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GENERALIZED VON NEUMANN-JORDAN CONSTANT FOR MORREY SPACES AND SMALL MORREY SPACES

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ABSTRACT. In this paper we calculate some geometric constants for Morrey spaces and small Morrey spaces, namely generalized Von Neumann-Jordan constant, modified Von Neumann-Jordan constants, and Zbáganu constant. All these constants measure the uniformly nonsquareness of the spaces. We obtain that their values are the same as the value of Von Neumann-Jordan constant for Morrey spaces and small Morrey spaces.

Key words and phrases: Generalized Von Neumann-Jordan constant; Morrey spaces; Small Morrey spaces.

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1. INTRODUCTION

We shall discuss some geometric constants for Banach spaces. Three geometric constants have been studied by Gunawan *et al.* [7], but there are other geometric constants which were introduced by other authors. Inspired by Clarkson [2], who introduced the Von Neumann-Jordan constant for a Banach space $(X, \|\cdot\|_X)$, Cui *et al.* [3] defined the generalized Von Neumann-Jordan constant $C_{NJ}^{(s)}(X)$ by

$$C_{NJ}^{(s)}(X) := \sup\left\{\frac{\|x+y\|_X^s + \|x-y\|_X^s}{2^{s-1}(\|x\|_X^s + \|y\|_X^s)} : x, y \in X\backslash\{0\}\right\},$$

for $1 \leq s < \infty$. Observe that $1 \leq C_{NJ}^{(s)}(X) \leq 2$ for any Banach space X.

Several years earlier, Alonso *et al.* [1] and Gao [4] studied the modified Von Neumann-Jordan constant, which is defined by

$$C'_{NJ}(X) := \sup\left\{\frac{\|x+y\|_X^2 + \|x-y\|_X^2}{4} : x, y \in X, \|x\|_X = \|y\|_X = 1\right\}.$$

Note that $1 \leq C'_{NJ}(X) \leq C_{NJ}(X) \leq 2$ is for any Banach space X. This constant was generalized by Yang *et al.* [12] to the following constant

$$\bar{C}_{NJ}^{(s)}(X) := \sup\left\{\frac{\|x+y\|_X^s + \|x-y\|_X^s}{2^s} : x, y \in X, \|x\|_X = \|y\|_X = 1\right\}.$$

It is proved in [12] that $\bar{C}_{NJ}^{(s)}(X) \leq C_{NJ}^{(s)}(X) \leq 2^{s-1}\left[1 + \left(2^{\frac{1}{s}}\left(C'_{NJ}(X)\right)^{\frac{1}{s}} - 1\right)\right]^{s-1}$ for any Banach space X.

Beside the above constants, we also know a constant called Zbáganu constant, which is defined by

$$C_Z(X) := \sup\left\{\frac{\|x+y\|_X\|x-y\|_X}{\|x\|_X^2 + \|y\|_X^2} : x, y \in X\backslash\{0\}\right\}.$$

This constant was introduced by Zbáganu [13] and studied further in [8, 9]. It is easy to see that $1 \leq C_Z(X) \leq C_{NJ}(X) \leq 2$ for any Banach space X.

In this paper, we want to calculate the values of those constants for Morrey spaces as well as for small Morrey spaces.

Let $1 \leq p \leq q < \infty$. The Morrey spaces $\mathcal{M}_q^p = \mathcal{M}_q^p(\mathbb{R}^n)$ is the set of all measurable function f such that

$$\|f\|_{\mathcal{M}_q^p} := \sup_{a\in\mathbb{R}^n, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}}\left(\int_B |f(x)|^p dx\right)^{\frac{1}{p}}$$

is finite. Here $B(a,r) = \{x \in \mathbb{R}^n : |x-a| < r\}$ dan $|B(a,r)|$ denotes its Lebesgue measure. Since $\mathcal{M}_q^p$ is a Banach space, it follows from [5] that

$$C_{NJ}^{(s)}(\mathcal{M}_q^p),\ C'_{NJ}(\mathcal{M}_q^p),\ \bar{C}_{NJ}^{(s)}(\mathcal{M}_q^p),\ C_Z(\mathcal{M}_q^p) \leq 2.$$

Meanwhile, the small Morrey $m_q^p = m_q^p(\mathbb{R}^n)$ is the set of all measurable function f such that

$$\|f\|_{m_q^p} := \sup_{a\in\mathbb{R}^n, r\in(0,1)} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}}\left(\int_B |f(x)|^p dx\right)^{\frac{1}{p}}$$

is finite. Small Morrey spaces are also Banach spaces [11] and, for all p and q, the small Morrey spaces properly contain the Morrey spaces.

The values of Von Neumann-Jordan constants for Morrey spaces and small Morrey spaces have been computed by Gunawan *et al.* [7] and Mu'tazili and Gunawan in [10], respectively.

In this paper, we want to calculate other geometric constants defined above. Our results are presented in the following sections.

2. MAIN RESULTS

The values of the geometric constants defined in the previous section for Morrey spaces and small Morrey spaces are presented in the following theorems.

Theorem 2.1. *For* $1 \leq p < q < \infty$ *and* $1 \leq s < \infty$*, we have*

$$C_{NJ}^{(s)}(\mathcal{M}_q^p) = C'_{NJ}(\mathcal{M}_q^p) = \bar{C}_{NJ}^{(s)}(\mathcal{M}_q^p) = C_Z(\mathcal{M}_q^p) = 2.$$

Proof. To prove the theorem, we consider the following functions: $f(x) = |x|^{-\frac{n}{q}}$, $g(x) = \chi_{(0,1)}(|x|)f(x)$, $h(x) = f(x) - g(x)$, and $k(x) = g(x) - h(x)$, where $x \in \mathbb{R}^n$. All functions are radial functions. Note that $f \in \mathcal{M}_q^p$, with

$$\|f\|_{\mathcal{M}_q^p} = c_n \left(1 - \frac{p}{q}\right)^{-\frac{1}{p}}.$$

One may also observe that

$$\|f\|_{\mathcal{M}_q^p} = \|g\|_{\mathcal{M}_q^p} = \|h\|_{\mathcal{M}_q^p} = \|k\|_{\mathcal{M}_q^p}.$$

It thus follows that

$$\begin{aligned} C_{NJ}^{(s)}(\mathcal{M}_q^p) &\geq \frac{\|f+k\|_{\mathcal{M}_q^p}^s + \|f-k\|_{\mathcal{M}_q^p}^s}{2^{s-1}(\|f\|_{\mathcal{M}_q^p}^s + \|k\|_{\mathcal{M}_q^p}^s)} \\ &= \frac{\|2g\|_{\mathcal{M}_q^p}^s + \|2h\|_{\mathcal{M}_q^p}^s}{2^{s-1}\left(\|f\|_{\mathcal{M}_q^p}^s + \|f\|_{\mathcal{M}_q^p}^s\right)} \\ &= \frac{2^s\|g\|_{\mathcal{M}_q^p}^s + 2^s\|h\|_{\mathcal{M}_q^p}^s}{2^{s-1}\left(2\|f\|_{\mathcal{M}_q^p}^s\right)} \\ &= 2. \end{aligned}$$

Since $C_{NJ}^{(s)}(\mathcal{M}_q^p) \leq 2$, we can conclude that $C_{NJ}^{(s)}(\mathcal{M}_q^p) = 2$.

Now, for the modified Von Neumann-Jordan constant, we consider $\frac{f}{\|f\|_{\mathcal{M}_q^p}}$ and $\frac{k}{\|k\|_{\mathcal{M}_q^p}} = \frac{k}{\|f\|_{\mathcal{M}_q^p}}$. We have

$$\begin{aligned} C'_{NJ}(\mathcal{M}_q^p) &\geq \frac{\|f+k\|_{\mathcal{M}_q^p}^2 + \|f-k\|_{\mathcal{M}_q^p}^2}{4\|f\|_{\mathcal{M}_q^p}^2} \\ &= \frac{\|2g\|_{\mathcal{M}_q^p}^2 + \|2h\|_{\mathcal{M}_q^p}^2}{4\|f\|_{\mathcal{M}_q^p}^2} \\ &= \frac{4\left(\|g\|_{\mathcal{M}_q^p}^2 + \|h\|_{\mathcal{M}_q^p}^2\right)}{4\|f\|_{\mathcal{M}_q^p}^2} \\ &= 2. \end{aligned}$$

This shows that $C'_{NJ}(\mathcal{M}_q^p) = 2$.

Similarly, for generalized modified Von Neumann-Jordan constant, we have

$$\begin{aligned}\bar{C}_{NJ}^{(s)}(\mathcal{M}_q^p) &\geq \frac{\|f+k\|_{\mathcal{M}_q^p}^s + \|f-k\|_{\mathcal{M}_q^p}^s}{2^s\|f\|_{\mathcal{M}_q^p}^s} \\ &= \frac{\|2g\|_{\mathcal{M}_q^p}^s + \|2h\|_{\mathcal{M}_q^p}^s}{2^s\|f\|_{\mathcal{M}_q^p}^s} \\ &= \frac{2^s\|g\|_{\mathcal{M}_q^p}^s + 2^s\|h\|_{\mathcal{M}_q^p}^s}{2^s\|f\|_{\mathcal{M}_q^p}^s} \\ &= 2.\end{aligned}$$

This also leads us to the conclusion that $\bar{C}_{NJ}^{(s)}(\mathcal{M}_q^p) = 2$.

Last, for the Zbáganu constant, we have

$$\begin{aligned}C_Z(\mathcal{M}_q^p) &\geq \frac{\|f+k\|_{\mathcal{M}_q^p}\|f-k\|_{\mathcal{M}_q^p}}{\|f\|_{\mathcal{M}_q^p}^2 + \|k\|_{\mathcal{M}_q^p}^2} \\ &= \frac{\|2g\|_{\mathcal{M}_q^p}\|2h\|_{\mathcal{M}_q^p}}{2\|f\|_{\mathcal{M}_q^p}^2} \\ &= 2,\end{aligned}$$

which implies that $C_Z(\mathcal{M}_q^p) = 2$, as desired. ∎

For small Morrey spaces, we have the following theorem.

Theorem 2.2. *Let* $1 \leq p < q < \infty$ *and* $1 \leq s < \infty$. *Then*

$$C_{NJ}^{(s)}(m_q^p) = C'_{NJ}(m_q^p) = \bar{C}_{NJ}^{(s)}(m_q^p) = C_Z(m_q^p) = 2.$$

Proof. The idea of the proof is similar to that of Theorem 2.1, but we use different functions. For $\varepsilon \in (0,1)$, we consider $f(x) = \chi_{(0,1)}(|x|)|x|^{-\frac{n}{q}},\ g(x) = \chi_{(0,\varepsilon)}(|x|)f(x),\ h(x) = f(x) - g(x)$, and $k(x) = g(x) - h(x)$, where $x \in \mathbb{R}^n$. Here g depends on ε, so that h and k also depend on ε. Since all functions are radial functions, it is not hard to compute their norms. First, we obtain that

$$\|f\|_{m_q^p} = \|g\|_{m_q^p} = \|k\|_{m_q^p} = c_n\left(1 - \frac{p}{q}\right)^{-\frac{1}{p}}.$$

Next, we observe that

$$\begin{aligned}
\|h\|_{m_q^p} &= \sup_{a\in\mathbb{R}^n, r\in(0,1)} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,r)} |f(x)\left(1-\chi_{(0,1)}(|x|)\right)|^p dx\right)^{\frac{1}{p}} \\
&\geq \sup_{r\in(0,1)} Cr^{n\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\int_{B(0,r)} |x|^p \left(1-\chi_{(0,1)}(|x|)\right) dx\right)^{\frac{1}{p}} \\
&= \sup_{r\in(\varepsilon,1)} Cr^{n\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\int_{B(\varepsilon,r)} |x|^p dx\right)^{\frac{1}{p}} \\
&= \sup_{r\in(\varepsilon,1)} Cr^{n\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\int_{\varepsilon}^{r} r^{\frac{-np}{q}} r^{n-1} dr\right)^{\frac{1}{p}} \\
&= \sup_{r\in(\varepsilon,1)} C\left[n\left(1-\frac{p}{q}\right)\right]^{-\frac{1}{p}} \left(1-r^{\frac{np}{q}-n}\varepsilon^{n-\frac{np}{q}}\right)^{\frac{1}{p}} \\
&= \|f\|_{m_q^p}\left(1-\varepsilon^{n-\frac{np}{q}}\right)^{\frac{1}{p}}.
\end{aligned}$$

We can now calculate the constants. First, let us observe the generalized Von Neumann-Jordan constant:

$$\begin{aligned}
C_{NJ}^{(s)}(m_q^p) &\geq \frac{\|f+k\|_{m_q^p}^s + \|f-k\|_{m_q^p}^s}{2^{s-1}\left(\|f\|_{m_q^p}^s + \|k\|_{m_q^p}^s\right)} \\
&= \frac{\|2g\|_{m_q^p}^s + \|2h\|_{m_q^p}^s}{2^{s-1}\left(2\|f\|_{m_q^p}^s\right)} \\
&\geq \frac{2^s\|f\|_{m_q^p}^s + 2^s\|f\|_{m_q^p}^s\left(1-\varepsilon^{n-\frac{np}{q}}\right)^{\frac{s}{p}}}{2^s\|f\|_{m_q^p}^s} \\
&= 1 + (1-\varepsilon^{n-\frac{np}{q}})^{\frac{s}{p}}.
\end{aligned}$$

Since we may choose ε to be arbitrary small, we conclude that $C_{NJ}^{(s)}(m_q^p) = 2$ (for we know that the constant cannot be larger than 2).

We now move to the modified Von Neumann-Jordan constant. Noting that $\|f\|_{m_q^p} = \|k\|_{m_q^p}$, we consider $\frac{f}{\|f\|_{m_q^p}}$ and $\frac{k}{\|f\|_{m_q^p}}$. We have

$$\begin{aligned}
C'_{NJ}(m_q^p) &\geq \frac{\|f+k\|_{m_q^p}^2 + \|f-k\|_{m_q^p}^2}{4\|f\|_{m_q^p}^2} \\
&= \frac{\|2g\|_{m_q^p}^2 + \|2h\|_{m_q^p}^2}{4\|f\|_{m_q^p}^2} \\
&\geq \frac{4\|f\|_{m_q^p}^2 + 4\|f\|_{m_q^p}^2\left(1-\varepsilon^{n-\frac{np}{q}}\right)^{\frac{2}{p}}}{4\|f\|_{m_q^p}^2} \\
&= 1 + (1-\varepsilon^{n-\frac{np}{q}})^{\frac{2}{p}}.
\end{aligned}$$

By using similar arguments as above, we conclude that $C'_{NJ}(m_q^p) = 2$.

For the generalization of the modified Von Neumann-Jordan constant, we observe that

$$\begin{aligned}
\bar{C}_{NJ}^{(s)}(m_q^p) &\geq \frac{\|f+k\|_{m_q^p}^s + \|f-k\|_{m_q^p}^s}{2^s\|f\|_{m_q^p}^s} \\
&= \frac{\|2g\|_{m_q^p}^s + \|2h\|_{m_q^p}^s}{2^s\|f\|_{m_q^p}^s} \\
&\geq \frac{2^s\|f\|_{m_q^p}^s + 2^s\|f\|_{m_q^p}^s \left(1-\varepsilon^{n-\frac{np}{q}}\right)^{\frac{s}{p}}}{2^s\|f\|_{m_q^p}^s} \\
&= 1 + (1-\varepsilon^{n-\frac{np}{q}})^{\frac{s}{p}}.
\end{aligned}$$

With the same arguments as above, we conclude that $\bar{C}_{NJ}^{(s)}(m_q^p) = 2$.

Last, for the Zbáganu constant, we have

$$\begin{aligned}
C_Z(m_q^p) &\geq \frac{\|f+k\|_{m_q^p}\|f-k\|_{m_q^p}}{\|f\|_{m_q^p}^2 + \|k\|_{m_q^p}^2} \\
&= \frac{4\|g\|_{m_q^p}\|h\|_{m_q^p}}{2\|f\|_{m_q^p}^2} \\
&\geq \frac{2\|f\|_{m_q^p}^2 \left(1-\varepsilon^{n-\frac{np}{q}}\right)^{\frac{1}{p}}}{\|f\|_{m_q^p}^2} \\
&= 2\left(1-\varepsilon^{n-\frac{np}{q}}\right)^{\frac{1}{p}},
\end{aligned}$$

By the same arguments, we conclude that $C_Z(m_q^p) = 2$. ∎

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