Aust. J. Math. Anal. Appl.
Vol. **18** (2021), No. 1, Art. 2, 12 pp.
AJMAA

EXISTENCE AND APPROXIMATION OF TRAVELING WAVEFRONTS FOR THE DIFFUSIVE MACKEY-GLASS EQUATION

C. RAMIREZ-CARRASCO* AND J. MOLINA-GARAY

Received 19 July, 2020; accepted 11 October, 2020; published 11 January, 2021.

FACULTAD DE CIENCIAS BÁSICAS, UNIVERSIDAD CATÓLICA DEL MAULE, TALCA, CHILE
*carloshrc1989@gmail.com
molina@imca.edu.pe

ABSTRACT. In this paper, we consider the diffusive Mackey-Glass model with discrete delay. This equation describes the dynamics of the blood cell production. We investigate the existence of traveling wavefronts solutions connecting the two steady states of the model. We develop an alternative proof of the existence of such solutions and we also demonstrate the existence of traveling wavefronts moving at minimum speed. The proposed approach is based on the use technique of upper-lower solutions. Finally, through an iterative procedure, we show numerical simulations that approximate the traveling wavefronts, thus confirming our theoretical results.

Key words and phrases: Traveling wavefronts; Diffusive Mackey-Glass model; Time delay; Upper-lower solutions; Numerical approximation.

2010 *Mathematics Subject Classification.* Primary 35K57, 34K99. Secondary 9210.

ISSN (electronic): 1449-5910

The authors would like to thank the support of doctoral scholarship of the Universidad Católica del Maule, Talca, Chile.

1. INTRODUCTION

The traveling wave theory was initiated in 1937 by Kolmogorov, Petrovskii, Piskunov [1] and Fisher [2]. Currently the literature on traveling waves solutions for delayed reaction-diffusion equations is extensive, for example see [3, 4, 6, 5, 7, 8, 9, 10]. The technique developed by Wu and Zou [5], based on a monotonous convergence scheme together with the standard upper-lower solutions technique to establish the existence of monotonous traveling waves for non-linear reaction-diffusion equation systems with and without quasimonotonicity and with discrete or distributed delay, it is widely used in [11, 12, 13, 14, 15, 16, 17, 18]. However, there are few studies that use such a technique to prove the existence of traveling wavefronts that move at minimum speed.

In this paper, we will investigate the existence of traveling wavefronts for the following partial differential equation with discrete delay:

$$u_t(t,x) = \Delta u(t,x) + \frac{pu(t-\tau,x)}{1+au^q(t-\tau,x)} - du(t,x), \tag{1.1}$$

where $x \in \mathbb{R}$, $t \geq 0$, $u \geq 0$, and all parameters are positive constants. The results obtained in [19] describe the oscillatory behavior of solutions about the positive equilibrium of Eq. (1.1) with Neumann boundary condition. Further, in [20] was investigated the existence of positive periodic solutions of Eq. (1.1) by using the Krasnosel'skii fixed point theorem. Eq. (1.1) without spatial dispersion, reduces to the following ordinary differential equation:

$$u'(t) = \frac{pu(t-\tau)}{1+au^q(t-\tau)} - du(t). \tag{1.2}$$

Eq. (1.2) was first suggested in 1977 by Mackey and Glass [21], to model the concentration of cells in the circulating blood and where τ is the time delay between the production of immature stem cells in bone marrow and their maturation for release in the circulating blood stream. This equation has been studied in [21, 22, 23, 24, 25, 26]. For example the numerical simulations of Eq. (1.2) by Mackey and Glass [21] indicated that there is a cascading sequence of bifurcating periodic solutions when the delay is increased, however when the delay is further increased the periodic solutions becomes aperiodic and chaotic.

A traveling wavefront solution to Eq. (1.1) is a special type of bounded positive continuous non-constant solution $u(t,x)$ having the form $u(t,x) = \phi(x+ct)$. The number $c > 0$ is called the wave speed of the propagation, and ϕ is a C^2-smooth function called the wave profile and satisfying $\phi(-\infty) = 0$, $\phi(+\infty) = k > 0$. The existence of the traveling wavefronts in Eq. (1.1) is equivalent to the presence of positive heteroclinic connections in an associated second order non-linear differential equation:

$$\phi''(z) - c\phi'(z) + \frac{p\phi(z-cr)}{1+a\phi^q(z-cr)} - d\phi(z) = 0, \quad z \in \mathbb{R}.$$

As far as the authors know, the existence of traveling wave fronts that propagate at the minimum speed ($c = c*$) for Eq. (1.1) has not been investigated. In this paper we give an alternative proof to the one carried out in [25] for the existence of traveling wavefronts solutions for $c > c*$. Moreover, using the ideas of [5, 27], we extend this result by proving the existence of traveling wavefronts moving at minimum speed and through the iterative procedure developed in [5], we show numerical simulations that approximate the traveling wavefronts.

Let us state now the main result of this paper.

Theorem 1.1. *There exists $c_* > 0$ such that for every $c \geq c_*$, Eq. (1.1) has a positive monotone traveling wavefront $u(t,x) = \phi(x+ct)$, connecting 0 with $k = ((p-d)/ad)^{1/q}$, if one of the following conditions holds:*

(a) $1 < \dfrac{p}{d} \leq \dfrac{q}{q-1}$ *if* $q > 1$

(b) $1 < \dfrac{p}{d} < +\infty$ *if* $0 < q \leq 1$.

The organization of this paper is as follows. In Section 2, we will introduce some notations, and present one of the main theorems of Wu and Zou given in [5] that will be employed in this paper. In section 3, we give an alternative proof to that developed in [25] to establish the existence of traveling waves moving at speed $c > c*$. In Section 4, we extend this result by proving the existence of traveling waves moving at minimum speed. Finally, in Section 5 we carried out numerical simulations to verify our theoretical results.

2. PRELIMINARIES

In this section, we introduce some important results, which will be used in our analysis. As we know, Wu and Zou in [5] developed a quite general and applicable theory to coupled reaction-diffusion systems with delay. For convenience, here we only present a simple version of their result.

Consider the scalar reaction-diffusion equation with a discrete delay given by Eq. (2.1)

$$u_t(t,x) = \Delta u(t,x) + f(u(t,x), u(t-\tau,x)), \tag{2.1}$$

where $x \in \mathbb{R}$, $t \geq 0$, $u \geq 0$ and f is a continuous function. Substituting the wave profile $u(t,x) = \phi(x+ct)$ into Eq. (2.1) and denoting $x+ct$ by z, we obtain Eq (2.2)

$$c\phi'(z) = \phi''(z) + f_c(\phi_z), \tag{2.2}$$

where $f_c : X_c = C([-c\tau, 0], \mathbb{R}) \longrightarrow \mathbb{R}$ is defined by $f_c(y_z) = f(y_z^c(0), y_z^c(-\tau))$ where $y_z^c(s) = y_z(cs) = y(z+cs)$ for all $s \in [-\tau, 0]$. We assume the following conditions on f:

- **(F1)** There exists $k > 0$ such that $f_c(\hat{0}) = f_c(\hat{k}) = 0$, and $f_c(\hat{u}) \neq 0$ for $0 < u < k$, where $\hat{u}$ denotes the constant function taking the values u on $[-c\tau, 0]$, i.e., $\hat{u}(s) = u$, $s \in [-c\tau, 0]$.
- **(F2)** (Quasimonotonicity). There exists $\beta \geq 0$ such that
$$f_c(\phi_z) - f_c(\psi_z) + \beta[\phi_z(0) - \psi_z(0)] \geq 0,$$
with $\phi_z, \psi_z \in X_c$ and $0 \leq \psi_z(s) \leq \phi_z(s) \leq k$, $s \in [-c\tau, 0]$.

We look for traveling wavefronts for Eq. (2.1) in the following profile set:

$$\Gamma = \{\phi \in C(\mathbb{R}, \mathbb{R}) : \phi \text{ is nondecreasing in } \mathbb{R},\ \phi(-\infty) = 0, \text{ and } \phi(+\infty) = k\}.$$

Next we define upper and lower solutions for Eq. (2.2).

Definition 2.1. [5] A continuous function $\rho \in C(\mathbb{R}, \mathbb{R})$ is called an upper solution of Eq. (2.2) if ρ' and ρ'' exist almost everywhere in $\mathbb{R}$, they are essentially bounded on $\mathbb{R}$ and if the following inequality holds:

$$\rho''(z) - c\rho'(z) + f_c(\rho_z) \leq 0, \quad z \in \mathbb{R}. \tag{2.3}$$

A lower solution for Eq. (2.2) is defined in a similar way by reversing the inequality in (2.3).

Now, we are in the position to state a scalar version of theorem 3.6 of [5].

Theorem 2.1. *Assume that* **(F1)** *and* **(F2)** *holds. Suppose that Eq. (2.2) has an upper solution* $\bar{\phi} \in \Gamma$ *and a lower solution* $\underline{\phi}$ *(which is not necessarily in* Γ*) satisfying:*

- **(S1)** $\underline{\phi}(z) \not\equiv 0, \quad z \in \mathbb{R}$
- **(S2)** $0 < \underline{\phi}(z) \leq \bar{\phi}(z) \leq k, \quad z \in \mathbb{R}$.

Then Eq. (2.2) has a solution in Γ *. That is, Eq. (2.1) has a traveling wavefront with speed* c.

3. EXISTENCE OF TRAVELING WAVEFRONTS: CASE $c > c_*$

Substituting the wave profile $u(t,x) = \phi(x+ct)$ in Eq. (1.1), we obtain the second order functional differential equation given by Eq. (3.1)

$$\phi''(z) - c\phi'(z) + f_c(\phi_z) = 0, \quad z \in \mathbb{R}, \tag{3.1}$$

with

$$\phi(-\infty) = 0, \quad \phi(+\infty) = k = \left(\frac{p-d}{ad}\right)^{1/q} \tag{3.2}$$

and

$$f_c(\phi_z) = \frac{p\phi_z(-cr)}{1 + a\phi_z^q(-cr)} - d\phi_z(0). \tag{3.3}$$

From now on we will denote the left side of Eq. (3.1) as the differential Operator (3.4)

$$L_\phi = \phi''(z) - c\phi'(z) + f_c(\phi_z). \tag{3.4}$$

Note that the following lemma 3.1 and 3.2 prove the hypotheses (**F1**) and (**F2**) respectively on f.

Lemma 3.1. *If $p/d > 1$, then there exists $k > 0$ such that $f_c(\hat{0}) = f_c(\hat{k}) = 0$, and $f_c(\hat{u}) \neq 0$ for $0 < u < k$, where $\hat{u}(s) = u$, $s \in [-c\tau, 0]$.*

Proof. Clearly $k = ((p-d)/ad)^{1/q} > 0$, since $p/d > 1$. So by computing the stationary states of Eq. (3.1), we get the result. ∎

Lemma 3.2. *If (a) or (b) holds, then for all $\beta \geq d$, f_c satisfies the quasimonotonicity condition.*

Proof. Let $\phi_z, \psi_z \in X_c$ be, such that $0 \leq \psi_z(s) \leq \phi_z(s) \leq k$, $s \in [-c\tau, 0]$. Then

$$f_c(\phi_z) - f_c(\psi_z) = p\left(\frac{\phi_z(-c\tau)}{1 + a[\phi_z(-c\tau)]^q} - \frac{\psi_z(-c\tau)}{1 + a[\psi_z(-c\tau)]^q}\right) - d\left(\phi_z(0) - \psi_z(0)\right).$$

We consider the function $g(y) = py/(1+ay^q)$. We notice that $g'(y) = p[1 + ay^q(1-q)]/(1 + ay^q)^2 \geq 0$, for all $y \in \mathbb{R}$, since $0 < q \leq 1$, in the case that $q > 1$, $g'(y) \geq 0$, for all $y \in [0, (aq-a)^{-1/q}]$ and $0 < k \leq (aq-a)^{-1/q}$, because $p/d \leq q/(q-1)$. So the function g is non-decreasig. Therefore $f_c(\phi_z) - f_c(\psi_z) + d[\phi(z) - \psi(z)] \geq 0$. ∎

In order to build a upper and lower solution, we linearize Eq. (3.1) around the equilibria 0 and we obtain Eq. (3.5)

$$\psi_0(\lambda, c) = \lambda^2 - c\lambda - d + pe^{-\lambda c\tau}. \tag{3.5}$$

We also linearize Eq. (3.1) around the equilibria $k = ((p-d)/ad)^{1/q}$ and we obtain Eq. (3.6)

$$\psi_1(\mu, c) = \mu^2 - c\mu - d + p_1 e^{-\mu c\tau} \tag{3.6}$$

where $p_1 = d[p - q(p-d)]/p > 0$, when (a) or (b) is satisfied.

Proposition 3.3. *There exists $c_* > 0$ such that for $c > c_*$, the equation $\psi_0(\lambda, c) = 0$ has two positive real roots, $0 < \lambda_1 < \lambda_2$ and $\psi_0(\lambda, c) > 0$ for all $\lambda \in \mathbb{R} \setminus [\lambda_1, \lambda_2]$.*

Proof. We notice that the function $\psi_0(\lambda, .)$ is concave up, since $\partial^2\psi_0/\partial\lambda^2 = 2 + pc^2\tau^2 e^{-\lambda c\tau} > 0$. We also have that $\psi_0(0, c) = p - d > 0$, $\psi_0(+\infty, c) = +\infty$, so if we choose $\lambda = c/2$ we have

$$\psi_0\left(\frac{c}{2}, c\right) = \left(\frac{c}{2}\right)^2 - c\left(\frac{c}{2}\right) - d + pe^{-\lambda\frac{c}{2}\tau} < p - d - \frac{c^2}{4},$$

thus, if $\psi_0(c/2, c) < 0$, then $c > 2\sqrt{p-d}$. Hence $c_* < 2\sqrt{p-d}$. ∎

Proposition 3.4. *For any $c > 0$, the equation $\psi_1(\mu, c) = 0$ has two real roots $\mu_1 < 0 < \mu_2$ and $\psi_1(\mu, c) > 0$ for all $\mu \in \mathbb{R} \setminus [\mu_1, \mu_2]$.*

Proof. We notice that the function $\psi_1(\mu, .)$ is concave up, since $\partial^2\psi_1/\partial\mu^2 > 0$. We also have that $\psi_1(0, c) = -dq(p - d)/p < 0$ and $\psi_1(\pm\infty, c) = +\infty$. Therefore the result is obtained. ∎

Proposition 3.5. *For any $c > c_*$ and $\eta \geq \eta_0 > 0$. The Function* (3.7)

$$\bar{\phi}(z) = \begin{cases} \dfrac{k(\eta - \mu_1)}{\lambda_1 - \mu_1 + \eta} e^{z\lambda_1} & \textit{si } z < 0 \\ k - \dfrac{\lambda_1 k}{\lambda_1 - \mu_1 + \eta} e^{z(\mu_1 - \eta)} & \textit{si } z \geq 0, \end{cases} \tag{3.7}$$

where $\eta_0 = \dfrac{2\mu_1 - c + \sqrt{(c - 2\mu_1)^2 + 4p_1 e^{-\mu_1 c\tau}}}{2}$, *is upper solution of Eq. (3.1).*

Proof. First notice that $\bar{\phi} \in \Gamma$, since $\bar{\phi}$ is nondecreasing, $\bar{\phi}(-\infty) = 0$ and $\bar{\phi}(+\infty) = k$, besides $\bar{\phi}(z) \to k(\eta - \mu_1)/(\lambda_1 - \mu_1 + \eta)$ when $z \to 0$ and $\bar{\phi}$ was built differentiable. Second, let us prove the inequality (2.3). In effect, let $z < 0$ be, then

$$\begin{aligned} L_{\bar{\phi}} &\leq \bar{\phi}''(z) - c\bar{\phi}'(z) + p\bar{\phi}(z - c\tau) - d\bar{\phi}(z) \\ &= \frac{k(\eta - \mu_1)e^{z\lambda_1}}{\lambda_1 - \mu_1 + \eta}(\lambda_1^2 - c\lambda_1 - d + pe^{-\lambda_1 c\tau}) \\ &= \frac{k(\eta - \mu_1)e^{z\lambda_1}}{\lambda_1 - \mu_1 + \eta}\psi_0(\lambda_1, c) = 0. \end{aligned}$$

On the other hand, if $z \geq 0$ then we have

$$\begin{aligned} L_{\bar{\phi}} &\leq \bar{\phi}''(z) - c\bar{\phi}'(z) + \frac{pk}{1 + ak^q} - d\bar{\phi}(z) \\ &= -\frac{\lambda_1 k e^{z(\mu_1 - \eta)}}{\lambda_1 - \mu_1 + \eta}[(\mu_1 - \eta)^2 - c(\mu_1 - \eta)] + \frac{pk}{1 + ak^q} - d\left(k - \frac{\lambda_1 k}{\lambda_1 - \mu_1 + \eta}e^{z(\mu - \eta)}\right) \\ &= -\frac{\lambda_1 K e^{z(\mu_1 - \eta)}}{\lambda_1 - \mu_1 + \eta}[(\mu_1 - \eta)^2 - c(\mu_1 - \eta) - d] + \frac{pk}{1 + ak^q} - dk \\ &= -\frac{\lambda_1 k e^{z(\mu_1 - \eta)}}{\lambda_1 - \mu_1 + \eta}[\psi_1(\mu_1, c) + \eta^2 + \eta(c - 2\mu_1) - p_1 e^{-\mu_1 c\tau}] \\ &= -\frac{\lambda_1 k e^{z(\mu_1 - \eta)}}{\lambda_1 - \mu_1 + \eta}[\eta^2 + \eta(c - 2\mu_1) - p_1 e^{-\mu_1 c\tau}], \end{aligned}$$

note that $\upsilon(\eta) = \eta^2 + \eta(c - 2\mu_1) - p_1 e^{-\mu_1 c\tau} \geq 0$, for all $\eta \geq \eta_0 > 0$, since η_0 is root for $\upsilon(\eta)$ then $L_{\bar{\phi}} \leq 0$. Therefore the result is obtained. ∎

Now, for us to build a lower solution of Eq. (3.1), we chose $\varepsilon > 0$ such that $\varepsilon < \lambda_1 < \lambda_1 + \varepsilon < \lambda_2$. Then we provide the following proposition.

Proposition 3.6. *For any $c > c_*$ and $M > 1$. The Function* (3.8)

$$\underline{\phi}(z) = \begin{cases} \dfrac{k(\eta - \mu_1)}{\lambda_1 - \mu_1 + \eta}(1 - Me^{\varepsilon z})e^{\lambda_1 z} & \textit{si } z < \frac{1}{\varepsilon}\ln\left(\frac{1}{M}\right) \\ 0 & \textit{si } z \geq \frac{1}{\varepsilon}\ln\left(\frac{1}{M}\right), \end{cases} \tag{3.8}$$

where $M > -\dfrac{apk_0 e^{-\lambda_1 c\tau(q+1)}(1 + e^{-\varepsilon c\tau})^{q+1}}{\psi_0(\lambda_1 + \varepsilon, c)}$, *with* $k_0 = \frac{k(\eta - \mu_1)}{\lambda_1 - \mu_1 + \eta}$; *is lower solution of Eq. (3.1).*

Proof. For $z \geq \ln(1/M)/\varepsilon$ is easy to see that $L_{\underline{\phi}} \geq 0$. Now let $z < \ln(1/M)/\varepsilon$ be, and we note that $\frac{1}{1+a\phi_z^q(-c\tau)} \geq 1 - a\phi_z^q(-c\tau)$ for $q > 0$. Then

$$\begin{aligned}
L_{\underline{\phi}} \geq& \underline{\phi}''(z) - c\underline{\phi}'(z) + p\underline{\phi}(z-c\tau)[1 - a\underline{\phi}^q(z-c\tau)] - d\underline{\phi}(z) \\
=& k_0 e^{z\lambda_1}[\lambda_1^2 - M(\lambda_1+\varepsilon)^2 e^{z\varepsilon}] - ce^{z\lambda_1}[\lambda_1 - M(\lambda_1+\varepsilon)e^{z\varepsilon}] \\
&+ pe^{(z-c\tau)\lambda_1}[1 - Me^{(z-c\tau)\varepsilon}][1 - ak_0(e^{(z-c\tau)\lambda_1}(1 - Me^{(z-c\tau)\varepsilon}))^q] - de^{z\lambda_1}(1 - Me^{z\varepsilon}) \\
=& k_0 e^{z\lambda_1}[(\lambda_1^2 - c\lambda_1 - d + pe^{-\lambda_1 c\tau}) - Me^{z\varepsilon}((\lambda_1+\varepsilon)^2 - c(\lambda_1+\varepsilon) + pe^{-(\lambda_1+\varepsilon)c\tau}) \\
&- ak_0 pe^{z\lambda_1 q} e^{-\lambda_1 c\tau(q+1)}(1 - Me^{\varepsilon(z-c\tau)})^{q+1})].
\end{aligned}$$

Furthermore $(1 - Me^{(z-c\tau)\varepsilon})^{q+1} < (1 + e^{-\varepsilon c\tau})^{q+1}$, since $1 - Me^{(z-c\tau)\varepsilon} > 0$, then

$$\begin{aligned}
L_{\underline{\phi}} \geq& k_0 e^{z\lambda_1}[(\lambda_1^2 - c\lambda_1 - d + pe^{-\lambda_1 c\tau}) - Me^{z\varepsilon}((\lambda_1+\varepsilon)^2 - c(\lambda_1+\varepsilon) - d + pe^{-(\lambda_1+\varepsilon)c\tau}) \\
&- ak_0 pe^{z\lambda_1 q} e^{-\lambda_1 c\tau(q+1)}(1 + e^{-\varepsilon c\tau})^{q+1}] \\
=& k_0 e^{z\lambda_1}[\psi_0(\lambda_1, c) - Me^{z\varepsilon}\psi_0(\lambda_1+\varepsilon, c) - ak_0 pe^{z\lambda_1 q} e^{-\lambda_1 c\tau(q+1)}(1 + e^{-\varepsilon c\tau})^{q+1}]
\end{aligned}$$

and choosing $0 < \varepsilon < \lambda_1 q$, we have

$$\begin{aligned}
L_{\underline{\phi}} \geq& k_0 e^{z(\lambda_1+\varepsilon)}\left[-M\psi_0(\lambda_1+\varepsilon, c) - ak_0 pe^{-\lambda_1 c\tau(q+1)}(1 + e^{-\varepsilon c\tau})^{q+1}\right] \\
=& k_0 e^{z(\lambda_1+\varepsilon)}(-\psi_0(\lambda_1+\varepsilon, c))\left[M + \frac{ak_0 pe^{-\lambda_1 c\tau(q+1)}(1 + e^{-\varepsilon c\tau})^{q+1}}{\psi_0(\lambda_1+\varepsilon, c)}\right].
\end{aligned}$$

We note that $\psi_0(\lambda_1+\varepsilon, c) < 0$ then $L_{\underline{\phi}} \geq 0$. Therefore the result is obtained. ∎

In order to apply Theorem 2.1, first note that the condition **(S1)** is immediate by the definition of $\underline{\phi}$ and it is also easy to verify the condition **(S2)**. Then, for the case $c > c_*$, we obtain our main result on the existence of monotone traveling wavefronts for Eq. (1.1).

4. EXISTENCE OF TRAVELING WAVEFRONTS: CASE $c = c_*$

In this section we will prove the existence of a traveling wavefront at minimum speed $c = c_*$. In order to do that, we construct a lower and a upper solution for Eq. (3.1).

Remark 4.1. There exist a minimum speed $c_* = c_*(\tau, p, d) > 0$ and a corresponding number $\lambda_* = \lambda(c_*) > 0$ satisfying Eq. (4.1)

$$p\tau e^{\frac{2 - c^2\tau - \sqrt{(c^2\tau-2)^2 + 4c^2\tau(d\tau+1)}}{2}} + 1 = \frac{c^2\tau - 2 + \sqrt{(c^2\tau-2)^2 + 4c^2\tau(d\tau+1)}}{c^2\tau}. \tag{4.1}$$

We notice that (λ_*, c_*) is solution the system

$$\begin{aligned}
h_1(\lambda, c) &= h_2(\lambda, c) \\
h_1'(\lambda, c) &= h_2'(\lambda, c),
\end{aligned}$$

where $h_1(\lambda, c) = pe^{-\lambda c\tau}$, $h_2(\lambda, c) = d + c\lambda - \lambda^2$, since and (λ_*, c_*) is the tangent point of h_1 and h_2 and $\partial\psi_0/\partial c < 0$ then Eq. (3.5) has a double root λ_*. See Figure 1.

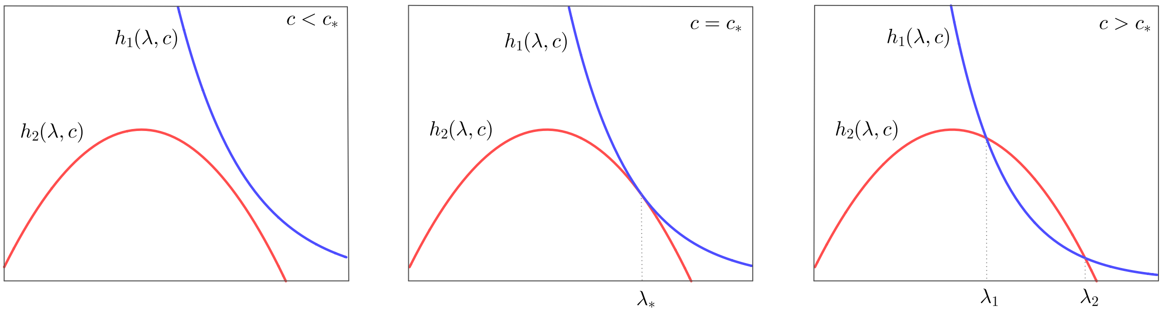

Figure 1: The graphs of h_1 and h_2 for $c < c_*$, $c = c_*$ and $c > c_*$, respectively.

Proposition 4.1. *For $c = c_*$ and $\eta_* \geq \eta_0 > 0$. The Function* (4.2)

$$\bar{\phi}_*(z) = \begin{cases} \dfrac{k(\eta_* - \mu_1)}{\lambda_* + 2(\eta_* - \mu_1)}(2 - \lambda_* z)e^{z\lambda_*} & si\ z < 0 \\ k - \dfrac{\lambda_* k}{\lambda_* + 2(\eta_* - \mu_1)}e^{z(\mu_1 - \eta_*)} & si\ z \geq 0, \end{cases} \tag{4.2}$$

where η_0 is the same as that given in the Proposition 3.5, is upper solution of Eq. (3.1).

Proof. First notice that $\bar{\phi}_* \in \Gamma$, since $\bar{\phi}_*$ is nondecreasing, $\bar{\phi}_*(-\infty) = 0$ and $\bar{\phi}_*(+\infty) = k$, also $\bar{\phi}_*(z) \to \frac{2k(\eta_* - \mu_1)}{\lambda_* + 2(\eta_* - \mu_1)}$ when $z \to 0$ and $\bar{\phi}_*$ was built differentiable. Second, let us prove the inequality (2.3). In effect, let $z < 0$ be, then

$$\begin{aligned} L_{\bar{\phi}_*} &\leq \bar{\phi}_*''(z) - c\bar{\phi}_*'(z) + p\bar{\phi}_*(z - c\tau) - d\bar{\phi}_*(z) \\ &= \frac{k(\eta_* - \mu_1)}{\lambda_* + 2(\eta_* - \mu_1)}[-\lambda_*^3 z e^{\lambda_* z} - c_*(\lambda_* e^{\lambda_* z} - \lambda_*^2 z e^{\lambda_* z}) + p\left(2 - \lambda_*(z - c_*\tau)\right) e^{\lambda_*(z - c_*\tau)} \\ &\quad - d(2 - \lambda_* z)e^{\lambda_* z}] \\ &= \frac{k(\eta_* - \mu_1)e^{\lambda_* z}}{\lambda_* + 2(\eta_* - \mu_1)}\left[-\lambda_*^3 z - c\lambda_* + c_*\lambda_*^2 z + p\left(2 - \lambda_*(z - c_*\tau)\right) e^{-\lambda_* c_* \tau} - d(2 - \lambda_* z)\right] \\ &= \frac{k(\eta_* - \mu_1)e^{\lambda_* z}}{\lambda_* + 2(\eta_* - \mu_1)}[-\lambda_* z(\lambda_*^2 - c\lambda - d + pe^{-\lambda_* c\tau}) - c\lambda_* - 2d + 2pe^{-\lambda_* c\tau} \\ &\quad + p\lambda_* c_*\tau e^{-\lambda_* c_* \tau}] \\ &= \frac{k(\eta_* - \mu_1)e^{\lambda_* z}}{\lambda_* + 2(\eta_* - \mu_1)}\left[(2 - \lambda_* z)(\lambda_*^2 - c\lambda - d + pe^{-\lambda_* c\tau}) - \lambda_*(2\lambda_* - c_* - pc_*\tau e^{-\lambda_* c_* \tau})\right] \\ &= \frac{k(\eta_* - \mu_1)e^{\lambda_* z}}{\lambda_* + 2(\eta_* - \mu_1)}\left[(2 - \lambda_* z)\psi_0(\lambda_*, c_*) - \lambda_*\psi_0'(\lambda_*, c_*)\right]. \end{aligned}$$

Observe that λ_* is double root of Eq. (3.5), then $\psi_0(\lambda_*, c_*) = \psi_0'(\lambda_*, c_*) = 0$, therefore $L_{\bar{\phi}} \leq 0$. On the other hand, if $z \geq 0$, then we have similarly how it was done in the Proposition 3.5. Therefore the result is obtained. ∎

Remark 4.2. In proposition 3.6, the construction of the lower solution for Eq. (3.1) depends on the existence of some $\varepsilon > 0$, such that $\varepsilon < \lambda_1 < \lambda_1 + \varepsilon < \lambda_2$, where λ_1 and λ_2 are the two positive real roots of Eq (3.5), however such construction does not apply to the case where $\lambda_1 = \lambda_2 = \lambda_*$ is a double root.

Proposition 4.2. *For* $c = c_*$ *and* $0 < b < \lambda_* q$. *There exists* $N > 0$ *such that the Function* (4.3)

$$\underline{\phi}_*(z) = \begin{cases} N(b^{-1}e^{bz-1} - z)e^{\lambda_* z} & \text{si } z < b^{-1} \\ 0 & \text{si } z \geq b^{-1}, \end{cases} \tag{4.3}$$

is lower solution of Eq. (3.1).

Proof. First notice that $\underline{\phi}_*$ was built differently and $\underline{\phi}_*(z) \to 0$ when $z \to b^{-1}$. Second, for the case $z \geq b^{-1}$ the result is easily obtained. Let us prove it in the case $z < b^{-1}$. Then we have

$$\begin{aligned}
L_{\underline{\phi}_*} \geq & \underline{\phi}_*''(z) - c\underline{\phi}_*'(z) + p\underline{\phi}_*(z - c\tau)[1 - a\underline{\phi}_*^q(z - c\tau)] - d\underline{\phi}_*(z) \\
= & Ne^{\lambda_* z}[(b^{-1}e^{bz-1}(\lambda_* + b)^2 - 2\lambda_* - \lambda_*^2 z) - c(b^{-1}e^{bz-1}(\lambda_* + b) - z\lambda_* - 1) \\
& + pe^{-\lambda_* c\tau}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)(1 - aN^q e^{\lambda_* q(z-c\tau)}(b^{-1}e^{b(z-c\tau-)1} - z + c\tau)^q) \\
& - d(b^{-1}e^{bz-1} - z)] \\
= & Ne^{\lambda_* z}[b^{-1}e^{bz-1}((\lambda_* + b)^2 - c(\lambda_* + b) - d) - z(\lambda_*^2 - c\lambda_* - d) - 2\lambda_* + c \\
& + pe^{-\lambda_* c\tau}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)(1 - aN^q e^{\lambda_* q(z-c\tau)}(b^{-1}e^{b(z-c\tau-)1} - z + c\tau)^q)] \\
= & Ne^{\lambda_* z}[b^{-1}e^{bz-1}((\lambda_* + b)^2 - c(\lambda_* + b) - d) - z(\lambda_*^2 - c\lambda_* - d) - 2\lambda_* + c \\
& + pe^{-\lambda_* c\tau}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)(1 - aN^q e^{\lambda_* q(z-c\tau)}(b^{-1}e^{b(z-c\tau-)1} - z + c\tau)^q)] \\
= & Ne^{\lambda_* z}[b^{-1}e^{bz-1}((\lambda_* + b)^2 - c(\lambda_* + b) - d) - z(\lambda_*^2 - c\lambda_* - d) - 2\lambda_* + c \\
& + pe^{-\lambda_* c\tau}b^{-1}e^{b(z-c\tau)-1} - pe^{-\lambda_* c\tau}z + pe^{-\lambda_* c\tau}c\tau \\
& - paN^q e^{-\lambda_* c\tau}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)^{q+1}e^{\lambda_* q(z-c\tau)}] \\
= & Ne^{\lambda_* z}[b^{-1}e^{bz-1}((\lambda_* + b)^2 - c(\lambda_* + b) - d + pe^{-c\tau(\lambda_* + b)}) - z(\lambda_*^2 - c\lambda_* - d + pe^{-\lambda_* c\tau}) \\
& - (2\lambda_* - c - pe^{-\lambda_* c\tau}c\tau) - paN^q e^{-\lambda_* c\tau}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)^{q+1}e^{\lambda_* q(z-c\tau)}] \\
= & Ne^{\lambda_* z}[b^{-1}e^{bz-1}\psi_0(\lambda_* + b, c_*) - z\psi_0(\lambda_*, c_*) - \psi_0'(\lambda_*, c_*) \\
& - paN^q e^{-\lambda_* c\tau(q+1)}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)^{q+1}e^{\lambda_* qz}] \\
= & Ne^{\lambda_* z}[b^{-1}e^{bz-1}\psi_0(\lambda_* + b, c_*) - paN^q e^{-\lambda_* c\tau(q+1)}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)^{q+1}e^{\lambda_* qz}] \\
= & Ne^{(\lambda_* + b)z}\left[b^{-1}e^{-1}\psi_o(\lambda_* + b, c_*) - paN^q e^{-\lambda_* c\tau(q+1)}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)^{q+1}\right. \\
& \left. .e^{(\lambda_* q - b)z}\right]
\end{aligned}$$

Note that the function $\xi(z) = (b^{-1}e^{b(z-c\tau)-1} - z + c\tau)^{q+1}e^{(\lambda_* q - b)z}$ is positive for all $z < b^{-1}$, $\lim_{z \to -\infty} \xi(z) = 0$ since $b < \lambda_* q$ and $\xi(b^{-1} + c\tau) = 0$. Then let $\xi_* = \max_{z < b^{-1}} \xi(z)$ and so we obtain

$$L_{\underline{\phi}_*} \geq Ne^{(\lambda_* + b)z}\left[b^{-1}e^{-1}\psi_o(\lambda_* + b, c_*) - paN^q e^{-\lambda_* c\tau(q+1)}\xi_*\right]$$

and $L_{\underline{\phi}_*} \geq 0$ if we choose $N^q \leq \frac{\psi_0(\lambda_* + b, c_*)}{bpa\xi_* e^{1 - \lambda_* c\tau(q+1)}}$. Therefore the result is obtained. ∎

Lemma 4.3. $\underline{\phi}_*(z)$ *and* $\bar{\phi}_*(z)$, *satisfy the condition* **(S2)**.

Proof. Note that

$$\lim_{z \to -\infty} \frac{\bar{\phi}_*(z)}{\underline{\phi}_*(z)} = \lim_{z \to -\infty} \frac{\frac{K(\eta_* - \mu_1)(2 - \lambda_* z)}{\lambda_* + 2(\eta_* - \mu_1)}e^{z\lambda_*}}{N(b^{-1}e^{bz-1} - z)e^{\lambda_* z}} = \frac{\frac{K(\eta_* - \mu_1)\lambda_*}{\lambda_* + 2(\eta_* - \mu_1)}}{N}.$$

Then, if we choose $N < \frac{K(\eta_* - \mu_1)\lambda_*}{\lambda_* + 2(\eta_* - \mu_1)}$, there exists $T \gg 1$ such that $\underline{\phi}_*(z) < \bar{\phi}_*(z)$ for all $z \in (-\infty, -T]$. It is easy to prove that $\bar{\phi}_*$ is an increasing function this implies that

$\min_{z\in[-T,b^{-1}]} \bar{\phi}_*(z) = \bar{\phi}_*(-T)$. Let $m_0 = \max_{z\in[-T,b^{-1}]} (b^{-1}e^{bz-1} - z)e^{\lambda_* z}$. If we choose N satisfying $Nm_0 < \bar{\phi}_*(-T)$, then we have

$$\underline{\phi}_*(z) < Nm_0 < \bar{\phi}_*(-T) < \bar{\phi}_*(z).$$

Therefore, if $N \leq \min\left\{\frac{k(\eta_*-\mu_1)\lambda_*}{\lambda_*+2(\eta_*-\mu_1)}, \frac{\bar{\phi}_*(-T)}{m_0}\right\}$, hypothesis **(S2)** is satisfied. ∎

In order to apply Theorem 2.1, we note that the condition **(S1)** is immediate by the definition of $\underline{\phi}_*$. Then, for the case $c = c_*$, we obtain our main result on the existence of monotone traveling wavefronts for Eq. (1.1).

5. NUMERICAL SIMULATIONS

In this section, we present some numerical simulations. We find that these numerical results confirm our theoretical results shown in section 3 and section 4. The computational results reported in this section, to numerically approximate the traveling wavefront, are based on the following monotonous iteration scheme that arise from the results given in [5].

(5.1)
$$\begin{cases} \phi_{n+1}(z) = \dfrac{p}{d(\alpha_2-\alpha_1)}\left[\displaystyle\int_{-\infty}^{z} e^{\alpha_1(z-s)}H(\phi_n(s))ds + \int_{z}^{+\infty} e^{\alpha_2(z-s)}H(\phi_n(s))ds\right], \; n \in \mathbb{N} \\ \phi_0(z) = \bar{\phi}(z) \quad (\text{or } \bar{\phi}_*(z)), \end{cases}$$

where $H(\phi_n(z)) = \frac{\phi_n(z-cr)}{1+a\phi_n^q(z-cr)}$ and $\alpha_1 < 0 < \alpha_2$ are roots of $x^2 - cx - 1 = 0$.

Now to approximate the traveling wavefront that moves at minimum speed $c = c_*$, we take some particular values for the parameters $a = e-1, \tau = 1, p = e$ and $d = 1$ then $c_*(\tau, p, d) = 1$. See Figure 2 and Figure 3.

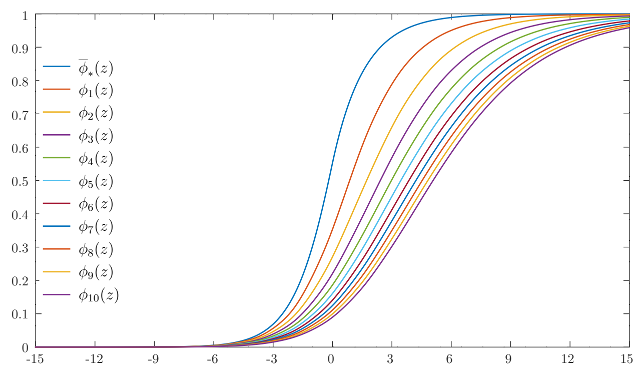

Figure 2: First 10 *iterations for* $q = 1$.

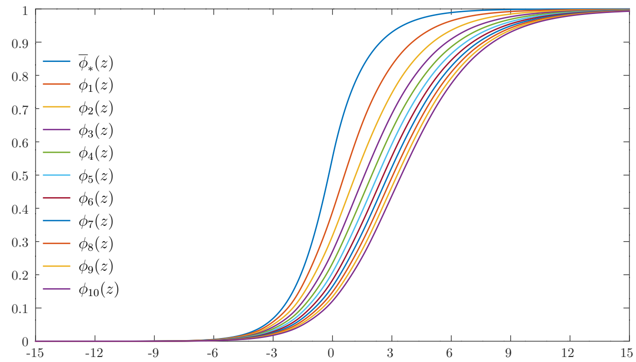

Figure 3: First 10 *iterations for* $q = 1.5$.

Now we approach the traveling wave fronts that move at a speed greater than the minimum $c > c_*$. In particular we choose the same previous values for the $a = e - 1$, $\tau = 1$, $p = e$ and $d = 1$ parameters. See Figure 4 and Figure 5.

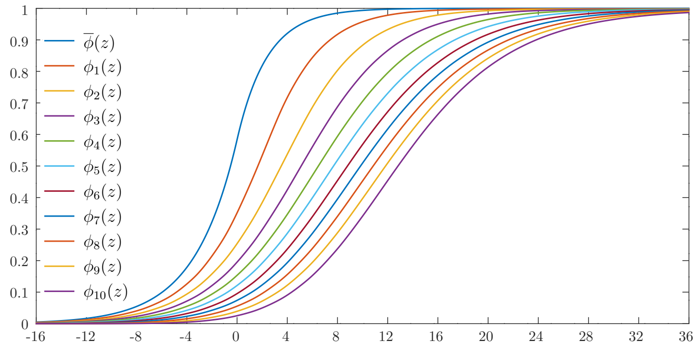

Figure 4: First 10 *iterations for* $c = 2$ *and* $q = 1$.

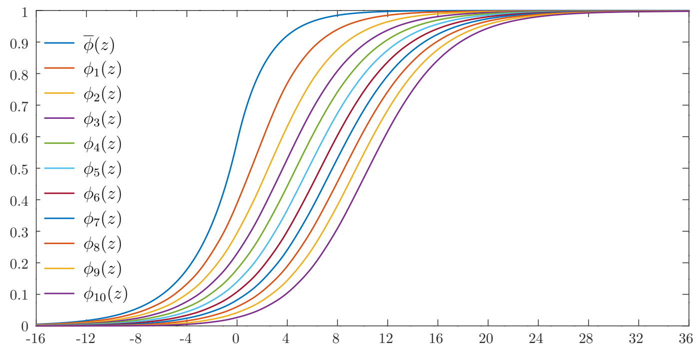

Figure 5: First 10 *iterations for* $c = 3$ *and* $q = 1.5$.

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