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ON ADMISSIBLE MAPPING VIA SIMULATION FUNCTION

ANANTACHAI PADCHAROEN[1] AND PAKEETA SUKPRASERT[2,*]

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[1]DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, RAMBHAI BARNI RAJABHAT UNIVERSITY, CHANTHABURI 22000, THAILAND.
anantachai.p@rbru.ac.th

[2]DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, FACULTY OF SCIENCE AND TECHNOLOGY, RAJAMANGALA UNIVERSITY OF TECHNOLOGY THANYABURI (RMUTT), THANYABURI, PATHUMTHANI 12110, THAILAND.
pakeeta_s@rmutt.ac.th

ABSTRACT. In this paper, we present some fixed point results in complete metric spaces by using generalized admissible mapping embedded in the simulation function. Its applications, Our results used to study the existence problem of nonlinear Hammerstein integral equations.

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1. INTRODUCTION

It is well known that functional analysis is made up of two main methods which are variational methods and fixed point methods. fixed point theory has been one of the most influential research topics in various field of engineering and science. The most incredible result in this direction was stated by Banach, known as the Banach contraction principle [1]. Many researcher studies this topic $e.g.$, [4, 5, 6, 7, 8].

Khojasteh et al. [2] introduced the notion of $\mathcal{Z}$-contraction by using a new class of auxiliary functions called simulation functions. This kind of functions have attracted much attention because they are useful to express a great family of contractivity conditions that were well known in the field of fixed point theory, which attracted the attention of many researchers to develop further $e.g.$, [11, 12, 13].

In this paper, we introduce the motion Generalized α-admssible-$\mathcal{Z}$-contraction and establish various fixed point theorems for such mappings in complete metric spaces. The presented theorems extend, generalize and improve many existing results in the literature, in particular the Banach contraction principle. Moreover, we obtain fixed point result is applied to guarantee the existence of solution of nonlinear Hammerstein integral equations.

2. PRELIMINARIES

Definition 2.1. [2] Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a mapping, then ζ is called a simulation function if it satisfies the following conditions:

($\zeta 1$) $\zeta(0,0) = 0$;
($\zeta 2$) $\zeta(t,s) < s - t$ for all $t, s > 0$;
($\zeta 3$) if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

We denote the set of all simulation functions by $\mathcal{Z}$.
The following functions $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ belongs to $\mathcal{Z}$.

Definition 2.2. [2] Let (X,d) be a metric space, $f : X \to X$ a mapping and $\zeta \in \mathcal{Z}$. Then f is called a $\mathcal{Z}$-contraction with respect to ζ, if the following condition is satisfied

$$\zeta(d(fx,fy), d(x,y)) \geq 0 \text{ for all } x, y \in X.$$

Remark 2.1. [2] It is clear from the definition of simulation function that $\zeta(t,s) < 0$ for all $t \geq s > 0$. Therefore if f is a $\mathcal{Z}$-contraction with respect to ζ, then

$$d(fx,fy) < d(x,y)$$

for all distinct $x, y \in X$.

Lemma 2.1. [2] *Let (X,d) be a metric space and $F : X \to X$ be a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then the fixed point of f in X is unique, provided it exists.*

Remark 2.2. [2] Every $\mathcal{Z}$-contraction is contractive and hence Banach contraction.

Theorem 2.2. [1] *Let (X,d) be a complete metric space. Then every contraction mapping has a unique fixed point. It is known as Banach contraction principle.*

Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) ψ is nondecreasing;

(ii) there exist $k_0 \in \mathbb{N}$ and $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that
$$\psi^{k+1}(t) \leq a\psi^k(t) + v_k$$
for $k \geq k_0$ and any $t \in \mathbb{R}^+$.

Lemma 2.3. [6] *If $\psi \in \Psi$, then the following hold:*
(i) $(\psi^n(t))_{n\in\mathbb{N}}$ *converges to* 0 *as* $n \to \infty$ *for all* $t \in \mathbb{R}^+$;
(ii) $\psi(t) < t$, *for any* $t \in \mathbb{R}^+$;
(iii) ψ *is continuous at* 0;
(iv) *the series* $\sum_{k=1}^{\infty} \psi^k(t)$ *converges for any* $t \in \mathbb{R}^+$.

In the literature such functions are called as either Bianchini-Grandolfi gauge functions (see [3, 4, 5]) or (c)-comparison functions (see [6]).

Definition 2.3. [7] Let $f : X \to X$ be a self mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then f is said to be α-admissible if
$$\alpha(x, y) \geq 1 \quad \text{implies} \quad \alpha(fx, fy) \geq 1.$$

Definition 2.4. [9] An α-admissible mapping f is said to be triangular α-admissible if
$$\alpha(x, z) \geq 1 \quad \text{and} \quad \alpha(z, y) \geq 1 \quad \text{imply} \quad \alpha(x, y) \geq 1.$$

Definition 2.5. [10] Let $f : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. We say that f is an α-orbital admissible if
$$\alpha(x, fx) \geq 1 \quad \text{implies} \quad \alpha(fx, f^2x) \geq 1.$$

Moreover, f is called a triangular α-orbital admissible if f is α-orbital admissible and
$$\alpha(x, y) \geq 1 \quad \text{and} \quad \alpha(y, fy) \geq 1 \text{implies} \quad \alpha(x, fy) \geq 1.$$

Definition 2.6. [11] Let f be a self mapping defined on a metric space (X, d). If there exist $\zeta \in \mathcal{Z}$ and $\alpha : X \times X \to [0, \infty)$ such that
$$\zeta(\alpha(x, y)d(fx, fy), d(x, y)) \geq 0 \quad \text{for all} \quad x, y \in X,$$
then we say that f is an α-admissible $\mathcal{Z}$-contraction with respect to ζ.

Theorem 2.4. *Let (X, d) be a complete metric space and let $f : X \to X$ be an α-admissible $\mathcal{Z}$-contraction with respect to ζ. Suppose that*
(i) f *is triangular α-orbital admissible;*
(ii) *there exists $x_0 \in X$ such that* $\alpha(x_0, fx_0) \geq 1$;
(iii) f *is continuous.*
Then there exists $u \in X$ such that $fu = u$.

3. MAIN RESULT

Definition 3.1. Let (X, d) be a metric space, $f : X \to X$ be a self mapping, there exist $\zeta \in \mathcal{Z}$ and $\alpha : X \times X \to [0, \infty)$. Then f is called generalized α-admissible-$\mathcal{Z}$-contraction with respect to ζ if the following condition is satisfied

$$\zeta(\alpha(x, fx)\alpha(y, fy)d(fx, fy), M(x, y)) \geq 0 \text{ for all distinct } x, y \in X, \tag{3.1}$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{4}\right\}. \tag{3.2}$$

Remark 3.1. It is clear from the definition of simulation function that $\zeta(t,s) < 0$ for all $t \geq s > 0$. Therefore f is a generalized α-$\mathcal{Z}$-contraction with respect to ζ, then

$$\alpha(x,fx)\alpha(y,fy)d(fx,fy) < M(x,y)$$

for all distinct $x, y \in X$.

Theorem 3.1. *Let (X,d) be a complete metric space, f is a generalized α-admissible-$\mathcal{Z}$-contraction with respect to ζ. Assume that*

(i) *f is admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
(iii) *for every sequence $\{x_n\}$ in X such that $\alpha(x_n, fx_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{x_n\}$ converges to x, then $\alpha(x, fx) \geq 1$;*
(iv) *$\alpha(x, fx) \geq 1,$ for all $x \in$ Fix(f).*

Then f has a unique fixed point x^* in X.

Proof. By (ii), let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. There exist $x_n \in X$ such that $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. Since f is α-admissible, we obtain

$$\alpha(fx_0, fx_1) = \alpha(x_1, x_2) \geq 1 \quad \text{implies} \quad \alpha(fx_1, fx_2) = \alpha(x_2, x_3) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x_n = x_{n+1} = fx_n$ and hence x_n is a fixed point of f. Therefore, we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then we get $d(x_n, x_{n+1}) > 0$, so by (3.1)

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n, fx_n)\alpha(x_{n-1}, fx_{n-1})d(fx_n, fx_{n-1}), M(x_n, x_{n-1})) \\ &= \zeta(\alpha(x_n, x_{n+1})\alpha(x_{n-1}, x_n)d(x_{n+1}, x_n), M(x_n, x_{n-1})) \\ &< M(x_n, x_{n-1}) - \alpha(x_n, x_{n+1})\alpha(x_{n-1}, x_n)d(x_{n+1}, x_n), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} &M(x_n, x_{n-1}) \\ &= \max\left\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), \frac{d(x_n, x_n) + d(x_{n-1}, x_{n+1})}{4}\right\} \\ &= \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{4}\right\}. \end{aligned}$$

The triangle inequality yields

$$\frac{d(x_{n-1}, x_{n+1})}{4} \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.$$

Therefore,

$$M(x_n, x_{n-1}) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\},$$

from (3.4), we get that

$$0 < \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} - \alpha(x_n, x_{n+1})\alpha(x_{n-1}, x_n)d(x_n, x_{n+1}) \quad . \tag{3.5}$$

The inequality (3.5) shows that

$$M(x_n, x_{n-1}) = d(x_{n-1}, x_n) \text{ for all } n \in \mathbb{N}.$$

Consequently, we have that

$$d(x_n, x_{n+1}) \leq \alpha(x_n, x_{n+1})\alpha(x_{n-1}, x_n)d(x_n, x_{n+1}) < d(x_n, x_{n-1}) \text{ for all } n \in \mathbb{N}. \tag{3.6}$$

Thus, we conclude that the sequence $\{d(x_{n-1}, x_n)\}$ is a monotonically decreasing sequence of non-negative reals and bounded from below by zero. So there is some $C \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = C.$$

We will show that

$$\lim_{n\to\infty} d(x_n, x_{n-1}) = 0. \tag{3.7}$$

If $C > 0$ then since f is a generalized α-admissible-$\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ therefore by ($\zeta 3$), we have

$$0 \leq \limsup_{n\to\infty} \zeta(\alpha(x_n, x_{n+1})\alpha(x_{n-1}, x_n)d(x_n, x_{n+1}), d(x_{n-1}, x_n)) < 0.$$

This is a contradiction. Then we conclude that $C = 0$, that is, $\lim_{n\to\infty} d(x_{n-1}, x_n) = 0$.

Now, we will show that sequence $\{x_n\}$ is a Cauchy sequence. Assume that $\{x_n\}$ is not a Cauchy sequence. Thus, for all $\epsilon > 0$, and sequences $\{x_{m_k}\}, \{x_{n_k}\}, m_k > n_k > k$ such that and $d(x_{m_k}, x_{n_k}) > \epsilon$ and $d(x_{m_k-1}, x_{n_k}) \leq \epsilon$ for all $m, n, k \in \mathbb{N}$. Therefore,by the triangle inequality, we have that

$$\begin{aligned} \epsilon < d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + \epsilon. \end{aligned} \tag{3.8}$$

Letting $k \to \infty$, using (3.7) and (3.8), we get

$$\lim_{n\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \tag{3.9}$$

By f is a generalized α-admissible-$\mathcal{Z}$-contraction with respect to ζ, we have

$$0 \leq \zeta(\alpha(x_{m_k-1}, x_{m_k})\alpha(x_{n_k-1}, x_{n_k})d(x_{m_k}, x_{n_k}), M(x_{m_k-1}, x_{n_k-1})).$$

It follows from condition ($\zeta 2$), we get

$$\begin{aligned} d(x_{m_k}, x_{n_k}) = d(fx_{m_k-1}, fx_{n_k-1}) &< M(x_{m_k-1}, x_{n_k-1}) \\ &= \max\Bigg\{ d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k}), \\ &\quad \frac{d(x_{m_k-1}, x_{n_k}) + d(x_{m_k}, x_{n_k-1})}{4} \Bigg\} \\ &\leq \max\Bigg\{ d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k}), \\ &\quad \frac{d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k-1})}{4} \Bigg\}. \end{aligned}$$

Letting $k \to \infty$, using (3.7) and (3.9), we get

$$\lim_{k\to\infty} M(x_{m_k-1}, x_{n_k-1}) = \epsilon. \tag{3.10}$$

By (3.9),(3.10) and the condition ($\zeta 3$), we get

$$0 \leq \limsup_{k\to\infty} \zeta(\alpha(x_{m_k-1}, x_{m_k})\alpha(x_{n_k-1}, x_{n_k})d(x_{m_k}, x_{n_k}), M(x_{m_k-1}, x_{n_k-1})) < 0.$$

This is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Thus $\lim_{m,n\to\infty} d(x_n, x_m)$ exists and is equal to 0. Since (X, d) is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0 \tag{3.11}$$

then

$$0 = \lim_{m,n\to\infty} d(x_m, x_n) = \lim_{n\to\infty} d(x_n, x^*) = d(x^*, x^*) \text{ and } \alpha(x^*, fx^*) \geq 1 \text{ (by (iii))}. \tag{3.12}$$

Moreover,

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n, fx_n)\alpha(x^*, fx^*)d(fx_n, fx^*), M(x_n, x^*)) \\ &= \zeta(\alpha(x_n, x_{n+1})\alpha(x^*, fx^*)d(x_{n+1}, fx^*), M(x_n, x^*)) \\ &< M(x_n, x^*) - \alpha(x_n, x_{n+1})\alpha(x^*, fx^*)d(x_{n+1}, fx^*), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} M(x_n, x^*) &= \max\left\{d(x_n, x^*), d(x^*, fx^*), d(x_n, fx_n), \frac{d(x^*, fx_n) + d(x_n, fx^*)}{4}\right\} \\ &\leq \max\left\{d(x_n, x^*), d(x^*, fx^*), d(x_n, x_{n+1}), \frac{d(x^*, x_{n+1}) + d(x_n, fx^*)}{4}\right\} \\ &\leq \max\left\{d(x_n, x^*), d(x^*, fx^*), d(x_n, x_{n+1}),\right. \\ &\qquad \left.\frac{d(x^*, x_n) + d(x_n, x_{n+1}) + d(x_n, x^*) + d(x^*, fx^*)}{4}\right\} \\ &= d(x^*, fx^*) \text{ for large } n. \end{aligned} \tag{3.14}$$

Consequently, we have

$$d(x_{n+1}, fx^*) = d(fx_n, fx^*) \leq \alpha(x_n, fx_n)\alpha(x^*, fx^*)d(fx_n, fx^*) < d(x^*, fx^*) \tag{3.15}$$

By (3.13), (3.15) and the condition ($\zeta 3$), we get

$$0 \leq \limsup_{k\to\infty} \zeta(\alpha(x_n, fx_n)\alpha(x^*, fx^*)d(fx_n, fx^*), M(x_n, x^*)) < 0.$$

This is a contradiction. Hence, Therefore x^* is a fixed point of f.

Suppose that x^* and u^* be two fixed point points of f and hence $x^*, u^* \in \mathrm{Fix}(f)$ which is a generalized α-admissible-$\mathcal{Z}$-contraction self-mappings of a metric space (X, d). By (3.1), we have that

$$0 \leq \zeta(\alpha(x^*, fx^*)\alpha(u^*, fu^*)d(fx^*, fu^*), M(x^*, u^*)), \tag{3.16}$$

where

$$M(x^*, u^*) = \max\left\{d(x^*, u^*), d(x^*, fx^*), d(u^*, fu^*), \frac{d(x^*, fu^*) + d(u^*, fx^*)}{4}\right\} = d(x^*, u^*). \tag{3.17}$$

This together with (3.24) shows that

$$0 \leq \zeta(\alpha(x^*, fx^*)\alpha(u^*, fu^*)d(fx^*, fu^*), M(x^*, u^*)) = \zeta(\alpha(x^*, x^*)\alpha(u^*, u^*)d(x^*, u^*), d(x^*, u^*)). \tag{3.18}$$

This is a contradiction. Thus, we have $x^* = u^*$. Hence f has a unique fixed point. ∎

Theorem 3.2. *Let (X, d) be a complete metric space, f is a generalized α-admissible-$\mathcal{Z}$-contraction with respect to ζ. Assume that*

(i) *f is admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
(iii) *X is α regular and for every sequence $\{x_n\}$ in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and we have $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$;*
(iv) *$\alpha(x, y) \geq 1$, for all $x, y \in Fix(f)$.*

Then f has a unique fixed point x^* in X.

Proof. By (ii), let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. There exist $x_n \in X$ such that $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. We have by Theorem 3.1, $\{x_n\}$ is a Cauchy sequence such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Thus $\lim_{m,n\to\infty} d(x_n, x_m)$ exists and is equal to 0. Since (X, d) is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0 \tag{3.19}$$

then

$$0 = \lim_{m,n\to\infty} d(x_m, x_n) = \lim_{n\to\infty} d(x_n, x^*) = d(x^*, x^*). \tag{3.20}$$

Since X is regular, therefore there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ for all $k \in \mathbb{N}$. Therefore

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_{n_k}, fx_{n_k})\alpha(x^*, fx^*)d(fx_{n_k}, fx^*), M(x_{n_k}, x^*)) \\ &= \zeta(\alpha(x_{n_k}, x_{n_k+1})\alpha(x^*, fx^*)d(x_{n_k+1}, fx^*), M(x_{n_k}, x^*)) \\ &< M(x_{n_k}, x^*) - \alpha(x_{n_k}, x_{n_k+1})\alpha(x^*, fx^*)d(x_{n_k+1}, fx^*), \end{aligned} \tag{3.21}$$

where
(3.22)

$$\begin{aligned} M(x_{n_k}, x^*) &= \max\left\{ d(x_{n_k}, x^*), d(x^*, fx^*), d(x_{n_k}, fx_{n_k}), \frac{d(x^*, fx_{n_k}) + d(x_{n_k}, fx^*)}{4} \right\} \\ &\leq \max\left\{ d(x_{n_k}, x^*), d(x^*, fx^*), d(x_{n_k}, x_{n_k+1}), \frac{d(x^*, x_{n_k+1}) + d(x_{n_k}, fx^*)}{4} \right\} \\ &\leq \max\Bigg\{ d(x_{n_k}, x^*), d(x^*, fx^*), d(x_{n_k}, x_{n_k+1}), \\ &\quad \frac{d(x^*, x_{n_k}) + d(x_{n_k}, x_{n_k+1}) + d(x_{n_k}, x^*) + d(x^*, fx^*)}{4} \Bigg\} \\ &= d(x^*, fx^*) \text{ for large } k. \end{aligned}$$

Consequently, we have

$$d(x_{n_k+1}, fx^*) = d(fx_{n_k}, fx^*) \leq \alpha(x_{n_k}, fx_{n_k})\alpha(x^*, fx^*)d(fx_{n_k}, fx^*) < d(x^*, fx^*) \tag{3.23}$$

for all $k \in \mathbb{N}$. By (3.13), (3.23) and the condition ($\zeta 3$), we get

$$0 \leq \limsup_{k\to\infty} \zeta(\alpha(x_n, fx_n)\alpha(x^*, fx^*)d(fx_n, fx^*), M(x_n, x^*)) < 0.$$

This is a contradiction. Hence, Therefore x^* is a fixed point of f. Suppose that x^* and u^* be two fixed point points of f and hence $x^*, u^* \in \text{Fix}(f)$ which is a generalized α-admissible-$\mathcal{Z}$-contraction self-mappings of a metric space (X, d). By (3.1), we have that

$$0 \leq \zeta(\alpha(x^*, fx^*)\alpha(u^*, fu^*)d(fx^*, fu^*), M(x^*, u^*)), \tag{3.24}$$

where

$$M(x^*, u^*) = \max\left\{d(x^*, u^*), d(x^*, fx^*), d(u^*, fu^*), \frac{d(x^*, fu^*) + d(x^*, fx^*)}{4}\right\} = d(x^*, u^*). \tag{3.25}$$

This together with (3.24) shows that

$$0 \leq \zeta(\alpha(x^*, fx^*)\alpha(u^*, fu^*)d(fx^*, fu^*), M(x^*, u^*)) = \zeta(\alpha(x^*, x^*)\alpha(u^*, u^*)d(x^*, u^*), d(x^*, u^*)). \tag{3.26}$$

This is a contradiction. Thus, we have $x^* = u^*$. Hence f has a unique fixed point. ∎

Corollary 3.3. *$f : X \to X$ be a self mapping, there exist $\zeta \in \mathcal{Z}$ and $\alpha : X \times X \to [0, \infty)$ be a function with $\alpha(x, y) = 1$ for all $x, y \in X$ such that*

$$\zeta(d(fx, fy), M(x, y)) \geq 0 \textit{ for all distinct } x, y \in X,$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{4}\right\}$$

Then f has a unique fixed point x^* in X.

4. APPLICATION

In this section, we present an application of Theorem to guarantee the existence and uniqueness problem of solutions for some kind of nonlinear Hammerstein integral equations.

We consider nonlinear Hammerstein integral equation as follows.

$$x(t) = g(t) + \int_0^t K(t, s)h(s, x(s))ds, \tag{4.1}$$

where the unknown function $x(t)$ takes real values.

Let $X = C([0, 1])$ be the space of all real continuous functions defined on $[0, 1]$. It is well known that $C([0, 1])$ endowed with the metric

$$d(x, y) = \|x - y\| = \max_{t \in [0,1]} |x(t) - y(t)| \tag{4.2}$$

is a complete metric space. Define a mapping $G : X \to X$ by

$$G(x)(t) = g(t) + \int_0^t K(t, s)h(s, x(s))ds, \quad \text{for all } \ t \in (0, 1). \tag{4.3}$$

Assumption 4.1

(1) $g \in C([0, 1] \times (-\infty, +\infty))$, $g \in X$ and $K \in C([0, 1]) \times ([0, 1])$ such that $K(t, s) \geq 0$;

(2) $h(t, \cdot) : (-\infty, +\infty) \to (-\infty, +\infty)$ is increasing for all $t \in (0, 1)$ such that

$$|h(t, x) - h(t, y)| < M(x, y) \ \text{ for all distinct } \ x, y \in X, t \in (0, 1),$$

where $M(x, y) = \max\left\{|x - y|, |x - Gx|, |y - Gy|, \dfrac{|x - Gy| + |y - Gx|}{4}\right\}$;

(3) $\max_{t,s \in [0,1]} |K(t, s)| \leq 1$.

Theorem 4.1. *Let $X = C([0,1]), (X,d), G, h, K(t,s)$ are satisfied in Assumption 4.1, then the nonlinear Hammerstein integral equation* (4.1) *has a unique solution $x^* \in C([0,1])$ and for each $x \in C([0,1])$ the iterative sequence $\{x_n = G^n x\}$ converges to the unique solution $x^* \in X$ of equation* (4.1)*.*

Proof. First, we show that the mapping $G : X \to X$ define by (4.2) is a Suzuki type $\mathcal{Z}$-contraction. From condition (2) and (3), for all distinct $x, y \in C([0,1])$, $t \in (0,1)$, we have

$$
\begin{aligned}
&|Gx(t) - Gy(t)| \\
&= \left|\int_0^t K(t,s)(h(s,x(s)) - h(s,y(s)))ds\right| \\
&\leq \int_0^t |K(t,s)||h(s,x(s)) - h(s,y(s))|ds \\
&\leq \int_0^t |h(s,x(s)) - h(s,y(s))|ds \\
&< \int_0^t \mathcal{M}(x(s),y(s))ds \\
&= \int_0^t \max\left\{|x(s)-y(s)|, |x(s)-Gx(s)|, |y(s)-Gy(s)|, \frac{|x(s)-Gy(s)| + |y(s)-Gx(s)|}{4}\right\}ds \\
&\leq \int_0^t \max\left\{d(x,y), d(x,Gx), d(y,Gy), \frac{d(x,Gy)+d(y,Gx)}{4}\right\}ds \\
&= M(x,y)\int_0^t ds \\
&= tM(x,y) \\
&\leq M(x,y).
\end{aligned}
$$

Hence, the mapping G is a generalized α-admissible-$\mathcal{Z}$-contraction with $\alpha(x,Gx)\alpha(y,Gy) = 1$ and hence Theorem 3.1 applies to G, which has a unique fixed point $x^* \in X$, $i.e., x^*$ is the unique solution of the nonlinear Hammerstein integral equations (4.1). For each $x \in X$, the sequence $\{x_n = G^n x\}$ converges to x^*. ∎

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