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NONLINEAR SYSTEM OF MIXED ORDERED VARIATIONAL INCLUSIONS INVOLVING XOR OPERATION

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ABSTRACT. In this work, we introduce and solve an NSMOVI frameworks system involving XOR operation with the help of a proposed iterative algorithm in real ordered positive Hilbert spaces. We discuss the existence of a solution of a considered system of inclusions involving XOR operation by applying the resolvent operator technique with XOR operation and also study the strong convergence of the sequences generated by the considered algorithm. Further, we give a numerical example in support of our considered problem which gives the grantee that all the proposed conditions of our main result are fulfilled.

Key words and phrases: Comparison; Convergence; Ordered; (α_A, λ)-XOR-weak-ANODD multi-valued mapping; System; XOR Operation.

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1. INTRODUCTION

A wide class of inclusion problems has been investigated to find the zeros of the monotone operator G from $\mathbb{R}^n$ to itself that is find $p \in \mathbb{R}^n$ such that $0 \in G(p)$. Many problems in management sciences, economics, operations research, physics, and applied sciences can be formulated as an inclusion problem $0 \in G(p)$, for a given multi-valued mapping G in Hilbert spaces. The resolvent operator elegant methods introduced to prove the existence of solution and developed some iterative procedures for several types of variational inclusions and their generalizations which provided us a powerful and novel framework for the study abroad class of nonlinear problems arising in optimization, convex programming problems, tomography, molecular biology, image restoring processing in applied and pure sciences (see, [1, 4, 7, 8, 9, 10, 11, 12, 14, 15, 16, 20, 21, 22, 25]).

In 1972, Amann [8] established for computing the solutions of nonlinear equations and fixed point theory with nonlinear mapping and applications have been studied with nonlinear increasing operators in real ordered Hilbert space or Banach spaces investigated by Du [13] which is applicable in nonlinear analysis and developed the methods to solve original mathematical problems. Future, many authors discussed and studied the idea of ordered nonlinear variational inequalities (inclusions) in different settings which is available in the literature. In 2008, Li and his coauthors have investigated and analyzed the ordered variational inequality problem to obtain $t \in \mathcal{B}$ such that $T(h(t)) \geq 0$ and after that introduced and considered a general nonlinear ordered variational inequalities problem to obtain $t \in \mathcal{B}$ such that $T(t) \oplus G(t, h(t)) \geq 0$ (h, T and $G(.,.)$ are nonlinear mappings), and discussed the existence and convergence results in real ordered Hilbert or Banach spaces with the help of restricted-accretive mapping techniques (see, [17, 18]).

Very recently, many authors have been considered and studied ordered equations (inclusions) problem which solved by using the different kinds of multi-valued mappings to find the solutions of nonlinear ordered equations (inclusions) with XOR operations in different settings (see [2, 3, 5, 17, 18, 19, 23, 24]).

Motivated and inspired by the above research described above, the aim of this work is proposed as follows. In section 2, contains certain basic results needed in this paper. In Section 3, we consider a nonlinear system of mixed ordered variational inclusions with XOR operation in real positive ordered Hilbert spaces with the help of the idea of XOR operation. We propose the iterative algorithms which are more general than the previous iterative algorithms involving XOR operation which is investigated by Li et al. [17, 18, 19]. In section 4, we discuss the existence of a solution of the considered problem and analyze the convergence criteria of the proposed algorithm. Finally, we demonstrate an example that ensures that all the assumptions of our consider problem are fulfilled.

2. PRELIMINARIES

In this article, we consider that $\mathcal{B}$ is a real ordered positive Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let d be the metric induced by the norm $\|\cdot\|$ and $2^{\mathcal{B}}$ (respectively, $CB(\mathcal{B})$) express the collection of all nonempty (respectively, bounded and closed) subsets of $\mathcal{B}$, and $D(\cdot, \cdot)$ is the Hausdorff metric on $CB(\mathcal{B})$ defined by

$$D(S,T) = \max\left\{\sup_{s\in S} d(s,T), \sup_{t\in T} d(S,t)\right\},$$

where $d(s,T) = \inf_{t\in T} d(s,t)$ and $d(S,t) = \inf_{s\in S} d(s,t)$.

Definition 2.1 ([13, 25]). *A non-empty subset P of $\mathcal{B}$ is called*

(i) *a normal cone if there exists a constant $\delta_P > 0$ such that for $0 \leq s \leq t$, we have $||s|| \leq \delta_P ||t||$;*
(ii) *for each $s, t \in \mathcal{B}$, $s \leq t$ if and only if $t - s \in P$;*
(iii) *s and t are said to be comparative to each other if and only if, we have either $s \leq t$ or $t \leq s$ and is denoted by $s \propto t$.*

Definition 2.2 ([7, 25]). *For arbitrary elements $s, t \in \mathcal{B}$, $lub\{s, t\}$ is denoted by least upper bound of the set $\{s, t\}$ and $glb\{s, t\}$ is denoted by greatest lower bound of the set $\{s, t\}$, respectively. Let $glb\{s, t\}$ and $lub\{s, t\}$ exist, binary operations $\vee, \wedge, \oplus$ and $\odot$ which called as AND, OR, XNOR and XOR operations, respectively are defined as follows:*

(i) $s \wedge t = glb\{s, t\}$;
(ii) $s \vee t = lub\{s, t\}$;
(iii) $s \odot t = (s - t) \wedge (t - s)$;
(iv) $s \oplus t = (s - t) \vee (t - s)$.

Lemma 2.1 ([13]). *For any positive integer n, $s \propto t_n$ and $t_n \to t^*$ as $n \to \infty$, then $s \propto t^*$.*

Lemma 2.2 ([13, 19, 25]). *Let $\odot$ and $\oplus$ be the XNOR and XOR operations, respectively. Then the following properties satisfied:*

(i) $s \odot s = 0$, $s \oplus t = t \oplus s = -(s\odot) = -(t \odot s)$;
(ii) $(\lambda s) \oplus (\lambda t) = |\lambda|(s \oplus t)$;
(iii) $0 \leq s \oplus t$, *if $s \propto t$;*
(iv) *if $s \propto t$, then $s \oplus t = 0$ if and only if $s = t$;*
(v) $(s + t) \odot (u + v) \geq (s \odot u) + (t \odot v)$;
(vi) *if s, t and w are comparative to each other, then $(s \oplus t) \leq s \oplus w + w \oplus t$;*
(vii) *if $s \propto t$, then $((s \oplus 0) \oplus (t \oplus 0)) \leq (s \oplus t) \oplus 0 = s \oplus t$;*
$(viii)$ $(ms) \oplus (ls) = |m - l|s = (m \oplus l)s$, *if $s \propto 0$, $\forall\, s, t, u, v, w \in \mathcal{B}$ and $m, l, \lambda \in \mathbb{R}$.*

Lemma 2.3 ([13]). *Let P be a normal cone with normal constant δ_P in real ordered positive Hilbert space, then for arbitrary $s, t \in \mathcal{B}$, the following properties hold:*

(i) $\|s \vee t\| \leq \|s\| \vee \|t\| \leq \|s\| + \|t\|$;
(ii) $\|s \oplus t\| \leq \|s - t\| \leq \delta_P |s \oplus t\|$;
(iii) *if $s \propto t$, then $\|s \oplus t\| = \|s - t\|$.*

Definition 2.3 ([19]). *A comparison mapping $S : \mathcal{B} \to \mathcal{B}$ is called*

(i) *a strongly comparison mapping, $s \propto t$ if and only if $S(s) \propto S(t)$, for all $s, t \in \mathcal{B}$;*
(ii) *a β-ordered compression mapping, if S is a comparison mapping and there exists $0 < \lambda_S < 1$ such that*

$$S(s) \oplus S(t) \leq \lambda_S (s \oplus t), \text{ for all } s, t \in \mathcal{B}.$$

(iii) *a ν-ordered non-extended mapping, if there exists $\nu > 0$ such that*

$$S(s) \oplus S(t) \geq \nu(s \oplus t), \text{ for all } s, t \in \mathcal{B}.$$

Definition 2.4. *A multi-valued mapping $F : \mathcal{B} \to CB(\mathcal{B})$ is called D-Lipschitz continuous, if for every $s, t \in \mathcal{B}$, $s \propto t$, there exists a constant δ_F such that*

$$D(F(s), F(t)) \leq \delta_F (s \oplus t), \text{ for all } s, t \in \mathcal{B}.$$

Definition 2.5 ([17, 19]). *Let $A : \mathcal{B} \to \mathcal{B}$ be a strong comparison mapping and ν-ordered non-extended mapping, and $M : \mathcal{B} \to 2^{\mathcal{B}}$ be a multi-valued mapping. Then M is said to be*

(i) *a weak comparison mapping, if for every* $v_s \in M(s)$, $s \propto v_s$, *and if* $s \propto t$, *then for every* $v_s \in M(s)$ *and* $v_t \in M(t)$, $v_s \propto v_t$, *for all* $s, t \in \mathcal{B}$;

(ii) *a* α_A*-weak-non-ordinary difference mapping with respect to* A, *if for every* $s, t \in \mathcal{B}$, *there exist* $\alpha_A > 0$ *and* $v_s \in M(A(s))$ *and* $v_t \in M(A(t))$ *such that*

$$(v_s \oplus v_t) \oplus \alpha_A(A(s) \oplus A(t)) = 0;$$

(iii) *a* λ*-XOR-ordered different weak compression mapping with respect to* A, *if for every* $s, t \in \mathcal{B}$, *there exists a constant* $\lambda > 0$ *and* $v_s \in M(A(s)), v_t \in M(A(t))$ *such that*

$$\lambda(v_s \oplus v_t) \geq s \oplus t;$$

(iv) *a* (α_A, λ)*-XOR-weak-ANODD multi-valued mapping, if* M *is a* α_A*-weak-non-ordinary difference mapping with respect to* A *and* λ*-XOR-ordered different weak compression mapping with respect to* A, *and* $[A \oplus \lambda M](\mathcal{B}) = \mathcal{B}$, *for* $\lambda, \beta, \alpha > 0$.

Definition 2.6. *Let* $A : \mathcal{B} \to \mathcal{B}$ *be a strongly comparison and* γ*-ordered non-extended mapping. Let* $M : \mathcal{B} \to 2^{\mathcal{B}}$ *be a* (α_A, λ)*-XOR-weak-ANODD multi-valued mapping. The resolvent operator* $\mathcal{J}^A_{\lambda,M} : \mathcal{B} \to \mathcal{B}$ *associated with* A *and* M *is defined by*

$$\mathcal{J}^A_{\lambda,M}(s) = [A \oplus \lambda M]^{-1}(s), \forall s \in \mathcal{B}, \tag{2.1}$$

where $\lambda > 0$ *is a constant.*

Lemma 2.4. *(*[6]*) Let* $A : \mathcal{B} \to \mathcal{B}$ *be a strongly comparison,* γ*-ordered non-extended mapping and* $M : \mathcal{B} \to 2^{\mathcal{B}}$ *be a* α_A*-weak-non-ordinary difference multi-valued mapping with respect to* A *with* $\lambda\alpha_A \neq 1$. *Then the resolvent operator* $\mathcal{J}^A_{\lambda,M} : \mathcal{B} \to \mathcal{B}$ *is well-defined and single-valued, for all* $\alpha, \lambda > 0$.

Lemma 2.5. *(*[6]*) Let* $M : \mathcal{B} \to 2^{\mathcal{B}}$ *be a* (α_A, λ)*-XOR-weak-ANODD multi-valued mapping with respect to* $\mathcal{J}^A_{\lambda,M}$. *Let* $A : \mathcal{B} \to \mathcal{B}$ *be a comparison and* γ*-ordered non-extended mapping with respect to* $\mathcal{J}^A_{\lambda,M}$, *for* $\mu \geq 1$ *and* $\lambda\alpha_A > \mu$. *Then the resolvent operator* $\mathcal{J}^A_{\lambda,M}$ *is comparison and the following condition holds:*

$$\mathcal{J}^A_{\lambda,M}(s) \oplus \mathcal{J}^A_{\lambda,M}(t) \leq \frac{\mu}{(\lambda\alpha_A \oplus \mu)}(s \oplus t), \ for \ all \ s, t \in \mathcal{B}.$$

3. EXISTENCE RESULT FOR NSMOVI PROBLEM AND ITERATIVE ALGORITHMS

For $i = 1, 2$, let $\mathcal{B}_i$ be the real ordered positive Hilbert spaces and P_i be the normal cones with normal constants δ_{P_i}. Let $A_i, f_i, g_i : \mathcal{B}_i \to \mathcal{B}_i$ and $T_i : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{B}_i$ be the single-valued mappings. Let $F_i : \mathcal{B}_i \to 2^{\mathcal{B}_i}$ be a multi-valued mapping and $M_i : \mathcal{B}_1 \times \mathcal{B}_2 \to 2^{\mathcal{B}_i}$ be a (α_A, λ)-XOR-weak-ANODD multi-valued mapping. We propose the following problem:

For each $\omega_i \geq 0$, find $(s, t) \in \mathcal{B}_1 \times \mathcal{B}_2$, for $y \in F_1(s),\ z \in F_2(t)$ such that

$$\left.\begin{array}{rcl} \omega_1 & \in & T_1(s - g_1(s), z) \oplus \rho_1 M_1(f_1(s), t), \\ \\ \omega_2 & \in & T_2(y, t \oplus g_2(t)) + \rho_2 M_2(s, f_2(t)). \end{array}\right\} \tag{3.1}$$

We call this problem as nonlinear system of mixed ordered variational inclusions involving XOR operation (in short, NSMOVI).

By applying the resolvent operator method, we establish an equivalence result for NSMOVI (3.1) and a nonlinear equations.

Lemma 3.1. *Let $(s,t) \in \mathcal{B}_1 \times \mathcal{B}_2$, $y \in F_1(s)$, $z \in F_2(t)$. Then (s,t,y,z) is a solution of NSMOVI (3.1) if and only if (s,t,y,z) satisfies the following relations:*

$$\left.\begin{aligned} & f_1(s) \oplus \mathcal{J}^{A_1}_{\lambda_1, M_1(.,t)}\Big((A_1of_1)(s) \oplus \tfrac{\lambda_1}{\rho_1}\big[\omega_1 \oplus T_1(s-g_1(s),z)\big]\Big) = 0, \\ & f_2(t) \oplus \mathcal{J}^{A_2}_{\lambda_2, M_2(s,.)}\Big((A_2of_2)(t) \oplus \tfrac{\lambda_2}{\rho_2}\left[\omega_2 - T_2(y, t\oplus g_2(t))\right]\Big) = 0. \end{aligned}\right\} \tag{3.2}$$

Proof. The proof is a direct consequence of the resolvent operator $\mathcal{J}^A_{\lambda,M}$ defined in Definition 2.6. ∎

Now, we construct the iterative algorithms based on Lemma 3.1 to find the approximate solutions of NSMOVI (3.1).

Iterative Algorithm 3.1. *For $i = 1, 2$, let $f_i, g_i : \mathcal{B}_i \to \mathcal{B}_i$ and $T_i : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{B}_i$ be the single-valued mappings. Let $F_i : \mathcal{B}_i \to 2^{\mathcal{B}_i}$ be a multi-valued mapping and $M_i : \mathcal{B}_1 \times \mathcal{B}_2 \to 2^{\mathcal{B}_i}$ be a (α_A, λ)-XOR-weak-ANODD multi-valued mapping.*

Choose $(s_0, t_0) \in \mathcal{B}_1 \times \mathcal{B}_2$ and choose $y_0 \in F_1(s_0)$ and $z_0 \in F_2(t_0)$. Let $s_{n+1} \propto s_n$ and $t_{n+1} \propto t_n$.

Step 1. Let

$$\begin{aligned} f_1(s_{n+1}) &= (1-\alpha_n-\beta_n)f_1(s_n) + \alpha_n \mathcal{J}^{A_1}_{\lambda_1, M_1(.,t_n)}\Big((A_1of_1)(s_n) \\ &\quad \oplus \frac{\lambda_1}{\rho_1}\big[\omega_1 \oplus T_1(s_n - g_1(s_n), z_n)\big]\Big), \end{aligned} \tag{3.3}$$

$$\begin{aligned} f_2(t_{n+1}) &= (1-\alpha_n-\beta_n)f_2(t_n) + \alpha_n \mathcal{J}^{A_2}_{\lambda_2, M_2(s_n,.)}\Big((A_2of_2)(t_n) \\ &\quad \oplus \frac{\lambda_2}{\rho_2}\big[\omega_2 - T_2(y_n, t_n \oplus g_2(t_n))\big]\Big), \end{aligned} \tag{3.4}$$

where α_n and β_n are non-negative constants such that $0 < \alpha_n + \beta_n \leq 1$ and $\limsup\limits_{n\geq 0} \alpha_n < 1$.

Step 2. Choose $y_{n+1} \in F_1(s_{n+1})$ and $z_{n+1} \in F_2(t_{n+1})$ such that

$$y_{n+1} \oplus y_n \leq \left(1 + (1+n)^{-1}\right) D_1(F_1(s_{n+1}), F_1(s_n)), \tag{3.5}$$

$$z_{n+1} \oplus z_n \leq \left(1 + (1+n)^{-1}\right) D_2(F_2(t_{n+1}), F_2(t_n)), \tag{3.6}$$

where $D_i(.,.)$ are the Hausdorff metrics on $CB(\mathcal{B}_i)$.

Step 3. If $s_{n+1}, t_{n+1}, y_{n+1}$ and z_{n+1} satisfiying (3.3) and (3.4) to a sufficient degree of accuracy, stop; otherwise, set $n = n+1$ and return to step 2.

Iterative Algorithm 3.2. *Choose $(s_0, t_0) \in \mathcal{B}_1 \times \mathcal{B}_2$ and choose $s_0 \in F_1(s_0)$ and $t_0 \in F_2(t_0)$. Let $s_{n+1} \propto s_n$ and $t_{n+1} \propto t_n$.*

Step 1. Let

$$\begin{aligned} f_1(s_{n+1}) &= (1-\alpha-\beta)f_1(s_n) + \alpha \mathcal{J}^{A_1}_{\lambda_1, M_1(.,t_n)}\Big((A_1of_1)(s_n) \\ &\quad \oplus \frac{\lambda_1}{\rho_1}\big[\omega_1 \oplus T_1(s_n - g_1(s_n), z_n)\big]\Big), \end{aligned} \tag{3.7}$$

$$\begin{aligned} f_2(t_{n+1}) &= (1-\alpha-\beta)f_2(t_n) + \alpha \mathcal{J}^{A_2}_{\lambda_2, M_2(s_n,.)}\Big((A_2of_2)(t_n) \\ &\quad \oplus \frac{\lambda_2}{\rho_2}\big[\omega_2 - T_2(y_n, t_n \oplus g_2(t_n))\big]\Big), \end{aligned} \tag{3.8}$$

where α *and* β *are non-negative constants such that* $0 < \alpha + \beta \leq 1$.

Step 2. Choose $y_{n+1} \in F_1(s_{n+1})$ *and* $z_{n+1} \in F_2(t_{n+1})$ *such that*

$$y_{n+1} \oplus y_n \leq \left(1 + (1+n)^{-1}\right) D_1(F_1(s_{n+1}), F_1(s_n)), \tag{3.9}$$

$$z_{n+1} \oplus z_n \leq \left(1 + (1+n)^{-1}\right) D_2(F_2(t_{n+1}), F_2(t_n)), \tag{3.10}$$

where $D_i(.,.)$ *are the Hausdorff metric on* $CB(\mathcal{B}_i)$.

Step 3. If $s_{n+1}, t_{n+1}, y_{n+1}$ *and* z_{n+1} *satisfying (3.7) and (3.8) to a sufficient degree of accuracy, stop; otherwise, set* $n = n+1$ *and return to step 2.*

Iterative Algorithm 3.3. *Choose* $(s_0, t_0) \in \mathcal{B}_1 \times \mathcal{B}_2$ *and choose* $y_0 \in F_1(s_0)$ *and* $z_0 \in F_2(t_0)$ *and let* $s_{n+1} \propto s_n$, $t_{n+1} \propto t_n$, $y_{n+1} \propto y_n$ *and* $z_{n+1} \propto z_n$.

Step 1. Let

$$\begin{aligned} f_1(s_{n+1}) &= (1-\alpha) f_1(s_n) + \alpha \mathcal{J}^{A_1}_{\lambda_1, M_1(.,t_n)}\Big((A_1 o f_1)(s_n) \oplus \frac{\lambda_1}{\rho_1}\big[\omega_1 \\ &\quad \oplus T_1(s_n - g_1(s_n), z_n)\big]\Big), \end{aligned} \tag{3.11}$$

$$\begin{aligned} f_2(t_{n+1}) &= (1-\alpha) f_2(t_n) + \alpha \mathcal{J}^{A_2}_{\lambda_2, M_2(s_n,.)}\Big((A_2 o f_2)(t_n) \oplus \frac{\lambda_2}{\rho_2}\big[\omega_2 \\ &\quad - T_2(y_n, t_n \oplus g_2(t_n))\big]\Big), \end{aligned} \tag{3.12}$$

where α *is non-negative constants such that* $0 < \alpha \leq 1$.

Step 2. Choose $y_{n+1} \in F_1(s_{n+1})$ *and* $z_{n+1} \in F_2(t_{n+1})$ *such that*

$$y_{n+1} \oplus y_n \leq \left(1 + (1+n)^{-1}\right) D_1(F_1(s_{n+1}), F_1(s_n)), \tag{3.13}$$

$$z_{n+1} \oplus z_n \leq \left(1 + (1+n)^{-1}\right) D_2(F_2(t_{n+1}), F_2(t_n)), \tag{3.14}$$

where $D_i(.,.)$ *are the Hausdorff metric on* $CB(\mathcal{B}_i)$.

Step 3. If $s_{n+1}, t_{n+1}, y_{n+1}$ *and* z_{n+1} *satisfying (3.11) and (3.12) to a sufficient degree of accuracy, stop; otherwise, set* $n = n+1$ *and return to step 2.*

4. MAIN RESULTS

In this section, we able to discuss the existence and convergence analysis of the proposed algorithms for NSMOVI (3.1).

Theorem 4.1. *For* $i = 1, 2$, *let* $A_i, f_i, g_i : \mathcal{B}_i \to \mathcal{B}_i$ *and* $T_i : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{B}_i$ *be the single-valued mappings such that* A_i *are comparison and* λ_{A_i}*-ordered copmression mappings,* f_i *are comparison,* λ_{f_i}*-ordered copmression and* ν_i*-ordered non-extended mappings,* g_i *are comparison and* λ_{g_i}*-ordered copmression mappings,* T_i *are comparison and ordered copmression mappings with respect to first and second arguments with constants* λ_{T_i} *and* λ'_{T_i}*, respectively. Let* $F_i : \mathcal{B}_i \to 2^{\mathcal{B}_i}$ *be the comparison and* δ_{F_i}*-ordered Lipschitz type continuous multi-valued mappings. Suppose* $M_i : \mathcal{B}_1 \times \mathcal{B}_2 \to 2^{\mathcal{B}_i}$ *is* $(\alpha_{A_i}, \lambda_i)$*-XOR-weak-ANODD multi-valued mapping with respect to* A_i *and* f_i*, and* D*-Lipschitz type continuous with constants* δ_{F_i}*. In addition, if* A_i, f_i, g_i, F_i, M_i *and* $\mathcal{J}^{A_i}_{\lambda_i, M_i(.,.)}$ *are compared to each other, and for all* $\omega_i \geq 0$*, the following conditions are satisfied:*

$$\delta_{P_1}\nu_2\mu_1\big(\rho_1\lambda_{A_1}\lambda_{f_1} \oplus \lambda_1\lambda_{T_1}(1+\lambda_{g_1})\big) < \Big[\rho_1\delta_{P_2}\xi_1(\lambda_1\alpha_{A_1} \oplus \mu_1)\nu_1\lambda'_{T_2}\delta_{F_1}\Big], \tag{4.1}$$

$$\delta_{P_2}\nu_1\mu_2\big(\rho_2\lambda_{A_2}\lambda_{f_2} \oplus \lambda_2\lambda_{T_2}(1 \oplus \lambda_{g_2})\big) < \Big[\rho_2\delta_{P_1}\xi_2(\lambda_2\alpha_{A_2} \oplus \mu_2)\nu_2\lambda'_{T_1}\delta_{F_2}\Big], \tag{4.2}$$

and

$$\mathcal{J}^{A_1}_{\lambda_1,M_1(s,.)}(p) \oplus \mathcal{J}^{A_1}_{\lambda_1,M_1(y,.)}(p), \quad \leq \quad \xi_1(s \oplus y), \tag{4.3}$$

$$\mathcal{J}^{A_2}_{\lambda_2,M_2(.,t)}(p) \oplus \mathcal{J}^{A_2}_{\lambda_2,M_2(.,w)}(p) \quad \leq \quad \xi_2(t \oplus w). \tag{4.4}$$

Then, the NSMOVI (3.1) *admits a solution* $(s,t) \in \mathcal{B}_1 \times \mathcal{B}_2$. *Moreover,* $s_n \to s$ *and* $t_n \to t$, *as* $n \to \infty$, *where* $\{s_n\}$ *and* $\{t_n\}$ *are the sequences defined in iterative Algorithm 3.1.*

Proof. By using Algorithm 3.1, Lemma 2.2 and Lemma 2.5, we have

$$\begin{aligned}
f_1(s_{n+1}) \oplus f_1(s_n) &= \Big[(1-\alpha_n-\beta_n)f_1(s_n) + \alpha_n \mathcal{J}^{A_1}_{\lambda_1,M_1(.,t_n)}\Big((A_1of_1)(s_n) \\
&\quad \oplus \frac{\lambda_1}{\rho_1}\left(\omega_1 \oplus T_1(s_n - g(s_n), z_n)\right)\Big)\Big] \oplus \Big[(1-\alpha_n-\beta_n)f_1(s_{n-1}) \\
&\quad + \alpha_n \mathcal{J}^{A_1}_{\lambda_1,M_1(.,t_{n-1})}\Big((A_1of_1)(s_{n-1}) \oplus \frac{\lambda_1}{\rho_1}\big(\omega_1 \oplus T_1(s_{n-1} \\
&\quad - g(s_{n-1}), z_{n-1})\big)\Big)\Big] \\
&\leq (1-\alpha_n-\beta_n)\left(f_1(s_n) \oplus f_1(s_{n-1})\right) \\
&\quad + \alpha_n \Big[J^{A_1}_{\lambda_1,M_1(.,t_n)}\Big((A_1of_1)(s_n) \oplus \frac{\lambda_1}{\rho_1}\big(\omega_1 \oplus T_1(s_n \\
&\quad - g(s_n), z_n)\big)\Big) \oplus J^{A_1}_{\lambda_1,M_1(.,t_{n-1})}\Big((A_1of_1)(s_{n-1}) \\
&\quad \oplus \frac{\lambda_1}{\rho_1}\big(\omega_1 \oplus T_1(s_{n-1} - g(s_{n-1}), z_{n-1})\big)\Big)\Big] \\
&\leq (1-\alpha_n)\lambda_{f_1}(s_n \oplus s_{n-1}) + \alpha_n \Big[\Big(J^{A_1}_{\lambda_1,M_1(.,t_n)}\Big((A_1of_1)(s_n) \\
&\quad \oplus \frac{\lambda_1}{\rho_1}\left(\omega_1 \oplus T_1(s_n - g(s_n), z_n)\right)\Big) \oplus J^{A_1}_{\lambda_1,M_1(.,t_n)} \\
&\quad \Big((A_1of_1)(s_{n-1}) \oplus \frac{\lambda_1}{\rho_1}\left(\omega_1 \oplus T_1(s_{n-1} - g(s_{n-1}), z_{n-1})\right)\Big)\Big) \\
&\quad + \Big(J^{A_1}_{\lambda_1,M_1(.,t_n)}\Big((A_1of_1)(s_{n-1}) \oplus \frac{\lambda_1}{\rho_1}\Big(\omega_1 \oplus T_1(s_{n-1} \\
&\quad - g(s_{n-1}), z_{n-1})\Big)\Big) \oplus J^{A_1}_{\lambda_1,M_1(.,t_{n-1})}\Big((A_1of_1)(s_{n-1}) \\
&\quad \oplus \frac{\lambda_1}{\rho_1}\Big(\omega_1 \oplus T_1(s_{n-1} - g(s_{n-1}), z_{n-1})\Big)\Big)\Big)\Big] \\
&\leq (1-\alpha_n)\lambda_{f_1}(s_n \oplus s_{n-1}) + \alpha_n \Big[\Theta_1\Big(\Big((A_1of_1)(s_n) \\
&\quad \oplus \frac{\lambda_1}{\rho_1}\left(\omega_1 \oplus T_1(s_n - g(s_n), z_n)\right)\Big) \oplus \Big((A_1of_1)(s_{n-1}) \\
&\quad \oplus \frac{\lambda_1}{\rho_1}\left(\omega_1 \oplus T_1(s_{n-1} - g(s_{n-1}), z_{n-1})\right)\Big)\Big)\Big] \\
&\quad + \xi_2(t_n \oplus t_{n-1})
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha_n)(s_n \oplus s_{n-1}) + \alpha_n \Theta_1 \Big[(A_1 o f_1)(s_n) \oplus (A_1 o f_1)(s_{n-1}) \\
&\quad \oplus \frac{\lambda_1}{\rho_1} \Big(T_1(s_n - g_1(s_n), z_n) \oplus T_1(s_{n-1} - g_1(s_{n-1}), z_{n-1}) \Big) \Big] \\
&\quad + \xi_2 (t_n \oplus t_{n-1}), \qquad (4.5)
\end{aligned}$$

where

$$\Theta_1 = \frac{\mu_1}{(\lambda_1 \alpha_{A_1} \oplus \mu_1)}.$$

Now,

$$\begin{aligned}
T_1(s_n - g_1(s_n), z_n) &\oplus T_1(s_{n-1} - g_1(s_{n-1}), z_{n-1}) \\
&\leq T_1(s_n - g_1(s_n), z_n) \oplus T_1(s_{n-1} - g_1(s_{n-1}), z_n) \\
&\quad + T_1(s_{n-1} - g_1(s_{n-1}), z_n) \oplus T_1(s_{n-1} - g_1(s_{n-1}), z_{n-1}) \\
&\leq \lambda_{T_1} \Big((s_n - g_1(s_n)) \oplus (s_{n-1} - g_1(s_{n-1})) \Big) + \lambda'_{T_1} (z_n \oplus z_{n-1}) \\
&\leq \lambda_{T_1} \Big[(s_n \oplus s_{n-1}) + \lambda_{g_1} (s_n \oplus s_{n-1}) \Big] \\
&\quad + \lambda'_{T_1} \Big(1 + (1+n)^{-1} \Big) \delta_{F_2} (t_n \oplus t_{n-1}) \\
&\leq \lambda'_{T_1} \Big(1 + (1+n)^{-1} \Big) \delta_{F_2} (t_n \oplus t_{n-1}) \\
&\quad + \big(\lambda_{T_1} (1 + \lambda_{g_1}) \big) (s_n \oplus s_{n-1}),
\end{aligned}$$

i.e.

$$\begin{aligned}
&T_1(s_n - g_1(s_n), z_n) \oplus T_1(s_{n-1} - g_1(s_{n-1}), z_{n-1}) \\
&\leq \big(\lambda_{T_1} (1 + \lambda_{g_1}) \big) (s_n \oplus s_{n-1}) + \lambda'_{T_1} \Big(1 + (1+n)^{-1} \Big) \delta_{F_2} (t_n \oplus t_{n-1}).
\end{aligned}$$

Eq. (4.5) becomes as

$$\begin{aligned}
f_1(s_{n+1}) \oplus f_1(s_n) &\leq (1-\alpha_n) \lambda_{f_1} (s_n \oplus s_{n-1}) + \alpha_n \Theta_1 \Big(\lambda_{A_1} \lambda_{f_1} (s_n \oplus s_{n-1}) \\
&\quad \oplus \frac{\lambda_1}{\rho_1} \big(\lambda_{T_1} (1 + \lambda_{g_1}) \big) (s_n \oplus s_{n-1}) \\
&\quad + \Big(\alpha_n \xi_2 \lambda'_{T_1} \Big(1 + (1+n)^{-1} \Big) \delta_{F_2} \Big) \Big) (t_n \oplus t_{n-1}) \\
&= \Big[(1-\alpha_n) \lambda_{f_1} + \alpha_n \Theta_1 \Big(\lambda_{A_1} \lambda_{f_1} \oplus \frac{\lambda_1}{\rho_1} \big(\lambda_{T_1} (1 + \lambda_{g_1}) \big) (s_n \oplus s_{n-1}) \\
&\quad + \Big(\alpha_n \xi_2 \lambda'_{T_1} \Big(1 + (1+n)^{-1} \Big) \delta_{F_2} \Big) \Big) (t_n \oplus t_{n-1}) \Big]. \qquad (4.6)
\end{aligned}$$

Eq. (4.6) becomes as

$$\begin{aligned}
s_{n+1} \oplus s_n &\leq \frac{1}{\nu_1} \Big[(1-\alpha_n) \lambda_{f_1} + \alpha_n \Theta_1 \Big(\lambda_{A_1} \lambda_{f_1} \oplus \frac{\lambda_1}{\rho_1} \big(\lambda_{T_1} (1 + \lambda_{g_1}) \big) (s_n \oplus s_{n-1}) \\
&\quad + \Big(\alpha_n \xi_2 \lambda'_{T_1} \Big(1 + (1+n)^{-1} \Big) \delta_{F_2} \Big) \Big) (t_n \oplus t_{n-1}) \Big].
\end{aligned}$$

By definition of normal cone, we have

$$\|s_{n+1} - s_n\| \leq \Big[(1-\alpha_n) \frac{\lambda_{f_1} \delta_{P_1}}{\nu_1} + \alpha_n \frac{\delta_{P_1} \Theta_1}{\nu_1} \Big(\lambda_{A_1} \lambda_{f_1} \oplus \frac{\lambda_1}{\rho_1} \big(\lambda_{T_1} (1 + \lambda_{g_1}) \big) \Big) \Big] \|s_n - s_{n-1}\|$$

$$
\begin{aligned}
&+\alpha_n \frac{\xi_2 \lambda'_{T_1} \delta_{F_2}}{\nu_1}\Big(1+(1+n)^{-1}\Big)\|t_n - t_{n-1}\| \\
\leq\ & \Big[1-\alpha_n\Big(1-\frac{\delta_{P_1}\Theta_1}{\nu_1}\Big(\lambda_{A_1}\lambda_{f_1} \oplus \frac{\lambda_1}{\rho_1}\big(\lambda_{T_1}(1+\lambda_{g_1})\big)\Big)\Big)\Big]\|s_n - s_{n-1}\| \\
&+\alpha_n \frac{\xi_2 \lambda'_{T_1} \delta_{F_2}}{\nu_1}\Big(1+(1+n)^{-1}\Big)\|t_n - t_{n-1}\|. \qquad (4.7)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
f_2(t_{n+1}) \oplus f_2(t_n) &= \Big[(1-\alpha_n-\beta_n)f_2(t_n) + \alpha_n \mathcal{J}^{A_2}_{\lambda_2, M_2(s_n,.)}\Big((A_2 o f_2)(t_n) \\
&\quad \oplus \frac{\lambda_2}{\rho_2}\Big(\omega_2 \oplus T_2(y_n, t_n \oplus g_2(t_n))\Big)\Big)\Big] \oplus \Big[(1-\alpha_n-\beta_n) \\
&\quad f_2(t_{n-1}) + \alpha_n \mathcal{J}^{A_2}_{\lambda_2, M_2(s_{n-1},.)}\Big((A_2 o f_2)(t_{n-1}) \oplus \frac{\lambda_2}{\rho_2} \\
&\quad \big(\omega_2 \oplus T_2(y_{n-1}, t_{n-1} \oplus g_2(t_{n-1}))\big)\Big)\Big] \\
&\leq (1-\alpha_n)\lambda_{f_2}(t_n \oplus t_{n-1}) + \alpha_n\Big[\Theta_2\Big(\lambda_{A_2}\lambda_{f_2}(t_n \oplus t_{n-1}) \\
&\quad \oplus \frac{\lambda_2}{\rho_2}\Big(T_1(y_n, t_n \oplus g_2(t_n)) \oplus T_2(y_{n-1}, t_{n-1} \oplus g_2(t_{n-1}))\Big)\Big)\Big] \\
&\quad +\alpha_n \xi_1 (s_n \oplus s_{n-1}), \qquad (4.8)
\end{aligned}
$$

where

$$
\Theta_2 = \frac{\mu_2}{(\lambda_2 \alpha_{A_2} \oplus \mu_2)}.
$$

Now,

$$
\begin{aligned}
T_2(y_n, t_n \oplus g_2(t_n)) &\oplus T_2(y_{n-1}, t_{n-1} \oplus g_2(t_{n-1})) \\
&= T_2(y_n, t_n \oplus g_2(t_n)) \oplus T_2(y_n, t_{n-1} \oplus g_2(t_{n-1})) \\
&\quad + T_2(y_n, t_{n-1} \oplus g_2(t_{n-1})) \oplus T_2(y_{n-1}, t_{n-1} \oplus g_2(t_{n-1})) \\
&\leq \lambda_{T_2}\Big[(t_n \oplus g_2(t_n)) \oplus (t_{n-1} \oplus g_2(t_{n-1}))\Big] \\
&\quad + \lambda'_{T_2}\Big(1+(1+n)^{-1}\Big)\delta_{F_1}(s_n \oplus s_{n-1}) \\
&\leq \lambda_{T_2}(1 \oplus \lambda_{g_2})(t_n \oplus t_{n-1}) + \lambda'_{T_2}\Big(1+(1+n)^{-1}\Big)\delta_{F_1}(s_n \oplus s_{n-1}).
\end{aligned}
$$

From (4.8), we have

$$
\begin{aligned}
f_2(t_{n+1}) \oplus f_2(t_n) &\leq (1-\alpha_n)\lambda_{f_2}(t_n \oplus t_{n-1}) + \alpha_n \Theta_2\Big(\lambda_{A_2}\lambda_{f_2} \\
&\quad \oplus \frac{\lambda_2}{\rho_2}\big(\lambda_{T_2}(1 \oplus \lambda_{g_2})\big)\Big)(t_n \oplus t_{n-1}) \\
&\quad + \alpha_n \xi_1 \lambda'_{T_2}\delta_{F_1}\Big(1+(1+n)^{-1}\Big)(s_n \oplus s_{n-1}) \\
&\leq \Big[(1-\alpha_n)\lambda_{f_2} + \alpha_n\Theta_2\Big(\lambda_{A_2}\lambda_{f_2} \oplus \frac{\lambda_2}{\rho_2}\big(\lambda_{T_2}(1 \oplus \lambda_{g_2})\big)\Big)\Big](t_n \oplus t_{n-1}) \\
&\quad + \alpha_n \xi_1 \lambda'_{T_2}\delta_{F_1}\Big(1+(1+n)^{-1}\Big)(s_n \oplus s_{n-1})
\end{aligned}
$$

By definition, we have

$$\begin{aligned} t_{n+1} \oplus t_n &\leq \frac{1}{\nu_2}\Big[f_2(t_{n+1}) \oplus f_1(t_n)\Big] \\ &\leq (1-\alpha_n)\frac{\lambda_{f_2}}{\nu_2} + \frac{\alpha_n \Theta_2}{\nu_2}\Big(\lambda_{A_2}\lambda_{f_2} \oplus \frac{\lambda_2}{\rho_2}\big(\lambda_{T_2}(1 \oplus \lambda_{g_2})\big)\Big)(t_n \oplus t_{n-1}) \\ &\quad +\alpha_n \frac{\xi_1 \lambda'_{T_2}\delta_{F_1}}{\nu_2}\Big(1+(1+n)^{-1}\Big)(s_n \oplus s_{n-1}). \end{aligned}$$

By definition of normal cone, we have

$$\begin{aligned} \|t_{n+1} - t_n\| &\leq \Big[(1-\alpha_n)\frac{\delta_{P_2}\lambda_{f_2}}{\nu_2} + \frac{\alpha_n \Theta_2 \delta_{P_2}}{\nu_2}\Big(\lambda_{A_2}\lambda_{f_2} \\ &\quad \oplus \frac{\lambda_2}{\rho_2}\big(\lambda_{T_2}(1 \oplus \lambda_{g_2})\big)\Big)\Big]\|t_n \oplus t_{n-1}\| \\ &\quad +\alpha_n \frac{\delta_{P_2}\xi_1 \lambda'_{T_2}\delta_{F_1}}{\nu_2}\Big(1+(1+n)^{-1}\Big)(s_n \oplus s_{n-1}) \\ &= \Big[1-\alpha_n\Big(1-\frac{\Theta_2\delta_{P_2}}{\nu_2}\Big(\lambda_{A_2}\lambda_{f_2} \oplus \frac{\lambda_2}{\rho_2}\big(\lambda_{T_2}(1 \oplus \lambda_{g_2})\big)\Big)\Big)\Big]\|t_n - t_{n-1}\| \\ &\quad +\alpha_n \frac{\delta_{P_2}\xi_1 \lambda'_{T_2}\delta_{F_1}}{\nu_2}\Big(1+(1+n)^{-1}\Big)\|s_n - s_{n-1}\|. \end{aligned} \tag{4.9}$$

From Eq. (4.7) and Eq. (4.9), we have

$$\begin{aligned} \|s_{n+1} - s_n\| + \|t_{n+1} - t_n\| &\leq \Big[1-\alpha_n\Big(1-\frac{\delta_{P_1}\Theta_1}{\nu_1}\Big(\lambda_{A_1}\lambda_{f_1} \oplus \frac{\lambda_1}{\rho_1}\big(\lambda_{T_1}(1+\lambda_{g_1})\big)\Big) \\ &\quad -\frac{\delta_{P_2}\xi_1\lambda'_{T_2}\delta_{F_1}}{\nu_2}\Big(1+(1+n)^{-1}\Big)\Big)\Big]\|s_n - s_{n-1}\| \\ &\quad +\Big[1-\alpha_n\Big(1-\frac{\delta_{P_2}\Theta_2}{\nu_2}\Big(\lambda_{A_2}\lambda_{f_2} \oplus \frac{\lambda_2}{\rho_2}\big(\lambda_{T_2}(1 \oplus \lambda_{g_2})\big)\Big) \\ &\quad -\frac{\delta_{P_1}\xi_2\lambda'_{T_1}\delta_{F_2}}{\nu_1}\Big(1+(1+n)^{-1}\Big)\Big)\Big]\|t_n - t_{n-1}\| \\ &\leq (1-\alpha_n(1-\Omega_n))\|s_{n+1}-s_n\| \\ &\quad +(1-\alpha_n(1-\Omega'_n))\|t_{n+1}-t_n\|, \end{aligned}$$

where

$$\Omega_n = \Big[\frac{\delta_{P_1}\Theta_1}{\nu_1}\Big(\lambda_{A_1}\lambda_{f_1} \oplus \frac{\lambda_1}{\rho_1}\big(\lambda_{T_1}(1+\lambda_{g_1})\big)\Big) - \frac{\delta_{P_2}\xi_1\lambda'_{T_2}\delta_{F_1}}{\nu_2}\Big(1+(1+n)^{-1}\Big)\Big]$$

and

$$\Omega'_n = \Big[\frac{\delta_{P_2}\Theta_2}{\nu_2}\Big(\lambda_{A_2}\lambda_{f_2} \oplus \frac{\lambda_2}{\rho_2}\big(\lambda_{T_2}(1 \oplus \lambda_{g_2})\big)\Big) - \frac{\delta_{P_1}\xi_2\lambda'_{T_1}\delta_{F_2}}{\nu_1}\Big(1+(1+n)^{-1}\Big)\Big]$$

Now,

$$\begin{aligned} \|s_{n+1} - s_n\| + \|t_{n+1} - t_n\| &\leq (1-\alpha_n(1-\Delta_n))\Big(\|s_n - s_{n-1}\| + \|t_n - t_{n-1}\|\Big), \\ &\leq (1-\alpha(1-\Delta_n))\Big(\|s_n - s_{n-1}\| + \|t_n - t_{n-1}\|\Big), \end{aligned} \tag{4.10}$$

where $\Delta_n = \max\{\Omega_n, \Omega_n'\}$ and $\alpha = \limsup_{n\geq 1} \alpha_n < 1$. If we set $\Delta = \max\{\Omega, \Omega'\}$, where

$$\Omega = \Big[\frac{\delta_{P_1}\Theta_1}{\nu_1}\Big(\lambda_{A_1}\lambda_{f_1} \oplus \frac{\lambda_1}{\rho_1}\big(\lambda_{T_1}(1+\lambda_{g_1})\big)\Big) - \frac{\delta_{P_2}\xi_1\lambda_{T_2}'\delta_{F_1}}{\nu_2}\Big]$$

and

$$\Omega' = \Big[\frac{\delta_{P_2}\Theta_2}{\nu_2}\Big(\lambda_{A_2}\lambda_{f_2} \oplus \frac{\lambda_2}{\rho_2}\big(\lambda_{T_2}(1\oplus\lambda_{g_2})\big)\Big) - \frac{\delta_{P_1}\xi_2\lambda_{T_1}'\delta_{F_2}}{\nu_1}\Big].$$

It follows that $\Delta_n \to \Delta$ as $n \to \infty$. From conditions (4.1) and (4.2) that $0 < \Delta < 1$. Therefore, by (4.10) and $0 < \alpha_n + \beta_n \leq 1$ implies that $\{s_n\}$ and $\{t_n\}$ are Cauchy sequences. Thus there exists $s, t \in \mathcal{B}$ such that $s_n \to s$ and $t_n \to t,$ as $n \to \infty$.

From (3.5) and (3.6), we have

$$\begin{aligned} y_{n+1} \oplus y_n &\leq \left(1+(1+n)^{-1}\right) D_1(F_1(s_{n+1}), F_1(s_n)) \\ &\leq \left(1+(1+n)^{-1}\right) \delta_{F_1}(s_{n+1} \oplus s_n) \end{aligned}$$

$$\begin{aligned} z_{n+1} \oplus z_n &\leq \left(1+(1+n)^{-1}\right) D_2(F_2(t_{n+1}), F_2(t_n)) \\ &\leq \left(1+(1+n)^{-1}\right) \delta_{F_2}(t_{n+1} \oplus t_n). \end{aligned}$$

By the definition of normal cone, we have

$$\|y_{n+1} - y_n\| \leq \delta_{P_1}\left(1+(1+n)^{-1}\right)\delta_{F_1}\|s_{n+1} - s_n\| \tag{4.11}$$

$$\|z_{n+1} - z_n\| \leq \delta_{P_2}\left(1+(1+n)^{-1}\right)\delta_{F_2}\|t_{n+1} - t_n\|. \tag{4.12}$$

It follows from (4.11) and (4.12) that $\{y_n\}$ and $\{z_n\}$ are also Cauchy sequences. Therefore, there exist $y \in \mathcal{B}_1$ and $z \in \mathcal{B}_2$ such that $y_n \to y$ and $z_n \to z,$ as $n \to \infty$. Next, we show that

$$y_n \to y \in F_1(s) \text{ and } z_n \to z \in F_2(t), \text{ as } n \to \infty.$$

Furthermore,

$$\begin{aligned} d(y, F_1(s)) &= \inf\{\|y - t\| : t \in F_1(s)\} \\ &\leq \|y - y_n\| + d(y_n, F_1(s)) \\ &\leq \|y - y_n\| + d(F_1(s_n), F_1(s)) \\ &\leq \|y - y_n\| + \delta_{F_1}(s_n \oplus s) \\ &\leq \|y - y_n\| + \delta_{F_1}\delta_{P_1}\|s_n - s\| \to 0, \text{ as } n \to \infty. \end{aligned}$$

Since $F_1(s)$ is closed, we have $y \in F_1(s)$. Similarly, we can show that $z \in F_2(t)$. Finally apply the continuity, s, t, y and z satisfy the following relations:

$$f_1(s) = J^{A_1}_{\lambda_1, M_1(.,t)}\left[(A_1 o f_1)(s) \oplus \frac{\lambda_1}{\rho_1}\left[\omega_1 \oplus T_1(s - g_1(s), z)\right]\right],$$

$$f_2(t) = J^{A_2}_{\lambda_2, M_2(s,.)}\left[(A_2 o f_2)(t) \oplus \frac{\lambda_2}{\rho_2}\left[\omega_2 \oplus T_2(y, t - g_2(s))\right]\right],$$

which implies that

$$\begin{aligned} f_1(s) \oplus J^{A_1}_{\lambda_1, M_1(.,t)}\left[(A_1 o f_1)(s) \oplus \frac{\lambda_1}{\rho_1}\left[\omega_1 \oplus T_1(s - g_1(s), z)\right]\right] &= 0, \\ f_2(t) \oplus J^{A_2}_{\lambda_2, M_2(s,.)}\left[(A_2 o f_2)(t) \oplus \frac{\lambda_2}{\rho_2}\left[\omega_2 \oplus T_2(y, t - g_2(s))\right]\right] &= 0. \end{aligned}$$

Therefore (s, t) are solution of NSMOVI (3.1). ∎

Theorem 4.2. *Let $A_i, f_i, g_i : \mathcal{B}_i \to \mathcal{B}_i$ and $T_i : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{B}_i$ be the single-valued mappings such that A_i are comparison and λ_{A_i}-ordered copmression mappings, f_i are comparison, λ_{f_i}-ordered copmression and ν_i-ordered non-extended mappings, g_i are comparison and λ_{g_i}-ordered copmression mappings, T_i are comparison and ordered copmression mappings with respect to first and second arguments with constants λ_{T_i} and λ'_{T_i}, respectively. Let $F_i : \mathcal{B}_i \to 2^{\mathcal{B}_i}$ be the comparison and δ_{F_i}-ordered Lipschitz type continuous multi-valued mappings. Suppose $M_i : \mathcal{B}_1 \times \mathcal{B}_2 \to 2^{\mathcal{B}_i}$ are $(\alpha_{A_i}, \lambda_i)$-XOR-weak-ANODD multi-valued mappings with respect to A_i and f_i, , and D-Lipschitz type continuous with constants δ_{F_i}. In addition, if A_i, f_i, g_i, F_i, M_i and $\mathcal{J}^{A_i}_{\lambda_i, M_i(.,.)}$ are compared to each other, and for all $\omega_i \geq 0$, the following conditions are satisfied:*

$$\rho_1\lambda_{A_1}\lambda_{f_1} \oplus \lambda_1\lambda_{T_1}(1+\lambda_{g_1})\big) < \Big[\frac{\rho_1\delta_{P_2}\xi_1(\lambda_1\alpha_{A_1} \oplus \mu_1)\nu_1\lambda'_{T_2}\delta_{F_1}}{\nu_2\mu_1}\Big] \min\Big\{\frac{1}{\delta_{P_1}}, 1\Big\}, \tag{4.13}$$

$$\big(\rho_2\lambda_{A_2}\lambda_{f_2} \oplus \lambda_2\lambda_{T_2}(1\oplus\lambda_{g_2})\big) < \Big[\frac{\rho_2\delta_{P_1}\xi_2(\lambda_2\alpha_{A_2} \oplus \mu_2)\nu_2\lambda'_{T_1}\delta_{F_2}}{\nu_1\mu_2}\Big] \min\Big\{\frac{1}{\delta_{P_2}}, 1\Big\}, \tag{4.14}$$

and

$$\mathcal{J}^{A_1}_{\lambda_1, M_1(s,.)}(p) \oplus \mathcal{J}^{A_1}_{\lambda_1, M_1(y,.)}(p) \leq \xi_1(s \oplus y), \tag{4.15}$$

$$\mathcal{J}^{A_2}_{\lambda_2, M_2(.,t)}(p) \oplus \mathcal{J}^{A_2}_{\lambda_2, M_2(.,w)}(p) \leq \xi_2(t \oplus w). \tag{4.16}$$

Then, the NSMOVI (3.1) admits a solution $(s,t) \in \mathcal{B}_1 \times \mathcal{B}_2$. Moreover, $s_n \to s$ and $t_n \to t$, as $n \to \infty$, where $\{s_n\}$ and $\{t_n\}$ are the sequences defined in iterative Algorithm 3.2.

Proof. The proof is same as Theorem 4.1 except Algorithm 3.2 is applied instead. ∎

Theorem 4.3. *Let $A_i, f_i, g_i : \mathcal{B}_i \to \mathcal{B}_i$ and $T_i : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{B}_i$ be the single-valued mappings such that A_i are comparison and λ_{A_i}-ordered copmression mappings, f_i are comparison, λ_{f_i}-ordered copmression and ν_i-ordered non-extended mappings, g_i are comparison and λ_{g_i}-ordered copmression mappings, T_i are comparison and ordered copmression mappings with respect to first and second arguments with constants λ_{T_i} and λ'_{T_i}, respectively. Let $F_i : \mathcal{B}_i \to 2^{\mathcal{B}_i}$ be the comparison and δ_{F_i}-ordered Lipschitz type continuous multi-valued mappings. Suppose $M_i : \mathcal{B}_1 \times \mathcal{B}_2 \to 2^{\mathcal{B}_i}$ are $(\alpha_{A_i}, \lambda_i)$-XOR-weak-ANODD multi-valued mapping with respect to A_i and f_i, , and D-Lipschitz type continuous with constants δ_{F_i}. In addition, if A_i, f_i, g_i, F_i, M_i and $\mathcal{J}^{A_i}_{\lambda_i, M_i(.,.)}$ are compared to each other, and for all $\omega_i \geq 0$, the following conditions are satisfied:*

$$\rho_1\lambda_{A_1}\lambda_{f_1} \oplus \lambda_1\lambda_{T_1}(1+\lambda_{g_1})\big) < \Big[\frac{\rho_1\delta_{P_2}\xi_1(\lambda_1\alpha_{A_1} \oplus \mu_1)\nu_1\lambda'_{T_2}\delta_{F_1}}{\nu_2\mu_1}\Big] \min\Big\{\frac{1}{\delta_{P_1}}, 1\Big\}, \tag{4.17}$$

$$\big(\rho_2\lambda_{A_2}\lambda_{f_2} \oplus \lambda_2\lambda_{T_2}(1\oplus\lambda_{g_2})\big) < \Big[\frac{\rho_2\delta_{P_1}\xi_2(\lambda_2\alpha_{A_2} \oplus \mu_2)\nu_2\lambda'_{T_1}\delta_{F_2}}{\nu_1\mu_2}\Big] \min\Big\{\frac{1}{\delta_{P_2}}, 1\Big\}, \tag{4.18}$$

and

$$\mathcal{J}^{A_1}_{\lambda_1, M_1(s,.)}(p) \oplus \mathcal{J}^{A_1}_{\lambda_1, M_1(y,.)}(p) \leq \xi_1(s \oplus y), \tag{4.19}$$

$$\mathcal{J}^{A_2}_{\lambda_2, M_2(.,t)}(p) \oplus \mathcal{J}^{A_2}_{\lambda_2, M_2(.,w)}(p) \leq \xi_2(t \oplus w). \tag{4.20}$$

Then, the NSMOVI (3.1) admits a solution $(s,t) \in \mathcal{B}_1 \times \mathcal{B}_2$. Moreover, $s_n \to s$ and $t_n \to t$, as $n \to \infty$, where $\{s_n\}$ and $\{t_n\}$ are the sequences defined in iterative Algorithm 3.3.

Proof. The proof is same as Theorem 4.1 except Algorithm 3.3 is applied instead. ∎

The following numerical example gives the guarantee that all the proposed conditions of Theorem 4.1 are satisfied.

Example 4.1. *For each $i \in \{1,2\}$, let $\mathcal{B}_i = \mathbb{R}_+ \cup \{0\}$ and $A_i, f_i, g_i : \mathcal{B}_i \to \mathcal{B}_i$ and $F_i : \mathcal{B}_i \to 2^{\mathcal{B}_i}$ be the mappings defined by*

$$g_i(s_i) = \frac{s_i}{103+i},\ f_i(s_i) = \frac{s_i}{5i},\ A_i(s_i) = \frac{2i}{15}s_i \text{ and } F_i(s_i) = \frac{i+3}{15}s_i\ \forall\ s_i \in \mathcal{B}_i.$$

Suppose that the mappings $T_i : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{B}_i$ and are defined by

$$T_i(s_1,s_2) = \frac{s_i}{12+i},\ \forall\ (s_1,s_2) \in \mathcal{B}_1 \times \mathcal{B}_2,$$

and the mappings $M_i : \mathcal{B}_1 \times \mathcal{B}_2 \to 2^{\mathcal{B}_i}$ are defined by

$$M_i(s_1,s_2) = \left\{\frac{s_i}{5i}\right\},\ \forall\ (s_1,s_2) \in \mathcal{B}_1 \times \mathcal{B}_2.$$

Now,

$$f_i(s_i) \oplus f_i(t_i) = \left(\frac{s_i}{5i} \oplus \frac{t_i}{5i}\right) \le \frac{2}{5i}(s_i \oplus t_i),$$

and

$$f_i(s_i) \oplus f_i(t_i) = \left(\frac{s_i}{5i} \oplus \frac{t_i}{5i}\right) \ = \ \frac{1}{5i}(s_i \oplus t_i) \ge \frac{1}{6i}(s_i \oplus t_i),$$

i.e., f_i are $\frac{2}{5i}$-ordered compression and $\frac{1}{6i}$-ordered non-extended appings. In the similar way, it is easy to check that g_i are $\frac{2}{103+i}$-ordered compression mappings, A_i are $\frac{i}{5}$-ordered compression mappings and F_i are $\frac{i}{3}$-ordered compression mappings, T_i ordered copmression mappings with respect to first and second arguments with constants $\frac{2}{12+i}$ and $\frac{2+i}{12+i}$, respectively.

In addition, it is easy to verify that M_i are $\frac{3}{2i^2}$-weak-non-ordinary-difference mappings with respect to f_i and λ_i-XOR-ordered different weak compression mapping with respect to A_i, where $\lambda_i \ge \frac{55}{i}$. For $\lambda_i \ge \frac{55}{i}$, $[A_i + \lambda_i M_i](\mathcal{B}_i) = \mathcal{B}_i$, which shows that M_i are $(\frac{3}{2i^2}, \lambda_i)$-XOR-weak-ANODD multi-valued mappings with respect to A_i and f_i, where $\lambda_i \ge \frac{55}{i}$. Hence, the resolvent operators $\mathcal{J}^{A_i}_{\lambda_i,M_i} : \mathcal{B}_i \to \mathcal{B}_i$ associated with A_i and M_i are of the form:

$$\mathcal{J}^{A_i}_{\lambda_i,M_i}(s_i) = \frac{s_i}{\left(\frac{2i}{15} \oplus \frac{\lambda_i}{5i}\right)} = \left(\frac{75i}{10i^2 \oplus 15\lambda_i}\right)s_i,\ \forall s_i \in \mathcal{B}_i.$$

It is clear that the resolvent operators defined above are single-valued and comparisons.
Now,

$$\begin{aligned}
\mathcal{J}^{A_i}_{\lambda_i,M_i}(s_i) \oplus \mathcal{J}^{A_i}_{\lambda_i,M_i}(t_i) &= \left(\frac{75i}{10i^2 \oplus 15\lambda_i}\right)s_i \oplus \left(\frac{75i}{10i^2 \oplus 15\lambda_i}\right)t_i \\
&= \left(\frac{75i}{10i^2 \oplus 15\lambda_i}\right)(s_i \oplus t_i) \\
&\le \frac{5i}{\lambda_i}(s_i \oplus t_i),\ \text{where } \lambda_i \ge \frac{55}{i},
\end{aligned}$$

Hence, the resolvent operators $\mathcal{J}^{A_i}_{\lambda_i,M_i}$ are $\frac{5i}{\lambda_i}$-Lipschitz type continuous mappings. Hence, all the proposed conditions of Theorem 4.1 are fulfilled.

Remark 4.1. We choose $\lambda_{f_i} = \frac{2}{5i}$, $\nu_i = \frac{1}{6i}$, $\lambda_{g_i} = \frac{2}{103+i}$, $\lambda_{A_i} = \frac{i}{5}$, $\delta_{F_i} = \frac{i}{3}$, $\lambda_{T_i} = \frac{2}{12+i}$, $\lambda'_{T_i} = \frac{2+i}{12+i}$, $\alpha_{A_i} = \frac{3}{2i^2}$, $\lambda_i = \frac{55}{i}$, $\mu_i = \frac{165}{i^2}$, $\Theta_i = 2$, $\xi_i = 7i$, $\omega_i = 0$, $\rho_i = 104i$ and $N = 1$, the conditions (4.1) and (4.2) of Theorem 4.1 are satisfied.

Remark 4.2. We remark that the generalizations of the iterative method presented in this paper need further research affords and the technique is helpful to solve the system of n-variational inclusions.

5. CONCLUSION

In this article, we introduced and analyzed an NSMOVI involving XOR operation and proved the existence of the solution to our main problem. We constructed the iterative algorithms based on the fixed point formulation with XOR operation and discussed the convergence of the iterative sequences generated by the proposed algorithms which suggested that algorithms converge to a solution of the proposed problem. Finally, we constructed a numerical example to show that all conditions are fulfilled for our main result in this paper. The obtained results in this article are an important and significant generalization to recent known results in nonlinear analysis and establish results that can be extended Banach spaces and other higher dimensional spaces. Note that it needs further research on the forward as well as a backward splitting method based on the inertial technique for solving ordered inclusion problems with XOR operation technique and also it needs to develop the algorithms to solve the image deblurring and image recovery problems by using the Tseng method and viscosity method in real ordered Hilbert spaces.

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