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EXISTENCE OF SOLUTION OF DIFFERENTIAL AND RIEMANN-LIOUVILLE EQUATION VIA FIXED POINT APPROACH IN COMPLEX VALUED b-METRIC SPACES.

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ABSTRACT. In this paper, we establish some fixed point and common fixed point results for a new type of generalized contractive mapping using the notion of C-class function in the framework of complex valued b-metric spaces. As an application, we establish the existence and uniqueness of a solution for Riemann-Liouville integral and ordinary differential equation in the framework of a complete complex valued b-metric spaces. The obtained results generalize and improve some fixed point results in the literature.

Key words and phrases: Generalized contraction type mapping; Common fixed point; Fixed point; Complex valued b-metric space.

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1. INTRODUCTION AND PRELIMINARIES

The theory of fixed point plays an important role in nonlinear functional analysis and is known to be very useful in establishing the existence and uniqueness theorems for nonlinear differential and integral equations. Banach [8] in 1922 proved the well celebrated Banach contraction principle in the frame work of metric spaces. The importance of the Banach contraction principle cannot be over emphasized in the study of fixed point theory and its applications. Due to its importance and fruitful applications, many authors have generalized this result by considering classes of nonlinear mappings which are more general than contraction mappings and in other classical and important spaces (see [1, 2, 14, 21] and the references therein). For example, Berinde [9, 10] introduced and studied a class of contractive mappings, which is defined as follows:

Definition 1.1. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a generalized almost contraction if there exist $\delta \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x, y \in X$.

Furthermore, in 2008, Suzuki [28] introduced a class of mappings satisfying condition (C), known as Suzuki-type generalized nonexpansive mapping and he proved some fixed point theorems for this class of mappings.

Definition 1.2. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to satisfy condition (C) if for all $x, y \in X$,

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

Theorem 1.1. *Let (X, d) be a compact metric space and $T : X \to X$ be a mapping satisfying*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) < d(x, y),$$

for all $x, y \in X$. Then T has a unique fixed point.

In 2014, Ansari [4] introduced the notion of C-class function, he proved some fixed point results using the concept of C-class function and also established that the C-class function is a generalization of a whole lot of contractive conditions.

Definition 1.3. [4] A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and the following axioms hold:

(1) $F(s, t) \leq s$ for all $s, t \in [0, \infty)$;
(2) $F(s, t) = s$ implies either $s = 0$ or $t = 0$.

Example 1.1. *The following functions $F : [0, \infty)^2 \to \mathbb{R}$ defined for all $s, t \in [0, \infty)$ by*

(1) $F(s, t) = s - t$, $F(s, t) = s$ *implies* $t = 0$;
(2) $F(s, t) = ms$, $0 < m < 1$, $F(s, t) = s$ *implies* $s = 0$;
(3) $F(s, t) = s\beta(s)$, $\beta : [0, \infty) \to [0, 1)$ *is a continuous function,* $F(s, t) = s$ *implies* $s = 0$.

For details about C-class functions see [4], and $\mathcal{C}$ denote the class of C-functions. In 2016, Chandok et al. [12] introduced a new type of contractive mappings using the notion of cyclic admissible mappings in the framework of metric spaces.

Definition 1.4. [12] Let $T : X \to X$ be a mapping and let $\alpha, \beta : X \to \mathbb{R}^+$ be two functions. Then T is called a cyclic (α, β)-admissible mapping, if

(1) $\alpha(x) \geq 1$ for some $x \in X$ implies that $\beta(Tx) \geq 1$,
(2) $\beta(x) \geq 1$ for some $x \in X$ implies that $\alpha(Tx) \geq 1$.

Definition 1.5. [12] Let (X, d) be a metric space and let $\alpha, \beta : X \to [0, \infty)$ be two mappings. We say that T is a TAC-contractive mapping, if for all $x, y \in X$,

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(d(Tx, Ty)) \leq f(\psi(d(x, y)), \phi(d(x, y))),$$

where ψ is a continuous and nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$, ϕ is continuous with $\lim_{n\to\infty} \phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$ and $f \in \mathcal{C}$.

Theorem 1.2. [12] *Let (X, d) be a complete metric space and let $T : X \to X$ be a cyclic (α, β)-admissible mapping. Suppose that T is a TAC contraction mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1, \beta(x_0) \geq 1$ and either of the following conditions hold:*

(1) *T is continuous,*
(2) *if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq 1$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq 1$.*

In addition, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in F(T)$ (where $F(T)$ denotes the set of fixed points of T), then T has a unique fixed point.

One of the interesting generalization of metric spaces is the concept of b-metric spaces introduced by Czerwik in [13]. He established the Banach contraction principle in this framework with the fact that b need not be continuous. Thereafter, several results has been extended from metric spaces to b-metric spaces, more so, a lot of results on the fixed point theory of various classes of mappings in the frame work of b-metric spaces has been established by different researchers in this area (see [11, 13, 19] and the references therein). Yamaod and Sintunawarat [30] introduced the notion of (α, β)-(ψ, φ)-contraction mapping in the frame work of b-metric spaces as follows:

Definition 1.6. Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $\alpha, \beta : X \to [0, \infty)$ be two given mappings. We say that $T : X \to X$ is an (α, β)-(ψ, φ)-contraction mapping if the following conditions holds: for all $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$ implies that

$$\psi(s^3 d(Tx, Ty)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)),$$

where $M_s(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}$ and $\psi, \varphi : [0, \infty) \to [0, \infty)$ are alternating distance functions.

Theorem 1.3. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ an (α, β)-(ψ, φ)-contraction mapping. Suppose that one of the following conditions holds:*

(1) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$,*
(2) *there exists $y_0 \in X$ such that $\alpha(y_0) \geq 1$,*

and the following holds:

(1) *T is continuous,*
(2) *T is cyclic (α, β)-admissible.*

Then T has a fixed point.

Recently, Babu et al. [6] generalized the result of Chandok et al. [12] by introducing a generalized TAC-contractive mapping in the frame work of b-metric spaces.

Definition 1.7. Let (X, d) be a b-metric space, $\alpha, \beta : X \to [0, \infty)$ be two given mappings and T be a self map on X. The mapping T is said to be generalized TAC-contrative map in b-metric spaces, if for all $x, y \in X$,

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(s^3 d(Tx, Ty)) \leq f(\psi(M_s(x, y)), \phi(M_s(x, y))),$$

where $M_s(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}$, ψ is an alternating distance function, ϕ is continuous with $\lim_{n\to\infty}\phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$ and $f \in \mathcal{C}$.

Theorem 1.4. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$. Let $T : X \to X$ be a generalized TAC-contraction mapping. Suppose the following conditions hold:*

1. *T is a cyclic (α,β)-admissible mapping,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,*
3. *T is continuous,*
4. *if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq 1$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq 1$.*

Then T has a fixed point.

In mathematics researchers try to come up with new algebraic structures in order to improve and extend results obtained in the literature. In [5] Azam et al. introduce the notion of complex valued metric space and established some common fixed point results for mapping satisfying generalized contractive conditions. Thereafter, several results and applications has been extended from metric spaces to complex valued metric spaces, more so, a lot of results on the fixed point theory and common fixed point results of various classes of mappings in the framework of complex valued metric spaces has been established by different researchers in this area (see[25, 26, 27] and the references therein).

The following symbols, notation and definition can be found in [5] will be useful in this study. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \quad \text{if and only if} \quad Re(z_1) \leq Re(z_2),\ Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

1. $Re(z_1) = Re(z_2), \quad Im(z_1) < Im(z_2);$
2. $Re(z_1) < Re(z_2), \quad Im(z_1) = Im(z_2);$
3. $Re(z_1) < Re(z_2), \quad Im(z_1) < Im(z_2);$
4. $Re(z_1) = Re(z_2), \quad Im(z_1) \leq Im(z_2).$

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(1), (2)$ and (3) is satisfied and we we write $z_1 \prec z_2$ if only (3) is satisfied. Note that

1. $a, b \in \mathbb{R}$ and $a \leq b$ implies that $az \precsim bz$ for all $z \in \mathbb{C}$;
2. $0 \precsim z_1 \precnsim z_2$ implies that $|z_1| < |z_2|$;
3. $z_1 \precsim z_2$ and $z_2 \prec z_z$ implies that $z_1 \prec z_2$.

Definition 1.8. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies:

1. $0 \precsim d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
2. $d(x,y) = d(y,x)$ for all $x, y \in X$;
3. $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric and (X,d) is called a complex valued metric space.

Example 1.2. *Let $X = \mathbb{C}$ and $d_i : X \times X \to \mathbb{C},\ i = 1,2,3$ be defined as*

1. $d(z_1,z_2) = |z_1 - z_2|$ *for all* $z_1, z_2 \in X$;
2. $d(z_1,z_2) = e^{ik}|z_1 - z_2|$ *for all* $z_1, z_2 \in X$ *and* $k \in \mathbb{R}$;
3. $d(z_1,z_2) = e^{i\theta}|z_1 - z_2|$ *for all* $z_1, z_2 \in X$ *and* $\theta \in (0, \frac{\Pi}{2})$.

Motivated by the concept of b-metric spaces and complex valued metric spaces [13, 5], Rao et al. in [18], introduced the notion of complex valued b-metric spaces and established some common fixed point results. Thereafter, several results and applications has been extended from metric spaces, b-metric spaces and complex valued metric spaces to complex valued b-metric spaces (see [18] and the reference therein). The notion of complex valued b-metric spaces generalize, improves and unifies results in metric spaces, b-metric spaces and complex valued metric spaces.

Definition 1.9. Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that the mapping $d_b : X \times X \to \mathbb{C}$, satisfies:

(1) $0 \precsim d_b(x, y)$ for all $x, y \in X$ and $d_b(x, y) = 0$ if and only if $x = y$;
(2) $d_b(x, y) = d_b(y, x)$ for all $x, y \in X$;
(3) $d_b(x, y) \precsim s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then d_b is called a complex valued b metric and (X, d_b) is called a complex valued metric space.

Example 1.3. [18] *Let $X = \mathbb{C}$ defined the mapping $d_b : X \times X \to \mathbb{C}$ by $d_b(z_1, z_2) = |z_1 - z_2|^2 + i|z_1 - z_2|^2$ for all $z_1, z_2 \in X$.*

Definition 1.10. Suppose that (X, d_b) is a complex valued b-metric space and $\{z_n\}$ is a sequence in X, then the sequence $\{z_n\}$

(1) converges to an element to and element $z_0 \in X$ if for every $0 \prec c \in \mathbb{C}$, there exist an integer $\mathbb{N}$ such that $d_b(z_n, z_0) \prec c$ for all $n \in \mathbb{N}$.
(2) is a Cauchy sequence if for every $0 \prec c \in \mathbb{C}$, there exist an integer $\mathbb{N}$ such that $d_b(z_n, z_m) \prec c$ for all $n, m \in \mathbb{N}$.

Definition 1.11. Suppose that (X, d_b) is a complex valued b-metric space, the space (X, d_b) is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 1.12. [16] Let X be a nonempty set and $S, T : X \to X$ be any two mappings.

(1) A point $x \in X$ is called:
 (a) coincidence point of S and T if $Sx = Tx$,
 (b) common fixed point of S and T if $x = Sx = Tx$.
(2) If $y = Sx = Tx$ for some $x \in X$, then y is called the point of coincidence of S and T.
(3) A pair (S, T) is said to be:
 (a) commuting if $TSx = STx$ for all $x \in X$,
 (b) weakly compatible if they commute at their coincidence points, that is $STx = TSx$, whenever $Sx = Tx$.

Motivated by the current research interest in this direction, the purpose of this work is to further develop the concept of C-class function and establish some fixed point and common fixed point results for a new type of generalized contractive mapping using the notion of C-class function in the framework of complex valued b-metric spaces. As an application, we establish the existence of a solution for Riemann-Liouville integral and ordinary differential equation in the framework of a complete complex valued b-metric spaces.

2. MAIN RESULT

In this section, we define a complex C-class function, established some common fixed point and fixed point results. Throughout this work, we will use d instead of d_b to denoted a complex valued b metric. We define

$$\mathbb{S} = \{z \in \mathbb{C} : 0 \precsim z\}$$

Definition 2.1. A mapping $F : \mathbb{S} \times \mathbb{S} \to \mathbb{C}$ is called a C-class function if it is continuous and the following axioms holds:

(1) $F(s, t) \preceq s$;
(2) $F(s, t) = s$ implies either $s = 0$ or $t = 0$

for all $t, s \in \mathbb{S}$.

In the course of this work, $\mathcal{C}$ denote the class of C-functions. Let Ψ denote the class of functions $\psi : \mathbb{S} \to \mathbb{S}$ satisfying the following condition:

(1) ψ is continuous;
(2) $\psi(t) \succ t$ for all $t \succ 0$ and $\psi(t) = 0$ if and only if $t = 0$.

Let Φ denote the class of functions $\phi : \mathbb{S} \to \mathbb{S}$ satisfying the following condition:

(1) ϕ is continuous;
(2) $\phi(t) \succ t$ for all $t \succ 0$ and $\phi(0) \succeq 0$.

Theorem 2.1. *Let (X, d_b) be a complex valued b-metric space with $s > 1$ and S, T be a self map on X satisfying*

$$\frac{1}{2s} d(x, Sx) \preceq d(x, y) \Rightarrow \psi(s^\epsilon d(Tx, Sy) \preceq F(\psi(M(x, y)), \phi(M(x, y))) \tag{2.1}$$

for all $x, y \in X$, where $\epsilon \geq 1, F \in \mathcal{C}, \phi \in \Phi, \psi \in \Psi$ and

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Tx)d(y, Sy)}{s + d(x, y)}\}.$$

Then the pair S and T have a unique common fixed point.

Proof. Let x_0 be any arbitrary point in X and we define $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for all $n = 0, 1, 2, \cdots$. Since $\frac{1}{2s} d(x_{2n+1}, Sx_{2n+1}) = \frac{1}{2s} d(x_{2n+1}, x_{2n+2}) \prec d(x_{2n+1}, x_{2n+2})$, using (2.1) and the properties of F, we have

$$\begin{aligned} \psi(s^\epsilon d(x_{2n+1}, x_{2n+2})) &= \psi((s^\epsilon d(Tx_{2n}, Sx_{2n+1}))) \\ &\preceq F(\psi(M(x_{2n}, x_{2n+1})), \phi(M(x_{2n}, x_{2n+1})) \\ &\preceq \psi(M(x_{2n}, x_{2n+1})), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} &M(x_{2n}, x_{2n+1}) \\ &= \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1}), \frac{d(x_{2n}, Tx_{2n})d(x_{2n+1}, Sx_{2n+1})}{s + d(x_{2n}, x_{2n+1})}\right\} \\ &= \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{s + d(x_{2n}, x_{2n+1})}\right\}, \end{aligned}$$

since $\frac{d(x_{2n}, x_{2n+1})}{s + d(x_{2n}, x_{2n+1})} \prec 1$, we have that $\frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{s + d(x_{2n}, x_{2n+1})} \prec d(x_{2n+1}, x_{2n+2})$, as such, we have that

$$M(x_{2n}, x_{2n+1}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}.$$

If we suppose that $M(x_{2n}, x_{2n+1}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$, we then have (2.2) becomes

$$\begin{aligned} \psi(s^\epsilon d(x_{2n+1}, x_{2n+2})) &\preceq \psi(d(x_{2n+1}, x_{2n+2})) \\ \Rightarrow s^\epsilon d(x_{2n+1}, x_{2n+2}) &\preceq d(x_{2n+1}, x_{2n+2}), \end{aligned}$$

which is a contradiction, as such we must have that

$$M(x_{2n}, x_{2n+1}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n}, x_{2n+1}),$$

we then have (2.2) becomes

$$\begin{aligned}\psi(s^{\epsilon}d(x_{2n+1},x_{2n+2})) &\preceq \psi(d(x_{2n},x_{2n+1}))\\ \Rightarrow s^{\epsilon}d(x_{2n+1},x_{2n+2}) &\preceq d(x_{2n},x_{2n+1}),\\ \Rightarrow d(x_{2n+1},x_{2n+2}) &\preceq \frac{1}{s^{\epsilon}}d(x_{2n},x_{2n+1}).\end{aligned}$$

Inductively, we have that

$$d(x_n,x_{n+1}) \preceq \left(\frac{1}{s^{\epsilon}}\right)^n d(x_0,x_1)$$

$$|d(x_n,x_{n+1})| \leq \left(\frac{1}{s^{\epsilon}}\right)^n |d(x_0,x_1)|.$$

Now for any $m > n$, where $m,n \in \mathbb{N}$, observe that

$$\begin{aligned}d(x_n,x_m) &\preceq sd(x_n,x_{n+1}) + sd(x_{n+1},x_m)\\ &\preceq sd(x_n,x_{n+1}) + s^2d(x_{n+1},x_{n+2}) + s^2d(x_{n+2},x_m)\\ &\preceq sd(x_n,x_{n+1}) + s^2d(x_{n+1},x_{n+2}) + s^3d(x_{n+2},x_{n+3}) + s^3d(x_{n+3},x_m)\\ &\preceq sd(x_n,x_{n+1}) + s^2d(x_{n+1},x_{n+2}) + s^3d(x_{n+2},x_{n+3})\\ &+ \cdots + s^{m-n}d(x_{m-1},x_m)\\ &\preceq s\left(\frac{1}{s^{\epsilon}}\right)^n d(x_0,x_1) + s^2\left(\frac{1}{s^{\epsilon}}\right)^{n+1} d(x_0,x_1) + s^3\left(\frac{1}{s^{\epsilon}}\right)^{n+2} d(x_0,x_1)\\ &+ \cdots + s^{m-n}\left(\frac{1}{s^{\epsilon}}\right)^{m-1} d(x_0,x_1)\\ &\preceq s\left(\frac{1}{s^{\epsilon}}\right)^n \left[1 + \left(\frac{1}{s^{\epsilon-1}}\right) + \cdots + \left(\frac{1}{s^{\epsilon-1}}\right)^{m-n-1}\right] d(x_0,x_1)\\ &\preceq s\left(\frac{1}{s^{\epsilon}}\right)^n \left(\frac{s^{\epsilon-1}}{s^{\epsilon-1}-1}\right) d(x_0,x_1).\end{aligned}$$

We then have

$$|d(x_n,x_m)| \leq s\left(\frac{1}{s^{\epsilon}}\right)^n \left(\frac{s^{\epsilon-1}}{s^{\epsilon-1}-1}\right)|d(x_0,x_1)|,$$

since $s > 1, \epsilon \geq 1$, taking limit as $m,n \to \infty$, we have

$$\lim_{n,m\to\infty} |d(x_n,x_m)| = 0. \tag{2.3}$$

Thus, then sequence $\{x_n\}$ is a complex valued b-Cauchy sequence. Since X is complete there exists $x \in X$ such that $\lim_{n\to\infty} |d(x_n,x)| = 0$. We also have $d(x_{n+1},x) \preceq sd(x_{n+1},x_n) + sd(x_n,x)$ using (2.3) and the fact that $\lim_{n\to\infty} |d(x_n,x)| = 0$, we have $\lim_{n\to\infty} |d(x_{n+1},x)| = 0$. Using a similar approach, we have $\lim_{n\to\infty} |d(x_{n+2},x)| = 0$. Now, we claim that

$$\frac{1}{2s}d(x_n,x_{n+1}) \preceq d(x_n,x) \text{ or } \frac{1}{2s}d(x_{n+1},x_{n+2}) \preceq d(x_{n+1},x).$$

Suppose on the contrary that there exists $m \geq 0$, such that

$$\frac{1}{2s}d(x_m,x_{m+1}) \succ d(x_m,x) \text{ or } \frac{1}{2s}d(x_{m+1},x_{m+2}) \succ d(x_{m+1},x). \tag{2.4}$$

Now observe that

$$\begin{aligned} 2sd(x_m, x) &\prec d(x_m, x_{m+1}) \\ &\preceq sd(x_m, x) + sd(x, x_{m+1}) \\ &\Rightarrow sd(x_m, x) \prec sd(x, x_{m+1}). \end{aligned} \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$sd(x_m, x) \prec sd(x, x_{m+1}) \prec \frac{1}{2}d(x_{m+1}, x_{m+2}). \tag{2.6}$$

Since $\frac{1}{2s}d(x_{m+1}, x_{m+2}) \leq d(x_{m+1}, x_{m+2})$, we have that

$$\begin{aligned} \psi(s^\epsilon d(x_{m+1}, x_{m+2})) &= \psi(s^\epsilon d(Tx_m, Sx_{m+1})) \\ &\preceq F(\psi(M(x_m, x_{m+1})), \phi(M(x_m, x_{m+1}))) \\ &\preceq \psi(M(x_m, x_{m+1})) \\ &\preceq \psi(d(x_m, x_{m+1})). \end{aligned} \tag{2.7}$$

We then have that (2.7) becomes

$$\psi(s^\epsilon d(x_{m+1}, x_{m+2})) \preceq \psi(d(x_m, x_{m+1})). \tag{2.8}$$

Using the properties of ψ, we have that

$$s^\epsilon d(x_{m+1}, x_{m+2}) \preceq d(x_m, x_{m+1}). \tag{2.9}$$

Using this fact, (2.4) and (2.5),we have

$$\begin{aligned} s^\epsilon d(x_{m+1}, x_{m+2}) &\preceq d(x_m, x_{m+1}) \\ &\preceq sd(x_m, x) + sd(x, x_{m+1}) \\ &\preceq sd(x, x_{m+1}) + sd(x, x_{m+1}) \\ &\prec \frac{1}{2}d(x_{m+1}, x_{m+2}) + \frac{1}{2}d(x_{m+1}, x_{m+2}) \\ &= d(x_{m+1}, x_{m+2}), \end{aligned} \tag{2.10}$$

which is a contradiction. Thus, we must have that

$$\frac{1}{2}d(x_n, x_{n+1}) \preceq d(x_n, x) \text{ or } \frac{1}{2}d(x_{n+1}, x_{n+2}) \preceq d(x_{n+1}, x).$$

We then have

$$\begin{aligned} &\psi(s^\epsilon d(x_{n+1}, Sx)) \\ &= \psi(s^\epsilon d(Tx_n, Sx)) \\ &\preceq F(\psi(M(x_n, x)), \phi(M(x_n, x))) \\ &\preceq \psi(M(x_n, x)) \\ &= \psi(\max\{d(x_n, x), d(x_n, x_{n+1}), d(x, Sx), \frac{d(x_n, Tx_n)d(x, Sx)}{s + d(x_n, x)}\}) \\ &\preceq \psi(\max\{d(x_n, x), sd(x_n, x) + sd(x, x_{n+1}), d(x, Sx), \frac{d(x_n, Tx_n)d(x, Sx)}{s + d(x_n, x)}\}), \end{aligned}$$

this implies that $s^\epsilon d(x_{n+1}, Sx) \preceq \max\{d(x_n, x), sd(x_n, x)+sd(x, x_{n+1}), d(x, Sx), \frac{d(x_n,Tx_n)d(x,Sx)}{s+d(x_n,x)}\}$ and taking limit as $n \to \infty$, we have

$$|d(x, Sx)| \leq \frac{1}{s^\epsilon}|d(x, Sx)|, \tag{2.11}$$

which is a contradiction, as such we must have that

$$Sx = x. \tag{2.12}$$

More so, we have

$$\begin{aligned}
&\psi(s^{\epsilon}d(Tx, x_{n+2}) \\
&= \psi(s^{\epsilon}d(Tx, Sx_{n+1})) \\
&\preceq F(\psi(M(x, x_{n+1})), \phi(M(x, x_{n+1}))) \\
&\preceq \psi(M(x, x_{n+1})) \\
&= \psi(\max\{d(x, x_{n+1}), d(x, Tx), d(x_{n+1}, Sx_{n+1}), \frac{d(x, Tx)d(x_{n+1}, Sx_{n+1})}{s + d(x_n, x)}\}) \\
&\preceq \psi(\max\{d(x_n, x), d(x_{n+1}, x_{n+2}), d(x, Tx), \frac{d(x, Tx)d(x_{n+1}, Sx_{n+1})}{s + d(x_n, x)}\}),
\end{aligned}$$

this implies that $s^{\epsilon}d(Tx, x_{n+2}) \preceq \max\{d(x_n, x), d(x_{n+1}, x_{n+2}), d(x, Tx), \frac{d(x,Tx)d(x_{n+1},Sx_{n+1})}{s+d(x_n,x)}\}$ and taking limit as $n \to \infty$, we have

$$|d(Tx, x)| \leq \frac{1}{s^{\epsilon}}|d(Tx, x)|, \tag{2.13}$$

which is a contradiction, as such we must have that

$$Tx = x. \tag{2.14}$$

From (2.12) and (2.14), we have that

$$x = Tx = Sx.$$

Hence x is the common fixed point for the pair S and T.

Let x and y be two common fixed points for the pair S and T, such that $x \neq y$. Now observe that

$$\begin{aligned}
\psi(s^{\epsilon}d(x, y)) &= \psi(s^{\epsilon}d(Tx, Sy)) \\
&\preceq F(\psi(M(x, y)), \phi(x, y)) \\
&\preceq \psi(M(x, y)) \\
&= \psi(d(x, y)),
\end{aligned}$$

which implies that $|d(x, y)| < \frac{1}{s^{\epsilon}}|d(x, y)|$, we get a contradiction, as such, we have $x = y$. ∎

Theorem 2.2. *Let (X, d_b) be a complex valued b-metric space with $s > 1$ and T be a self map on X satisfying*

$$\frac{1}{2s}d(x, Tx) \preceq d(x, y) \Rightarrow \psi(s^{\epsilon}d(Tx, Ty) \preceq F(\psi(M(x, y)), \phi(M(x, y))) \tag{2.15}$$

for all $x, y \in X$, where $\epsilon \geq 1, F \in \mathcal{C}, \phi \in \Phi, \psi \in \Psi$ and

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Tx)d(y, Ty)}{s + d(x, y)}\}.$$

Then T has a unique fixed point.

Proof. The prove follows a similar approach as in Theorem 2.1 as such we omit it. ∎

Corollary 2.3. *Let (X, d_b) be a complex valued b-metric space with $s > 1$ and S, T be a self map on X satisfying*

$$\psi(s^\epsilon d(Tx, Sy) \preceq \psi(M(x, y)) - \phi(M(x, y)) \tag{2.16}$$

for all $x, y \in X$, where $\epsilon \geq 1, \phi \in \Phi, \psi \in \Psi$ and

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Tx)d(y, Sy)}{s + d(x, y)}\}.$$

Then the pair S and T have a unique common fixed point.

Corollary 2.4. *Let (X, d_b) be a complex valued b-metric space with $s > 1$ and S, T be a self map on X satisfying*

$$\psi(s^\epsilon d(Tx, Ty) \preceq \psi(M(x, y)) - \phi(M(x, y)) \tag{2.17}$$

for all $x, y \in X$, where $\epsilon \geq 1, \phi \in \Phi, \psi \in \Psi$ and

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Tx)}{s + d(x, y)}\}.$$

Then T has a unique fixed point.

Corollary 2.5. *Let (X, d_b) be a complex valued b-metric space with $s > 1$ and S, T be a self map on X satisfying*

$$\frac{1}{2s}d(x, Sx) \preceq d(x, y) \Rightarrow \psi(s^\epsilon d(Tx, Sy) \preceq k\psi(M(x, y)) \tag{2.18}$$

for all $x, y \in X$, where $k \in (0, 1), \epsilon \geq 1, \psi \in \Psi$ and

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Tx)d(y, Sy)}{s + d(x, y)}\}.$$

Then the pair S and T have a unique common fixed point.

Corollary 2.6. *Let (X, d_b) be a complex valued b-metric space with $s > 1$ and T be a self map on X satisfying*

$$\frac{1}{2s}d(x, Tx) \preceq d(x, y) \Rightarrow \psi(s^\epsilon d(Tx, Ty) \preceq k\psi(M(x, y)) \tag{2.19}$$

for all $x, y \in X$, where $k \in (0, 1), \epsilon \geq 1, \psi \in \Psi$ and

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Tx)d(y, Ty)}{s + d(x, y)}\}.$$

Then T has a unique fixed point.

Corollary 2.7. *Let (X, d_b) be a complex valued b-metric space with $s > 1$ and S, T be a self map on X satisfying*

$$sd(Tx, Sy) \preceq kd(x, y) \tag{2.20}$$

for all $x, y \in X$, where $k \in (0, 1)$. Then the pair S and T have a unique common fixed point.

Corollary 2.8. *Let (X, d_b) be a complex valued b-metric space with $s > 1$ and T be a self map on X satisfying*

$$s^2d(Tx, Ty) \preceq d(x, y) \tag{2.21}$$

for all $x, y \in X$. Then T has a unique fixed point.

3. APPLICATION

3.1. **Application to Riemann-Liouville Equation.** In this section, we establish the existence of a solution of a Riemann-Liouville of the form:

$$^{RL}cI_t^u x(t) = \Gamma(u)\int_c^t (t-s)^{u-1}x(s)ds, \mathcal{R}(u) > 0 \tag{3.1}$$

where $u \in \mathbb{C}, x(t) \in X = C([0,1],\mathbb{R})$ and $t, s \in [0,1]$ which is the fractional integral. Let $X = C([0,1],\mathbb{R})$ be the space of continuous function, and $d : X \times X \to \mathbb{C}$ be defined as

$$d(u,v) = \left[\max_{t\in[0,1]} \|u(t) - v(t)\|\sqrt{1+a^2}e^{i\tan^{-1}a}\right]^2$$

where $s = 2$. It is well-known that (X,d) is a complete complex valued b-metric space. Define $T : X \to X$ by

$$Tx(t) = \Gamma(u)\int_c^t (t-s)^{u-1}x(s)ds. \tag{3.2}$$

Theorem 3.1. $X = C([0,1],\mathbb{R})$ *and suppose that*

$$\left[max_{t\in[0,1]}\frac{1}{\Gamma(u+1)}\frac{(t-s)^{u-1}(t-c)^u}{|(t-s)^{u-1}|}\right]^2 \preceq \frac{1}{4}$$

then Equation 3.1 has a solution.

Proof. It is well-known that $x \in X$ is a fixed point of T if and only if x is a solution of problem (3.1). Note that at some pointl Now observe that for all $u, v \in X$, we have that

$$
\begin{aligned}
& d(Tx, Ty) \\
&= \left[\max_{t\in[0,1]} |Tx(t) - Ty(t)|\sqrt{1+a^2}e^{i\tan^{-1}a}\right]^2 \\
&= \left[\max_{t\in[0,1]} \left|\frac{1}{\Gamma(u)}\int_c^t (t-s)^{u-1}x(s) - \frac{1}{\Gamma(u)}\int_c^t (t-s)^{u-1}y(s)\right|\sqrt{1+a^2}e^{i\tan^{-1}a}\right]^2 \\
&\precsim \left[\max_{t\in[0,1]} \frac{1}{\Gamma(u)}\left|\int_c^t (t-s)^{u-1}ds\right|(|x(s)-y(s)|)\sqrt{1+a^2}e^{i\tan^{-1}a}\right]^2 \\
&\preceq \left[\max_{t\in[0,1]} \frac{1}{\Gamma(u)}\int_c^t |(t-s)^{u-1}ds|(|x(s)-y(s)|)\sqrt{1+a^2}e^{i\tan^{-1}a}\right]^2 \\
&\preceq \left[\max_{t\in[0,1]} \frac{1}{\Gamma(u)}\frac{(t-s)^{u-1}}{|(t-s)^{u-1}|}\int_c^t |(t-s)^{u-1}ds|(|x(s)-y(s)|)\sqrt{1+a^2}e^{i\tan^{-1}a}\right]^2 \\
&\preceq \left[\max_{t\in[0,1]} -\frac{1}{\Gamma(u)}\frac{(t-s)^{u-1}}{|(t-s)^{u-1}|}\int_{t-c}^0 q^{u-1}dq(|x(s)-y(s)|)\sqrt{1+a^2}e^{i\tan^{-1}a}\right]^2, \text{ where q= t-s} \\
&\preceq \left[\max_{t\in[0,1]} -\frac{1}{\Gamma(u)}\frac{(t-s)^{u-1}}{|(t-s)^{u-1}|}\left(\frac{q^u}{u}\right)_{t-c}^0 (|x(s)-y(s)|)\sqrt{1+a^2}e^{i\tan^{-1}a}\right]^2 \\
&= \left[\max_{t\in[0,1]} \frac{1}{\Gamma(u)}\frac{(t-s)^{u-1}}{|(t-s)^{u-1}|}\frac{(t-c)^u}{u}(|x(s)-y(s)|)\sqrt{1+a^2}e^{i\tan^{-1}a}\right]^2 \\
&\preceq \left[\max_{t\in[0,1]} \frac{1}{\Gamma(u+1)}\frac{(t-s)^{u-1}(t-c)^u}{|(t-s)^{u-1}|}(|x(s)-y(s)|)\sqrt{1+a^2}e^{i\tan^{-1}a}\right]^2 \\
&= \left[\max_{t\in[0,1]}(|x(s)-y(s)|)\sqrt{1+a^2}e^{i\tan^{-1}a}\right]^2\left[\max_{t\in[0,1]} \frac{1}{\Gamma(u+1)}\frac{(t-s)^{u-1}(t-c)^u}{|(t-s)^{u-1}|}\right]^2 \\
&\preceq d(x,y)\frac{1}{4} \\
&= d(x,y)\frac{1}{s^2}.
\end{aligned}
$$

Thus, we have that $s^2 d(Tx, Ty) \precsim d(x, y)$. Clearly, all conditions in Corollary 2.8 are satisfied and guarantees the existence of the fixed point $x \in X$. Thus, x is the solution of the integral equation 3.1. ∎

3.2. **Application to Differential Equation.** In this section, we establish the existence of a solution of a differential equation of the form:

$$
\begin{aligned}
& u'(t) = f(t, u(t)), \quad t \in I = [0, 1] \\
& u(0) = u(1),
\end{aligned} \tag{3.3}
$$

where $f : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. It is easy to see that (3.3) is can be rewritten as

$$
\begin{aligned}
& u'(t) + 2u(t) = f(t, u(t)) + 2u(t), \quad t \in I = [0, 1] \\
& u(0) = u(1),
\end{aligned} \tag{3.4}
$$

which is equivalent to

$$u(t) = \int_0^1 G(t,s)[f(s,u(s)) + 2u(s)]ds.$$

The Green function $G(t,s)$ associated with (3.3) is given by

$$G(t,s) = \begin{cases} \frac{e^{2(1+s-t)}}{e^2-1} & \text{if} \quad 0 \le s \le t \le 1 \\ \frac{e^{2(s-t)}}{e^2-1} & \text{if} \quad 0 \le t \le s \le 1. \end{cases}$$

It is easy to see that $\max_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{1}{2}$. Let $X = C([0,1],\mathbb{R}^n)$ be the space of continuous function, $u : [0,1] \to \mathbb{R}^n$ and $\|(u_1, u_2, \cdots, u_n)\| = \max\{|u_1|, |u_2|, \cdots, |u_n|\}$ and $d : X \times X \to \mathbb{C}$ be defined as

$$d(u,v) = \left[\max_{t\in[0,1]} \|u(t) - v(t)\| \sqrt{1+a^2} e^{i\tan^{-1} a}\right]^2$$

where $s = 2$. It is well-known that (X,d) is a complete complex valued b-metric space. Define $T : X \to X$ as

$$Tu(t) = \int_0^1 G(t,s)[f(s,u(s)) + 2u(s)]ds$$

Theorem 3.2. $X = C([0,1],\mathbb{R}^n)$ *and suppose that*

$$\|f(t,u) + 2u(s) - f(t,v) + 2v(s)\| \precsim \|u(s) - v(s)\|, \tag{3.5}$$

then Equation 3.3 has a solution.

Proof. It is well-known that $u \in X$ is a fixed point of T if and only if u is a solution of problem (3.3). Now observe that for all $u, v \in X$, we have that

$$\begin{aligned} & d(Tu,Tv) \\ &= \left[\max_{t\in[0,1]} \|Tu(t) - Tv(t)\| \sqrt{1+a^2} e^{i\tan^{-1} a}\right]^2 \\ &= \left[\max_{t\in[0,1]} \left\| \int_0^1 G(t,s)[f(s,u(s)) + 2u(s) - f(s,v(s)) - 2v(s)]ds \right\| \sqrt{1+a^2} e^{i\tan^{-1} a}\right]^2 \\ &\precsim \left[\max_{t\in[0,1]} \int_0^1 G(t,s)|u(s) - v(s)|ds \sqrt{1+a^2} e^{i\tan^{-1} a}\right]^2 \\ &= \left[\max_{t\in[0,1]} |u(s) - v(s)| \sqrt{1+a^2} e^{i\tan^{-1} a}\right]^2 \left(\max_{t\in[0,1]} \int_0^1 G(t,s)ds\right)^2 \\ &= d(u,v)\frac{1}{4} \\ &= d(u,v)\frac{1}{s^2}. \end{aligned}$$

Thus, we have that $s^2 d(Tu,Tv) \precsim d(u,v)$. Clearly, all conditions in Corollary 2.8 are satisfied and guarantees the existence of the fixed point $x \in X$. Thus, x is the solution of the integral equation 3.3. ∎

REFERENCES

[1] C. T. AAGE and J. N .SALUNKE, Fixed points for weak contractions in G-metric spaces, *A*ppl. Math. E-Notes, **12** (2012), pp. 23–28.

[2] H. A. ABASS, A. A. MEBAWONDU and O.T. MEWOMO, Some results for a new three steps iteration scheme in Banach spaces, *Bull. Transilv. Univ. Brasov Ser. III*, **11** (2018), pp. 1–18.

[3] A. AGHAJANI, M. ABBAS and J. ROSHAN, Common fixed points of generalized weak contractive mappings in partially ordered G_b-metric spaces, *Filomat*, **28** (2014), pp. 1087–1101.

[4] A. H. ANSARI, Note on φ-ψ-contractive type mappings and related fixed point, *P*roceedings of the 2nd Regional Conference on Mathematics and Applications, Payame Noor University, Tonekabon, Iran, (2014), pp. 377–380.

[5] A. AZAM, B. FiISHER and M. KHAN, Common fixed point theorems in complex valued metric spaces, *Numer. Funct. Anal. Optim.*, **32** (2011), pp. 243–253.

[6] G. V. R. BABU and T. M. DULA, Fixed points of generalized TAC-contractiv mappings in b-metric spaces, *Matematicki Vesnik*, **69** (2017), pp. 75–88.

[7] G. V. R. BABU and P. D. SAILAJA, A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces, *Thai J. of Math.*, **9** (2011), pp. 1–10.

[8] S. BANACH, Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales, *Fundamenta Mathematicae*, **3** (1922), pp. 133–181.

[9] V. BERINDI, Approximating fixed points of weak contractions using the Picard iteration, *Nonlinear Anal. Forum*, **9** (2004), pp. 43–53.

[10] V. BERINDI, General constructive fixed point theorem for Ciric-type almost contractions in metric spaces, *Carpath. J. Math.*, **24** (2008), pp. 10–19.

[11] M. BORICEANU, M. BOTA and A. PETRUSEL, Mutivalued fractals in b-metric spaces, *Cent. Eur. J. Math.*, **8** (2010), pp. 367–377.

[12] S. CHANDOK, K. TAS, A and H. ANSARI, Some fixed point results for TAC-type contractive mappings, *J. Function Spaces*, (2016) Article ID 1907676, pp. 1–6.

[13] S. CZERWIK, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostraviensis*, **1** (1993), pp. 5–11.

[14] D. S. JAGGI, Some unique fixed point theorems, *Indian J. Pure and Appl. Math.*, **8** (1977), pp. 223-230.

[15] G. JUNGCK, Commuting mappings and fixed points, *Amer. Math. Monthly*, 83 (4) (1976), pp. 261–263.

[16] G. JUNGCK and N. HUSSAIN, Compatible maps and invariant approximations, *J. Math. Anal. Appl.*, **325** (2007), pp. 1003-1012.

[17] N. HUSSAIN, M. A. KUTBI and P. SALIMI, Fixed point theory in α-complete metric spaces with applications, *Abstr. Appl. Anal.*, textbf2014 (2014), Article ID 280817.

[18] J. K. P. R. RAO, P. R. SWAMY and J. R. PRASAD, A common fixed point theorem in complex valued b-metric spaces, *Bull. Maths. Stat. res.*, **1** (2013), pp. 1–8.

[19] J. R. ROSHAN, V. PARVANEH and Z. KADELBURG, Common fixed point theorems for weakly isotone increasing mappings in ordered b-metric spaces, *J. Nonlinear Sci. Appl.*, **7** (2014), pp. 229-245.

[20] P. SALIMI, A. LATIF and N. HUSSAIN, Modified α-ψ-contraction mapping with applications, *Fixed Point Theory Appl.*, **2013** (2013), pp. 1–19.

[21] P. SALIMI and V. PASQUALE, A result of Suzuki type in partial G-metric spaces, *Acta Math. Sci. Ser. B (Engl. Ed.)*, **34** (2014), pp. 274–284.

[22] B. SAMET, C. VETRO and P. VETRO, Fixed point theorem for α-ψ-contractive type mappings, *Nonlinear Anal.*, **75** (2012), pp. 2154–2165.

[23] S. L. SINGH, R. KAMAL, M. DE LA SEN and R. CHUGH, A fixed point theorem for generalized weak contractions, *Filomat*, **29** (2015), pp. 1481–1490.

[24] W. SINTUNAVARAT, Nonlinear integral equations with new admissibility types in b-metric spaces, *J. Fixed Point Theory Appl.*, DOI 10.1007/s11784-015-0276-6, pp. 1–20.

[25] W. SINTUNAVARAT, Y. J. CHO and P. KUMAM, Urysohn integral equations approach by common fixed point in complex valued metric spaces, *J. Adv. Difference Equ.* **49** (2013), pp. 1–14.

[26] W. SINTUNAVARAT and P. KUMAM, Generalized common fixed point theorems in complex valued metric spaces and applications, *J. Inequal. Appl.*, **84** (2012), pp. 1–12.

[27] W. SINTUNAVARAT, Z. M. BAHADUR and M. SARWAR, Common solution of Urysohn integral equations with the help of common fixed point results in complex valued metric spaces, *Rev. R. Acad. Cienc. Exactas F'is. Nat. Ser. A Math. RACSAM*, **111** (2017), pp. 531–545.

[28] T. SUZUKI, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, *J. Math. Anal. Appl.*, **340** (2008), pp. 1088–1095.

[29] T. SUZUKI, A new type of fixed point theorem in metric spaces, *Nonlinear Anal.*, **71** (2009), pp. 5313–5317.

[30] O. YAMAODA and W. SINTUNAVARAT, Fixed point theorems for $(\alpha, \beta) - (\psi, \varphi)$-contractive mappings in b-metric spaces with some numerical results and applications, *J. Nonlinear Sci. Appl.*, **9** (2016), pp. 22–33.