Aust. J. Math. Anal. Appl.
Vol. **18** (2021), No. 1, Art. 16, 23 pp.
AJMAA

POINTWISE CONVERGENCE OF FOURIER-TYPE SERIES WITH EXPONENTIAL WEIGHTS

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Received 8 December, 2016; accepted 3 December, 2020; published 20 April, 2021.

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ABSTRACT. Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in C^1(\mathbb{R}) : \mathbb{R} \to [0, \infty)$ be an even function. We consider the exponential weights $w(x) = e^{-Q(x)}$, $x \in \mathbb{R}$. In this paper we obtain a pointwise convergence theorem for the Fourier-type series with respect to the orthonormal polynomials $\{p_n(w^2; x)\}$.

Key words and phrases: Exponential weights, Partial sum of Fourier-type series.

2010 *Mathematics Subject Classification.* 42A20 .

ISSN (electronic): 1449-5910

1. INTRODUCTION AND THEOREMS

Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in C^1(\mathbb{R}) : \mathbb{R} \to [0, \infty)$ be an even function. We consider the weights $w(x) := \exp(-Q(x))$. Then we suppose that $\int_0^\infty x^n w^2(x)dx < \infty$ for all $n = 0, 1, 2, \ldots$. First we need the following definition from [4]. We say that $f : \mathbb{R} \to \mathbb{R}^+$ is quasi-increasing if there exists $C > 0$ such that $f(x) \leq Cf(y), 0 < x < y$.

Definition 1.1. Let $Q : \mathbb{R} \to \mathbb{R}^+$ be a continuous even function satisfying the following properties:
(a) $Q'(x)$ is continuous in $\mathbb{R}$ and $Q(0) = 0$.
(b) $Q''(x)$ exists and is positive in $\mathbb{R} \setminus \{0\}$.
(c) $\lim_{x\to\infty} Q(x) = \infty$.
(d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0$$

is quasi-increasing in $(0, \infty)$ with

$$T(x) \geq \Lambda > 1, \quad x \in \mathbb{R}^+ \setminus \{0\}.$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{|Q'(x)|}{Q(x)}, \quad a.e. \quad x \in \mathbb{R} \setminus \{0\}.$$

Furthermore, if there exists a compact subinterval $J (\ni 0)$ of $\mathbb{R}$ and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{|Q'(x)|}{Q(x)}, \quad a.e. \quad x \in \mathbb{R} \setminus J,$$

then we write $w = \exp(-Q) \in \mathcal{F}(C^2+)$. If $T(x)$ is bounded, then w is called the Freud-type weight, and if $T(x)$ is unbounded, then w is called the Erdös-type weight.

A typical example in $\mathcal{F}(C^2+)$ is given as follows:

Example 1.1 ([4, Example 2] and [1, Theorem 3.1])**.** (a) ([4]) For $\alpha > 1$ and a non-negative integer ℓ, we put

$$Q(x) = Q_{\ell,\alpha}(x) := \exp_\ell(|x|^\alpha) - \exp_\ell(0),$$

where for $\ell \geq 1$,

$$\exp_\ell(x) := \exp(\exp(\exp(\cdots \exp x)\ldots)) \quad (\ell\text{-times})$$

and $\exp_0(x) := x$.
(b) ([1]) More precisely, we define for $\alpha + m > 1$, $m \geqslant 0$, $l \geq 1$ and $\alpha \geqslant 0$

$$Q_{l,\alpha,m}(x) = |x|^m \{\exp_l(|x|^\alpha) - \alpha^* \exp_l(0)\},$$

where $\alpha^* = 0$ if $\alpha = 0$, otherwise $\alpha^* = 1$. We note that $Q_{l,0,m}$ gives a Freud-type weight.
(c) ([1]) We define

$$Q_\alpha(x) = (1 + |x|)^{|x|^\alpha} - 1, \quad \alpha > 1.$$

We construct the orthonormal polynomials $p_n(x) = p_n(w^2, x)$ of degree n for $w^2(x)$, that is,

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)w^2(x)dx = \delta_{mn} \quad (\text{Kronecker delta}).$$

Let $fw \in L_1(\mathbb{R})$. The Fourier series of f is defined by

$$\tilde{f}(x) := \sum_{k=0}^{\infty} a_k f p_k(x), \quad a_k(f) := \int_{-\infty}^{\infty} f(t)p_k(t)w^2(t)dt.$$

We denote the partial sum of $\tilde{f}(x)$ by

$$s_n(f,x) := s_n(w^2, f, x) := \sum_{k=0}^{n-1} a_k(f) p_k(x).$$

The partial sum $s_n(f,x)$ admits the representation

$$s_n(f,x) = \int_{-\infty}^{\infty} f(t) K_n(x,t) w^2(t) dt,$$

where

$$K_n(x,t) = \sum_{k=0}^{n-1} p_k(x) p_k(t).$$

Since

$$\int_{-\infty}^{\infty} K_n(x,t) w^2(t) dt = 1,$$

we have

$$s_n(f,x) - f(x) = \int_{-\infty}^{\infty} K_n(x,t)(f(t) - f(x)) w^2(t) dt. \tag{1.1}$$

The Christoffel-Darboux formula asserts that

$$K_n(x,t) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)p_{n-1}(t) - p_{n-1}(x)p_n(t)}{x-t}, \quad p_n(x) =: \gamma_n x^n + \dots. \tag{1.2}$$

In this paper we will show a pointwise convergence for the partial sum $s_n(f,x)$ of $\tilde{f}(x)$. Let $g : \mathbb{R} \to \mathbb{R}$ be a function having bounded variation on every compact interval. The measure introduced by g on Borel subset of $\mathbb{R}$ will be denoted by $|dg|$. For any interval (finite or infinite) I, we define

$$V_\delta(I,g) := \int_I w^\delta(t) |dg(t)|,$$

where $0 < \delta \leq 1$ is fixed. Let $T(x)$ be defined by Definition 1.1, and be also continuous at $x = 0$. Let $\mathcal{B}_\delta(T^{1/4})$ denote the class of all functions g satisfying that $T^{1/4}g$ has bounded variation on $\mathbb{R}$, that is, $V_\delta(\mathbb{R}, T^{1/4}g) < \infty$. We need the Mhaskar-Rakhmanov-Saff numbers a_x;

$$x = \frac{2}{\pi} \int_0^1 \frac{a_x u Q'(a_x u)}{(1-u^2)^{1/2}} du, \quad x > 0.$$

Mhaskar [5] got the following pointwise convergence theorem. If w is a Freud-type weight, then we write $\mathcal{B}_1 := \mathcal{B}_1\left(T^{1/4}\right)$, and if w is an Erdös-type weight, then we let $0 < \delta < 1$.

Mhaskar Theorem ([5, Theorem 9.1.2]**).** Let $w = \exp(-Q)$ be a Freud-type weight such that Q'' is increasing on $(0,\infty)$, $f \in \mathcal{B}_1$, and let x be a point of continuity of f. Then for $n \geq cxQ'(x)$,

$$\begin{aligned} & |s_n(f,x) - f(x)| \\ \leq \; & C \exp\left(cxQ'(x)\right) \left\{ \frac{1}{n} \sum_{k=1}^{n} V_1\left(\left[x - \frac{a_n}{k}, x + \frac{a_n}{k}\right], f\right) + \int_{|u| \geq c_1 a_n} w(u) |df(u)| \right\}. \end{aligned}$$

In particular, the sequence $\{s_n(f,x)\}$ converges to $f(x)$.

We extend Mhaskar Theorem to the case of $w = \exp(-Q) \in \mathcal{F}(C^2+)$ as follows:

Theorem 1.1. *Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$, and let $T(x)$ be continuous at $x = 0$. Let $f \in \mathcal{B}_\delta\left(T^{1/4}\right)$. When x is a point of continuity of f, there exist C, c_1 and $\beta > 0$ such that $|x| \le a_{dn}/6$, where d is defined in (2.11) below.*

$$\begin{aligned}
&|s_n(f,x) - f(x)| \\
&\le C \exp\left(Q(x)\right)\exp\left(\beta x Q'(x)\right)\left\{V_1\left(\left[x - \frac{a_n}{n}, x + \frac{a_n}{n}\right], f\right)\right. \\
&\left. + \frac{1}{n}\sum_{k=1}^{n} V_\delta\left(\left[x - \frac{a_n}{k}, x + \frac{a_n}{k}\right], f\right) + \frac{1}{n}\int_{|u| \ge ca_n} w(u)\left|d\left(T^{1/4}f\right)(u)\right|\right\}.
\end{aligned} \tag{1.3}$$

Hence we have

(1.4)

$$\begin{aligned}
|s_n(f,x) - f(x)| &\le C \exp\left(Q(x)\right)\exp\left(\beta x Q'(x)\right)\left\{V_1\left(\left[x - \frac{a_n}{n}, x + \frac{a_n}{n}\right], f\right)\right. \\
&+ \sqrt{\frac{a_n}{n}} V_\delta\left(\left[x - a_n, x + a_n\right], f\right) + V_\delta\left(\left[x - \sqrt{\frac{a_n}{n}}, x + \sqrt{\frac{a_n}{n}}\right], f\right) \\
&\left. + \frac{1}{n}\int_{|u| \ge ca_n} w(u)\left|d\left(T^{1/4}f\right)(u)\right|\right).
\end{aligned}$$

In particular, the sequence $\{s_n(f,x)\}$ converges to $f(x)$.

For any nonzero real valued functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if there exist the constants $C_1, C_2 > 0$ independent of x such that $C_1 g(x) \le f(x) \le C_2 g(x)$ for all x. Similarly, for any two sequences of positive numbers $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{=1}^{\infty}$ we define $c_n \sim d_n$. Throughout this paper $C, C_1, C_2, ...$ denote positive constants independent of n, x, t or polynomials $P_n(x)$. The same symbol does not necessarily denote the same constant in different occurences.

2. LEMMAS

To prove Theorem 1.1, we need some lemmas. In this paper we treat $w = \exp(-Q) \in \mathcal{F}(C^2+)$.

Lemma 2.1. (1) [4, Lemma 3.5 (3.27)-(3.29)] *For fixed $L > 0$ and uniformly for $t > 0$,*

$$a_{Lt} \sim a_t, \qquad T(a_{Lt}) \sim T(a_t) \quad \text{and} \quad Q^{(j)}(a_{Lt}) \sim Q^{(j)}(a_t), \quad j = 0, 1.$$

(2) [4, Lemma 3.4 (3.18),(3.17), Lemma 3.8 (3.42)] *For $t > 0$,*

$$Q(a_t) \sim \frac{t}{\sqrt{T(a_t)}}, \quad Q'(a_t) \sim \frac{t\sqrt{T(a_t)}}{a_t}$$

and for $x \in [0, a_n/2]$,

$$Q'(x) \sim \frac{n}{a_n}\left(\frac{x}{a_n}\right)^{\Lambda - 1},$$

where $\Lambda > 1$ is defined in Definition 1.1 (d).

(3) [4, Lemma 3.6 (3.36)] *There exists L_0 such that for any fixed $L \ge L_0$ and uniformly $n > 0$*

$$\frac{Q(a_{Ln})}{Q(a_n)} \ge 2.$$

(4) [4, Lemma 3.11 (a),(b)] *Given fixed $\alpha > 0$, we have uniformly for $t > 0$,*

$$\left|1 - \frac{a_{\alpha t}}{a_t}\right| \sim \frac{1}{T(a_t)},$$

and we have for $t > 0$,

$$\left|1 - \frac{a_t}{a_s}\right| \sim \frac{1}{T(a_t)} \left|1 - \frac{t}{s}\right|, \quad \frac{1}{2} \le \frac{t}{s} \le 2.$$

We define for $u > 0$,

$$\varphi_u(x) = \begin{cases} \frac{a_u}{u} \frac{1-\frac{|x|}{a_{2u}}}{\sqrt{1-\frac{|x|}{a_u}+\delta_u}}, & |x| \le a_u; \\ \varphi_u(a_u), & a_u < |x|, \end{cases}$$

where

$$\delta_u = \{uT(a_u)\}^{-2/3}.$$

Let $0 < p < \infty$. The L_p Christoffel functions $\lambda_{n,p}(w;x)$ with a weight w are defined as follows;

$$\lambda_{n,p}(w;x) := \inf_{P \in \mathcal{P}_{n-1}} \int_{-\infty}^{\infty} |Pw|^p(u)du/|P|^p(x).$$

Then we have

$$\lambda_{n,2}(w;x) = \frac{1}{K_n(x,x)} = \frac{1}{\sum_{j=0}^{n-1} p_k(w^2,x)}$$

(see [4, (9.14),(9.15)]). We denote the zeros of the orthonormal polynomial $p_n(w^2,x)$ by $x_{n,n} < x_{n-1,n} < \ldots < x_{1,n}$. Then we define the Christoffel numbers $\lambda_{k,n}, k = 1,2,\ldots,n$ such as $\lambda_{k,n} := \lambda_{n,2}(w, x_{k,n})$.

Lemma 2.2 ([4, Theorem 9.3 (c)]). *Let* $0 < p < \infty$.
(1) *Let* $L > 0$. *Then uniformly for* $n \ge 1$ *and* $|x| \le a_n(1 + L\delta_n)$, *we have*

$$\lambda_{n,p}(w;x) \sim \varphi_n(x)w^p(x).$$

(2) *Moreover, uniformly for* $n \ge 1$ *and* $x \in \mathbb{R}$,

$$\lambda_{n,p}(w;x) \ge C\varphi_n(x)w^p(x).$$

Lemma 2.3. (1)[4, Corollary 13.4, (12.20)] *Uniformly for* $n \ge 1, 1 \le k \le n-1$,

$$x_{k,n} - x_{k+1,n} \sim \varphi_n(x_{k,n}).$$

Moreover,

$$\varphi_n(x_{k,n}) \sim \varphi_n(x_{k+1,n}), k = 1,2,\ldots,n-1.$$

(2) [2, Lemma 3.4 (d)] *Let* $\max\{|x_{k,n}|, |x_{k+1,n}|\} \le a_{n/2}$. *Then we have for* $x_{k+1,n} \le x \le x_{k,n}$

$$w(x_{k,n}) \sim w(x_{k+1,n}) \sim w(x).$$

So, for given $C > 0$ *and* $|x| \le a_{n/3}$, *if* $|x - x_{k,n}| \le C\varphi_n(x)$, *then we have*

$$w(x) \sim w(x_{k,n}).$$

Lemma 2.4 ([6, Lemma 3.4]). *For a certain constant* $C > 0$,

$$\frac{a_n}{n} \frac{1}{\sqrt{T(x)}} \varphi_n^{-1}(x) \le C.$$

We define

$$\Phi(x) := \frac{1}{(1+Q(x))^{2/3}T(x)}.$$

Here, we note that for $0 < d \le |x|$,

$$\Phi(x) \sim \frac{Q^{1/3}(x)}{xQ'(x)}.$$

Moreover we define

$$\Phi_n(x) := \max\left\{1 - \frac{|x|}{a_n}, \delta_n\right\}, \quad n = 1, 2, 3 \ldots .$$

Then we have the following:

Lemma 2.5 ([2, Lemma 3.3])**.** *For* $n \geq 1$*, we have*

$$\Phi(x) \leq C\Phi_n(x), \quad x \in \mathbb{R}.$$

Lemma 2.6 ([4, Theorem 1.17, Theorem 1.18])**.** *Uniformly for* $n \geq 1$ *we have*

$$\sup_{x\in\mathbb{R}} |p_n(x)w(x)|x^2 - a_n^2|^{1/4} \sim 1, \tag{2.1}$$

and for $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$*,*

$$\sup_{x\in\mathbb{R}} |p_n(x)w(x)| \sim a_n^{-1/2}(nT(a_n))^{1/6}. \tag{2.2}$$

Lemma 2.7. (1) ([4, Corollary 13.4]) *For the maximum zero* $x_{1,n}$ *of* $p_n(x)$ *we have*

$$1 - \frac{x_{1,n}}{a_n} \sim \delta_n. \tag{2.3}$$

(2) ([4, Lemma 13.9]) *Uniformly for* $n \geq 1$*,*

$$\frac{\gamma_{n-1}}{\gamma_n} \sim a_n.$$

Lemma 2.8. *We have*

$$\sup_{x\in\mathbb{R}} |p_n(x)w(x)\Phi^{1/4}(x)| \leq C \sup_{x\in\mathbb{R}} |p_n(x)w(x)\Phi_n^{1/4}(x)| \leq Ca_n^{-1/2}.$$

Proof. The first inequality follows from Lemma 2.6. We show the second inequality. Noting (2.3), from (2.1), we have

$$C \geq \sup_{|x|\leq x_{1,n}} |p_n(x)|w(x)|x^2 - a_n^2|^{1/4} \sim \sup_{|x|\leq x_{1,n}} |p_n(x)|w(x)a_n^{1/2}\Phi_n^{1/4}(x),$$

so,

$$a_n^{-1/2} \geq C \sup_{|x|\leq x_{1,n}} |p_n(x)|w(x)\Phi_n^{1/4}(x).$$

On the other hand, from (2.2) we see that

$$a_n^{-1/2} \geq C \sup_{|x|\geq x_{1,n}} |p_n(x)|w(x)\delta_n^{1/4} \sim \sup_{|x|\geq x_{1,n}} |p_n(x)|w(x)\Phi_n^{1/4}(x).$$

Therefore, we have the result. ∎

Lemma 2.9. *Let* $r > 1$ *be fixed. Then there exists* $0 < C \leq 1$ *such that uniformly* $t > 0$*,*

$$a_{rt} \geq a_t \left(1 + \frac{C}{T(a_t)}\right). \tag{2.4}$$

Proof. By Lemma 2.1 (4), there exists $C_1 > 0$ such that for $t > 0$,

$$\left|1 - \frac{a_t}{a_{st}}\right| \geq \frac{C_1}{T(a_t)}\left|1 - \frac{1}{s}\right|, \quad \frac{1}{2} \leq s \leq 2.$$

So, for $1 < r \leq 2$ we easily have

$$a_{rt} - at \geq \frac{C_1 a_{rt}}{T(a_t)}\left|1 - \frac{1}{r}\right| \geq \frac{C_1 a_t}{T(a_t)}\left|1 - \frac{1}{r}\right|.$$

Now we can take $0 < C \leq 1$ as $C = \min\left\{1, C_1\left(1 - \frac{1}{r}\right)\right\}$. Then we have $0 < C \leq 1$ such that

$$a_{rt} \geq a_t + \frac{C_1 a_{rt}}{T(a_t)} = a_t\left(1 + \frac{C}{T(a_t)}\right),$$

that is, (2.4) holds with $r > 1$. ∎

Lemma 2.10. *Let $r > 1$ be fixed and $0 < p \leq \infty$. There exist $C_1, C_2 > 0$ such that for $m \geq 1$ and $P \in \mathcal{P}_m$,*

$$\|(Pw)(x)\|_{L_p(|x|\geq a_{rm})} \leq C_2 \exp\left(-C_1 \frac{m}{\sqrt{T(a_m)}}\right) \|Pw\|_{L_p(|x|\leq a_m)}.$$

Proof. Let $r > 1$ and $0 < C \leq 1$ be fixed in Lemma 2.9. By [3, Theorem 6.4] there exist C_3, $C_4 > 0$ such that for $m \geq 1, \tau \in (0, \frac{1}{T(a_m)}]$ and polynomial $P \in \mathcal{P}_m$,

$$\|(Pw)(x)\|_{L_p(|x|\geq a_m(1+\tau))} \leq C_4 \exp\left(-C_3 m T(a_m)\tau^{3/2}\right) \|Pw\|_{L_p(|x|\leq a_m)}. \tag{2.5}$$

Now, we can take $\tau := \frac{C}{T(a_m)}$, $0 < C \leq 1$, then by Lemma 2.9 we have

$$a_m(1+\tau) = a_m\left(1 + \frac{C}{T(a_m)}\right) \leq a_{rm}. \tag{2.6}$$

So, from (2.5) we have for some $C_1 > 0$,

$$\begin{aligned}\|(Pw)(x)\|_{L_p(|x|\geq a_{rm})} &\leq \|(Pw)(x)\|_{L_p(|x|\geq a_m(1+\tau))} \\ &\leq C_4 \exp\left(-C_3 C \frac{m}{\sqrt{T(a_m)}}\right) \|(Pw)(x)\|_{L_p(|x|\leq a_m)}.\end{aligned}$$

Here we take $C_1 := C_3 C$ and $C_2 := C_4$. ∎

Lemma 2.11 ([4, Theorem 10.3])**.** *Let $P \in \mathcal{P}_n$. When $0 < q \leq p \leq \infty$, we have for some $C > 0$,*

$$\|wP\|_{L_q(\mathbb{R})} \leq C a_n^{\frac{1}{q}-\frac{1}{p}} \|wP\|_{L_p(\mathbb{R})},$$

and when $0 < p \leq q \leq \infty$, we have for some $C > 0$,

$$\|wP\|_{L_q(\mathbb{R})} \leq C \left(\frac{n\sqrt{T(a_n)}}{a_n}\right)^{\frac{1}{p}-\frac{1}{q}} \|wP\|_{L_p(\mathbb{R})}.$$

Lemma 2.12 ([4, Theorem 1.9 infinite-finite range inequality])**.** *Let $0 < p \leq \infty$ and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that for some $\varepsilon > 0$, and $n > 0, P \in P \in \mathcal{P}_n$,*

$$\|Pw\|_{L_p(a_{rn}\leq|x|)} \leq C_1 \exp(-C_2 n^{\varepsilon}) \|Pw\|_{L_p(|x|\leq a_n)}.$$

Lemma 2.13. *Let p,$q > 0$ and let $r > 1$. Then there exist constants C, $C_1 > 0$ such that for $P \in \mathcal{P}_{[\frac{n}{2r}]}$*

$$\left\{\int_{|t|\geq a_{n/2}} |(Pw)(t)|^q dt\right\}^{1/q} \leq C_1 \exp\left(-\frac{C}{4r}\frac{n}{\sqrt{T(a_n)}}\right) \left\{\int_{|t|\leq a_{n/2}} |(Pw)(t)|^p dt\right\}^{1/p}.$$

Proof. Let $m := \left[\frac{n}{2r}\right]$, then we see $2rm \le n$, and if we take n large enough, then we have $\frac{n}{4r} \le m$. Therefore, using Lemma 2.10 and Lemma 2.11, for $P \in \mathcal{P}_m$,

$$\begin{aligned}
&\|(Pw)(x)\|_{L_q(a_{n/2}\le|x|)} \le \|(Pw)(x)\|_{L_q(a_{rm}\le|x|)} \\
&\le C_1 \exp\left(-C\frac{m}{\sqrt{T(a_m)}}\right)\|(Pw)(x)\|_{L_q(|x|\le a_m)} \\
&\le C_1 \exp\left(-C\frac{m}{n}\frac{n}{\sqrt{T(a_m)}}\right)\|(Pw)(x)\|_{L_q(\mathbb{R})} \\
&\le C_1 \exp\left(-\frac{C}{4r}\frac{n}{\sqrt{T(a_n)}}\right)\begin{cases} a_m^{\frac{1}{q}-\frac{1}{p}}\|wP\|_{L_p(\mathbb{R})}, & 0<q<p\le\infty, \\ \left(\frac{m\sqrt{T(a_m)}}{a_m}\right)^{\frac{1}{p}-\frac{1}{q}}\|wP\|_{L_p(\mathbb{R})}, & 0<p\le q\le\infty \end{cases} \\
&\le C_1 \exp\left(-\frac{C_2}{4r}\frac{n}{\sqrt{T(a_n)}}\right)\|wP\|_{L_p(\mathbb{R})},
\end{aligned}$$

because for any fixed $\varepsilon > 0$

$$\max\left\{a_m^{\frac{1}{q}-\frac{1}{p}}, \left(\frac{m\sqrt{T(a_m)}}{a_m}\right)^{\frac{1}{p}-\frac{1}{q}}\right\} \le \exp\left(\frac{\varepsilon n}{\sqrt{T(a_n)}}\right).$$

Now, we may estimate $\|wP\|_{L_p(\mathbb{R})}$. Noting (2.6), we have using Lemma 2.12 (infinite-finite range inequality),

$$\begin{aligned}
\|wP\|_{L_p(\mathbb{R})} &\le \|wP\|_{L_p(|x|\le a_{rm})} + \|wP\|_{L_p(a_{rm}<|x|)} \\
&\le \|wP\|_{L_p(|x|\le a_{rm})} + C_1\|wP\|_{L_p(|x|\le a_m)} \\
&\le C_2\|wP\|_{L_p(|x|\le a_{n/2})}, \quad (\because rm \le n/2).
\end{aligned}$$

∎

Lemma 2.14. *Let* $0<\delta<1$, $f \in \mathcal{B}_\delta\left(T^{1/4}\right)$*, and let* $x \in \mathbb{R}$ *be fixed. For* $t \in \mathbb{R}$*, we have*

$$\begin{aligned}
&w^\delta(x+t)\left|\left(T^{1/4}f\right)(x+t) - \left(T^{1/4}f\right)(x)\right| \\
&\le \begin{cases} \exp\left(cxQ'(x)\right)V_\delta\left([x,x+t]^*, T^{1/4}f\right), & \text{if} \quad xt<0 \text{ and } |t|<2|x|, \\ V_\delta\left([x,x+t]^*, T^{1/4}f\right), & \text{otherwise}, \end{cases}
\end{aligned}$$

and especially, when $f(x)=0$*, we also have*

$$\begin{aligned}
&w^\delta(x+t)\left|f(x+t)\right| \\
&\le \begin{cases} \exp\left(cxQ'(x)\right)V_\delta\left([x,x+t]^*, T^{1/4}f\right), & \text{if} \quad xt<0 \text{ and } |t|<2|x|, \\ V_\delta\left([x,x+t]^*, T^{1/4}f\right), & \text{otherwise}, \end{cases}
\end{aligned}$$

where

$$[a,b]^* := \begin{cases} [a,b] & \text{if } a\le b \\ [b,a] & \text{if } a \ge b. \end{cases}$$

Proof. Let $xt \ge 0$. Then

$$\begin{aligned}
&w^\delta(x+t)\left|\left(T^{1/4}f\right)(x+t) - \left(T^{1/4}f\right)(x)\right| \le w^\delta(x+t)\int_{[x,x+t]^*}\left|d\left(T^{1/4}f\right)(u)\right| \\
&\le \int_{[x,x+t]^*} w^\delta(u)\left|d\left(T^{1/4}f\right)(u)\right| \le V_\delta\left([x,x+t]^*, T^{1/4}f\right).
\end{aligned} \tag{2.7}$$

Next, let $xt < 0$ and $|t| \geq 2|x|$. Then, for $x \leq u \leq x + t (t > 0)$ or $x + t \leq u \leq x (t < 0)$ we have $w^\delta(u) \geq w^\delta(x + t)$ because of $|u| \leq |x + t|$. So we have (2.7). Finally, we consider the case of $xt < 0$ and $|t| < 2|x|$. Let $u \in [x, x + t]$ $(t > 0)$ or $u \in [x + t, x]$ $(t < 0)$. If $u \leq |x+t|$, then we simply have

$$w^\delta(x + t) \leq w^\delta(u).$$

So we have the result as (2.7). Let $|x + t| < |u|$. We see that

$$|Q(u) - Q(x + t)| \leq |t||Q'(x)| < 2|x||Q'(x)| = 2xQ'(x).$$

Hence,

$$Q(u) - 2xQ'(x) \leq Q(x + t),$$

so

$$w^\delta(x + t) \leq \exp(2\delta xQ'(x))w^\delta(u).$$

Therefore, as (2.7) we have the result. If $f(x) = 0$, then we see

$$w^\delta(x + t) |f(x + t)| \leq w^\delta(x + t) \left|\left(T^{1/4} f\right)(x + t) - \left(T^{1/4} f\right)(x)\right|.$$

So, from the above result we have the second inequality. ∎

Let

$$\chi_x(t) := \begin{cases} 1, & \text{if } t \leq x, \\ 0, & \text{otherwise}. \end{cases}$$

Lemma 2.15 ([5, Corollary 1.2.6])**.** *Let $x \in \mathbb{R}$ be a fixed number, and let integer k be found so that $0 \leq k \leq n + 1$ and $x \in (x_{k+1,n}, x_{k,n}]$. Then there exist $P := P_x, R := R_x \in \mathcal{P}_{2n-1}$ such that*

$$R(t) \leq \chi_x(t) \leq P(t), \quad t \in \mathbb{R}, \tag{2.8}$$

and

$$\int_{-\infty}^{\infty} |P(t) - R(t)|w^2(t)dt \leq \lambda_{k+1,n} + \lambda_{k,n}. \tag{2.9}$$

Here, P_x and R_x mean the polynomials $P_{(x_{k+1,n},x_{k,n}]}$, $R_{(x_{k+1,n},x_{k,n}]}$ defined by the interval $(x_{k+1,n}, x_{k,n}]$ which contains x)).

Lemma 2.16 (cf. [5, Lemma 4.1.3])**.** *For $x \in \mathbb{R}$ and $n = 1, 2, \cdots$, we have*

$$E_{1,n}(w; \chi_x) \leq C\frac{a_n}{n}w(x).$$

Proof. Using Lemma 2.15, we estimate $E_{1,n}(w^2; \chi_x)$. First, let $|x| \leq a_{n/3}$. Let k be an integer such that $x \in [x_{k+1,n}, x_{k,n}] \subset [-a_n, a_n]$. Here we note $x_{1,n} < a_n$ (see (2.3)). By Lemma 2.15, we get polynomials P and R satisfying (2.8) and (2.9), so that

$$\begin{aligned} E_{1,n}(w^2; \chi_x) &\leq \int_{-\infty}^{\infty} [P(t) - \chi_x(t)]\, w^2(t)dt + \int_{-\infty}^{\infty} [\chi_x(t) - R(t)]\, w^2(t)dt \\ &\leq \lambda_{k+1,n} + \lambda_{k,n}. \end{aligned} \tag{2.10}$$

By Lemma 2.3, we have $\varphi_n(x_{k+1,n}) \sim \varphi_n(x_{k,n}) \sim \varphi_n(x)$ and $w(x_{k+1,n}) \sim w(x_{k,n}) \sim w(x)$. Therefore, by Lemma 2.2 (1), we have

$$\lambda_{k,n} \leq C\varphi_n(x)w^2(x), \quad \lambda_{k+1,n} \leq C\varphi_n(x)w^2(x).$$

Consequently, (2.10) shows that for $|x| \leq a_{n/3}$,

$$E_{1,n}(w^2; \chi_x) \leq C\varphi_n(x)w^2(x) \sim \frac{a_n}{n}\sqrt{1 - \frac{|x|}{a_n}}w^2(x) \leq C\frac{a_n}{n}w^2(x).$$

Now, when $|x| \leq a_{n/3}$, we estimate $E_{1,n}(w^2;\chi_x)$ with $w(x) = \exp(-Q(x))$. Since the Mhaskar-Saff number for the weight $w^{1/2}$ equals to a_{2n}, using the above estimate, we have

$$E_{1,n}(w;\chi_x) = E_{1,n}((w^{1/2})^2;\chi_x) \leq C\frac{a_{2n}}{n}w(x) \leq C\frac{a_n}{n}w(x).$$

We consider the case of $|x| \geq a_{n/3}$. Then, by Lemma 2.1 (2) and the increasingness of $Q'(x)$, we have

$$Q'(x) \geq Q'(a_{n/3}) \geq C\frac{n\sqrt{T(a_{n/3})}}{2a_{n/3}} \geq C\frac{n\sqrt{T(a_n)}}{a_n}.$$

Thus,

$$\begin{aligned} E_{1,n}(w;\chi_x) &\leq \int_{-\infty}^{\infty}(1-\chi_x(t))w(t)dt = \int_x^{\infty}\exp(-Q(t))dt \\ &\leq \frac{-1}{Q'(x)}\int_x^{\infty}(-Q'(t))\exp(-Q(t))dt = \frac{w(x)}{Q'(x)} \\ &\leq C\frac{a_n}{n\sqrt{T(a_n)}}w(x) \leq C\frac{a_n}{n}w(x). \end{aligned}$$

∎

We use the notation

$$\Phi^*(x) := \begin{cases} 1, & \text{if } T(x) \text{ is bounded} \\ \Phi(x), & \text{if } T(x) \text{ is unbounded.} \end{cases}$$

Lemma 2.17. *Let*

$$\Lambda_n(t) := \int_t^{\infty} p_n(v)w^2(v)dv, \quad t \in \mathbb{R},$$

and we suppose that $0 < \delta \leq 1$, *where* $\delta = 1$ *if* $T(x)$ *is bounded, and* $0 < \delta < 1$ *if* $T(x)$ *is unbounded. Then there exist constants* $d > 0$ *and* $C > 0$ *such that*

$$|\Lambda_n(t)| \leq C\frac{\sqrt{a_n}}{n}w^{\delta}(t), \quad |t| \leq a_{dn}. \tag{2.11}$$

Proof. We consider the case of n large enough. We use $r > 1$ and $P \in \mathcal{P}_{\left[\frac{n}{2r}\right]}$ in Lemma 2.13. By Lemma 2.16, we have that for $t \in \mathbb{R}$ and $n = 1, 2, \ldots,$

$$E_{1,n}(w^{\delta};\chi_t) \leq C\frac{a_{n/\delta}}{n}w^{\delta}(t) \leq C\frac{a_n}{n}w^{\delta}(t).$$

So there exists $P \in \mathcal{P}_m$, $m = \left[\frac{n}{2r}\right]$ such that

$$\int_{\mathbb{R}} |\chi_t(u) - P(u)|w^{\delta}(u)du \leq C\frac{a_n}{n}w^{\delta}(t). \tag{2.12}$$

(Here, we note that for n large enough, $\frac{1}{4}n \leq rm \leq \frac{1}{2}n$). Hence, by the orthogonal polynomial p_n and $P - 1 \in \mathcal{P}_{n-1}$, we have

$$\begin{aligned} &|\Lambda_n(t)| = \left|\int_{-\infty}^{\infty}(1-\chi_t(u))(p_nw^2)(u)du\right| \\ &= \left|\int_{-\infty}^{\infty}(\chi_t(u) - P(u) + P(u) - 1)(p_nw^2)(u)du\right| = \left|\int_{-\infty}^{\infty}(\chi_t(u) - P(u))(p_nw^2)(u)du\right| \\ &\leq \int_{|u|\leq a_{n/2}} \left|(\chi_t(u) - P(u))(p_nw^2)(u)\right| du + \int_{|u|\geq a_{n/2}} \left|(\chi_t(u) - P(u))(p_nw^2)(u)\right| du \\ &=: J_1 + J_2. \end{aligned}$$

By Lemma 2.8 and (2.12) we see

$$\begin{aligned} J_1 &= \int_{|u|\le a_{n/2}} |(\chi_t(u)-P(u))\Phi_n^{*\,-1/4}(u)w(u)\Phi_n^{*\,1/4}(u)p_n(u)w(u)|du \\ &\le C\frac{1}{\sqrt{a_n}}\int_{|u|\le a_{n/2}} |(\chi_t(u)-P(u))\Phi_n^{*\,-1/4}(u)w(u)|du \\ &\le C\frac{1}{\sqrt{a_n}}\int_{|u|\le a_{n/2}} |(\chi_t(u)-P(u))w^{\delta}(u)|du \le C\frac{\sqrt{a_n}}{n}w^{\delta}(t). \end{aligned} \tag{2.13}$$

Here we used the fact that the Mhaskar-Rakhmanov-Saff number for the weight $w^{\delta}(x)$ is $a_{n/\delta}$.

Next, we estimate J_2. From (2.12),

$$\begin{aligned} &\int_{|u|\le a_{n/2}} |1-P(u)|\,w(u)du \\ &\le \int_{|u|\le a_{n/2}} |\chi_t(u)-P(u)|w(u)du + \int_{|u|\le a_{n/2}} |1-\chi_t(u)|w(u)du \le C. \end{aligned}$$

Similarly,

$$\int_{|u|\le a_{n/2}} |P(u)|\,w(u)du \le \int_{|u|\le a_{n/2}} |\chi_t(u)-P(u)|w(u)du + \int_{|u|\le a_{n/2}} |\chi_t(u)|w(u)du \le C.$$

So

$$\max\left\{\int_{|u|\le a_{n/2}} |1-P(u)|\,w(u)du, \int_{|u|\le a_{n/2}} |P(u)|\,w(u)du\right\} \le C. \tag{2.14}$$

Since $1-P, P\in\mathcal{P}_{[\frac{n}{2r}]}$, using Lemma 2.10, Lemma 2.11 with $p=1$, $q=2$ and (2.14), we have

$$\begin{aligned} &\left\{\int_{|u|\ge a_{n/2}} |1-P(u)|^2\,w^2(u)du\right\}^{1/2} \le \left\{\int_{|u|\ge a_{r[\frac{n}{2r}]}} |1-P(u)|^2\,w^2(u)du\right\}^{1/2} \\ &\le C_1\exp\left(-C\frac{n}{\sqrt{T(a_n)}}\right)\left\{\int_{|u|\le a_{r[\frac{n}{2r}]}} |1-P(u)|^2\,w^2(u)du\right\}^{1/2} \\ &\le C_1\exp\left(-C\frac{n}{\sqrt{T(a_n)}}\right)\left\{\int_{|u|\le a_{n/2}} |1-P(u)|^2\,w^2(u)du\right\}^{1/2} \\ &\le C_1\exp\left(-C\frac{n}{\sqrt{T(a_n)}}\right)\left\{\frac{n\sqrt{T(a_n)}}{a_n}\right\}^{1/2}\int_{|u|\le a_{n/2}} |(1-P(v))w(v)|dv \\ &\le C_1\exp\left(-C_2\frac{n}{\sqrt{T(a_n)}}\right), \end{aligned}$$

and similarly

$$\left\{\int_{|u|\ge a_{n/2}} |P(u)|^2\,w^2(u)du\right\}^{1/2} \le C_1\exp\left(-C_2\frac{n}{\sqrt{T(a_n)}}\right).$$

So we have

$$\max\left\{\left\{\int_{|u|\ge a_{n/2}}|1-P(u)|^2w^2(u)du\right\}^{1/2},\left\{\int_{|u|\ge a_{n/2}}|P(u)|^2w^2(u)du\right\}^{1/2}\right\}$$
$$\le C_1\exp\left(-C_2\frac{n}{\sqrt{T(a_n)}}\right). \tag{2.15}$$

Now, using the Schwarz inequality, we see

$$\begin{aligned}
J_2 &\le \left(\int_{|u|\ge a_{n/2}}|\chi_t(u)-P(u)|^2w^2(u)du\right)^{1/2}\left(\int_{-\infty}^{\infty}p_n^2(u)w^2(u)du\right)^{1/2}\\
&= \left(\int_{|u|\ge a_{n/2}}|\chi_t(u)-P(u)|^2w^2(u)du\right)^{1/2}\\
&\le C_1\begin{cases}\left(\int_{-\infty}^{-a_{n/2}}|1-P(u)|^2w^2(u)du+\int_{a_{n/2}}^{\infty}|P(u)|^2w^2(u)du\right)^{1/2}, & |t|\le a_{n/2},\\
\left(\int_{-\infty}^{-a_{n/2}}|1-P(u)|^2w^2(u)du+\int_{a_{n/2}}^{t}|1-P(u)|^2w^2(u)du\right. & \\
\qquad\left.+\int_{t}^{\infty}|P(u)|^2w^2(u)du\right)^{1/2}, & t>a_{n/2},\\
\left(\int_{-\infty}^{t}|1-P(u)|^2w^2(u)du+\int_{t}^{-a_{n/2}}|1-P(u)|^2w^2(u)du\right. & \\
\qquad\left.+\int_{a_{n/2}}^{\infty}|P(u)|^2w^2(u)du\right)^{1/2}, & t<-a_{n/2}\end{cases}\\
&\le C_1\begin{cases}\left(\int_{-\infty}^{-a_{n/2}}|1-P(u)|^2w^2(u)du+\int_{a_{n/2}}^{\infty}|P(u)|^2w^2(u)du\right)^{1/2}, & |t|\le a_{n/2},\\
\left(\int_{-\infty}^{-a_{n/2}}|1-P(u)|^2w^2(u)du+\int_{a_{n/2}}^{\infty}|1-P(u)|^2w^2(u)du\right. & \\
\qquad\left.+\int_{a_{n/2}}^{\infty}|P(u)|^2w^2(u)du\right)^{1/2}, & t>a_{n/2},\\
\left(\int_{-\infty}^{-a_{n/2}}|1-P(u)|^2w^2(u)du+\int_{-\infty}^{-a_{n/2}}|1-P(u)|^2w^2(u)du\right. & \\
\qquad\left.+\int_{a_{n/2}}^{\infty}|P(u)|^2w^2(u)du\right)^{1/2}, & t<-a_{n/2},\end{cases}\\
&\le C_1\max\left\{\left(\int_{|u|\ge a_{n/2}}|1-P(u)|^2w^2(u)du\right)^{1/2},\left(\int_{|u|\ge a_{n/2}}|P(u)|^2w^2(u)du\right)^{1/2}\right\}\\
&\le C_1\exp\left(-C_2\frac{n}{\sqrt{T(a_n)}}\right)\text{ by (2.15).}
\end{aligned} \tag{2.16}$$

Here we will show that there exists $0<d<1$ such that

$$\exp\left(-\frac{C_2}{2}\frac{n}{\sqrt{T(a_n)}}\right)\le w(t),\quad |t|\le a_{dn}. \tag{2.17}$$

In fact, by Lemma 2.1 (2), there exists C_1 such that

$$\frac{C_2}{2}\frac{n}{\sqrt{T(a_n)}}\ge\frac{C_2}{2}C_1Q(a_n).$$

If $\frac{C_2}{2}C_1 \geq 1$, then we can take $d = 1$, because of

$$\exp\left(-\frac{C_2}{2}\frac{n}{\sqrt{T(a_n)}}\right) \leq \exp\left(-Q(a_n)\right) \leq \exp\left(-Q(t)\right), \quad |t| \leq a_n.$$

Let $\frac{C_2}{2}C_1 < 1$. By [4, Lemma 3.4 (3.4)] there exist C_3, C_4 such that for $s/r \geq 1$,

$$\left(\frac{s}{r}\right)^{\max\{\Lambda, C_3 T(r)\}} \leq \frac{Q(s)}{Q(r)} \leq \left(\frac{s}{r}\right)^{C_4 T(r)}.$$

Using it, and Lemma 2.1 (4) we have

$$\frac{Q(a_{2t})}{Q(a_t)} \geq \left(\frac{a_{2t}}{a_t}\right)^{\max\{\Lambda, C_3 T(a_t)\}} \geq \left(1 + \frac{C_2}{T(a_t)}\right)^{\max\{\Lambda, C_3 T(a_t)\}} \geq C_5 > 1.$$

Then for a positive integer k,

$$\frac{Q(a_{2^k t})}{Q(a_t)} \geq C_5^k.$$

We take k as $\frac{C_2}{2}C_1 C_5^k \geq 1$, and set $d := 1/2^k$. If we put $n = 2^k t$, then we have

$$\frac{C_2}{2}C_1 Q(a_n) \geq \frac{C_2}{2}C_1 C_5^k Q(a_{dn}) \geq Q(a_{dn}).$$

Consequently we have (2.17). Therefore, from (2.16) we have $0 < d \leq 1$ such that for some $C_1 > 0$,

$$J_2 \leq C_1 \exp\left(-C_2\frac{n}{\sqrt{T(a_n)}}\right) \leq C_1 \exp\left(-\frac{C_2}{2}\frac{n}{\sqrt{T(a_n)}}\right) w(t), \quad |t| \leq a_{dn}. \tag{2.18}$$

Here, we see that for n large enough,

$$\exp\left(-\frac{C_2}{2}\frac{n}{\sqrt{T(a_n)}}\right) \leq C_1\frac{\sqrt{a_n}}{n}. \tag{2.19}$$

In fact, there exists $\varepsilon > 0$ such that

$$\frac{n}{\sqrt{T(a_n)}} \geq n^{\varepsilon},$$

and so

$$\frac{C_2}{2}\frac{n}{\sqrt{T(a_n)}} \geq \frac{C_2}{2}n^{\varepsilon} \geq \log\left(\frac{1}{C_1}n\right) \geq \log\left(\frac{1}{C_1}\frac{n}{\sqrt{a_n}}\right),$$

that is, we have (2.19). Hence (2.18) means

$$J_2 \leq C_1\frac{\sqrt{a_n}}{n}w(t), \quad |t| \leq a_{dn}. \tag{2.20}$$

Consequently, from (2.13) and (2.20) we have the result (2.11). ∎

Proof of Theorem 1.1. We will consider only for $x \geq 0$, and for the other cases we omit the proof, because it can be shown similarly. In fact, we can do it as follows. Let us define for $j = 2, 3, 4, 5$,

$$I_j =: \begin{cases} I_j(+), & x \geq 0 \\ I_j(-), & x < 0. \end{cases}$$

Then when $x < 0$, we can show $I_3(-)$ as $I_2(+)$, $I_5(-)$ as $I_4(+)$, $I_2(-)$ as $I_3(+)$, and $I_4(-)$ as $I_5(+)$. Now, let $x \geq 0$ and $w = \exp(-Q) \in \mathcal{F}(C^2+)$. Let $x \in \mathbb{R}$ be fixed, then we consider

$$0 \leq x \leq a_{dn}/6,$$

where a_{dn} is defined in (2.11). From (1.1) we see that

$$s_n(f,x) - f(x) = \int_{-\infty}^{\infty} K_n(x,t)\left(f(t) - f(x)\right)w^2(t)dt.$$

Without loss of generality, we may assume that $f(x) = 0$, and exchange $f(t) - f(x)$ with $f(t)$. Then we may estimate

$$|s_n(f,x)| = \left|\int_{-\infty}^{\infty} K_n(x,t)f(t)w^2(t)dt\right| = \left|\int_{-\infty}^{\infty} K_n(x,x+t)f(x+t)w^2(x+t)dt\right|.$$

We consider a_{dn} in (2.11). For $|t| \ge \frac{a_n}{n}$ and a fixed x, we define

$$a_{dn}^{\sharp} := a_{dn} - x, \quad a_{dn}^{\flat} := a_{dn} + x. \tag{2.21}$$

Noting $f(x) = 0$, and by (1.1), we split $s_n(f,x)$ in five terms as follows:

$$s_n(f,x) = \int_{-\infty}^{\infty} K_n(x,x+t)f(x+t)w^2(x+t)dt =: \sum_{k=1}^{5} I_k,$$

where, with $H(t) := K_n(x,x+t)f(x+t)w^2(x+t)$,

$$I_1 := \int_{|t|\le\frac{a_n}{n}} H(t)dt, \quad I_2 := \int_{-a_{dn}^{\flat}}^{-a_n/n} H(t)dt, \quad I_3 := \int_{a_n/n}^{a_{dn}^{\sharp}} H(t)dt,$$
$$I_4 := \int_{-\infty}^{-a_{dn}^{\flat}} H(t)dt, \quad I_5 := \int_{a_{dn}^{\sharp}}^{\infty} H(t)dt.$$

First, we estimate I_1. Using the Schwarz inequality and the estimates on the Christoffel functions from Lemma 2.2 and Lemma 2.4,

$$\begin{aligned}
&K_n^2(x,x+t) \le K_n(x,x)K_n(x+t,x+t)\\
&\le C\varphi_n^{-1}(x)\varphi_n^{-1}(x+t)w^{-2}(x)w^{-2}(x+t)\\
&= C\sqrt{T(x)}\sqrt{T(x+t)}\frac{1}{\sqrt{T(x)}}\frac{1}{\sqrt{T(x+t)}}\varphi_n^{-1}(x)\varphi_n^{-1}(x+t)w^{-2}(x)w^{-2}(x+t)\\
&\le C\left(\frac{n}{a_n}\right)^2\sqrt{T(x)}\sqrt{T(x+t)}w^{-2}(x)w^{-2}(x+t).
\end{aligned}$$

Therefore, we have

$$H(t) \le C\frac{n}{a_n}T^{1/4}(x)w^{-1}(x)f(x+t)T^{1/4}(x+t)w(x+t). \tag{2.22}$$

Hence, we have

$$\begin{aligned}
|I_1| &\le C\frac{n}{a_n}T^{1/4}(x)w^{-1}(x)\int_{|t|\le a_n/n}\left|f(x+t)T^{1/4}(x+t)w(x+t)\right|dt\\
&\le C\frac{n}{a_n}T^{1/4}(x)w^{-1}(x)\int_{|t|\le a_n/n} w(x+t)\int_{[x,x+t]^*}\left|d\left(T^{1/4}f\right)(u)\right|
\end{aligned}$$

(note $f(x) = 0$). Here, since we suppose $|x| \le a_{dn}/6$, we see $\varphi_n(x) \sim a_n/n$. By Lemma 2.3 (b) we have $w(x+t) \sim w(x)$, so

$$\begin{aligned}
|I_1| &\le C\frac{n}{a_n}T^{1/4}(x)w^{-1}(x)\int_{|t|\le a_n/n}\int_{[x,x+t]^*} w^{\delta}(u)\,|df(u)|\,dt\\
&\le C\frac{n}{a_n}T^{1/4}(x)w^{-1}(x)\int_{|t|\le a_n/n} V_1\left([x,x+t]^*, T^{1/4}f\right)dt\\
&\le CT^{1/4}(x)\exp\left(Q(x)\right)V_1\left(\left[x - \frac{a_n}{n}, x + \frac{a_n}{n}\right], T^{1/4}f\right).
\end{aligned} \tag{2.23}$$

Next, we estimate I_3. It is sufficient to prove that for some $\beta > 0$ and $C > 0$,

$$|I_3| \le C \exp\left(Q(x)\right)\left(\exp\left(\beta x Q'(x)\right)\right)\frac{1}{n}\sum_{k=1}^{n} V_\delta\left(\left[x, x+\frac{a_n}{k}\right], f\right). \tag{2.24}$$

Then we have

$$|I_3| \le C \exp\left(Q(x)\right)\left(\exp\left(\beta x Q'(x)\right)\right)\frac{1}{n}\sum_{k=1}^{n} V_\delta\left(\left[x, x+\frac{a_n}{k}\right], T^{1/4}f\right). \tag{2.25}$$

(note $f(x) = 0$). By (1.2) we have

$$K_n(x, x+t) = \frac{\gamma_{n-1}}{\gamma_n}\frac{p_{n-1}(x)p_n(x+t) - p_n(x)p_{n-1}(x+t)}{t}.$$

Using it, we estimate I_3. We see that

$$I_3 := \frac{\gamma_{n-1}}{\gamma_n}\left\{p_{n-1}(x)I_{3,1} - p_n(x)I_{3,2}\right\},$$

where

$$I_{3,1} := \int_{a_n/n}^{a^\sharp_{dn}} p_n(x+t)\frac{f(x+t)}{t}w^2(x+t)dt,$$
$$I_{3,2} := \int_{a_n/n}^{a^\sharp_{dn}} p_{n-1}(x+t)\frac{f(x+t)}{t}w^2(x+t)dt.$$

From $\gamma_{n-1}/\gamma_n \sim a_n$ (see Lemma 2.7 (2)) and Lemma 2.8, we have

$$|I_3| \le C a_n^{1/2}\Phi^{*-1/4}(x)w^{-1}(x)\left\{|I_{3,1}| + |I_{3,2}|\right\}.$$

We use $\Lambda_n(x)$ in Lemma 2.17. Applying integration by parts, we have

$$I_{3,1} = \frac{n}{a_n}f\left(x+\frac{a_n}{n}\right)\Lambda_n\left(x+\frac{a_n}{n}\right) - \frac{1}{a^\sharp_{dn}}f(x+a^\sharp_{dn})\Lambda_n(x+a^\sharp_{dn})$$
$$-\int_{a_n/n}^{a^\sharp_{dn}}\frac{\Lambda_n(x+t)|df(x+t)|}{t} + \int_{a_n/n}^{a^\sharp_{dn}}\frac{\Lambda_n(x+t)f(x+t)}{t^2}dt.$$

When $0 < t \le a^\sharp_{dn}$, we see that $|x+t| \le a_{dn}$ (see (2.21)). Hence, by (2.11) we have

$$\begin{aligned}|I_{3,1}| &\le C\Bigg\{\frac{1}{\sqrt{a_n}}\left|f\left(x+\frac{a_n}{n}\right)\right|w^\delta\left(x+\frac{a_n}{n}\right) + \frac{\sqrt{a_n}}{na^\sharp_{dn}}|f(a_{dn})|w^\delta(a_{dn})\\ &\quad+\frac{\sqrt{a_n}}{n}\int_{a_n/n}^{a^\sharp_{dn}}\frac{w^\delta(x+t)|df(x+t)|}{t} + \frac{\sqrt{a_n}}{n}\int_{a_n/n}^{a^\sharp_{dn}}\frac{w^\delta(x+t)|f(x+t)|}{t^2}dt\Bigg\}\\ &\le C\Bigg\{\frac{1}{\sqrt{a_n}}\left|f\left(x+\frac{a_n}{n}\right)\right|w^\delta\left(x+\frac{a_n}{n}\right) + \frac{1}{n\sqrt{a_n}}|f(a_{dn})|w^\delta(a_{dn})\\ &\quad+\frac{\sqrt{a_n}}{n}\int_{a_n/n}^{a^\sharp_{dn}}\frac{w^\delta(x+t)|df(x+t)|}{t} + \frac{\sqrt{a_n}}{n}\int_{a_n/n}^{a^\sharp_{dn}}\frac{w^\delta(x+t)|f(x+t)|}{t^2}dt\Bigg\}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\sqrt{a_n}|I_{3,1}| &\le C\Bigg\{\left|f\left(x+\frac{a_n}{n}\right)\right|w^\delta\left(x+\frac{a_n}{n}\right) + \frac{1}{n}|f(a_{dn})|w^\delta(a_{dn})\\ &\quad+\frac{a_n}{n}\int_{a_n/n}^{a^\sharp_{dn}}\frac{w^\delta(x+t)|df(x+t)|}{t} + \frac{a_n}{n}\int_{a_n/n}^{a^\sharp_{dn}}\frac{w^\delta(x+t)|f(x+t)|}{t^2}dt\Bigg\}.\end{aligned} \tag{2.26}$$

As in the estimation of I_1,

$$\begin{aligned} w^\delta(x+t)|f(x+t)| &\leq w^\delta(x+t)\int_x^{x+\frac{a_n}{n}} |df(u)| \leq \int_x^{x+\frac{a_n}{n}} w^\delta(u)|df(u)| \\ &\leq CV_\delta\left(\left[x, x+\frac{a_n}{n}\right], f\right) \leq C\frac{1}{n}\sum_{k=1}^n V_\delta\left(\left[x, x+\frac{a_n}{k}\right], f\right) \end{aligned} \tag{2.27}$$

by decrease of $V_\delta\left(\left[x, x+\frac{a_n}{k}\right], f\right)$ with respect to k. Using the second inequality of Lemma 2.14 with $f(a_{dn}) - f(x) = f(a_{dn})$,

$$\frac{1}{n}w^\delta(a_{dn})|f(a_{dn})| \leq C\frac{1}{n}V_\delta\left([x, a_{dn}], f\right) \leq C\frac{1}{n}V_\delta\left([x, a_n], T^{1/4}f\right). \tag{2.28}$$

Similarly,

$$\int_{a_n/n}^{a^\sharp_{dn}} \frac{w^\delta(x+t)|f(x+t)|}{t^2}dt \leq C\int_{a_n/n}^{a^\sharp_{dn}} \frac{V_\delta\left([x, x+t], T^{1/4}f\right)}{t^2}dt. \tag{2.29}$$

Let $u := \frac{a_n}{t}$. Then noting the fact that $V_\delta\left(\left[x, x+\frac{a_n}{n}\right], f\right)$ is a decreasing function of u, we have

$$\begin{aligned} \int_{a_n/n}^{a^\sharp_{dn}} \frac{V_\delta\left([x, x+t], T^{1/4}f\right)}{t^2}dt &= \frac{1}{a_n}\int_{a_n/a^\sharp_{dn}}^{n} V_\delta\left(\left[x, x+\frac{a_n}{u}\right], T^{1/4}f\right)dt \\ &\leq \frac{1}{a_n}\sum_{k=1}^n V_\delta\left(\left[x, x+\frac{a_n}{k}\right], T^{1/4}f\right). \end{aligned}$$

Therefore, from (2.29) we have

$$\int_{a_n/n}^{a^\sharp_{dn}} \frac{w^\delta(x+t)|f(x+t)|}{t^2}dt \leq C\frac{1}{a_n}\sum_{k=1}^n V_\delta\left(\left[x, x+\frac{a_n}{k}\right], T^{1/4}f\right). \tag{2.30}$$

We estimate the remaining term in (2.26). Using integration by parts and (2.30), we have

$$\begin{aligned} &\int_{a_n/n}^{a^\sharp_{dn}} \frac{w^\delta(x+t)|df(x+t)|}{t} \\ &= \frac{1}{a^\sharp_{dn}}V_\delta\left(\left[x, x+a^\sharp_{dn}\right], f\right) - \frac{n}{a_n}V_\delta\left(\left[x, x+\frac{a_n}{n}\right], f\right) + \int_{a_n/n}^{a^\sharp_{dn}} \frac{w^\delta(x+t)|f(x+t)|}{t^2}dt \\ &\leq C\frac{1}{a_n}\sum_{k=1}^n V_\delta\left(\left[x, x+\frac{a_n}{k}\right], T^{1/4}f\right). \end{aligned} \tag{2.31}$$

Hence, substituting (2.27), (2.28), (2.30) and (2.31) into (2.26), we get

$$\begin{aligned} \sqrt{a_n}|I_{3,1}| &\leq C\frac{1}{n}\left\{\sum_{k=1}^n V_\delta\left(\left[x, x+\frac{a_n}{k}\right], f\right) + V_\delta\left([x, a_n], T^{1/4}f\right)\right. \\ &\qquad\left. + \sum_{k=1}^n V_\delta\left(\left[x, x+\frac{a_n}{k}\right], T^{1/4}f\right)\right\} \\ &\leq C\frac{1}{n}\sum_{k=1}^n V_\delta\left(\left[x, x+\frac{a_n}{k}\right], T^{1/4}f\right). \end{aligned} \tag{2.32}$$

For $I_{3,2}$ we obtain the estimate as (2.32), so we have for a constant $\beta > 0$,

$$\begin{aligned}|I_3| &\le C\Phi^{*-1/4}(x)w^{-1}(x)\frac{1}{n}\sum_{k=1}^{n} V_\delta\left(\left[x, x+\frac{a_n}{k}\right], T^{1/4}f\right)\\ &\le C\exp\left(Q(x)\right)\exp\left(\beta x Q'(x)\right)\sum_{k=1}^{n} V_\delta\left(\left[x, x+\frac{a_n}{k}\right], T^{1/4}f\right),\end{aligned}$$

here we used $\Phi^{*-1/4}(x) \le C\exp\left(\beta x Q'(x)\right)$. Hence, we have (2.24), and so we have (2.25). Next, we estimate I_5. Using (2.22), we have

$$\begin{aligned}|I_5| &\le C\frac{n}{a_n}T^{1/4}(x)w^{-1}(x)\int_{a_{dn}^\sharp}^{\infty} w(x+t)|(T^{1/4}f)(x+t)|dt\\ &\le C\frac{n}{a_n}T^{1/4}(x)w^{-1}(x)\left\{I_{5,1}+I_{5,2}\right\},\end{aligned} \tag{2.33}$$

where

$$\begin{aligned}I_{5,1} &:= \int_{a_{dn}^\sharp}^{\infty} w(x+t)\int_{0}^{a_{dn}^\sharp} |d(T^{1/4}f)(x+u)|dt,\\ I_{5,2} &:= \int_{a_{dn}^\sharp}^{\infty} w(x+t)\int_{a_{dn}^\sharp}^{t} |d(T^{1/4}f)(x+u)|dt\end{aligned}$$

(note $(T^{1/4}f)(x+t) = \int_0^t d(T^{1/4}f)(x+u)$). Since $a_{dn} - x = a_{dn}^\sharp \le t$, we see for every $t \ge a_{dn}^\sharp$

$$Q'(x+t) \ge Q'(a_{dn}) \sim Q'(a_n) \ge C\left(n\sqrt{T(a_n)}\right)/a_n$$

(by Lemma 2.1 (1), (2)). Then for every $u \ge a_{dn}^\sharp$

$$\begin{aligned}\textstyle\int_u^\infty w(x+t)dt &\le C\frac{a_n}{n\sqrt{T(a_n)}}\int_u^\infty Q'(x+t)w(x+t)dt\\ &= C\frac{a_n}{n\sqrt{T(a_n)}}w(x+u).\end{aligned} \tag{2.34}$$

Therefore, with integration by parts and (2.34),

$$\begin{aligned}|I_{5,2}| &= -\int_u^\infty w(x+t)dt\int_{a_{dn}^\sharp}^{u}\left|d(T^{1/4}f)(x+u)\right|\\ &\quad+\int_{a_{dn}^\sharp}^{\infty}\int_u^\infty w(x+t)dt\left|d(T^{1/4}f)(x+u)\right|\\ &\le C\frac{a_n}{n\sqrt{T(a_n)}}\int_{a_{dn}^\sharp}^{\infty} w(x+u)|d(T^{1/4}f)(x+u)|\\ &\le C\frac{a_n}{n\sqrt{T(a_n)}}V_1\left(\left[x+a_{dn}^\sharp, \infty\right], T^{1/4}f\right)\\ &= C\frac{a_n}{n\sqrt{T(a_n)}}V_1\left(\left[a_{dn}, \infty\right], T^{1/4}f\right),\end{aligned} \tag{2.35}$$

and using (2.34) with $u = a_{dn}^\sharp$,

$$|I_{5,1}| \le C\frac{a_n}{n\sqrt{T(a_n)}}w(x+a_{dn}^\sharp)\int_0^{a_{dn}^\sharp}|d(T^{1/4}f)(x+u)|. \tag{2.36}$$

Therefore, from $x+\frac{a^\sharp_{dn}}{2}=\frac{2x+a^\sharp_{dn}}{2}\le\frac{3a_{dn}}{4}$, there exists $\varepsilon>0$ such that
(2.37)
$$\begin{aligned}&w(x+a^\sharp_{dn})\int_0^{\frac{a^\sharp_{dn}}{2}}|d(T^{1/4}f)(x+u)|\le\frac{w(x+a^\sharp_{dn})}{w(x+\frac{a^\sharp_{dn}}{2})}\int_0^{\frac{a^\sharp_{dn}}{2}}w(x+u)|d(T^{1/4}f)(x+u)|\\&\le\frac{w(a_{dn})}{w(\frac{3a_{dn}}{4})}V_1\left(\left[x,x+\frac{a_{dn}}{2}\right],T^{1/4}f\right)\le C\exp(-cn^\varepsilon)V_1\left([x,x+a_n],T^{1/4}f\right).\end{aligned}$$

The last inequality holds as follows. First, we treat the case of which w is a Freud-type weight. Then we consider $q_n>0$ satisfying $q_nQ'(q_n)=n$. Then we can show $q_n\le a_n$, because

$$q_nQ'(q_n)=n=\frac{2}{\pi}\int_0^1\frac{a_nuQ'(a_nu)}{(1-u^2)}^{1/2}du\le a_nQ'(a_n)\frac{2}{\pi}\int_0^1\frac{1}{(1-u^2)}^{1/2}du\le a_nQ'(a_n).$$

Hence we have $q_n\le a_n$. Now,

$$Q(a_{dn})-Q\left(\frac{3a_{dn}}{4}\right)=Q'(\xi_n)\frac{a_{dn}}{4}\ge Q\left(\frac{3a_{dn}}{4}\right)\frac{a_{dn}}{4}\ge Q\left(\frac{3q_{dn}}{4}\right)\frac{q_{dn}}{4}.$$

From [4, (3.1.8)] there exists $\lambda>1$ such that $q_n\ge\lambda q_{n/2}$, so for $k=1,2,3,\dots$ we see $q_n\ge\lambda^kq_{n/2^k}$. Therefore for $k\ge1$ such as $3\lambda^k/4\ge1$, we see

$$\begin{aligned}Q(a_{dn})-Q\left(\tfrac{3a_{dn}}{4}\right)&\ge Q'\left(\frac{3q_{dn}}{4}\right)\frac{q_{dn}}{4}\ge Q'\left(\frac{3\lambda^kq_{n/2^k}}{4}\right)\frac{\lambda^kq_{n/2^k}}{4}\\&\ge\frac{1}{3}Q'\left(q_{n/2^k}\right)q_{n/2^k}=\frac{dn}{3\cdot2^k}.\end{aligned}$$

Therefore, there exists $c>0$ such that

$$\frac{w(a_{dn})}{w(\frac{3a_{dn}}{4})}\le e^{-cn}. \tag{2.38}$$

Let $T(x)$ be unbounded. Let $0<\alpha<1$. Then by Lemma 2.1 (4) we have

$$\lim_{n\to\infty}\frac{a_{\alpha n}}{a_n}=1.$$

So, for any fixed $L>1$, if we take n large enough, then we have $\frac{3a_{dn}}{4}\le a_{dn/L}$. From Lemma 2.1 (3) for fixed $L>1$ large enough we have

$$\frac{Q(a_{dn})}{Q(a_{dn/L})}\ge2,$$

that is,

$$Q(a_{dn})\ge2Q(a_{dn/L})\ge Q\left(\frac{3a_{dn}}{4}\right)+Q\left(a_{dn/L}\right).$$

Consequently, using Lemma 2.1 (1), (2) and (5), there exists $\varepsilon>0$ such that

$$Q(a_{dn})-Q\left(\frac{3a_{dn}}{4}\right)\ge Q(a_{dn/L})\ge c\frac{dn/L}{\sqrt{T(a_{dn/L})}}\ge cn^{\varepsilon/2},$$

that is,

$$\frac{w(a_{dn})}{w(\frac{3a_{dn}}{4})}=\exp^{-\left(Q(a_{dn})-Q\left(\frac{3a_{dn}}{4}\right)\right)}\le e^{-cn^{\varepsilon/2}}. \tag{2.39}$$

Therefore we have the last inequality in (2.37). Since we consider only n such that $|x| \le a_{dn}/6$ when $a^\sharp_{dn}/2 \le u \le a^\sharp_{dn}$, we have $0 \le x+u \le x+a^\sharp_{dn} = a_{dn}$. Moreover, there exists $c_1 > 0$ such that

(2.40)
$$\begin{aligned} w(x+a^\sharp_{dn}) \int_{a^\sharp_{dn}/2}^{a^\sharp_{dn}} \left| d\left(T^{1/4}f\right)(x+u)\right| &\le \int_{a^\sharp_{dn}/2}^{a^\sharp_{dn}} w(x+u)\left| d\left(T^{1/4}f\right)(x+u)\right| \\ &\le CV_1\left(\left[(x+a^\sharp_{dn}/2,\infty\right], T^{1/4}f\right) \le CV_1\left([c_1a_n,\infty], T^{1/4}f\right). \end{aligned}$$

Substituting (2.37), (2.40) into (2.36), we have

(2.41)
$$|I_{5,1}| \le C\frac{a_n}{n\sqrt{T(a_n)}}\left\{\exp(-cn^{\varepsilon/2})V_1\left([x,x+a_n], T^{1/4}f\right) + V_1\left([c_1a_n,\infty), T^{1/4}f\right)\right\}.$$

Together with (2.35), (2.41) and (2.33) we have for a constant $c_1 > 0$,

(2.42)
$$\begin{aligned} |I_5| &\le C\frac{T^{1/4}(x)}{\sqrt{T(a_n)}w(x)}\left\{\exp(-cn^{\varepsilon/2})V_1\left([x,x+a_n], T^{1/4}f\right) + V_1\left([c_1a_n,\infty), T^{1/4}f\right)\right\} \\ &\le Cw^{-1}(x)\left\{\exp(-cn^{\varepsilon/2})V_1\left([x,x+a_n], T^{1/4}f\right) + V_1\left([c_1a_n,\infty), T^{1/4}f\right)\right\} \\ &\le Cw^{-1}(x)\left\{\frac{1}{n}V_1\left([x,x+a_n], T^{1/4}f\right) + V_1\left([c_1a_n,\infty), T^{1/4}f\right)\right\}. \end{aligned}$$

We can obtain an estimate of I_2 as I_3. But we need to notice slightly. Let us define

$$I_2 := \frac{\gamma_{n-1}}{\gamma_n}\left\{p_{n-1}(x)I_{2,1} - p_n(x)I_{2,2}\right\},$$

where

$$\begin{aligned} I_{2,1} &:= \int_{-a^\flat_{dn}}^{-a_n/n} p_n(x+t)\frac{f(x+t)}{t}w^2(x+t)dt, \\ I_{2,2} &:= \int_{-a^\flat_{dn}}^{-a_n/n} p_{n-1}(x+t)\frac{f(x+t)}{t}w^2(x+t)dt. \end{aligned}$$

Then we have

$$|I_2| \le Ca_n^{1/2}T^{1/4}(x)\Phi^{*-1/4}(x)w^{-1}(x)\left\{|I_{2,1}| + |I_{2,2}|\right\}.$$

The formula corresponding to (2.26) is

(2.43)
$$\begin{aligned} \sqrt{a_n}|I_{2,1}| \le C\Bigg\{ &\left|f\left(x-\frac{a_n}{n}\right)\right| w^\delta\left(x-\frac{a_n}{n}\right) + \frac{1}{n}|f(-a_{dn})|w^\delta(-a_{dn}) \\ &+\frac{a_n}{n}\int_{-a^\flat_{dn}}^{-a_n/n} \frac{w^\delta(x+t)|df(x+t)|}{t} + \frac{a_n}{n}\int_{-a^\flat_{dn}}^{-a_n/n}\frac{w^\delta(x+t)|f(x+t)|}{t^2}dt\Bigg\}. \end{aligned}$$

As in the estimation of I_1, we have

(2.44)
$$w^\delta\left(x-\frac{a_n}{n}\right)\left|f\left(x-\frac{a_n}{n}\right)\right| \le C\frac{1}{n}\sum_{k=1}^{n} V_\delta\left(\left[x-\frac{a_n}{k},x\right], f\right).$$

Since $a_n \ge a_{dn} \ge 2x > 0$, using Lemma 2.14 with $f(-a_{dn}) - f(x) = f(-a_{dn})$,

(2.45)
$$\frac{1}{n}|f(-a_{dn})|w^\delta(-a_{dn}) \le C\frac{1}{n}V_\delta([-a_{dn},x], f) \le C\frac{1}{n}V_\delta([-a_n,x], f).$$

From Lemma 2.14 again

(2.46)
$$\int_{-a^\flat_{dn}}^{-a_n/n}\frac{w^\delta(x+t)|f(x+t)|}{t^2}dt \le C\exp(cxQ'(x))\int_{-a^\flat_{dn}}^{-a_n/n}\frac{V_\delta([x+t,x], T^{1/4}f)}{t^2}dt.$$

Here we note that in the case of $|t| \le 2x, xt < 0$ we need the factor $\exp(cxQ'(x))$. Let $u := -\frac{a_n}{t}$. Then, noting the fact that $V_\delta(\left[x - \frac{a_n}{u}, x\right], f)$ is a decreasing function of u, we have

$$\begin{aligned}\int_{-a^\flat_{dn}}^{-a_n/n} \frac{V_\delta([x+t,x], T^{1/4}f)}{t^2} dt &= \frac{1}{a_n}\int_n^{\frac{a_n}{a^\flat_{dn}}} V_\delta\left(\left[x - \frac{a_n}{u}, x\right], T^{1/4}f\right) du \\ &\le \frac{1}{a_n}\sum_{k=1}^n V_\delta\left(\left[x - \frac{a_n}{k}, x\right], T^{1/4}f\right).\end{aligned}$$

Therefore, with (2.46) we have

$$\int_{-a^\flat_{dn}}^{-a_n/n} \frac{w^\delta(x+t)|f(x+t)|}{t^2} dt \le C\frac{\exp(cxQ'(x))}{a_n}\sum_{k=1}^n V_\delta\left(\left[x - \frac{a_n}{k}, x\right], T^{1/4}f\right). \tag{2.47}$$

We estimate the remaining term in (2.43). Using integration by parts and (2.47), we have

$$\begin{aligned}\int_{-a^\flat_{dn}}^{-a_n/n} \frac{w^\delta(x+t)|df(x+t)|}{t} &= \frac{1}{a^\flat_{dn}} V_\delta\left(\left[x - a^\flat_{dn}, x\right], f\right) \\ &\quad - \frac{n}{a_n} V_\delta\left(\left[x - \frac{a_n}{n}, x\right], f\right) + \int_{-a^\flat_{dn}}^{-a_n/n} \frac{w^\delta(x+t)|f(x+t)|}{t^2} dt \\ &\le C\frac{\exp(cxQ'(x))}{a_n}\sum_{k=1}^n V_\delta\left(\left[x - \frac{a_n}{k}, x\right], T^{1/4}f\right).\end{aligned} \tag{2.48}$$

Hence, substituting (2.44) (2.45), (2.47) and (2.48) into (2.43), we get

$$\begin{aligned}\sqrt{a_n}|I_{2,1}| &\le C\frac{1}{n}\Bigg\{\sum_{k=1}^n V_\delta\left(\left[x - \frac{a_n}{k}, x\right], T^{1/4}f\right) + V_\delta\left([x, x-a_n], T^{1/4}f\right) \\ &\quad + \exp(cxQ'(x))\sum_{k=1}^n V_\delta\left(\left[x - \frac{a_n}{k}, x\right], T^{1/4}f\right)\Bigg\} \\ &\le C\frac{\exp(cxQ'(x))}{n}\sum_{k=1}^n V_\delta\left(\left[x - \frac{a_n}{k}, x\right], T^{1/4}f\right).\end{aligned} \tag{2.49}$$

For $t < 0$, $|t| \le a^\flat_{dn}$ we can also obtain (2.49) similarly. For $I_{2,2}$ we obtain the estimate as (2.49), so we have for α, $\beta > 0$,

$$\begin{aligned}|I_2| &\le CT^{1/4}(x)\Phi^{*-1/4}(x)w^{-1}(x)\frac{\exp(cxQ'(x))}{n}\sum_{k=1}^n V_\delta\left(\left[x - \frac{a_n}{k}, x\right], T^{1/4}f\right) \\ &\le C\exp\left(Q(x)\right)\exp\left(\beta xQ'(x)\right)\frac{1}{n}\sum_{k=1}^n V_\delta\left(\left[x - \frac{a_n}{k}, x\right], T^{1/4}f\right).\end{aligned} \tag{2.50}$$

The estimate of I_4 is also obtained as I_5. Using (2.22), we have

$$\begin{aligned}|I_4| &\le C\frac{n}{a_n}T^{1/4}(x)w^{-1}(x)\int_{-\infty}^{-a^\flat_{dn}} w(x+t)\left|(T^{1/4}f)(x+t)\right| dt \\ &\le C\frac{n}{a_n}T^{1/4}(x)w^{-1}(x)\left\{I_{4,1} + I_{4,2}\right\},\end{aligned} \tag{2.51}$$

where

$$I_{4,1} := \int_{-\infty}^{-a^\flat_{dn}} w(x+t)\int_{-a^\flat_{dn}}^{0} \left|d(T^{1/4}f)(x+u)\right| dt,$$

$$I_{4,2} := \int_{-\infty}^{-a^\flat_{dn}} w(x+t)\int_{t}^{-a^\flat_{dn}} \left|d(T^{1/4}f)(x+u)\right| dt$$

(note $f(x+t)=\int_0^t df(x+u)$). Since $0\le x\le a_{dn}/6$, we see that for every $t\le -a^{\flat}_{dn}$

$$|Q'(x+t)|\ge |Q'(-a_{dn})|\sim Q'(a_n)\ge C\frac{n\sqrt{T(a_n)}}{a_n}$$

(by Lemma 2.1 (2)). Then for every $u\le -a^{\flat}_{dn}$

$$\begin{aligned}\int_{-\infty}^{u} w(x+t)dt &\le -C\frac{a_n}{n\sqrt{T(a_n)}}\left|\int_{-\infty}^{u} w(x+t)Q'(x+t)dt\right| \\ &= C\frac{a_n}{n\sqrt{T(a_n)}}w(x+u).\end{aligned} \tag{2.52}$$

Therefore, with integration by parts and (2.52),

$$\begin{aligned}|I_{4,2}| &= C\left[\int_{-\infty}^{t} w(x+s)ds\int_{t}^{-a^{\flat}_{dn}}\left|d(T^{1/4}f)(x+u)\right|\right]_{t=-\infty}^{-a^{\flat}_{dn}} \\ &\quad+\int_{t=-\infty}^{-a^{\flat}_{dn}}\int_{-\infty}^{t} w(x+s)ds\left|d(T^{1/4}f)(x+t)\right| \\ &\le C\frac{a_n}{n\sqrt{T(a_n)}}\int_{t=-\infty}^{-a^{\flat}_{dn}} w(x+t)\left|d(T^{1/4}f)(x+t)\right| \\ &\le C\frac{a_n}{n\sqrt{T(a_n)}}V_1\left(\left[-\infty,x-a^{\flat}_{dn}\right],T^{1/4}f\right)=C\frac{a_n}{n\sqrt{T(a_n)}}V_1\left(\left[-\infty,-a_{dn}\right],T^{1/4}f\right).\end{aligned} \tag{2.53}$$

Moreover, using (2.52) with $u=-a^{\flat}_{dn}$,

$$|I_{4,1}|\le C\frac{a_n}{n\sqrt{T(a_n)}}w(x-a^{\flat}_{dn})\int_{-a^{\flat}_{dn}}^{0}|d(T^{1/4}f)(x+u)|. \tag{2.54}$$

Therefore, from $x-a^{\flat}_{dn}/2\le x+u<0$, there exists $\varepsilon>0$ such that

$$\begin{aligned}&w(x-a^{\flat}_{dn})\int_{-a^{\flat}_{dn}/2}^{0}\left|d(T^{1/4}f)(x+u)\right| \\ &\le \frac{w(x-a^{\flat}_{dn})}{w(x-a^{\flat}_{dn}/2)}\int_{-a^{\flat}_{dn}/2}^{0} w(x+u)\left|d(T^{1/4}f)(x+u)\right| \\ &\le \frac{w(a_{dn})}{w(3a_{dn}/4)}V_1\left([x-a_{dn}/2,x],T^{1/4}f\right)\le C\exp\left(-cn^{\varepsilon/2}\right)V_1\left([x-a_n,x],T^{1/4}f\right).\end{aligned} \tag{2.55}$$

The last inequality holds as (2.38) and (2.39). When $-a^{\flat}_{dn}\le u\le -a^{\flat}_{dn}/2$, we see

$$x-a^{\flat}_{dn}\le x+u\le x-a^{\flat}_{dn}/2<0,$$

so we have

$$w(x-a^{\flat}_{dn})\le w(x+u).$$

Hence, there exists $c_1>0$ such that

$$\begin{aligned}&w(x-a^{\flat}_{dn})\int_{-a^{\flat}_{dn}}^{-a^{\flat}_{dn}/2}\left|d(T^{1/4}f)(x+u)\right|\le\int_{-a^{\flat}_{dn}}^{-a^{\flat}_{dn}/2} w(x+u)\left|d(T^{1/4}f)(x+u)\right| \\ &\le V_1\left(\left[-\infty,x-a^{\flat}_{dn}\right],T^{1/4}f\right)\le CV_1\left([-\infty,-c_1a_n],T^{1/4}f\right).\end{aligned} \tag{2.56}$$

Substituting (2.55) and (2.56) into (2.54), we have

$$
\begin{aligned}
|I_{4,1}| \ &\leq C\frac{a_n}{n\sqrt{T(a_n)}}\bigg\{\exp\left(-cn^{\varepsilon/2}\right)V_1\left(\left[x-a_n,x\right],T^{1/4}f\right)\\
&\qquad\qquad +V_1\left(\left[-\infty,-c_1a_n\right],T^{1/4}f\right)\bigg\}.
\end{aligned}
\tag{2.57}
$$

Together with (2.53), (2.57) and (2.51), we have for a constant $c_1>0$,

$$
\begin{aligned}
|I_4| &\leq C\frac{T^{1/4}(x)}{\sqrt{T(a_n)}w(x)}\bigg\{\exp\left(-cn^{\varepsilon/2}\right)V_1\left(\left[x-a_n,x\right],T^{1/4}f\right)\\
&\qquad\qquad +V_1\left(\left[-\infty,-c_1a_n\right],T^{1/4}f\right)\bigg\}\\
&\leq Cw^{-1}(x)\bigg\{\exp\left(-cn^{\varepsilon/2}\right)V_1\left(\left[x-a_n,x\right],T^{1/4}f\right)\\
&\qquad\qquad +V_1\left(\left[-\infty,-c_1a_n\right],T^{1/4}f\right)\bigg\}\\
&\leq Cw^{-1}(x)\left\{\frac{1}{n}V_1\left(\left[x-a_n,x\right],T^{1/4}f\right)+V_1\left(\left[-\infty,-c_1a_n\right],T^{1/4}f\right)\right\}.
\end{aligned}
\tag{2.58}
$$

Consequently, from (2.23), (2.25), (2.42), (2.50) and (2.58) we have (1.3), that is,

$$
\begin{aligned}
&|s_n(f,x)| \leq \sum_{k=1}^{5}|I_k| \leq CT^{1/4}(x)\exp\left(Q(x)\right)V_1\left(\left[x-\frac{a_n}{n},x+\frac{a_n}{n}\right],T^{1/4}f\right)\\
&+C\exp(Q(x))\exp(\beta xQ'(x))\frac{1}{n}\sum_{k=1}^{n}V_\delta\left(\left[x-\frac{a_n}{k},x+\frac{a_n}{k}\right],T^{1/4}f\right)\\
&+Cw^{-1}(x)\bigg\{\frac{1}{n}V_1\left(\left[x-a_n,x+a_n\right],T^{1/4}f\right)+V_1\left(\left[-\infty,-c_1a_n\right],T^{1/4}f\right)\\
&\qquad\qquad +V_1\left(\left[c_1a_n,\infty\right],T^{1/4}f\right)\bigg\}\\
&\leq C\bigg[T^{1/4}(x)w^{-1}(x)V_1\left(\left[x-\frac{a_n}{n},x+\frac{a_n}{n}\right],T^{1/4}f\right)\\
&\quad+\exp(Q(x))\exp(\beta xQ'(x))\frac{1}{n}\sum_{k=1}^{n}V_\delta\left(\left[x-\frac{a_n}{k},x+\frac{a_n}{k}\right],T^{1/4}f\right)\\
&\quad+w^{-1}(x)\left\{V_1\left(\left[-\infty,-c_1a_n\right],T^{1/4}f\right)+V_1\left(\left[c_1a_n,\infty\right],T^{1/4}f\right)\right\}\bigg]\\
&\leq C\exp(Q(x))\exp(\beta xQ'(x))\bigg\{V_1\left(\left[x-\frac{a_n}{n},x+\frac{a_n}{n}\right],T^{1/4}f\right)\\
&\quad+\frac{1}{n}\sum_{k=1}^{n}V_\delta\left(\left[x-\frac{a_n}{k},x+\frac{a_n}{k}\right],T^{1/4}f\right)+\int_{|u|\geq ca_n}w(u)\left|d\left(T^{1/4}f\right)(u)\right|\bigg\}.
\end{aligned}
$$

We need to show (1.4). Let $m := \left[\sqrt{a_n n}\right]$, then we see

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n} V_\delta\left(\left[x-\frac{a_n}{k}, x+\frac{a_n}{k}\right], T^{1/4}f\right) \\
&= \frac{1}{n}\sum_{k=1}^{m} V_\delta\left(\left[x-\frac{a_n}{k}, x+\frac{a_n}{k}\right], T^{1/4}f\right) + \frac{1}{n}\sum_{k=m}^{n} V_\delta\left(\left[x-\frac{a_n}{k}, x+\frac{a_n}{k}\right], T^{1/4}f\right) \\
&\le \frac{m}{n} V_\delta\left([x-a_n, x+a_n], T^{1/4}f\right) + \frac{1}{n}\sum_{k=m}^{n} V_\delta\left(\left[x-\frac{a_n}{m}, x+\frac{a_n}{m}\right], T^{1/4}f\right) \\
&\le \sqrt{\frac{a_n}{n}} V_\delta\left([x-a_n, x+a_n], T^{1/4}f\right) + V_\delta\left(\left[x-\sqrt{\frac{a_n}{n}}, x+\sqrt{\frac{a_n}{n}}\right], T^{1/4}f\right).
\end{aligned}$$

Moreover, we have

$$V_\delta([x-\varepsilon, x+\varepsilon], T^{1/4}f) \le C\int_{x-\varepsilon}^{x+\varepsilon} \left|d\left(T^{1/4}f\right)(t)\right|,$$

and so

$$\lim_{\varepsilon\to 0} V_\delta([x-\varepsilon, x+\varepsilon], T^{1/4}f) = 0.$$

Furthermore, it is clear that

$$\lim_{n\to\infty}\sqrt{\frac{a_n}{n}} V_\delta([x-a_n, x+a_n], T^{1/4}f) \le \lim_{n\to\infty}\sqrt{\frac{a_n}{n}} V_\delta(\mathbb{R}, T^{1/4}f) = 0,$$

and

$$\lim_{n\to\infty}\int_{|u|\ge ca_n} w(u)\left|d\left(T^{1/4}f\right)(u)\right| = 0.$$

Consequently, the proof of (1.4) is complete. ∎

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