \end{aligned} \tag{1.4}$$

If f is given by (1.1) we get:

$$\begin{aligned} T^m_{w,p,\alpha,\beta,\lambda,\delta} f(z) &= (z-w)^p + \sum_{n=p+1}^{\infty} [1 + \delta(n(n-1) - p(p-1)) + (n-p)(\lambda-\alpha)\beta]^m (z-w)^n, \\ &(\alpha > 0,\ \beta, \mu, \lambda \geq 0,\ p \in \mathbb{N} \text{ and } n \in \mathbb{N}_0), \end{aligned} \tag{1.5}$$

where $f \in A(p)$, $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$. This generalizes many operators as follows:

(i) When $p = 1$, we get $T^m_{w,\alpha,\beta,\lambda} f(z) = (z-w) + \sum\limits_{n=2}^{\infty} [1 + (n-1)\beta(\lambda-\alpha) + n\delta]^m a_n (z-w)^n$, (see Al-Hawary et al. [2]).

(ii) When $p = 1$ and $w = \delta = 0$, we get $T^m_{\alpha,\beta,\lambda} f(z) = z + \sum\limits_{n=2}^{\infty} [1 + (n-1)\beta(\lambda-\alpha)]^m a_n z^n$, (see Darus and Ibrahim [7]).

(iii) When $w = \alpha = \delta = 0$ and $\beta = 1$, we get $T^m_\lambda = (z-w) + \sum\limits_{n=2}^{\infty} [1 + (n-1)\lambda]^m a_n (z-w)^n$, (see Al-Oboud [3]).

(iv) When $\alpha = \delta = 0$ and $\lambda = \beta = 1$, we get $T^m f(z) = (z-w) + \sum\limits_{n=2}^{\infty} n^m a_n (z-w)^n$, (see Acu and Owa [1]).

(vi) When $w = \alpha = \delta = 0$ and $\lambda = \beta = 1$, we get $T^m f(z) = z + \sum\limits_{n=2}^{\infty} n^m a_n z^n$,(see Salagean [10]).

Let $G^w_m(\mu, \lambda, \alpha, \beta, \delta, b, p)$ denote the subclass of $A(p)$ consisting of function f which satisfy

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left[p(1-\mu)\frac{T^m_{w,p,\alpha,\beta,\lambda,\delta} f(z)}{z-w} + \mu\left(T^m_{w,p,\alpha,\beta,\lambda,\delta} f(z)\right)' - p(z-w)^{p-1}\right]\right\} > 0 \tag{1.6}$$

where $T^m_{w,p,\alpha,\beta,\lambda,\delta} f(z)$ is given by (1.5). This satisfies the following inequality

$$\left| \frac{p(1-\mu)\frac{T^m_{w,p,\alpha,\beta,\lambda,\delta} f(z)}{z-w} + \mu(T^m_{w,p,\alpha,\beta,\lambda,\delta} f(z))' - p(z-w)^{p-1}}{p(1-\mu)\frac{T^m_{w,p,\alpha,\beta,\lambda,\delta} f(z)}{z-w} + \mu(T^m_{w,p,\alpha,\beta,\lambda\delta} f(z))' + 2b - p(z-w)^{p-1}} \right| < 1. \tag{1.7}$$

where $z \in U, \mu \geq 0, p \in \mathbb{N}, n \in \mathbb{N}_0, b \in \mathbb{C} - \{0\}$.

We note that

- $G_m^w(\mu,\lambda,\alpha,\beta,\delta,b,1) = G(\mu,\lambda,\alpha,\beta,b)$
  $= \{f \in A(p) : \mathrm{Re}\{1 + \frac{1}{b}[(1-\mu)\frac{Tf(z)}{z-w} + \mu(T^m_{w,p,\alpha,\beta,\lambda,\delta} f(z))' - 1]\} > 0, z \in U\}$.
- $G_0^w(\mu,\lambda,\alpha,\beta,\delta,b,1) = G(\lambda,b)$
  $= \{f \in A(p) : \mathrm{Re}\{1 + \frac{1}{b}[(1-\mu)\frac{f(z)}{z-w} + \mu(f(z))' - 1]\} > 0, z \in U\}$.
- $G_m^0(1,1,1,0,1,b,1) = R_m(b) = \{f \in A(p) : \mathrm{Re}\{1 + \frac{1}{b}[(T^m f(z))' - 1]\} > 0, z \in U\}$.
- $G_m^w(0,1,1,0,1,b,1) = G_m(b) = \{f \in A(p) : \mathrm{Re}\{1 + \frac{1}{b}[\frac{T^m f(z)}{z-w} - 1]\} > 0, z \in U\}$.
- $G_0^w(0,1,1,0,1,b,1) = G(b) = \{f \in A(p) : \mathrm{Re}\{1 + \frac{1}{b}[\frac{f(z)}{z-w} - 1]\} > 0, z \in U\}$.
- $G_0^0(0,1,1,0,1-\alpha,1,1) = G_\alpha = \{f \in A(p) : \mathrm{Re}\,\frac{f(z)}{z-w} > \alpha, 0 \le \alpha < 1, z \in U\}$.
- $G_0^0(0,1,1,0,1-\alpha,1,1) = R_\alpha = \{f \in A(p) : \mathrm{Re}(f(z))' > \alpha, 0 \le \alpha < 1, z \in U\}$.

For more details see [4, 5, 6, 8, 9]. Recently Yousef and Salleh [11] investigated a subclass of complex-valued harmonic univalent functions defined by a generalized linear operator, and some properties such as coefficient bounds of this class are obtained. In this paper, coefficient inequalities, distortion theorem, and closure theorems of functions belonging to the class $G_m^w(\mu,\lambda,\alpha,\beta,\delta,b,p)$ are obtained.

2. COEFFICIENT INEQUALITIES

In this section we find the coefficient inequality for the class $G_m^w(\mu,\lambda,\alpha,\beta,\delta,b,p)$.

Theorem 2.1. *The function $f(z)$ defined by (1.1) satisfies the condition*

$$\sum_{n=p+1}^{\infty} [p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m |a_n| \le |b| \tag{2.1}$$

where $\mu \ge 0, p \in \mathbb{N}, n \in \mathbb{N}_0, \alpha > 0$ and $\beta + \lambda > 0$ if and only if $f(z) \in G_m^w(\mu,\lambda,\alpha,\beta,\delta,b,p)$. The result (2.1) is sharp for functions of the form:

$$f(z) = (z-w) + \frac{|b|}{[p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m}(z-w)^n$$
$$(n \ge p+1; m \in \mathbb{N}_0).$$

Proof. Suppose that the inequality (2.1) holds. Then we have for $z \in U$ and $|z| < 1$

$$\begin{aligned}
&\left|p(1-\mu)\frac{T^m_{w,p,\alpha,\beta,\lambda,\delta}f(z)}{z-w}+\mu(T^m_{w,p,\alpha,\beta,\lambda,\delta}f(z))'-p(z-w)^{p-1}\right| \\
&-\left|p(1-\mu)\frac{T^m_{w,p,\alpha,\beta,\lambda,\delta}f(z)}{z-w}+\mu(T^m_{w,p,\alpha,\beta,\lambda,\delta}f(z))'-p(z-w)^{p-1}+2b\right| \\
&=\left|\sum_{n=p+1}^{\infty}[p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m\right| \\
&|a_n|\left|(z-w)^{n-1}\right| \\
&-\left|2b+\sum_{n=p+1}^{\infty}[p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m\right| \\
&|a_n|\left|(z-w)^{n-1}\right| \\
&\leq\sum_{n=p+1}^{\infty}[p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m \\
&|a_n|\left|(z-w)^{n-1}\right| \\
&-2\left|b\right|-\sum_{n=p+1}^{\infty}[p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m \\
&|a_n|\left|(z-w)^{n-1}\right| \\
&\leq\sum_{n=p+1}^{\infty}[p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m\left|a_n\right|-\left|b\right| \\
&\leq 0
\end{aligned}$$

where $T^m_{w,p,\alpha,\beta,\lambda,\delta}f(z)$ is given by (1.5). This implies

$$\sum_{n=p+1}^{\infty}[p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m\left|a_n\right|\leq\left|b\right|,$$

which shows that $f\in G^w_m(\mu,\lambda,\alpha,\beta,b,p)$. For the converse, assume that

$$\left|\frac{p(1-\mu)\frac{T^m_{w,p,\alpha,\beta,\lambda,\delta}f(z)}{z-w}+\mu(T^m_{w,p,\alpha,\beta,\lambda,\delta}f(z))'-p(z-w)^{p-1}}{p(1-\mu)\frac{T^m_{w,p,\alpha,\beta,\lambda,\delta}f(z)}{z-w}+\mu(T^m_{w,p,\alpha,\beta,\lambda\delta}f(z))'+2b-p(z-w)^{p-1}}\right|<1. \tag{2.2}$$

This implies

$$\left|\frac{P}{Q}\right|<1$$

where

$$P=\sum_{n=p+1}^{\infty}[p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m a_n(z-w)^{n-1}$$

and

$$Q=2b+\sum_{n=p+1}^{\infty}[p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m a_n(z-w)^{n-1}.$$

Since $\text{Re}(z-w) \leq |z-w|$ for all $(z-w)$, it follows from (2.2) that

$$\text{Re}\left(\frac{R}{S}\right) < 1$$

where

$$R = \sum_{n=p+1}^{\infty} [p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m |a_n| (z-w)^{n-1}$$

and

$$S = 2b + \sum_{n=p+1}^{\infty} [p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m |a_n| (z-w)^{n-1}.$$

Choose values of $(z-w)$ on the real axis, and let $|z-w| \to 1^-$ through the real values, we obtain

$$\begin{aligned} &\sum_{n=p+1}^{\infty} [p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m |a_n| \\ \leq\ & 2b + \sum_{n=p+1}^{\infty} [p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m |a_n|. \end{aligned}$$

This gives the required condition given by (2.1). ∎

Corollary 2.2. *Let the function f defined by (1.1) be in the class $G_m^w(\mu, \lambda, \alpha, \beta, \delta, b, p)$, then we have*

$$|a_n| \leq \frac{|b|}{[p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m}, n \geq p+1.$$

3. GROWTH AND DISTORTION THEOREMS

A growth and distortion property for function f to be in the class $G_m^w(\mu, \lambda, \alpha, \beta, \delta, b, p)$ is given as follows:

Theorem 3.1. *If the function $f(z)$ be defined by (1.1) is in the class $G_m^w(\mu, \lambda, \alpha, \beta, \delta, b, p)$.Then for $|z-w| = r < 1$, we have*

$$\begin{aligned} &|r^p| - \frac{|b|\,|r|^n}{(p+\mu)\,[1+2p\delta+\beta(\lambda-\alpha)]^m} \leq |f(z)| \\ &\leq |r^p| + \frac{|b|\,|r|^n}{(p+\mu)\,[1+2p\delta+\beta(\lambda-\alpha)]^m} \end{aligned}$$

and

$$\begin{aligned} &pr^{p-1} - r^p \frac{(p+1)\,|b|}{(p+\mu)\,[1+2p\delta+\beta(\lambda-\alpha)]^m} \leq |f'(z)| \\ \leq\ & pr^{p-1} + r^p \frac{(p+1)\,|b|}{(p+\mu)\,[1+2p\delta+\beta(\lambda-\alpha)]^m}. \end{aligned}$$

Proof. Let $f \in G_m(\mu, \lambda, \alpha, \beta, \delta, b, p)$, then by (2.1) we have

$$\sum_{n=p+1}^{\infty} [p+\mu(n-p)][1+\delta(n(n-1)-p(p-1))+(n-p)(\lambda-\alpha)\beta]^m |a_n| \leq |b|.$$

By substituting $n = p + 1$, we obtain

$$\sum_{n=p+1}^{\infty} [p + \mu][1 + \delta((p + 1)p - p(p - 1)) + (\lambda - \alpha)\beta]^m |a_n| \le |b| .$$

This implies

$$\sum_{n=p+1}^{\infty} |a_n| \le \frac{|b|}{(p + \mu)[1 + 2p\delta + \beta(\lambda - \alpha)]^m}.$$

From equation (1.1) we have

$$\begin{aligned} |f(z)| &\le |z - w|^p + \sum_{n=p+1}^{\infty} |a_n| |z - w|^n \\ &\le r^p + \frac{|b|}{(p + \mu) [1 + 2p\delta + \beta(\lambda - \alpha)]^m} r^n. \end{aligned}$$

Similarly we can prove that

$$|f(z)| \ge r^p - \frac{|b|}{(p + \mu) [1 + 2p\delta + \beta(\lambda - \alpha)]^m} r^n.$$

Also

$$\begin{aligned} |f'(z)| &= \left| p(z - w)^{p-1} + \sum_{n=p+1}^{\infty} n a_n (z - w)^{n-1} \right| \\ &\le p |z - w|^{p-1} + \sum_{n=p+1}^{\infty} n |a_n| |z - w|^{n-1} \le p r^{p-1} + r^p \sum_{n=p+1}^{\infty} n |a_n| \\ &\le p r^{p-1} + r^p \frac{(p + 1) |b|}{(p + \mu) [1 + 2p\delta + \beta(\lambda - \alpha)]^m}. \end{aligned}$$

Similarly we can prove that

$$|f'(z)| \ge p r^{p-1} - r^p \frac{(p + 1) |b|}{(p + \mu) [1 + 2p\delta + \beta(\lambda - \alpha)]^m}.$$

∎

4. INCLUSION PROPERTIES

The inclusion properties for the class $G_m^w (\mu, \lambda, \alpha, \beta, \delta, b, p)$ are given by the following theorem.

Theorem 4.1. *Let the hypothesis of (2.1) be satisfied. Then*

$$\begin{aligned} G_m^w(\mu_2, \lambda, \alpha, \beta, \delta, b, p) &\subseteq G_m^w(\mu_1, \lambda, \alpha, \beta, \delta, b, p) \\ G_m^w(\mu, \lambda_2, \alpha, \beta, \delta, b, p) &\subseteq G_m^w(\mu, \lambda_1, \alpha, \beta, \delta, b, p) \\ G_m^w(\mu, \lambda, \alpha_2, \beta, \delta, b, p) &\subseteq G_m^w(\mu, \lambda, \alpha_1, \beta, \delta, b, p) \\ G_m^w(\mu, \lambda, \alpha, \beta_2, \delta, b, p) &\subseteq G_m^w(\mu, \lambda, \alpha, \beta_1, \delta, b, p) \\ G_m^w(\mu, \lambda, \alpha, \beta, \delta_2, b, p) &\subseteq G_m^w(\mu, \lambda, \alpha, \beta, \delta_1, b, p) \end{aligned}$$

where $\alpha_2 \ge \alpha_1, \beta_2 \ge \beta_1, \lambda_2 \ge \lambda_1, \mu_2 \ge \mu_1$ *and* $\delta_2 \ge \delta_1$.

Proof. Let $f \in G_m^w(\mu, \lambda, \alpha, \beta, \delta, b, p)$. Then by using (2.1) we have

$$\sum_{n=p+1}^{\infty} [p + \mu(n-p)][1 + \delta(n(n-1) - p(p-1)) + (n-p)(\lambda - \alpha)\beta]^m |a_n| \leq |b|.$$

If $\mu_2 \geq \mu_1$, then $p + \mu_2(n-p) \geq p + \mu_1(n-p)$ such that

$$\sum_{n=p+1}^{\infty} [p + \mu_2(n-p)][1 + \delta(n(n-1) - p(p-1)) + (n-p)(\lambda - \alpha)\beta]^m |a_n|$$
$$\geq \sum_{n=p+1}^{\infty} [p + \mu_1(n-p)][1 + \delta(n(n-1) - p(p-1)) + (n-p)(\lambda - \alpha)\beta]^m |a_n|.$$

This shows that

$$\sum_{n=p+1}^{\infty} [p + \mu_1(n-p)][1 + \delta(n(n-1) - p(p-1)) + (n-p)(\lambda - \alpha)\beta]^m |a_n|$$
$$\leq \sum_{n=p+1}^{\infty} [p + \mu_2(n-p)][1 + \delta(n(n-1) - p(p-1)) + (n-p)(\lambda - \alpha)]^m |a_n| \leq |b|$$

or

$$\sum_{n=p+1}^{\infty} [p + \mu_1(n-p)][1 + \delta(n(n-1) - p(p-1)) + (n-p)(\lambda - \alpha)\beta]^m |a_n| \leq |b|.$$

Hence $f \in G_m^w(\mu, \lambda, \alpha, \beta, \delta, b, p)$, which shows that

$$G_m^w(\mu_2, \lambda, \alpha, \beta, \delta, b, p) \subseteq G_m^w(\mu_1, \lambda, \alpha, \beta, \delta, b, p).$$

Similarly, let $f \in G_m^w(\mu, \lambda, \alpha, \beta, \delta, b, p)$, then by using (2.1) we have $\delta_2 \geq \delta_1$. This implies that

$$[1 + \delta_2(n(n-1) - p(p-1)) + (n-p)(\lambda - \alpha)]^m$$
$$\geq [1 + \delta_1(n(n-1) - p(p-1)) + (n-p)(\lambda - \alpha)]^m,$$

so

$$\sum_{n=p+1}^{\infty} [p + \mu(n-p)][1 + \delta_2(n(n-1) - p(p-1)) + (n-p)(\lambda - \alpha)\beta]^m |a_n|$$
$$\geq \sum_{n=p+1}^{\infty} [p + \mu(n-p)][1 + \delta_1(n(n-1) - p(p-1)) + (n-p)(\lambda - \alpha)\beta]^m |, a_n|$$

and hence

$$\sum_{n=p+1}^{\infty} [p + \mu(n-p)][1 + \delta_1(n(n-1) - p(p-1)) + (n-p)(\lambda - \alpha)\beta]^m |a_n| \leq |b|.$$

This proves that $f \in G_m^w(\mu, \lambda, \alpha, \beta, \delta, b, p)$ and finally implies that

$$G_m^w(\mu, \lambda, \alpha, \beta, \delta_2, b, p) \subseteq G_m^w(\mu, \lambda, \alpha, \beta, \delta_1, b, p).$$

Employing a similar procedure we can prove that

$$G_m^w(\mu, \lambda_2, \alpha, \beta, \delta, b, p) \subseteq G_m^w(\mu, \lambda_1, \alpha, \beta, \delta, b, p),$$
$$G_m^w(\mu, \lambda, \alpha_2, \beta, \delta, b, p) \subseteq G_m^w(\mu, \lambda, \alpha_1, \beta, \delta, b, p)$$

and

$$G_m^w(\mu, \lambda, \alpha, \beta_2, \delta, b, p) \subseteq G_m^w(\mu, \lambda, \alpha, \beta_1, \delta, b, p).$$

∎

5. INTEGRAL OPERATORS

In this section, we consider integral transforms of functions in the class $G_m^w(\mu, \lambda, \alpha, \beta, \delta, b, p)$.

Theorem 5.1. *If the function $f(z)$ given by (1.1) is in the class $G_m^w(\mu, \lambda, \alpha, \beta, \delta, b, p)$, and let c be a real number such that $c > -p$, then the function $I(z)$ defined by*

$$I(z) = \frac{c+p}{(z-w)^c} \int_w^z (t-w)^{c-1} f(t)dt \tag{5.1}$$

also belongs to the class $G_m^w(\mu, \lambda, \alpha, \beta, \delta, b, p)$.

Proof. From (5.1), it follows that $I(z) = (z-w)^p + \sum_{n=p+1}^{\infty} b_n (z-w)^n$, where $b_n = \left(\frac{c+p}{n+c}\right) a_n$. Therefore

$$\begin{aligned}
&\sum_{n=p+1}^{\infty} [p + \mu(n-p)][1 + \delta(n(n-1) - p(p-1)) + (n-p)(\lambda - \alpha)]^m b_n \\
&= \sum_{n=p+1}^{\infty} [p + \mu(n-p)][1 + \delta(n(n-1) - p(p-1)) + (n-p)(\lambda - \alpha)]^m \left(\frac{c+p}{n+c}\right) a_n \\
&\leq \sum_{n=p+1}^{\infty} [p + \mu(n-p)][1 + \delta(n(n-1) - p(p-1)) + (n-p)(\lambda - \alpha)]^m a_n \\
&\leq |b|,
\end{aligned}$$

since $f(z) \in G_m^w(\mu, \lambda, \alpha, \beta, \delta, b, p)$. Hence by Theorem 2.1, $I(z) \in G_m^w(\mu, \lambda, \alpha, \beta, \delta, b, p)$. ∎

6. CONCLUSION

In this work, the class of the p-valent functions $G_m^w(\mu, \lambda, \alpha, \beta, \delta, b, p)$ is introduced embedded with the generalized differential operator $T_{w,p,\alpha,\beta,\lambda,\delta}^m f(z)$. In the first section, the inequality (2.1) is sharp and also the function $f(z)$ belongs to the class $G_m^w(\mu, \lambda, \alpha, \beta, \delta, b, p)$ if and only if the inequality (2.1) holds. Upon the above result, distortion theorems and inclusion properties hold. Moreover coefficient estimates and growth theorem are studied and an integral operator $I(z)$ is introduced. Furthermore, the integral operator $I(z)$ belongs to the class $G_m^w(\mu, \lambda, \alpha, \beta, \delta, b, p)$.

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